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An operator on Hilbert space is called hypertransitive if the orbit of every nonzero vector is dense. Here we determine some new classes of nonhypertransitive operators.

1. Introduction

Let $\mathscr{H}$ be a separable, infinite-dimensional complex Hilbert space, and denote the algebra of all bounded linear operators on $\mathscr{H}$ by $\mathscr{L}(\mathscr{H})$. If $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, the countable (finite or infinite) set $\{T^n x\}_{n=0}^{\infty}$ is called the *orbit* of x *under* T, and is denoted by $\mathbb{O}(x, T)$. If $\mathbb{O}(x, T)$ is dense in $\mathscr{H}$, then x is called a *hypercyclic vector* for T. The question of which operators in $\mathscr{L}(\mathscr{H})$ have hypercyclic vectors has been much studied. See, for example, the recent survey article [Grosse-Erdmann 1999] and its extensive bibliography. An operator T in $\mathscr{L}(\mathscr{H})$ is called *transitive* if T has no invariant subspace (closed linear manifold) other than $\{0\}$ and $\mathscr{H}$, and is called *hypertransitive* if every nonzero vector in $\mathscr{H}$ is hypercyclic for T. Presently one doesn't know whether there exist transitive or hypertransitive operators in $\mathscr{L}(\mathscr{H})$. (It is obvious that every hypertransitive operator is transitive, and Read [1988] has constructed an operator on the Banach space (ℓ_1) that is hypertransitive.) Denote the set of all nontransitive operators in $\mathscr{L}(\mathscr{H})$ by (NIS) and the set of all nonhypertransitive operators in $\mathscr{L}(\mathscr{H})$ by (NHT). The *invariant subspace problem* is the open question whether (NIS) $= \mathscr{L}(\mathscr{H})$, and the *hypertransitive operator problem* is the open question whether (NHT) $= \mathscr{L}(\mathscr{H})$. (The hypertransitive operator problem is sometimes referred to as the nontrivial invariant closed set problem, but we eschew this terminology.)

It is the purpose of this paper to contribute to the theory of nonhypertransitive operators by showing that several classes of operators which are not known to be subsets of (NIS) are at least subsets of (NHT). There are many results of this sort. For example, it is obvious that every power bounded operator belongs to (NHT).

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But the class of power bounded operators is not invariant under compact perturbations, and if one hopes eventually to show that (NHT) $= \mathcal{L}(\mathcal{H})$ (as we believe), it seems clear that the route will be via results of the form that $(\mathscr{C}) \subset$ (NHT), where $(\mathscr{C})$ is a class of operators that is *invariant under compact perturbations*. Most of the results below are of this nature.

We begin by reviewing some standard notation and terminology. The set of (strictly) positive integers is denoted by $\mathbb{N}$. The open unit disc in $\mathbb{C}$ is denoted by $\mathbb{D}$ and the unit circle $\partial\mathbb{D}$ by $\mathbb{T}$. The ideal of compact operators in $\mathcal{L}(\mathcal{H})$ denotes by $\mathbb{K}(\mathcal{H})$, or simply $\mathbb{K}$, and π denotes the quotient map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$. For T in $\mathcal{L}(\mathcal{H})$ we write $\sigma(T)$ for the spectrum of T, $\sigma_e(T) = \sigma(\pi(T))$, $\sigma_{le}(T) = \sigma_l(\pi(T))$ (the left spectrum of $\pi(T)$), and $\sigma_{re}(T) = \sigma_r(\pi(T))$. We write $r(T)$ for the spectral radius of T, $r_e(T) = r(\pi(T))$, and $\|T\|_e = \|\pi(T)\|$. We also write $w(T)$ for the numerical radius of T:

$$w(T) = \sup\{|(Tx, x)| : x \in \mathcal{H}, \|x\| = 1\}.$$

Our first proposition is completely elementary and needs no proof.

Proposition 1.1. *If $T \in \mathcal{L}(\mathcal{H})$ and if at least one of the equalities*

$$\sigma(T) = \sigma_e(T) = \sigma_{le}(T) = \sigma_{re}(T) \tag{1–1}$$

fails to hold, then T or T^* has point spectrum and thus $T \in$ (NIS) $\subset$ (NHT). *Furthermore, if there exist nonzero vectors x and y in $\mathcal{H}$ such that either the sequence $\{(T^n x, y)\}_{n\in\mathbb{N}}$ is not dense in $\mathbb{C}$ or the sequence $\{\|T^n x\|\}_{n\in\mathbb{N}}$ is not dense in $\mathbb{R}_+$, then $T \in$* (NHT). *Finally, $T \in$* (NHT) *if and only if some (hence every) operator similar to T belongs to* (NHT).

The following deep fact is due to V. Lomonosov [1991]. For a different, perhaps easier, proof, see [Chevreau et al. 1998].

Theorem 1.2 (Lomonosov). *Suppose $\mathcal{A}$ is a proper subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed in the weak operator topology. Then there exist nonzero vectors x and y in $\mathcal{H}$ such that, for each $A \in \mathcal{A}$,*

$$|(A^n x, y)| \le \|A^n\|_e \quad \text{for } n \in \mathbb{N}. \tag{1–2}$$

This result was improved by Simonic [1996], using techniques similar in spirit to those of [Chevreau et al. 1998], as follows.

