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The retrosection theorem asserts that every closed Riemann surface of genus $g \geq 1$ can be uniformized by a Schottky group of rank g. Here we define and topologically classify Klein–Schottky groups; these are the freely acting extended Kleinian groups whose orientation-preserving subgroup is a Schottky group. These groups yield uniformizations of all nonorientable closed Klein surfaces.

1. Klein–Schottky Groups

One can define a Schottky group of genus g to be a geometrically finite Kleinian group isomorphic to a free group of rank g and without parabolic transformations. It is known that every Schottky group G is a Kleinian group of the second kind [Maskit 1967], with connected region of discontinuity $\Omega(G) \neq \varnothing$, where $\Omega(G)/G$ is a closed Riemann surface of genus g. As a consequence of the retrosection theorem [Bers 1975; Koebe 1910], every closed Riemann surface of genus $g \geq 1$ can be uniformized by a Schottky group of rank g. An *extended Kleinian group* is a group of hyperbolic isometries of $\mathbb{H}^3$, necessarily including orientation-reversing ones, containing a Kleinian group as a subgroup of index 2. We say that an extended Kleinian group is *freely acting* if the stabilizer of any point of its region of discontinuity (in the Riemann sphere) is trivial. A freely acting extended Kleinian group may have elements that fix points in the interior of hyperbolic space.

A *Klein–Schottky group* is a freely acting extended Kleinian group whose orientation-preserving subgroup is a Schottky group. If the orientation-preserving half of the Klein–Schottky group G has rank g, we say that G is a Klein–Schottky group of rank g.

If G is a Klein–Schottky group of rank g, then $\Omega(G)/G$ is a closed Klein surface S of (algebraic) genus g. The orientation cover S^+ of S can of course be realized as $S^+ = \Omega(G)/G^+$, where G^+ is the orientation-preserving half of G.

We mention here, and will use without further mention, that the topological type of a closed Klein surface is completely determined by its genus. That is, if S_1^+ and

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S_2^+ are closed Riemann surfaces of the same genus g and there is for $i = 1, 2$ an orientation-reversing involution τ_i acting without fixed points on S_i^+, then there is a orientation-preserving homeomorphism $\alpha : S_1^+ \to S_2^+$ conjugating τ_1 to τ_2.

A *glide reflection* is an orientation-reversing isometry (of $\mathbb{H}^3$) with exactly two fixed points, necessarily on the sphere at infinity; every glide reflection is conjugate, in the full group of isometries of $\mathbb{H}^3$, to one of the form $z \to \lambda\bar{z}$, where λ is real and $\lambda > 1$. An *imaginary reflection* is an orientation-reversing isometry of order 2 with exactly one fixed point, which is required to be in the interior of hyperbolic space; every imaginary reflection is conjugate to the transformation $z \to -1/\bar{z}$.

If G is a Klein–Schottky group and $\tau \in G - G^+$, then $\tau^2 \in G^+$. Since Schottky groups only contain loxodromic transformations, with the exception of the identity, τ cannot be either a square root of a parabolic transformation or the square root of an elliptic transformation of order at least two. Also, if τ has order 2, it cannot be a reflection, for every reflection has a circle of fixed points, and we have required that G act freely on $\Omega(G) = \Omega(G^+)$, which for a Schottky group is known to be connected and everywhere dense in the sphere. It follows that G may only contain glide reflections and imaginary reflections as orientation-reversing elements.

Deformations. To the best of our knowledge, there is no ready reference for the theory of quasiconformal deformations of extended Kleinian groups. This theory is not significantly different from the usual theory of quasiconformal deformations of Kleinian groups; we present a brief outline here.

Let G be an extended Kleinian group, and let G^+ be its orientation-preserving half. A *Beltrami differential* for G is an L^∞ function $\mu(z)$ on the extended complex plane satisfying $\|\mu\|_\infty < 1$ and

$$(*) \qquad \begin{aligned} \mu \circ \gamma(z)\frac{\overline{\gamma'(z)}}{\gamma'(z)} &= \mu(z) \quad \text{for } \gamma \in G^+, \\ \mu \circ \gamma(z)\frac{\overline{\gamma'(z)}}{\gamma'(z)} &= \bar{\mu}(z) \quad \text{for } \gamma \in G - G^+, \end{aligned}$$

where on the second line γ' is the derivative of γ with respect to $\bar{z}$.

As in the usual theory, if μ is a Beltrami differential for G, there are solutions $w_{\bar{z}} = \mu(z)w_z$ to the Beltrami equation that are quasiconformal homeomorphisms of the extended complex plane; further, if w_1 and w_2 are two such solutions, there is a Möbius transformation A such that $w_2 = A \circ w_1$. It follows that there is a unique such solution, called w^μ, once one specifies its values at three distinct (limit) points of G; these three limit points are usually taken to be 0, 1 and ∞.

It is classical that if $\gamma \in G^+$, then $w^\mu \circ \gamma \circ (w^\mu)^{-1}$ is again a Möbius transformation. One sees this by observing that $w^\mu \circ \gamma$ satisfies the same Beltrami equation as does w^μ. That is, for $\gamma \in G^+$, we compute

$$(w^\mu \circ \gamma)_z = (w^\mu_z \circ \gamma)\gamma_z + (w^\mu_{\bar z} \circ \gamma)\bar\gamma_z = (w^\mu_z \circ \gamma)\gamma',$$
$$(w^\mu \circ \gamma)_{\bar z} = (w^\mu_z \circ \gamma)\gamma_{\bar z} + (w^\mu_{\bar z} \circ \gamma)\bar\gamma_{\bar z} = (w^\mu_{\bar z} \circ \gamma)\overline{\gamma'}.$$

We conclude that

$$\frac{(w^\mu \circ \gamma)_{\bar z}}{(w^\mu \circ \gamma)_z} = \frac{(w^\mu_{\bar z} \circ \gamma)\overline{\gamma'}}{(w^\mu_z \circ \gamma)\gamma'} = \frac{(\mu \circ \gamma)\overline{\gamma'}}{\gamma'} = \mu.$$

Then, since $w^\mu \circ \gamma$ and w^μ satisfy the same Beltrami equation, there is a Möbius transformation γ^μ such that $w^\mu \circ \gamma = \gamma^\mu \circ w^\mu$.

