Pacific Journal of Mathematics

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TOMOMI KAWAMURA

Volume 220 No. 2 June 2005

PACIFIC JOURNAL OF MATHEMATICS
Vol. 220, No. 2, 2005

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A'Campo, Gibson and Ishikawa constructed links associated to immersed compact one-manifolds in a two-dimensional disk, and determined the gordian numbers for links of immersed arcs. We determine the gordian numbers for links associated with certain immersed circles.

1. Introduction

A'Campo [1999] constructed links of divides as an extension of the class of algebraic links, that is, links of singularities of algebraic curves. By his argument, a link is associated to any immersed intervals (and circles) in a disk as follows.

Let $D = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid \|x\|^2 = x_1^2 + x_2^2 \leq 1\}$ be the unit disk in the real plane $\mathbb{R}^2$. A *divide* P is a generic relative immersion in the unit disk $(D, \partial D)$ of a finite number of 1-manifolds, that is, copies of the unit interval $(I, \partial I)$ and the unit circle [A'Campo 1998a, 1999; Hirasawa 2002; Ishikawa 2001b]. We also call the image of such an immersion a *divide*. A *branch* of P is an image of the copies; we shall call each image of the copies of the interval an *interval branch*, and each image of the copies of the circle a *circle branch*. We shall call a divide with only circle branches a *circle divide*, and call a divide with only interval branches an *interval divide*. In this paper, we mainly consider circle divides.

Let $T_x X$ be the tangent space at a point x of a manifold X, and TX be the tangent bundle over a manifold X. We identify the 3-sphere S^3 with the set

$$ST\mathbb{R}^2 = \{(x, u) \in T\mathbb{R}^2 \mid x \in \mathbb{R}^2, u \in T_x\mathbb{R}^2, \|x\|^2 + \|u\|^2 = 1\}.$$

The *link of a divide* P is the set given by

$$L(P) = \{(x, u) \in ST\mathbb{R}^2 \mid x \in P,\ u \in T_x P\}.$$

MSC2000: 57M25, 57M27.

Keywords: divide link, free divide link, gordian number, four-dimensional clasp number.

The author was partially supported by Grant-in-Aid for Young Scientists (B) (No. 15740044), from the Ministry of Education, Culture, Sports, Science and Technology.

We orient the 3-sphere and the link $L(P)$ as follows. We identify the tangent bundle $T\mathbb{R}^2 = \mathbb{R}^4$ with the 2-dimensional complex space $\mathbb{C}^2$ by the map

$$\big((x_1, x_2), (u_1, u_2)\big) \mapsto \big(x_1 + \sqrt{-1}u_1, x_2 + \sqrt{-1}u_2\big).$$

The tangent bundle $T\mathbb{R}^2$ is oriented by the complex orientation of $\mathbb{C}^2$, and the 3-sphere is naturally oriented by the complex orientation of the 4-ball

$$\big\{(x, u) \in T\mathbb{R}^2 \mid x \in \mathbb{R}^2, u \in T_x\mathbb{R}^2, \|x\|^2 + \|u\|^2 \leq 1\big\}.$$

Let $[a, b]$ be a small interval with $a < b$. Let $\phi : [a, b] \to D$ be an embedding whose image lies on P. We orient a part of the link $L(P)$ as the image of the map

$$t \mapsto \left(\phi(t), \frac{\sqrt{1 - \|\phi(t)\|^2}}{\|\dot{\phi}(t)\|}\dot{\phi}(t)\right),$$

where $\dot{\phi}(t)$ is the differential of $\phi(t)$. We can extend this orientation to $L(P)$. A *divide link* is the oriented link ambient isotopic to the link of some divide.