Theorem 1.3 (Simonic). *Suppose $\mathcal{A}$ is a proper subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed in the weak operator topology. Then there exist unit vectors x and y in $\mathcal{H}$ such that for each $A \in \mathcal{A}$,*

$$|\mathrm{Re}(A^n x, y)| \le \|\mathrm{Re}\, A^n\|_e(x, y) \quad \text{for } n \in \mathbb{N}.$$

Corollary 1.4. *Suppose $T \in \mathcal{L}(\mathcal{H})$ and for each $n \in \mathbb{N}$, $T^n = H_n + iK_n$ where H_n and K_n are Hermitian. If either of the sequences $\{\|H_n\|_e\}$ or $\{\|K_n\|_e\}$ is bounded, then $T \in$ (NHT). In particular, if T is essentially power bounded, then $T \in$ (NHT).*

The next important fact that we need is a nice theorem of B. Beauzamy [1988, Theorem 2.A.7 and Remark 2.A.11]. Since its proof is only outlined in the original article, we give a complete proof.

Theorem 1.5 (Beauzamy). *Let $\{T_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of operators in $\mathcal{L}(\mathcal{H})$, let ε be any positive number, and let $\{1/\delta_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers in (ℓ_2). Then there exists a nonzero vector $y \in \mathcal{H}$ such that*

$$\|y\| \le (1+\varepsilon)\left(\sum_{n\in\mathbb{N}} \frac{1}{\delta_n^2}\right)^{1/2} \quad \textit{and} \quad \|T_n y\| \ge (1-\varepsilon)\frac{\|T_n\|_e}{\delta_n} \quad \textit{for } n \in \mathbb{N}.$$

Proof. Since $\|T_k\|_e = r_e((T_k^* T_k)^{1/2})$ for each $k \in \mathbb{N}$, there exists an orthonormal sequence $\{e_n^{(k)}\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that

$$\lim_n \|T_k e_n^{(k)}\| = \|T_k\|_e \quad \text{for } k \in \mathbb{N}. \tag{1–3}$$

We will choose, by induction, a sequence $\{e_{n_k}^{(k)}\}_{k=1}^\infty$ consisting of an appropriate vector from each sequence $\{e_n^{(k)}\}_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$; then we will show that the vector

$$y = \sum_{n\in\mathbb{N}} \frac{e_{n_k}^{(k)}}{\delta_k}$$

has the required properties. Using (1–3) we begin by choosing $e_{n_1}^{(1)}$ so far out in the sequence $\{e_n^{(1)}\}_{n\in\mathbb{N}}$ that

$$\left\|T_1 e_{n_1}^{(1)}\right\| > (1-\varepsilon)\,\|T_1\|_e,$$

and consequently that

$$\left\|T_1\left(\frac{e_{n_1}^{(1)}}{\delta_1}\right)\right\| > (1-\varepsilon)\frac{\|T_1\|_e}{\delta_1}. \tag{1–4}$$

Now suppose that $e_{n_i}^{(i)}$, for $i = 1, \dots, j$, have been chosen from the sequence $\{e_n^{(i)}\}_{n\in\mathbb{N}}$ so that

$$\left\|T_k \sum_{i=1}^{j} \frac{e_{n_i}^{(i)}}{\delta_i}\right\|^2 > (1-\varepsilon)^2 \frac{\|T_k\|_e^2}{\delta_k^2} \quad \text{for } k = 1, \dots, j \tag{1–5}$$

and

$$(1+\varepsilon)^2 \sum_{i=m}^{j} \frac{1}{\delta_i^2} > \left\|\sum_{i=m}^{j} \frac{e_{n_i}^{(i)}}{\delta_i}\right\|^2 > (1-\varepsilon)^2 \sum_{i=m}^{j} \frac{1}{\delta_i^2} \quad \text{for } m = 1, \dots, j. \tag{1–6}$$

Since for each $k \in \mathbb{N}$ the sequence $\{e_n^{(k)}\}_{n\in\mathbb{N}}$ converges weakly to 0, the same is true of every sequence of the form $\{T_m e_n^{(k)}\}_{n\in\mathbb{N}}$, for $m \in \mathbb{N}$ and $k \in \mathbb{N}$. It follows easily from this and (1–5) that

$$\liminf_n \left\| T_k\left(\frac{e_n^{(j+1)}}{\delta_{j+1}} + \sum_{i=1}^{j} \frac{e_{n_i}^{(i)}}{\delta_i}\right)\right\|^2 > (1-\varepsilon)^2 \frac{\|T_k\|_e^2}{\delta_k^2} \quad \text{for } k = 1, \ldots, j,$$

$$\begin{aligned}\lim_n \left\| T_{j+1}\left(\frac{e_n^{(j+1)}}{\delta_{j+1}} + \sum_{i=1}^{j} \frac{e_{n_i}^{(i)}}{\delta_i}\right)\right\|^2 &= \frac{\|T_{j+1}\|_e^2}{\delta_{j+1}^2} + \left\| T_{j+1}\left(\sum_{i=1}^{j} \frac{e_{n_i}^{(i)}}{\delta_i}\right)\right\|^2 \\ &> (1-\varepsilon)^2 \frac{\|T_{j+1}\|_e^2}{\delta_{j+1}^2},\end{aligned}$$

$$\lim_n \left\| \frac{e_n^{(j+1)}}{\delta_{j+1}} + \sum_{i=m}^{j} \frac{e_{n_i}^{(i)}}{\delta_i}\right\|^2 = \frac{1}{\delta_{j+1}^2} + \left\| \sum_{i=m}^{j} \frac{e_{n_i}^{(i)}}{\delta_i}\right\|^2 \quad \text{for } m = 1, \ldots, j.$$