Similar computations for γ an orientation-reversing element of G yield

$$(\bar w^\mu \circ \gamma)_z = (\bar w^\mu_z \circ \gamma)\gamma_z + (\bar w^\mu_{\bar z} \circ \gamma)\bar\gamma_z = (\bar w^\mu_{\bar z} \circ \gamma)\bar\gamma_z,$$
$$(\bar w^\mu \circ \gamma)_{\bar z} = (\bar w^\mu_z \circ \gamma)\gamma_{\bar z} + (\bar w^\mu_{\bar z} \circ \gamma)\bar\gamma_{\bar z} = (\bar w^\mu_z \circ \gamma)\gamma_{\bar z}.$$

We want to conclude that $\bar w^\mu \circ \gamma$ satisfies the same Beltrami equation as does w^μ, from which we will conclude that there is an orientation-reversing Möbius transformation γ^μ so that $w^\mu \circ \gamma = \gamma^\mu \circ w^\mu$. Hence we compute as follows:

$$\frac{(\bar w^\mu \circ \gamma)_{\bar z}}{(\bar w^\mu \circ \gamma)_z} = \frac{(\bar w^\mu_z \circ \gamma)\gamma_{\bar z}}{(\bar w^\mu_{\bar z} \circ \gamma)\bar\gamma_z} = \frac{(\overline{w^\mu_{\bar z} \circ \gamma})\gamma_{\bar z}}{(\overline{w^\mu_z \circ \gamma})\bar\gamma_z} = \frac{(\overline{\mu \circ \gamma})\gamma_{\bar z}}{\bar\gamma_z} = \mu.$$

Let G be an extended Kleinian group, with an invariant component Δ of its set of discontinuity, where G acts freely on Δ. Assume that $S = \Delta/G$ is a closed Klein surface and let S' be a topologically equivalent Klein surface; i.e., S and S' have the same genus. As is well known, we can replace the topological homeomorphism with a diffeomorphism. Lift this diffeomorphism to obtain an orientation-preserving diffeomorphism, say $\alpha : S^+ \to (S')^+$, from the orientation cover of S to the orientation cover of S', that conjugates the orientation-reversing diffeomorphism of S^+ onto that of $(S')^+$. Since S^+ is closed, α is quasiconformal. A lift of α to Δ defines a Beltrami differential μ satisfying equations $(*)$. Hence the solution w^μ of the Beltrami equation conjugates G into an extended Kleinian group G', with an invariant component, Δ', where $\Delta'/G' = S'$, and $w^\mu : \Delta \to \Delta'$ covers α.

It follows that the theory of quasiconformal deformation spaces of extended Kleinian groups is essentially the same as the corresponding theory of deformation spaces of Kleinian groups, except of course that these spaces are real analytic, rather than complex analytic. In particular, as in the complex case, if μ satisfies $(*)$ for all $g \in G$, then so does $t\mu$, for t real. It follows that the deformation space of G is connected; see Section 2 below.

From here on, we will use without further mention the basic fact that if there is a topological equivalence between the nonorientable surfaces, S and S', and there is an extended Kleinian group G uniformizing S, then there is a quasiconformal deformation G' of G uniformizing S'.

Construction. One can easily construct several families of Klein–Schottky groups as combination theorem free products of cyclic groups generated by glide reflections and imaginary reflections. One could also use cyclic groups generated by loxodromic (including hyperbolic) transformations, but these turn out to be redundant; see the sections on decomposition and reduction to normal form starting on page 318.

Let m and n be nonnegative integers, with $m+n>0$. Let D be a region in the extended complex plane bounded by $m+2n$ disjoint circles. Label these circles as $B_1,\dots,B_m$, $C_1, C'_1,\dots,C_n, C'_n$. For $i=1,\dots,m$ and $j=1,\dots,n$, let α_i be an imaginary reflection mapping the inside of B_i to its outside and let β_j be a glide reflection mapping the inside of C_j to the outside of C'_j. It is easy to see that $G=\langle\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n\rangle$ is a combination theorem free product of the cyclic groups generated by these generators.

If $m>0$, we can obtain the orientation-preserving half of G by looking at the region R^+ bounded by the $2(m-1)+4n$ circles

$$B_1,\dots,B_{m-1},C_1,\dots,C'_n,\alpha_m(B_1),\dots,\alpha_m(B_{m-1}),\alpha_m(C_1),\dots,\alpha_m(C'_n)$$

and observing that for $j=1,\dots,m-1$ we have $\alpha_m\circ\alpha_j(B_j)=\alpha_m(B_j)$, with $\alpha_m\circ\alpha_j(R^+)\cap R^+=\varnothing$; for $i=1,\dots,n$, $\alpha_m\circ\beta_i(C_i)=\alpha_m(C'_i)$, with $\alpha_m\circ\beta_i(R^+)\cap R^+=\varnothing$; and for $i=1,\dots,n$, $\alpha_m\circ\beta_i^{-1}(C'_i)=\alpha_m(C_i)$, with $\alpha_m\circ\beta_i^{-1}(R^+)\cap R^+=\varnothing$. We conclude that G^+ is a classical Schottky group of rank $m+2n-1$, with free generators $\alpha_m\circ\alpha_1,\dots,\alpha_m\circ\alpha_{m-1},\alpha_m\circ\beta_1,\dots,\alpha_m\circ\beta_n,\alpha_m\circ\beta_1^{-1},\dots,\alpha_m\circ\beta_n^{-1}$.

Similarly, if $m=0$, then $n>0$, so we can look at the region R^+ bounded by the $4n-2$ circles

$$C_1,C'_1,\dots,C_{n-1},C'_{n-1},\beta_n(C_1),\beta_n(C'_1),\dots,b_n(C_{n-1}),\beta_n(C'_{n-1}),C_n,\beta_n(C'_n).$$

In this case, we obtain $\beta_1,\dots,\beta_{n-1},\beta_n\circ\beta_1\circ\beta_n^{-1},\dots,\beta_n\circ\beta_{n-1}\circ\beta_n^{-1},\beta_n^2$ as $2n-1$ generators for G^+.

In any case, we have shown that G^+ is a classical Schottky group, having rank $m+2n-1$. It follows that G is a Klein–Schottky group of rank $m+2n-1$. The quotient space $S=\Omega(G)/G$ is a closed Klein surface of topological genus $m+2n$. We call this group, G, and any quasiconformal deformation of G, an (m,n)-Klein–Schottky group.