The *gordian number* (or *unknotting number*) of an r-component link L is the minimum number of crossing changes needed to obtain the trivial link. The *4-dimensional clasp number* of L is the minimum number of double points for r transversely-immersed disks in D^4 with boundary L and with only finite double points as singularities [Kawamura 2002b; 2002a]. A'Campo [1998a] showed that the gordian number (and also, as we note in Section 3, the 4-dimensional clasp number) of any interval divide link is equal to the number of double points of the divide. Gibson and Ishikawa [2002] defined free divides and their links, and gave an unknotting algorithm for free divides with only interval branches; we review their arguments in Section 6. In this paper, we give an 'unknotting operation' for certain circle divides, and determine the gordian number and the 4-dimensional clasp number of the links of such divides.

We shall call embedded circles which all lie outside of all others, as illustrated in Figure 1, a *trivial circle divide*.

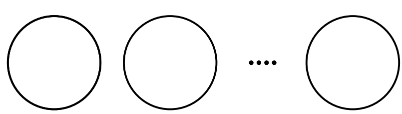

Figure 1. A trivial circle divide.

Theorem 1.1. *Suppose that a circle divide P can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk. Then the gordian number and the* 4*-dimensional clasp number of the link of P are equal to the sum of the number of circle branches and the number of double points of P.*

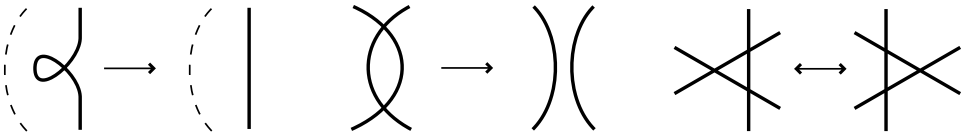

Figure 2. Transformations of divides. Dashed curves represent the boundary of the unit disk.

Ishikawa has asked, in a private communication to the author, whether the above theorem can be applied to slalom divides or ordered Morse divides; we answer this question in Sections 4 and 5.

In Section 6 we review the definition of a free divide and its link defined by Gibson and Ishikawa [2002], and extend Theorem 1.1 to certain free divides.

There are some circle divides to which we cannot apply Theorem 1.1, and we study these in Section 7.

2. Hirasawa's visualization of links of divides

Hirasawa [2002] has given an algorithm to draw a diagram of the link of a given divide, which we review and restate in this section.

We fix the coordinates (x_1, x_2) of D. A regular isotopy of P in the space of generic immersions does not change the isotopy type of the link $L(P)$, hence we may assume that divides are linear with slope ± 1 except near the 'corners', where a branch quickly changes its slope from ± 1 to ∓ 1. For a divide P we draw a link diagram by the following algorithm.

(1) For each branch B of P, we draw the boundary of a 'very small' neighborhood of B in the disk D, and assign it the clockwise orientation, as illustrated in Figure 3. In particular, we draw a 'hairpin curve' around each point of ∂P as illustrated on the right of Figure 3, where the dashed curve represents ∂D.

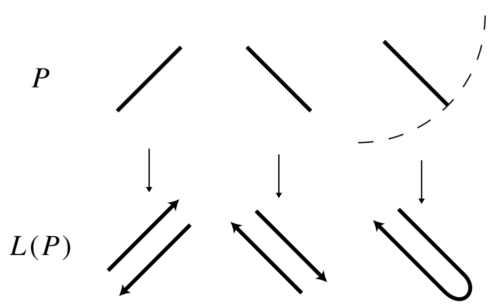

Figure 3. The first step of Hirasawa's visualization.

(2) Around double points, x_2-maximal points, and x_2-minimal points of P, we modify the diagram as illustrated in Figure 4, where dashed curves represent the boundary of D.

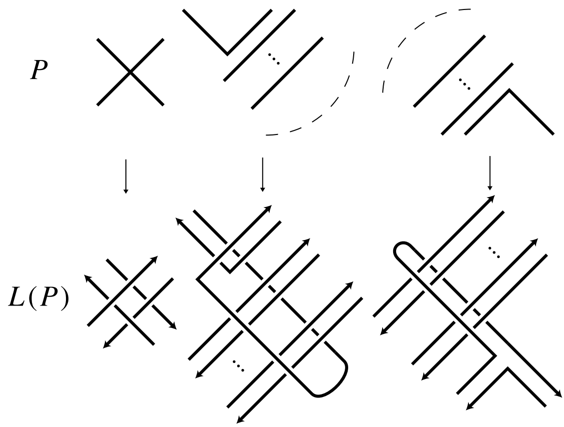

Figure 4. The second step of Hirasawa's visualization.