From these relations and (1–6) one sees immediately that it is possible to choose $e_{n_{j+1}}^{(j+1)}$ so far out in the sequence $\{e_n^{(j+1)}\}_{n\in\mathbb{N}}$ that

$$\left\| T_k \sum_{i=1}^{j+1} \frac{e_{n_i}^{(i)}}{\delta_i}\right\|^2 > (1-\varepsilon)^2 \frac{\|T_k\|_e^2}{\delta_k^2} \quad \text{for } k = 1, \ldots, j+1 \tag{1–7}$$

and

$$(1+\varepsilon)^2 \sum_{i=m}^{j+1} \frac{1}{\delta_i^2} > \left\| \sum_{i=m}^{j+1} \frac{e_{n_i}^{(i)}}{\delta_i}\right\|^2 > (1-\varepsilon)^2 \sum_{i=m}^{j+1} \frac{1}{\delta_i^2} \quad \text{for } m = 1, \ldots, j+1. \tag{1–8}$$

Thus, by induction, we construct a sequence $\{e_{n_i}^{(i)}\}_{i\in\mathbb{N}}$ such that (1–7) and (1–8) hold for every $j \in \mathbb{N}$. Next observe from (1–8) and the fact that the sequence $\{1/\delta_i\}_{j\in\mathbb{N}}$ lies in ℓ_2 that the sequence $y_j := \sum_{i=1}^{j} e_{n_i}^{(i)}/\delta_i$ is Cauchy, and thus converges strongly to a vector $y = \sum_{i\in\mathbb{N}} e_{n_i}^{(i)}/\delta_i$ satisfying

$$(1-\varepsilon)^2 \sum_{i\in\mathbb{N}} \frac{1}{\delta_i^2} \le \|y\|^2 \le (1+\varepsilon)^2 \sum_{i\in\mathbb{N}} \frac{1}{\delta_i^2}.$$

Moreover, for every fixed $k \in \mathbb{N}$, the sequence $\{T_k y_j\}_{j\in\mathbb{N}}$ converges to $T_k y$, and as a consequence of (1–7) (where $j > k$), we get $\|T_k y\| \ge (1-\varepsilon)\|T_k\|_e/\delta_k$ for $k \in \mathbb{N}$, as desired. □

Corollary 1.6 (Beauzamy). *Suppose $T \in \mathcal{L}(\mathcal{H})$ and satisfies $r_e(T) > 1$. Then there exists a vector y in $\mathcal{H}$ such that the sequence $\{\|T^n y\|\}_{n\in\mathbb{N}}$ diverges to ∞. In particular, if $r_e(T) > 1$, then $T \in$* (NHT).

Proof. Let $\lambda \in \sigma_e(T)$ be such that $|\lambda| > 1$. Then of course, $\lambda^n \in \sigma_e(T^n)$ for each $n \in \mathbb{N}$, and if we let $\delta_n = n$ we see that $\|T^n\|_e / \delta_n \ge |\lambda|^n / n \to +\infty$. The existence of the desired vector y follows at once from Theorem 1.5, and the relation $T \in$ (NHT) is a consequence of this and Proposition 1.1. $\square$

Our last tool is a nice theorem from [Ansari 1995].

Theorem 1.7 (Ansari). *An element T of $\mathscr{L}(\mathscr{H})$ lies in* (NHT) *if and only if some power T^n does* (*where $n \in \mathbb{N}$*), *if and only if any power T^n does.*

Proof. It is shown in [Ansari 1995] that if T has a hypercyclic vector, then T^n (for $n \in \mathbb{N}$) has exactly the same set of hypercyclic vectors as T. Thus if T is hypertransitive, so is every power T^n. On the other hand, if some T^n is hypertransitive, then obviously T is also, since $\mathbb{O}(T, x) \supset \mathbb{O}(T^n, x)$ for all x in $\mathscr{H}$. $\square$

Remark 1.8. By virtue of Theorem 1.7, many of the results to follow and Corollary 1.4 have obvious (but serious!) improvements, which, nevertheless, for the most part we leave to the reader to formulate. That of Corollary 1.4 reads as follows: *With the notation as in Corollary 1.4, if there exists a $k_0 \in \mathbb{N}$ such that either of the sequences $\{\|H_{nk_0}\|_e\}_{n\in\mathbb{N}}$ or $\{\|K_{nk_0}\|_e\}_{n\in\mathbb{N}}$ is bounded, then $T \in$* (NHT).