Uniqueness. It follows from the combination theorem that, if G is an (m,n)-Klein–Schottky group, then, as an abstract group, $G=A_1*\cdots*A_m*B_1*\cdots*B_n$, where each A_i has order 2 and each B_j is infinite cyclic. It follows that G has exactly m conjugacy classes of nontrivial finite subgroups.

Proposition 1.1. *Let G be an (m, n)-Klein–Schottky group, and let G' be an (m', n')-Klein–Schottky group. Then G and G' represent topologically equivalent Klein surfaces if and only if $m + 2n = m' + 2n'$.*

Proof. We remarked above that G^+ is a Schottky group of rank $m + 2n - 1$. It follows that $\Omega(G^+)/G^+$ and $\Omega((G')^+)/(G')^+$ are topologically equivalent if and only if $m+2n = m'+2n'$, from which it follows that $\Omega(G)/G$ and $\Omega(G')/G'$ are topologically equivalent if and only if $m + 2n = m' + 2n'$. □

Proposition 1.2. *Let G be an (m, n)-Klein–Schottky group, and let G' be an (m', n')-Klein–Schottky group. Then G and G' are topologically equivalent uniformizations of some topological Klein surface if and only if $(m, n) = (m', n')$.*

Proof. If $(m, n) = (m', n')$, then, since G and G' are both combination theorem free products of m cyclic groups generated by imaginary reflections, and n cyclic groups generated by glide reflections, it follows from the free product combination theorem that G and G' are topologically equivalent.

Conversely, if G and G' are topologically equivalent, then, by Proposition 1.1, $m+2n = m'+2n'$. Also, since they are topologically equivalent, they are isomorphic, so they have the same number of nonconjugate cyclic subgroups of order 2; it follows that m=m', and then $n = n'$. □

It is well known that we can replace the topological equivalence in the above with quasiconformal equivalence, and so obtain the following.

Proposition 1.3. *Let G be an (m, n)-Klein–Schottky group, and let G' be an (m', n')-Klein–Schottky group. Then G and G' are quasiconformally equivalent Klein–Schottky groups if and only if $(m, n) = (m', n')$.*

Characterization. It is well known that Schottky groups can be characterized as follows. Let S^+ be a fixed closed Riemann surface of genus $g \geq 1$, and let $\sigma : \tilde{S}^+ \to S^+$ be a regular covering of S^+, where $\tilde{S}^+$ is planar. Then the group of deck transformations for this covering is a Schottky group if and only if this covering is a lowest planar regular covering of S^+ (see [Maskit 1988, p. 317]). The next two propositions say that the Klein–Schottky groups are the lowest regular planar coverings of closed Klein surfaces.

Proposition 1.4. *There is no planar regular cover of a Klein surface that is lower than the covering defined by a Klein–Schottky group.*

Proof. Let G be a Klein–Schottky group covering the closed Klein surface S, and let $\sigma : \Omega(G) \to S = \Omega(G)/G$ be the natural cover map. Assume that there is a regular covering, $\tau : \tilde{S} \to S$, where $\tilde{S}$ is planar, and that there is a cover map $\rho : \Omega(G) \to \tilde{S}$, so that $\sigma = \tau\rho$. We need to show that ρ is a homeomorphism.

If necessary, we replace ρ by $\gamma\rho$, for an appropriately chosen γ, so that ρ is orientation-preserving.

We start with the observation that the cover map from any orientable covering surface to S necessarily factors through the orientation cover S^+ of S. It follows that σ and τ both factor through S^+. Then we can write $\sigma = \pi\sigma_1$ and $\tau = \pi\tau_1$, where $\pi : S^+ \to S$ is the orientation cover, and σ_1 and τ_1 are chosen so as to preserve orientation.

It follows from $\pi\sigma_1 = \pi\tau_1\rho$ that either $\sigma_1 = \tau_1\rho$ or $\omega\sigma_1 = \tau_1\rho$, where ω is an orientation-reversing deck transformation in the cover group of $\pi : S^+ \to S$. Since σ_1, τ_1 and ρ all preserve orientation, and ω does not, it must be the case that $\sigma_1 = \tau_1\rho$. Since the Schottky cover is the lowest planar regular covering of S^+, ρ is a homeomorphism. □

Proposition 1.5. *Let $p : \tilde{S} \to S$ be a planar regular cover of S with the property that there is no lower planar regular covering. Then the deck group of this cover can be realized as a Klein–Schottky group.*

Proof. As above, every planar regular cover of S factors through S^+, from which it follows that $\tilde{S}$, as a cover of S^+, is a lowest planar regular cover; hence it is topologically equivalent to a Schottky cover. It is well known that every planar regular cover of a Riemann surface can be realized by a Kleinian group; hence there is a Schottky group G^+, so that the natural projection from $\Omega(G^+)$ to S^+ is conformally equivalent to the given cover from $\tilde{S}$ to S^+.

In fact, as soon as we conformally map $\tilde{S}$ onto a subset of the Riemann sphere, the group of deck transformations is automatically conjugated into a Schottky group. It is known that if G^+ is a Schottky group, then $\Omega(G^+)$ is of class O_{AD}; that is, it admits no holomorphic function with finite Dirichlet norm (see [Ahlfors and Sario 1960, p. 241]). It follows from this (see p. 200 of the same reference) that every conformal map from $\Omega(G^+)$ into the extended complex plane is fractional linear. Then the complex conjugate of such a conformal map is necessarily an orientation-reversing Möbius transformation; in particular, every element of $G - G^+$ is an orientation-reversing Möbius transformation. Thus, after this conjugation, the deck group of the original cover of S by $\tilde{S}$ is a Klein–Schottky group. □

Decomposition. Throughout this section, G is a given Klein–Schottky group, and G^+ is its orientation-preserving half, so that G^+ is a Schottky group of some rank g. Write $S = \Omega(G)/G$ and $S^+ = \Omega(G)/G^+$, so that S^+ is the orientation cover of S. Also, write the cover maps as $\sigma : \Omega(G) \to S$, $\sigma^+ : \Omega(G) \to S^+$, and $\pi : S^+ \to S$.

Since S is nonorientable, the loops on S fall into two classes; the orientation-preserving loops, which lift to loops on S^+, and the orientation-reversing ones, which lift to open arcs on S^+. Of course, the square of an orientation-reversing loop is orientation-preserving and lifts to a loop on S^+.