Proposition 2.1 [Hirasawa 2002]. *For any divide P, the diagram obtained by the above algorithm represents the link of P.* □

In the next section, we apply this algorithm to show that the link of a circle divide can be unknotted by changing as many crossings as the sum of the number of circle branches and the number of double points under the condition supposed in Theorem 1.1.

3. Gordian numbers of circle divide links

If we transform a divide as illustrated in Figure 2, left, we say that we *remove an outermost kink*. We call the second and third moves in Figure 2 a *divide self-tangency move* and a *divide triangle move* respectively. In this section, we study these three transformations and prove Theorem 1.1.

By using Hirasawa's visualization algorithm, we draw link moves associated to these transformations. Removing an outermost kink is associated with the local link move illustrated in Figure 5, top left. As Hirasawa showed [2002], this local move is obtained by changing one crossing. A divide self-tangency move is associated with the local link move illustrated in Figure 5, top right, which is obtained by changing two crossings. A divide triangle move is associated with the local link move illustrated in Figure 5, bottom. This local move does not change the link type.

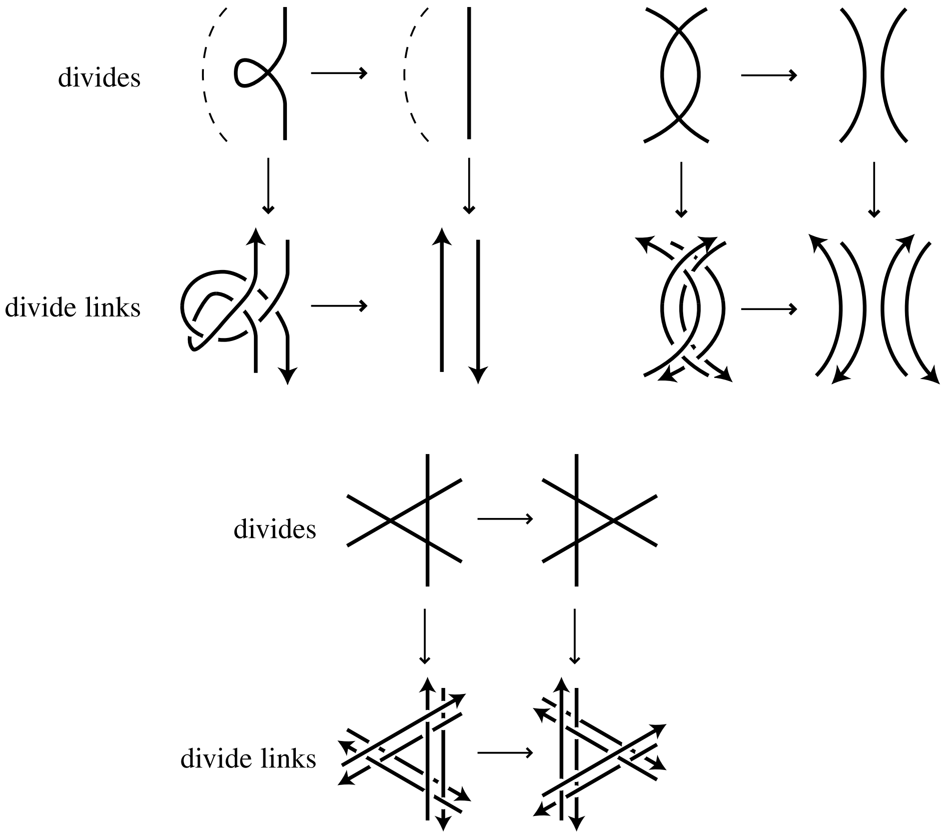

Figure 5. Link moves associated with: removal of an outermost kink (top left; dashed curves represent the boundary of D); divide self-tangency move (top right); and divide triangle move (bottom).