2. Consequences

We now employ the tools of Section 1 to derive the nonhypertransitivity of operators in some additional classes. Our first result is the following, which probably should also be credited to Beauzamy [1987].

Theorem 2.1. *If $T \in \mathscr{L}(\mathscr{H})$ and $r(T) \neq 1$, then $T \in$* (NHT).

Proof. By Proposition 1.1, we may suppose that $r(T) = r_e(T)$. If $r_e(T) > 1$, the result follows from Corollary 1.6, and if $r_e(T) < 1$, then T is similar to a strict contraction, and the result follows from Proposition 1.1 and Corollary 1.4. $\square$

We recall the notion of *essential numerical range* $W_e(T)$ of an operator $T \in \mathscr{L}(\mathscr{H})$. This has several characterizations (see [Fillmore et al. 1972]), one of which is that

$$W_e(T) = \{\lambda \in \mathbb{C} : \exists \text{ an o.n. sequence } \{x_n\} \text{ in } \mathscr{H} \text{ such that } (Tx_n, x_n) \to \lambda\};$$

likewise $w_e(T)$, the *essential numerical radius* of T, is defined as

$$w_e(T) = \sup\{|\lambda| : \lambda \in W_e(T)\}.$$

It is obvious that $w_e(T + K) = w_e(T)$ for all K in $\mathbb{K}$. We need the following additional facts, which are counterparts of well-known results about the relations between $\|T\|$, $w(T)$, and $r(T)$.

Proposition 2.2. *For every $T \in \mathscr{L}(\mathscr{H})$*

(a) $\|T\|_e \leq 2w_e(T)$;

(b) $\|T\|_e = w_e(T)$ *implies* $r_e(T) = w_e(T)$;

(c) (*the power inequality*) $w_e(T^n) \le (w_e(T))^n$ *for every* $n \in \mathbb{N}$.

Proof. Using the characterization of $W_e(T)$ given above, one can adapt the usual proofs of the corresponding relations between $\|T\|$, $w(T)$, and $r(T)$ to obtain (a), (b) and (c). (Alternatively, (c) is known to hold for elements in any C^*-algebra; see [Bonsall and Duncan 1973, page 51]. □

Theorem 2.3. *Suppose that* $T \in \mathscr{L}(\mathscr{H})$ *and that there exists some* $n \in \mathbb{N}$ *such that two of the three numbers* $r_e(T^n) \le w_e(T^n) \le \|T^n\|_e$ *coincide. Then for every invertible* $S \in \mathscr{L}(\mathscr{H})$ *and every* $K \in \mathbb{K}$, $STS^{-1} + K \in$ (NHT).

Proof. Fix an invertible S_0 in $\mathscr{L}(\mathscr{H})$ and $K_0 \in \mathbb{K}$, and observe that

$$(S_0TS_0^{-1} + K_0)^n = S_0T^nS_0^{-1} + K_1 = S_0(T^n + K_2)S_0^{-1}, \tag{2–1}$$

where $K_1, K_2 \in \mathbb{K}$ and $K_2 = S_0^{-1}K_1S_0$. By Theorem 1.7 it suffices to show that $(S_0TS_0^{-1} + K_0)^n \in$ (NHT), and from (2–1) and Proposition 1.1 we see that this is equivalent to showing that $A = T^n + K_2 \in$ (NHT). Moreover, $r_e(A) = r_e(T^n)$ and similarly for $w_e(\,\cdot\,)$ and $\|\cdot\|_e$, so by hypothesis, two of the numbers $r_e(A)$, $w_e(A)$, and $\|A\|_e$ coincide. If $r_e(A) = \|A\|_e$, then also $r_e(A) = w_e(A)$, and if $w_e(A) = \|A\|_e$, then by Proposition 2.2, $r_e(A) = w_e(A)$ also, which shows that it suffices to consider the case $r_e(A) = w_e(A)$. If $r_e(A) > 1$, then $A \in$ (NHT) by Corollary 1.6, so we may also suppose that $r_e(A) \le 1$. Then, using Proposition 2.2 twice more, we obtain

$$\|A^m\|_e \le 2w_e(A^m) \le 2(w_e(A))^m = 2(r_e(A))^m \le 2 \quad \text{for } m \in \mathbb{N}.$$

Thus A is essentially power bounded; the result now follows from Corollary 1.4. □

As a corollary we obtain:

Theorem 2.4. *If* $T \in \mathscr{L}(\mathscr{H})$ *and satisfies* $\|T^{n_0}\| = r(T^{n_0})$ *for some positive integer* n_0, *then* $STS^{-1} \in$ (NHT) *for every invertible* S *in* $\mathscr{L}(\mathscr{H})$.

Proof. Since $(STS^{-1})^{n_0} = ST^{n_0}S^{-1}$, by Proposition 1.1 and Theorem 1.7 it suffices to show that $T^{n_0} \in$ (NHT). By Proposition 1.1 we may suppose that $r_e(T^{n_0}) = r(T^{n_0})$. Thus

$$r(T^{n_0}) = r_e(T^{n_0}) \le \|T^{n_0}\|_e \le \|T^{n_0}\| = r(T^{n_0}),$$

and the result is immediate from Theorem 2.3. □

Recall that an operator T in $\mathscr{L}(\mathscr{H})$ is called *essentially hyponormal* if $\pi(TT^*) \le \pi(T^*T)$. Obviously if T is hyponormal and $K \in \mathbb{K}$, then $T + K$ is essentially hyponormal. In particular, if N is normal, all operators of the form $N + K$ are essentially hyponormal.