Since $\Omega(G)$ is not simply connected, the planarity theorem [Maskit 1965] says we can find a set of simple disjoint homotopically distinct loops $W_1, \ldots, W_s$ on S, and positive integers $a_1, \ldots, a_s$, where a_i is even whenever W_i is orientation-reversing, so that each $W_i^{a_i}$ lifts to a loop on $\Omega(G)$ and so that

$$\rho : \Omega(G) \to S$$

is the highest regular covering of S for which these loops all lift to loops.

It is clear that we can assume that each a_i is the minimal power with the property that $W_i^{a_i}$ lifts to a loop.

We first observe that if W_i is orientation-preserving, then $a_i = 1$. For, if not, the planar regular covering of S, whose defining subgroup is the smallest normal subgroup containing W_i to the first power, and all the other W_j to the power a_j, would be strictly lower than the cover $\sigma : \Omega(G) \to S$, which cannot be. Of course, if $a_i = 1$, then W_i is necessarily orientation-preserving. Similarly, if W_i is orientation-reversing, then $a_i = 2$, since a_i is necessarily even and G contains no elements of finite order greater than 2.

Since the cover map σ factors through S^+, each of the loops $W_i^{a_i}$ lifts to a loop on S^+. We can assume that this set of loops $W_1, \ldots, W_s$ is minimal on S, from which it follows that the set of lifts is minimal on S^+.

We next use the standard decomposition technique [Maskit 1973; Abikoff and Maskit 1977] to decompose G into a combination theorem free product. We start with W_1, and let $\mathcal{W}_1$ be the set of all lifts of W_1 to $\Omega(G)$. Since the W_i are all simple and disjoint, $\mathcal{W}_1$ consists of a set of simple disjoint loops. The loops of $\mathcal{W}_1$ divide $\Omega(G)$ into (open) regions; let R_1 be one of these, and let G_1 be its stabilizer. Since each simple loop of $\mathcal{W}_1$ is necessarily homotopically nontrivial in $\Omega(G)$, $G_1 \neq G$. Hence there is some element $\alpha_1 \in G - G_1$ mapping some boundary component of R_1 onto a (perhaps different) boundary component of R_1. If α_1 maps a boundary component of R_1 onto itself, label this boundary component as B_1; otherwise, label it C_1.

In the former case, we have $\alpha_1^2 \in G_1$, and α_1^2 preserves B_1; this can occur only if α_1 is an imaginary reflection. Using the third combination theorem [Maskit 1993], we see that G is the free product of G_1 and the group of order 2, $\langle \alpha_1 \rangle$.

In the second case, where C_1 and $C_1' = \alpha_1(C_1) \neq C_1$, we use the second combination theorem [Maskit 1993] to observe that G is a combination theorem HNN extension of G_1, and, since the stabilizer of C_1 in G_1 is necessarily trivial, that G is the free product of G_1 and the infinite cyclic group, $\langle \alpha_1 \rangle$.

Next consider $S_1 = \Omega(G_1)/G_1$, with the loops $W_2, \ldots, W_s$ marked on it; these are the projections to S_1 of the liftings of these loops from S to $\Omega(G) \subset \Omega(G_1)$.

It is clear that the loops $W_2^{a_2}, \dots, W_s^{a_s}$ lift to loops in $\Omega(G_1)$, and that every loop on S_1 that lifts to a loop is freely homotopic to some product of conjugates of these loops. Hence we can iterate the above procedure. We have shown:

Theorem 1.6. *Let G be a Klein–Schottky group. Then G is a free product, in the sense of combination theorems, of a finite number of cyclic groups, where each of these cyclic groups is generated by either a loxodromic transformation, a glide reflection or an imaginary reflection.*

Reduction to normal form. Let G be a Klein–Schottky group defined as a free product of m imaginary reflections, n_1 glide reflections and n_2 loxodromic transformations. That is, there is a fundamental domain R for G, bounded by $m+2n_1+2n_2$ disjoint simple loops,

$$B_1, \dots, B_m, C_1, C_1', \dots, C_{n_1}, C_{n_1}', D_1, D_1', \dots, D_{n_2}, D_{n_2}',$$

and there are generators, $\beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_{n_1}, \delta_1, \dots, \delta_{n_2}$, where β_i, for $i = 1 \dots, m$, is an imaginary reflection mapping B_i onto itself; γ_i, for $i = 1, \dots, m_1$, is a glide reflection mapping C_i onto C_i', with $\gamma_i(R) \cap R = \varnothing$; and δ_i, for $i = 1, \dots, n_2$, is a loxodromic transformation mapping D_i onto D_i', with $\delta_i(R) \cap R = \varnothing$. Our goal is to find a perhaps different set of loops, and perhaps different set of generators for the same group G, but with the same properties, so that for this new set of generators, $n_2 = 0$.

Since G contains at least one orientation-reversing element, either $m \neq 0$ or $n_1 \neq 0$.

If $n_1 \neq 0$, then it is a now standard procedure (see [Chuckrow 1968]) to find new loops, so that we can replace the generators γ_1 and δ_1 by the generators γ_1 and $\gamma_1\delta_1$. Chuckrow's procedure is to draw a simple path v inside R from the base point on C_1 to the base point on D_1', and then replace C_1 by a simple loop, $\tilde{C}_1$, that is freely homotopic to $C_1 \cdot v \cdot D_1' \cdot v^{-1}$ (if necessary, replace D_1 and D_1' by their inverses, so that there is such a simple loop). Then note that $\tilde{C}_1$, $\gamma_1(\tilde{C}_1)$, D_1 and $\gamma_1 \circ \delta_1(D_1)$ are four disjoint simple loops bounding a common region. This procedure works equally well for orientation-reversing transformations as for orientation-preserving ones.

After this move, we have increased n_1 by one and decreased n_2 by one. After a finite number of such steps, we will have reached the point where $n_2 = 0$.