The link of a trivial circle divide is a split sum of as many copies of the Hopf link as the divide has circle branches. It is well known that the gordian number of a Hopf link is one, which leads to the following lemma.

Lemma 3.1. *Let P be a circle divide, and suppose that it can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk, where dashed curves represent the boundary of D. Then the gordian number of the link of P is at most the sum of the number of circle branches and the number of double points of P.* □

Let L be an oriented link, and let $F \subset D^4$ be a smooth, oriented 2-manifold with $\partial F = L$, where D^4 is the 4-ball bounded by S^3. We suppose that F has no closed components, but do not assume that F is connected. We denote by $\chi_s(L)$

the greatest value of the Euler characteristic $\chi(F)$ for such 2-manifolds $F \subset D^4$, and call this invariant the *slice Euler characteristic*.

Lemma 3.2 [Kawamura 2002a]. *For any r-component link, the 4-dimensional clasp number is not greater than the gordian number, and is not less than $\frac{1}{2}(r-\chi_s)$, where χ_s is the slice Euler characteristic of the given link.* □

In [Kawamura 2002c], the author proved the following result by showing the quasipositivity of divide links and free divide links and applying a result due to Rudolph [1993].

Proposition 3.3 [Kawamura 2002c]. *Let P be a divide or a free divide. Let r_1 be the number of interval branches of P, and δ be the number of double points of P. Then the slice Euler characteristic of the link of P is $r_1 - 2\delta$.* □

We explain the definition of a free divide and its links in Section 6.

Theorem 3.4. *Let P be a divide or a free divide. The gordian number and the 4-dimensional clasp number of the link of P are not less than the sum of the number of double points and the number of circle branches of P.*

Proof. Each interval branch of P represents one component of the link of P, and each circle branch of P represents two components of the link of P. By Lemma 3.2 and Proposition 3.3, the gordian number and the 4-dimensional clasp number of the link of P are not less than the sum of the number of double points and the number of circle branches of P. □

Ishikawa [2001a, Theorem 5.10], also showed this result, though he commented on only the gordian number.

Furthermore, by the results due to A'Campo [1998a] and Gibson and Ishikawa [2002], the 4-dimensional clasp number and the gordian number of the link of an interval divide or a free interval divide are equal to the number of the double points, as the author remarked in [Kawamura 2002c, 2004].

Proof of Theorem 1.1. Let P be a circle divide which can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk. The gordian number and the 4-dimensional clasp number of the link of P are (by Lemma 3.1) not greater than, and (by Theorem 3.4) not less than, the sum of the number of circle branches and the number of double points of P. □

4. Slalom circle divides

For a connected divide P, let the complementary regions $D - P$ be colored black and white in chessboard fashion. Let Γ (respectively Γ') be the graph constructed by taking for each black (respectively white) region a vertex, and for each double point of P an edge connecting the vertices which are taken for black (respectively white) regions abutting at that point. The divide P is a *slalom divide* if either Γ and Γ' is a tree (a *slalom tree* T_P) [A'Campo 1998b; Hirasawa 2002]. A slalom divide is a connected divide such that every arc of P except near double points is accessible from the boundary of the unit disk D without passing through P [Ishikawa 2001b]. We call a slalom divide with only circle branches a *slalom circle divide*. Both divides illustrated in Figure 6 are slalom circle divides.

Removing an outermost kink from a slalom circle divide P is associated with removing a vertex of degree 1 and the edge adjacent to it from T_P. Therefore any slalom circle divide can be transformed into a trivial circle divide by a finite sequence of moves removing outermost kinks, and diffeomorphisms of the unit disk. Then for any slalom circle divide, the gordian number and the 4-dimensional clasp number of its link are equal to the sum of the number of double points and the number of circle branches.