Theorem 2.5. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ and that there exists $n \in \mathbb{N}$ such that T^n is either a Toeplitz operator or an essentially hyponormal operator. Then every operator of the form $STS^{-1} + K$ where S is invertible in $\mathcal{L}(\mathcal{H})$ and $K \in \mathbb{K}$ is nonhypertransitive.*

Proof. Suppose first that T^n is a Toeplitz operator. Then, as is well-known, $\sigma(T^n)$ is connected and $r(T^n) = \|T^n\|$. Thus

$$r(T^n) = r_e(T^n) \leq \|T^n\|_e \leq \|T^n\| = r(T^n),$$

and in this case the result follows from Theorem 2.3. On the other hand, if T^n is essentially hyponormal, then $r_e(T^n) = \|T^n\|_e$ and again the result follows from Theorem 2.3. □

Corollary 2.6. *Every operator in $\mathcal{L}(\mathcal{H})$ of the form $N+K$, where N is normal and $K \in \mathbb{K}$, is nonhypertransitive.*

3. Some additional results

Recall that an operator B in $\mathcal{L}(\mathcal{H})$ is called *block-diagonal* if B is unitarily equivalent to a direct sum $\bigoplus_{n\in\mathbb{N}} B_n$, where each B_n acts on a finite dimensional Hilbert space, and an operator Q in $\mathcal{L}(\mathcal{H})$ is called *quasidiagonal* if Q can be written as a sum $Q = B + K$ where B is block-diagonal and $K \in \mathbb{K}$ [Halmos 1970]. The classes of block-diagonal and quasidiagonal operators in $\mathcal{L}(\mathcal{H})$ will be denoted by (BD) and (QD), respectively.

Example 3.1. For $n \in \mathbb{N}$ an *n-normal operator T* in $\mathcal{L}(\mathcal{H})$ is, by definition, unitarily equivalent to an $n \times n$ matrix (N_{ij}), acting on the direct sum $\mathcal{H}^{(n)}$ of n copies of $\mathcal{H}$, with operator entries $N_{ij} \in \mathcal{L}(\mathcal{H})$ that are mutually commuting normal operators. One knows from [Pearcy and Salinas 1975] that there exists an ordered orthonormal basis $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}}$ such that $N_{ij} = D_{ij} + K_{ij}$ for all $i, j = 1, \ldots, n$, where the matrix for D_{ij} relative to $\mathcal{E}$ is a diagonal matrix and $K_{ij} \in \mathbb{K}$. Thus the $n \times n$ operator matrix $D = (D_{ij}) \in \mathcal{L}(\mathcal{H}^{(n)})$ is a block-diagonal operator and the $n \times n$ operator matrix $K = (K_{ij})$ is a compact operator in $\mathcal{L}(\mathcal{H}^{(n)})$. Thus $T = D + K$ is a quasidiagonal operator of a particularly simple sort. (The block-diagonal operator D has all of its diagonal blocks of size $n \times n$, whereas the most general block-diagonal operator has arbitrary (finite) sized diagonal blocks.) On the other hand, if a block-diagonal operator $B = \bigoplus_{n\in\mathbb{N}} B_n$ has the property that the dimensions of the spaces on which the B_n operate are bounded, then there exists an operator of the form $0 \oplus B$ which is an n-normal operator. Thus the classes of n-normal operators, (as n varies over $\mathbb{N}$) are intimately related to (QD), and to show that (QD) $\subset$ (NHT) one must first show that compact perturbations of n-normal operators lie in (NHT). One knows from [Hoover 1971] that every nonscalar n-normal operator has a nontrivial hyperinvariant subspace and thus lies in (NIS) $\subset$ (NHT), but presently the authors are unable to show that arbitrary compact perturbations of n-normal operators always lie in (NHT). For a concrete example of one such, see Example 4.5.

We next show that compact perturbations of some special classes of operators are subsets of (NHT). For this purpose, we need the following result from linear algebra, which needs no proof.

Proposition 3.2. *Suppose that $n \in \mathbb{N}$ and that M is an arbitrary $n \times n$ complex matrix, which we regard as an element of $\mathcal{L}(\mathbb{C}^n)$. Then either M is power bounded or there exists a constant $c_M > 0$ such that $\|M^n\| \geq c_M \cdot n$ for each $n \in \mathbb{N}$.*

As a consequence:

Proposition 3.3. *Suppose $n \in \mathbb{N}$, that $M \in \mathcal{L}(\mathbb{C}^n)$ is an $n \times n$ matrix, and that $A \in \mathcal{L}(\mathcal{H})$ satisfies $r_e(A) = \|A\|_e$. Then the tensor product $T = A \otimes M \in \mathcal{L}(\mathcal{H} \otimes \mathbb{C}^n)$ has the property that either T is essentially power bounded or there exists $c_T > 0$ such that $\|T^k\|_e \geq c_T k$ for all $k \in \mathbb{N}$. Thus if S is invertible in $\mathcal{L}(\mathcal{H})$ and $K \in \mathbb{K}$, then $STS^{-1} + K \in$ (NHT).*