In the other case, where we start with $n_1 = 0$ and $n_2 \neq 0$, we must have $m \neq 0$ since G is extended Kleinian. Draw a simple path v, which except for its endpoints lies entirely in the fundamental domain R, from the base point x on B_1 to the base point y on D_1. It is easy to find a simple β_1-invariant loop, call it B_1', that separates D_1 and $\beta_1(D_1')$ from D_1' and $\beta_1(D_1)$. Roughly speaking, B_1' can be found by following v from x to y; then following D_1 (or its inverse) back to y; then v^{-1}

back to x; then B_1 (or B_1^{-1}) to $\beta_1(x)$; then $\beta_1(v)$ to $\beta_1(D_1)$; then $\beta_1(D_1)$ (or its inverse) back to $\beta_1(x)$; then the inverse of $\beta_1(v)$ back to $\beta_1(x) \in B_1$; then continue the circuit of B_1 back to x.

We now have a new fundamental region R', bounded by B_1', D_1 and $\beta_1(D_1')$, and new generators, β_1 and $\beta_1\delta_1$, which both reverse orientation. Hence we have decreased the number of orientation-preserving generators, and increased the number of orientation-reversing generators.

Theorem 1.7. *Let G be a Klein–Schottky group. Then there are unique integers, m and n, so that G is an (m, n)-Klein–Schottky group.*

Proof. The procedure above shows that any Klein–Schottky group G can be written as an (m, n)-Klein–Schottky group. We saw in Proposition 1.2 that the integers m and n are uniquely determined by the group. □

2. Spaces of Klein–Schottky Groups

For each genus $g \geq 0$, the usual space of Schottky groups, $\mathcal{S}_g$, is the space of all sets of g generators of Schottky groups of genus g, modulo conjugation by the Möbius group. This is the same as starting with a particular set of free generators for a Schottky group, G_0, and considering its quasiconformal deformations up to conjugation by Möbius transformations. It is well known that, complex analytically, $\mathcal{S}_g$ is a point for $g = 0$, the punctured unit disc for $g = 1$, and a domain of holomorphy of complex dimension $3g - 3$, which can be realized as a subdomain of $\mathbb{C}^{3g-3}$, for $g \geq 2$; see [Hejhal 1975; Kra and Maskit 1981].

If we fix a pair of nonnegative integers, (m, n), so that $g = m + 2n - 1$, we then define $\mathcal{KS}_g^+(m, n)$ to be the locus of those Schottky groups that can be extended to a Klein–Schottky group of genus g and type (m, n). We call it the *extended Klein–Schottky space of genus g and type* (m, n). We can also view this space as follows. For each pair (m, n), we choose a particular set of generators for a particular (m, n)-Klein–Schottky group G_0, and we choose generators for G_0^+, the orientation-preserving half of G_0, as on page 316. Then $\mathcal{KS}_g^+(m, n)$ is the space of quasiconformal deformations of G_0, modulo conjugation by an orientation-preserving Möbius transformation. Each such deformation is of course also a deformation of G_0^+, and so can be regarded as a point in $\mathcal{S}_g$.

It is well known that for $g \geq 2$ this yields $\mathcal{KS}_g^+(m, n)$ as a submanifold of real dimension $3g - 3$. We have already remarked on page 315 that each $\mathcal{KS}_g^+(m, n)$ is connected. We also remark that it is easy to see that, for every $g \geq 1$, we can write g in the form $g = m + 2n - 1$ in exactly $r(g) = [(g + 3)/2]$ distinct ways, where $[k]$ is the greatest integer less than or equal to k. This shows that there are exactly $r(g)$ spaces of Klein–Schottky groups in the space of Schottky groups of genus g.

We should also define $\mathcal{KS}_g(m,n)$, the space of Klein–Schottky groups of genus g and type (m,n). A point in $\mathcal{KS}_g(m,n)$ is, as above, a quasiconformal deformation of G_0, but here two such deformations are equivalent if one is a conjugate of the other by a perhaps orientation-reversing Möbius transformation. One easily sees that, for $g \geq 2$, $\mathcal{KS}_g^+(m,n)$ is a branched two-sheeted covering of $\mathcal{KS}_g(m,n)$.

In this section, we discuss the intersections of the $r(g)$ spaces of extended Klein–Schottky groups within the Schottky space of genus g. We also discuss the topology of these spaces for low genus.

In general, two extended Klein–Schottky spaces of genus g will intersect if and only if there is a Schottky group of genus g that has two distinct extensions to Klein–Schottky groups, where these extensions are of the two different types.

The cases $\mathbf{g=0}$ and $\mathbf{g=1}$. For $g=0$, the only possibility is that $(m,n)=(1,0)$, and $\mathcal{KS}_g^+(1,0)$, is a single point.

For $g=1$, it is easy to observe that the two spaces, $\mathcal{KS}_1^+(2,0)$ and $\mathcal{KS}_1^+(0,1)$, coincide. If $G \in \mathcal{KS}_1(2,0)$, then the product of the two generators, which are both imaginary reflections, is hyperbolic; further, every hyperbolic transformation can be written as a product of two imaginary reflections. In the case that $(m,n)=(0,1)$, the square of the generator is hyperbolic; further, every hyperbolic transformation has a unique orientation-reversing square root. We have shown that every hyperbolic transformation has an extension to both a group in $\mathcal{KS}_1^+(2,0)$ and a group in $\mathcal{KS}_1^+(0,1)$. We conclude that we can view both of these spaces of Klein–Schottky groups as being the space of hyperbolic Möbius transformations modulo conjugation, which we can identify with, for example, the open unit interval. We could also more directly identify $\mathcal{KS}_1^+(2,0)$ with the positive reals given as the distance in $\mathbb{H}^3$ between the fixed points of the (imaginary reflection) generators, and we could identify $\mathcal{KS}_1^+(0,1)$ with the multiplier of the (glide reflection) generator, which we could take as lying in the open unit interval. We also note that every hyperbolic transformation is conjugate to one of the form $z \mapsto \lambda z$, with λ real and positive, from which we conclude that $\mathcal{KS}_1^+(2,0) = \mathcal{KS}_1(2,0)$, and that $\mathcal{KS}_1^+(0,1) = \mathcal{KS}_1(0,1)$.

We can view the universal covering of these spaces — that is, the space of Klein bottles, marked by generators for the fundamental group — as the space of complex numbers τ that are pure imaginary and positive. Then the universal covering group is generated by the transformations $z \mapsto \bar{z} + 1/2$ and $z \mapsto z+\tau$. The first exponential map, $w_1 = e^{2\pi i z}$, transforms this universal covering group to the group in $\mathcal{KS}_1(2,0)$, where the product of the generators is the transformation $w_1 \mapsto e^{2\pi i \tau} w_1$. The second exponential map, $w_2 = e^{2\pi i z/\tau}$, transforms this universal covering group to the group in $\mathcal{KS}_1(0,1)$, where the square of the generator is the transformation $w_2 \mapsto e^{2\pi i/\tau} w_2$.

The case $g = 2$. In genus 2 there are exactly two distinct Klein–Schottky spaces inside the Schottky space $\mathcal{S}_2$; they are $\mathcal{KS}_2^+(1,1)$ and $\mathcal{KS}_2^+(3,0)$.

Theorem 2.1. $\mathcal{KS}_2^+(1,1) \cap \mathcal{KS}_2^+(3,0) = \varnothing$.

Proof. Assume to the contrary that both Klein–Schottky subspaces of the space of Schottky groups of genus 2 intersect. It follows that there is a Schottky group J of genus 2, a Klein–Schottky group $G_1 = \langle \beta, \gamma \rangle$ of genus 2 and type $(1,1)$, where β is an imaginary reflection and γ is a glide reflection, and there is a Klein–Schottky group $G_2 = \langle \beta_1, \beta_2, \beta_3 \rangle$ of genus 2 and type $(3,0)$, where β_i is in imaginary reflection, $i = 1,2,3$, so that $G_1^+ = J = G_2^+$. In particular, all three of these groups are nonelementary and have the same limit set.

We remark that if η is an imaginary reflection and $p \in \mathbb{H}^3$ is its unique fixed point, then any hyperbolic line $L \subset \mathbb{H}^3$ containing p will be invariant, but not pointwise fixed, under η; that is, η will interchange the two endpoints of L, while fixing p. It follows that if the fixed points of the generators β_1, β_2 and β_3 all lie on the same hyperbolic line $L \subset \mathbb{H}^3$, then the pair of end points of L will be invariant under G_2, but not necessarily pointwise fixed, and hence, since the index of G_2 in J is 2, this pair of endpoints of L will also be invariant under J. Since J is nonelementary, this is impossible. We conclude that the fixed points of these three generators of G_2 are distinct and not collinear in $\mathbb{H}^3$; in particular, their three fixed points span a hyperbolic plane $P \subset \mathbb{H}^3$. Since each β_i is conjugate to the transformation (in the 3-ball), $x \mapsto -x$, $\beta_i(P) = P$, and, in its action on P, β_i is a half-turn; that is, an elliptic transformation of order 2. It follows that $J = G_2^+$, is a Fuchsian group acting on P, and preserving both sides of P. Since J is discrete and free on two hyperbolic generators, it is of the second kind, representing either a sphere with three holes, or a torus with one hole. Since G_2 is generated by three half-turns whose fixed points are noncollinear, $J = G_2^+$ represents a torus with one hole.

Next, since G_1 is a $\mathbb{Z}_2$-extension of the nonelementary Fuchsian group J, it also keeps the plane P invariant. It follows that β and γ both preserve P. In general, a glide reflection preserves exactly two planes; for one of these planes, it interchanges the two sides, and for the other, it preserves both sides. Of course, β interchanges the two sides of P, and every element of J preserves both sides of P. Since $\beta\gamma \in J$, and β interchanges the two sides of P, so does γ. Hence γ, which is orientation- reversing, preserves orientation on P; that is, it acts as a hyperbolic transformation. Since G_1 is the free product of the subgroups generated by β and γ, the fixed point of β does not lie on the axis of γ.

The natural generators for $J = G_1^+$ are $\beta\gamma$ and $\beta\gamma^{-1}$. Since J represents a torus with a hole, the axes of these two generators must cross. However, we can easily construct these axes as follows. Draw the line R through the fixed point of β orthogonal to the axis of γ; denote reflection in R by ρ. Draw the line R_1 orthogonal to R and passing through the fixed point of β; denote reflection in R_1 by ρ_1. Then $\beta = \rho\rho_1 = \rho_1\rho$. Draw the lines R_2 and R_3, and denote reflections in these lines by ρ_2 and ρ_3, respectively, so that $\gamma = \rho\rho_2 = \rho_3\rho$. Since J is purely hyperbolic, R_1 is disjoint from both R_2 and R_3 including at the circle at infinity. Now the axis of $\beta\gamma$ is the common perpendicular of R_2 and R_1, and the axis of $\beta\gamma^{-1}$ is the common perpendicular of R_3 and R_1. Since these two axes are both orthogonal to R_1, they do not cross. We have reached the desired contradiction. $\square$

We have in fact shown that we can identify the space $\mathcal{KS}_2^+(3, 0)$ with the space of Fuchsian groups representing a torus with a hole, as follows. Let $(\beta_1, \beta_2, \beta_3)$ be imaginary reflections generating a $(3, 0)$-Klein–Schottky group, G. Then we saw above that there is a G-invariant plane P so that P/G is a sphere with one hole and three orbifold points of order 2; every such group has a unique torsion-free subgroup representing a torus with a hole. Conversely, if F' is a Fuchsian group of the second kind representing a torus with a hole, then F' has a unique $\mathbb{Z}_2$ extension F, where $\mathbb{H}^2/F$ is a disc with three orbifold points of order 2. Choose elliptic generators $(\beta_1', \beta_2', \beta_3')$ for F. Then we can conjugate F so that it acts on the upper half-plane; and then we can regard F as acting on the plane $P \subset \mathbb{H}^3$, whose boundary on the sphere at infinity is the real axis; and, for $i = 1, 2, 3$, define β_i by requiring that $\beta_i|P = \beta_i'|P$, and that β_i interchanges the two half-spaces bounded by P. Then $G = \langle\beta_1, \beta_2, \beta_3\rangle$ is a group in $\mathcal{KS}_2^+(3, 0)$.

Proposition 2.2. *There is a natural real-analytic diffeomorphism between the space* $\mathcal{KS}_2^+(3, 0)$ *and the space of fuchsian groups of the second kind representing a disc with three orbifold points of order* 2; *which in turn is diffeomorphic to the space of Fuchsian groups representing a torus with one hole*; *in particular*, *the space* $\mathcal{KS}_2^+(3, 0)$ *is topologically a cell of real dimension* 3.