Proof. As before, $STS^{-1} + K = S(T + K_1)S^{-1}$ where $K_1 \in \mathbb{K}$, so it suffices to prove that $T + K_1 \in$ (NHT). If $\|A\|_e = 0$, then T is compact and the desired result is obvious, so by writing $T = pA \otimes (1/p)M$, we may suppose that $r_e(A) = \|A\|_e = 1$. Since $T^k = A^k \otimes M^k$ for $k \in \mathbb{N}$, it follows easily that $\|T^k\|_e = \|M^k\|$ and thus that if T is not essentially power bounded, then M is not power bounded, and the result follows from Proposition 3.2 and Theorem 1.5 by setting $\delta_k = k^{-2/3}$ for $k \in \mathbb{N}$. $\square$

Theorem 3.4. *Suppose that $n \in \mathbb{N}$, that $N = (N_{ij})$ is an n-normal operator in $\mathcal{L}(\mathcal{H}^{(n)})$ as in Example 3.1, and that each $\sigma_e(N_{ij})$ is finite, for $i, j = 1, \ldots, n$. Then $SNS^{-1} + K \in$ (NHT) for every invertible $S \in \mathcal{L}(\mathcal{H}^{(n)})$ and every $K \in \mathbb{K}(\mathcal{H}^{(n)})$.*

Proof. As mentioned in Example 3.1, there is a compact operator (K_{ij}) in $\mathcal{L}(\mathcal{H}^{(n)})$ such that $N = (D_{ij}) + (K_{ij})$ where the D_{ij} are mutually commuting diagonal normal operators in $\mathcal{L}(\mathcal{H})$. Now recall (from [Pearcy 1978, Chapter 2], for example) that for each $i, j = 1, \ldots, n$, there is a diagonal normal operator E_{ij} and a diagonal compact operator L_{ij} such that $D_{ij} = E_{ij} + L_{ij}$ and the E_{ij} are mutually commuting diagonal normal operators of uniform infinite multiplicity (i.e., each E_{ij} is a finite direct sum of scalar multiples of identity operators on infinite dimensional subspaces). Thus

$$N = (D_{ij}) + (K_{ij}) = (E_{ij}) + (L_{ij}) + (K_{ij}),$$

and writing $K \in \mathcal{L}(\mathcal{H}^{(n)})$ for the compact operator $(L_{ij}) + (K_{ij})$, we get $N = (E_{ij}) + K$. We next observe that there exists an operator $J \in \mathcal{L}(\mathcal{H}^{(n)})$ of finite rank such that $(E_{ij}) + J$ is unitarily equivalent to an operator matrix of the form $(\lambda_{ij} 1_{\mathcal{H}}) \in \mathcal{L}(\mathcal{H}^{(n)})$ where the $\lambda_{ij} \in \mathbb{C}$. We sketch the argument in the simplest case in which $n = 2$ and the (diagonal, mutually commuting) normal operators

E_{11}, E_{12}, E_{21}, and E_{22} each has two distinct eigenvalues. We regard E_{ij} as a function $e_{ij} : \mathbb{N} \to \sigma(E_{ij})$. Trivially, $\mathbb{N}$ may be partitioned into at most $2^{2\times 2} = 16$ disjoint subsets $\mathbb{N}_r$ such that each of the four numbers e_{11}, e_{12}, e_{21}, and e_{22} is constant on each $\mathbb{N}_r$.

If all these sets $\mathbb{N}_r$ are infinite (corresponding to an infinite subset of the diagonalizing orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for $\mathscr{H}$), the result is clear with no finite rank perturbation needed. If, on the other hand, some particular $\mathbb{N}_r$ is a finite set, change the definition of the functions e_{ij} on this set so that in the partition of $\mathbb{N}$ corresponding to the perturbed functions, the corresponding set is no longer finite. Clearly each such finite set of the partition can be removed by a finite rank perturbation of the E_{ij}.

Finally, we have

$$N \simeq (\lambda_{ij} 1_{\mathscr{H}}) + (K - J),$$

where $K - J$ is compact and $(\lambda_{ij} 1_{\mathscr{H}}) \simeq (\lambda_{ij}) \otimes 1_{\mathscr{H}}$, so the result follows from Proposition 3.3 by setting $A = 1_{\mathscr{H}}$ and $T = N + J - K$. $\square$

4. Some examples and open questions

We now set forth some examples that illustrate difficulties with the problem of showing that (NHT) $= \mathscr{L}(\mathscr{H})$.

Looking at Theorem 1.2, one might reasonably hope that if T is essentially power bounded and $0 \in \sigma_{le}(T)$, then there would exist a nonzero vector $x \in \mathscr{H}$ such that the sequence $\{\|T^n x\|\}_{n\in\mathbb{N}}$ is bounded. Unfortunately the following example shows that this is not the case.

Example 4.1. For $k \in \mathbb{N}$, let $\mathscr{H}_k$ be a copy of $\mathscr{H}$, and set $\mathscr{H}^{(\infty)} = \bigoplus_{k\in\mathbb{N}} \mathscr{H}_k$. Let, for each $k \in \mathbb{N}$, $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathscr{H}_k$, and for each such k, let U_k be the unilateral weighted shift on $\mathscr{H}_k$ defined by

$$U_k f_j^{(k)} = \begin{cases} (1/k) f_2^{(k)} & \text{if } j = 1, \\ (1 + 1/(k+j)) f_{j+1}^{(k)} & \text{if } j \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Define $U = \bigoplus_{k\in\mathbb{N}} U_k$, and note that since $\|U f_1^{(k)}\| = 1/k$ for all $k \in \mathbb{N}$ and the set $\{f_1^{(k)}\}_{k\in\mathbb{N}}$ is orthonormal, $0 \in \sigma_{le}(U)$. Moreover, it is easy to see that U is a compact perturbation of the weighted forward unilateral shift U_f of infinite multiplicity which has 0 for its first weight and 1 for all other weights. It follows that $\|U\|_e = 1$ and hence that U is essentially power bounded. On the other hand, it is an easy consequence of the fact that the series $\sum_{n\in\mathbb{N}} \ln(1 + 1/n)$ diverges to ∞ that for each $k \in \mathbb{N}$ and each nonzero vector $x_k \in \mathscr{H}_k$, the sequence $\{\|U_k^n x_k\|\}_{n\in\mathbb{N}}$ diverges to ∞. Thus the orbit $\mathbb{O}(x, U)$ of any nonzero x in $\mathscr{H}^{(\infty)}$ under U also diverges in norm to ∞, as promised.

Example 4.2. One naturally looks for transforms of an operator T that preserve the property of membership in (NHT). In this connection, recall that if $T = UP$ is the polar decomposition of T, then *the Aluthge transform* of T is the operator $\tilde{T} = P^{1/2}UP^{1/2}$. This transform has been much studied; see [Jung et al. 2000], for example. In the present context, *if* $\tilde{T}$ *lies in* (NHT), *then so does* T. *Moreover*, *if* T *is a quasiaffinity and* T *lies in* (NHT), *then so does* $\tilde{T}$. These facts follow easily from the equations

$$(\tilde{T})^n = P^{1/2}T^{n-1}UP^{1/2} \quad \text{and} \quad T^n = UP^{1/2}(\tilde{T})^{n-1}P^{1/2} \quad \text{for } n \in \mathbb{N}.$$

Example 4.3. Call a countable set $\{T_n\}_{n\in\mathbb{N}}$ of operators in $\mathcal{L}(\mathcal{H})$ hypertransitive if the set $\{T_n x\}_{n\in\mathbb{N}}$ is dense in $\mathcal{H}$ for every nonzero x in $\mathcal{H}$. It is an easy exercise to see that there exist hypertransitive sets $\{F_n\}_{n\in\mathbb{N}}$ where each F_n has finite rank.

Example 4.4. Regarding Theorem 2.3, it is easy to see that, indeed, there are operators T which do not satisfy $r_e(T) = w_e(T)$ but for which some power T^n does satisfy $r_e(T^n) = w_e(T^n)$. For example, any noncompact operator T such that $T^2 = 0$ has this property.

Example 4.5. Let H be a Hermitian 2-normal operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ represented as an operator matrix

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where H_1 and H_2 are commuting Hermitian operators in $\mathcal{L}(\mathcal{H})$ satisfying $\sigma(H_i) = \sigma_e(H_i) = [0, 1]$, $i = 1, 2$, and for each $n \in \mathbb{N}$, let $\mathcal{H}_n^{(2)}$ be a copy of $\mathcal{H} \oplus \mathcal{H}$. Write $\mathcal{H}_\infty^{(2)} = \bigoplus_{n=1}^\infty \mathcal{H}_n^{(2)}$, and consider the 2-normal operator

$$T = \bigoplus_{n=2}^{\infty} \begin{pmatrix} (1-1/n)1_{\mathcal{H}} & (1/\sqrt{n})1_{\mathcal{H}} \\ 0 & (1-1/n)1_{\mathcal{H}} \end{pmatrix}$$

acting on $\bigoplus_{n=2}^\infty \mathcal{H}_n^{(2)}$. It is an exercise in calculus to see that for all $k \in \mathbb{K}$, $\|T^k\|_e$ satisfies

$$\text{(4–1)} \qquad \frac{k}{(2k-1)^{1/2}}\left(\frac{2k-2}{2k-1}\right)^{k-1} \le \|T^k\|_e \le 1 + \frac{k}{(2k-1)^{1/2}}\left(\frac{2k-2}{2k-1}\right)^{k-1}.$$

Now consider the 2-normal operator $H \oplus T$ acting on $\mathcal{H}_\infty^{(2)}$, let K be an arbitrary compact operator in $\mathcal{L}(\mathcal{H}_\infty^{(2)})$, and set $A = K + (H \oplus T)$. Then $\sigma_e(A) = \sigma_e(H \oplus T) = [0, 1]$, but because of (4–1), no power of A is essentially power bounded. Moreover $\|A^k\|_e = \|T^k\|_e \sim \sqrt{k}$, so the growth is too slow for Theorem 1.5 to be applied. Thus if K is such that $\sigma(A) = \sigma_e(A)$, none of the techniques above would seem to apply to the operator A.