We also remark that if $G \in \mathcal{KS}_2^+(3, 0)$, the reflection in the invariant plane P spanned by the fixed points of the generators commutes elementwise with G; this shows that $\mathcal{KS}_2^+(3, 0) = \mathcal{KS}_2(3, 0)$.

In contrast to the above, one expects that $\pi_1(\mathcal{KS}_2^+(1, 1))$ is infinite cyclic, for there is a natural embedding of $\mathcal{KS}_2^+(1, 1)$ into a thickened cylinder. A point in $\mathcal{KS}_2^+(1, 1)$ is given by a pair of generators: (β, γ), defined up to conjugation in the group of all isometries of $\mathbb{H}^3$. We consider these groups as operating in the upper half-space, with coordinates (z, t), $z \in \mathbb{C}, t > 0$. Given any pair of generators, (β, γ) for a group in $\mathcal{KS}_2^+(1, 1)$, normalize so that $\gamma(z, t) = k(\bar{z}, t)$, where k is real and $k > 1$. We can further normalize so that the fixed point of β lies in the plane passing through $(0, 1)$, orthogonal to the axis of γ. In fact, we can further normalize so that the coordinate (z, t) of the fixed point of β is such that $\Re(z) \geq 0$.

Since the fixed point of β cannot lie on the axis of γ, we can uniquely specify this pair of generators, up to conjugation, by the multiplier k, lying in the interval, $(1, \infty)$, and by the location of the fixed point of β, which we can regard as lying in a punctured disc. Since every such point with k sufficiently large, and with the fixed point of β at (z, t), where t is sufficiently small, corresponds to a point in $\mathcal{KS}_2^+(1, 1)$, we see that there is at least one nontrivial loop in $\mathcal{KS}_2(1, 1)$.

The general case. We saw above that the two different spaces of Klein–Schottky groups, inside the space of Schottky groups of genus 2, do not intersect. The following two theorems show that, for every odd genus, there are intersecting spaces of Klein–Schottky groups. We expect that in even genus, as in genus 2, distinct Klein–Schottky spaces are disjoint; this will be explored elsewhere.

Theorem 2.3. *Let a and b be positive integers, and let $g = 2(a + b) - 1$. Then*

$$\mathcal{KS}_g^+(2a, b) \cap \mathcal{KS}_g^+(2b, a) \neq \varnothing.$$

Proof. Consider a collection of $a+b+1$ pairwise disjoint circles, each orthogonal to the real line and bounding a common domain. Denote these circles as S_0, S_1, $\dots$, S_{a+b}. Let $\tau(z) = \bar{z}$ and, for each $j = 0, \dots, a + b$, let σ_j be an imaginary reflection keeping invariant both the real axis and the circle S_j. Then τ commutes with each σ_j, $j = 0, \dots, a + b$.

For, $j = 1, \dots, a + b$, let σ_j' denote the imaginary reflection in the circle $S_j' = \sigma_0(S_j)$. We also let $\rho_j = \tau\sigma_0\sigma_j$. Let G_1 be the Klein–Schottky group of type $(2a, b)$ generated as follows:

$$G_1 = \langle \sigma_1, \dots, \sigma_a, \sigma_1', \dots, \sigma_a', \rho_{a+1}, \dots, \rho_{a+b} \rangle.$$

Let G_2 be the Klein–Schottky group of type $(2b, a)$ generated as follows:

$$G_2 = \langle \sigma_{a+1}, \dots, \sigma_{a+b}, \sigma_{a+1}', \dots, \sigma_{a+b}', \rho_1, \dots, \rho_a \rangle.$$

We reflect in the circle S_1 to obtain the following generators for the orientation-preserving half of G_1:

$$G_1^+ = \left\langle \sigma_1\sigma_2, \dots, \sigma_1\sigma_a, \sigma_1\sigma_1', \dots, \sigma_1\sigma_a', \sigma_1\rho_{a+1}, \dots, \sigma_1\rho_{a+b}, \sigma_1\rho_{a+1}^{-1}, \dots, \sigma_1\rho_{a+b}^{-1} \right\rangle.$$

We likewise reflect in the circle S_{a+1} to obtain the following generators for the orientation-preserving half of G_2:

$$G_2^+ = \left\langle \sigma_{a+1}\sigma_{a+2}, \dots, \sigma_{a+1}\sigma_{a+b}, \sigma_{a+1}\sigma_{a+1}', \dots, \sigma_{a+1}\sigma_{a+b}', \sigma_{a+1}\rho_1, \dots, \sigma_{a+1}\rho_a, \sigma_{a+1}\rho_1^{-1} \dots, \sigma_{a+1}\rho_a^{-1} \right\rangle.$$

We claim that $G_1^+ = G_2^+$. Since we can interchange the roles of a and b, it obviously suffices to show that $G_2^+ \subset G_1^+$. Since, for $j = 1, \dots, b$,

$$(\sigma_1 \rho_{a+1})^{-1}(\sigma_1 \rho_{a+j}) = \sigma_{a+1}\sigma_{a+j};$$

we conclude that the generators of G_2^+ of the form $\sigma_{a+1}\sigma_{a+j}$ are contained in G_1^+. We next observe that, since τ commutes with every σ_i,

$$(\sigma_1 \rho_{a+1})^{-1}(\sigma_1 \rho_{a+j}^{-1}) = \sigma_{a+1}\sigma'_{a+j};$$

we conclude that the generators of G_2^+, $\sigma_{a+1}\sigma'_{a+j}$, $j = 1 \dots, b$, are also in G_1^+. Now, using the fact that τ commutes with σ_0 and that

$$(\sigma_1 \rho_{a+1})^{-1}\sigma_1\sigma_i = \sigma_{a+1}\tau\sigma_0\sigma_i = \sigma_{a+1}\rho_i,$$

we conclude that the generators of G_2^+, $\sigma_{a+1}\rho_i$, $i = 1, \dots, a$, also lie in G_1^+. Finally, we observe that

$$(\sigma_{a+1}\rho_{a+1})^{-1}\sigma_1\sigma'_i = \sigma_{a+1}\sigma_i\sigma_0\tau = \sigma_{a+1}\rho_i^{-1},$$

from which we conclude that the generators of G_2^+, $\sigma_{a+1}\rho_i^{-1}$, $i = 1, \dots, a$ also lie in G_1^+. We have shown that $G_2^+ \subset G_1^+$. □

Theorem 2.4. *Let a be a positive integer, and let $g = 2a - 1$. Then*

$$\mathcal{HS}_g^+(2a, 0) \cap \mathcal{HS}_g^+(0, a) \neq \varnothing.$$

Proof. As above, we start with $a + 1$ circles, $S_0, \dots, S_a$ all orthogonal to the real axis, and all bounding a common domain. For each $i = 0, \dots, a$, let σ_i be an imaginary reflection keeping both S_i and the real axis invariant. For $i = 1, \dots, a$, let $S'_i = \sigma_0(S_i)$, and let $\sigma'_i = \sigma_0\sigma_i\sigma_0$.