Finally, we mention some open problems about nonhypertransitive operators that would seem to be of interest.

Problem 4.6. Does every compact perturbation of an n-normal operator belong to (NHT)?

Problem 4.7. If $T \in \mathcal{L}(\mathcal{H})$ is invertible and hypertransitive, must T^{-1} be hypertransitive too? (Recall that presently we don't know whether $T \in$ (NIS) implies that $T^{-1} \in$ (NIS) also, but we do know that T has a hypercyclic vector if and only if T^{-1} does.)

Problem 4.8. If $T \in \mathcal{L}(\mathcal{H})$ is such that for all nonzero x and y in $\mathcal{H}$, the set $\{(T^n x, y)\}_{n\in\mathbb{N}}$ is dense in $\mathbb{C}$, must T be hypertransitive? (Recall that an orbit $\mathbb{O}(T, x)$ is either dense in $\mathcal{H}$ or nowhere dense in $\mathcal{H}$, by a recent result of P. Bourdon and N. Feldman.)

Problem 4.9. Is it true that every hypertransitive operator T in $\mathcal{L}(\mathcal{H})$ must satisfy $\sigma(T) \subset \mathbb{T}$? Note that if T is invertible and $\sigma(T) \cap \mathbb{D} \neq \varnothing$, then at least $T^{-1} \in$ (NHT) by Corollary 1.6.

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References

[Ansari 1995] S. I. Ansari, "Hypercyclic and cyclic vectors", *J. Funct. Anal.* **128**:2 (1995), 374–383. MR 96h:47002 Zbl 0853.47013

[Beauzamy 1987] B. Beauzamy, "Opérateurs de rayon spectral strictement supérieur à 1", *C. R. Acad. Sci. Paris Sér. I Math.* **304**:10 (1987), 263–266. MR 88d:47004 Zbl 0606.47003

[Beauzamy 1988] B. Beauzamy, *Introduction to operator theory and invariant subspaces*, North-Holland Math. Library **42**, North-Holland, Amsterdam, 1988. MR 90d:47001 Zbl 0663.47002

[Bonsall and Duncan 1973] F. F. Bonsall and J. Duncan, *Numerical ranges, II*, London Math. Soc. Lecture Note Series **10**, Cambridge Univ. Press, New York, 1973. MR 56 #1063 Zbl 0262.47001

[Chevreau et al. 1998] B. Chevreau, W. S. Li, and C. Pearcy, "A new Lomonosov lemma", *J. Operator Theory* **40**:2 (1998), 409–417. MR 2000b:47014 Zbl 0999.47006

[Fillmore et al. 1972] P. A. Fillmore, J. G. Stampfli, and J. P. Williams, "On the essential numerical range, the essential spectrum, and a problem of Halmos", *Acta Sci. Math.* (*Szeged*) **33** (1972), 179–192. MR 48 #896 Zbl 0246.47006

[Grosse-Erdmann 1999] K.-G. Grosse-Erdmann, "Universal families and hypercyclic operators", *Bull. Amer. Math. Soc.* (*N.S.*) **36**:3 (1999), 345–381. MR 2000c:47001 Zbl 0933.47003

[Halmos 1970] P. R. Halmos, "Ten problems in Hilbert space", *Bull. Amer. Math. Soc.* **76** (1970), 887–933. MR 42 #5066 Zbl 0204.15001

[Hoover 1971] T. B. Hoover, "Hyperinvariant subspaces for n-normal operators", *Acta Sci. Math.* (*Szeged*) **32** (1971), 109–119. MR 46 #7930 Zbl 0228.47026

[Jung et al. 2000] I. B. Jung, E. Ko, and C. Pearcy, "Aluthge transforms of operators", *Integral Equations Operator Theory* **37**:4 (2000), 437–448. MR 2001i:47035 Zbl 0996.47008

[Lomonosov 1991] V. Lomonosov, "An extension of Burnside's theorem to infinite-dimensional spaces", *Israel J. Math.* **75**:2-3 (1991), 329–339. MR 93h:47007 Zbl 0777.47005

[Pearcy 1978] C. M. Pearcy, *Some recent developments in operator theory*, CBMS Regional Conference Series in Mathematics **36**, American Mathematical Society, Providence, R.I., 1978. MR 58 #7120 Zbl 0444.47001

[Pearcy and Salinas 1975] C. Pearcy and N. Salinas, "The reducing essential matricial spectra of an operator", *Duke Math. J.* **42**:3 (1975), 423–434. MR 52 #1354 Zbl 0336.46059

[Read 1988] C. J. Read, "The invariant subspace problem for a class of Banach spaces. II. Hypercyclic operators", *Israel J. Math.* **63**:1 (1988), 1–40. MR 90b:47013 Zbl 0782.47002

[Simonič 1996] A. Simonič, "An extension of Lomonosov's techniques to non-compact operators", *Trans. Amer. Math. Soc.* **348**:3 (1996), 975–995. MR 96j:47005 Zbl 0856.47007

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