Continuing as above, let $\tau(z) = \bar{z}$; note that τ commutes with every σ_i and with every σ'_i. Also, for $i = 1, \dots, a$, set $\rho_i = \tau\sigma_0\sigma_i$.

Define

$$G_1 = \langle \sigma_1, \dots, \sigma_a, \sigma'_1 \dots, \sigma'_a \rangle,$$
$$G_2 = \langle \rho_1, \dots, \rho_a \rangle.$$

Reflecting in the circle S_1, we can write the generators for Schottky groups that are the orientation-preserving halves of these groups as

$$G_1^+ = \langle \sigma_2\sigma_1, \dots, \sigma_a\sigma_1, \sigma'_1\sigma_1, \dots, \sigma'_a\sigma_1 \rangle,$$
$$G_2^+ = \langle \rho_1\rho_1, \dots, \rho_a\rho_1, \rho_2^{-1}\rho_1, \dots, \rho_a^{-1}\rho_1 \rangle.$$

One easily computes that, for $i = 1, \dots, a$, $\rho_i\rho_1 = \sigma'_i\sigma_1$, and that, for $i = 2, \dots, a$, $\rho_i^{-1}\rho_1 = \sigma_i\sigma_1$. Hence $G_1^+ = G_2^+$. □

Corollary 2.5. *If* $g \equiv 1 \mod 4$ *, then every Klein–Schottky space of genus* g *intersects at least one other Klein–Schottky space.*

Proof. If G is a (m, n)-Klein–Schottky group of genus g, then $g = m + 2n - 1$; hence, if g is odd, then m is even. That is, if g is odd, then every Klein–Schottky group of genus g is a $(2a, b)$-Klein–Schottky group.

One sees at once that $(2a, b) = (2b, a)$ if and only if $a = b$. However, if $a = b$, then $g = 2(a + b) - 1 \equiv 3 \mod 4$. Therefore, for $g \equiv 1 \mod 4$, and for all possible choices of a and b, the spaces $\mathcal{KS}_g^+(2a, b)$ and $\mathcal{KS}_g^+(2b, a)$ are distinct and intersect. □

Remark. The universal covering space of Schottky space is the Teichmüller space, which is topologically a cell. The group of deck transformations for this covering is a subgroup of the mapping class group, or Teichmüller modular group [Maskit 1971]. We can likewise recognize the universal covering of a Klein–Schottky space as either (i) a real subspace of the Teichmüller space, or (ii) as the space of extended Fuchsian groups representing Klein surfaces, of fixed genus, modulo conjugation by the full group of isometries of the hyperbolic plane. Using Fenchel–Nielsen coordinates, one easily sees, using this second representation, that the universal covering space of an extended Klein–Schottky space is topologically a cell. Using the first representation, one obtains that the fundamental group of an extended Klein–Schottky space is a subgroup of the fundamental group of Schottky space. Generators for the fundamental group of Schottky space are known [Luft 1978]; it is not clear if these results also hold for extended Klein–Schottky spaces.

References

[Abikoff and Maskit 1977] W. Abikoff and B. Maskit, "Geometric decompositions of Kleinian groups", *Amer. J. Math.* **99**:4 (1977), 687–697. MR 58 #1139 Zbl 0374.30018

[Ahlfors and Sario 1960] L. V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Mathematical Series **26**, Princeton Univ. Press, Princeton, NJ, 1960. MR 22 #5729 Zbl 0196.33801

[Bers 1975] L. Bers, "Automorphic forms for Schottky groups", *Advances in Math.* **16** (1975), 332–361. MR 51 #13218 Zbl 0327.32011

[Chuckrow 1968] V. Chuckrow, "On Schottky groups with applications to kleinian groups", *Ann. of Math.* (2) **88** (1968), 47–61. MR 37 #2987 Zbl 0186.40603

[Hejhal 1975] D. A. Hejhal, "On Schottky and Teichmüller spaces", *Advances in Math.* **15** (1975), 133–156. MR 50 #10328 Zbl 0307.32018

[Koebe 1910] P. Koebe, "Über die Uniformisierung der algebraischen Kurven II", *Math. Ann.* **69** (1910), 1–81. JFM 41.0480.02

[Kra and Maskit 1981] I. Kra and B. Maskit, "The deformation space of a Kleinian group", *Amer. J. Math.* **103**:5 (1981), 1065–1102. MR 83e:32026 Zbl 0476.32029

[Luft 1978] E. Luft, "Actions of the homeotopy group of an orientable 3-dimensional handlebody", *Math. Ann.* **234**:3 (1978), 279–292. MR 58 #18461 Zbl 0364.57011

[Maskit 1965] B. Maskit, "A theorem on planar covering surfaces with applications to 3-manifolds", *Ann. of Math.* (2) **81** (1965), 341–355. MR 30 #2472 Zbl 0151.33003

[Maskit 1967] B. Maskit, "A characterization of Schottky groups", *J. Analyse Math.* **19** (1967), 227–230. MR 36 #3981 Zbl 0168.06201

[Maskit 1971] B. Maskit, "Self-maps on Kleinian groups", *Amer. J. Math.* **93** (1971), 840–856. MR 45 #544 Zbl 0227.32007

[Maskit 1973] B. Maskit, "Decomposition of certain Kleinian groups", *Acta Math.* **130** (1973), 243–263. MR 53 #8414 Zbl 0258.30018

[Maskit 1988] B. Maskit, *Kleinian groups*, Grundlehren der Math. Wissenschaften **287**, Springer, Berlin, 1988. MR 90a:30132 Zbl 0627.30039

[Maskit 1993] B. Maskit, "On Klein's combination theorem, IV", *Trans. Amer. Math. Soc.* **336**:1 (1993), 265–294. MR 93e:30088 Zbl 0777.30028

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