5. Ordered Morse circle divides

An *ordered Morse divide* is a divide whose x_1-coordinate has at most one maximal value and at most one minimal value as a function [Couture and Perron 2000]. We shall call an ordered Morse divide with only circle branches an *ordered Morse circle divide*. Couture and Perron [2000] showed that the link of any ordered Morse divide is the closure of a positive braid, and that any algebraic link is the link of some ordered Morse divide. In [Kawamura 2002a], the author showed that the gordian number and the 4-dimensional clasp number for any closed positive braid with r components are both equal to $\frac{1}{2}(r - \chi_s)$. By combining this result with Proposition 3.3, these invariants of an ordered Morse circle divide are both equal to the sum of the number of circle branches and the number of the double points.

In this section, as an alternative proof of the above result, we show that the link of any ordered Morse circle divide can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk.

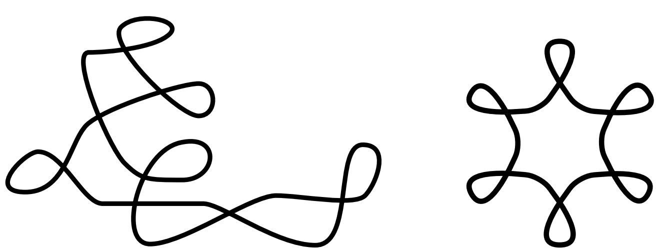

Figure 6. Slalom circle divides.

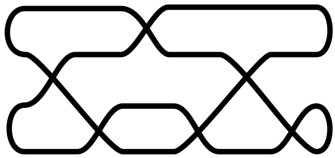

Figure 7. An ordered Morse circle divide.

Let P be an ordered Morse circle divide. Except near x_1-maximal or x_1-minimal points, P may be regarded as a braid diagram where all crossings are replaced with double points. Let δ_j be a symbol of a double point of P where the j-th lowest x_2-value arc intersects with the $(j+1)$-th lowest x_2-value arc. Let $a_1, \dots, a_k$ be the x_1-values of all double points of P. We may assume that distinct pairs of double points never have the same x_1-values, and that $a_1 < a_2 < \cdots < a_k$. If δ_{j_i} is the symbol of the double point with x_1-value a_i, then we may represent P as the word $\delta_{j_1} \dots \delta_{j_k}$. Let $\mathbf{D}_n$ be the set of all such products of $\delta_1, \dots, \delta_{n-1}$ with relations

$$\delta_i \delta_j = \delta_j \delta_i \qquad \text{if } |i-j| > 1, \tag{1}$$

$$\delta_i \delta_j \delta_i = \delta_j \delta_i \delta_j \qquad \text{if } |i-j| = 1. \tag{2}$$

The second relation is given by a divide triangle move. For example, the ordered Morse circle divide illustrated in Figure 7 is represented as the word $\delta_2\delta_1\delta_3\delta_1\delta_2\delta_1$ where $\delta_1, \delta_2, \delta_3 \in \mathbf{D}_4$. In the following, we regard $\mathbf{D}_k$ as the subset of $\mathbf{D}_n$ by the natural injection when $k < n$.

Proposition 5.1. *Any ordered Morse circle divide can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk.*

The argument of the proof is partially similar to that due to Rudolph [1983] where he gave a upper bound of the gordian number for any knot represented as the closure of a positive braid. Before the proof, we prepare some lemmas.

Lemma 5.2. *Let P be a connected, ordered Morse circle divide, and $W_P \in \mathbf{D}_{2m}$ be the word representing P. where m is the number of x_1-maximal points of P. We suppose $m > 1$ and that W_P does not include two consecutive letters equal under the relations of $\mathbf{D}_{2m}$. If W_P does not include δ_{2m-1}, then we can obtain an ordered Morse divide with $(m-1)$ x_1-maximal points from P by a finite sequence of divide triangle moves and removing an outermost kink.*

Proof. Since the given divide P is connected, W_P contains δ_{2m-2} at least once. We suppose that W_P contains δ_{2m-2} at least twice. Then W_P is of the form $\alpha_0\delta_{2m-2}w_0\delta_{2m-2}\beta_0$, where $w_0 \in \mathbf{D}_{2m-2}$ is nontrivial. By relation (1) and the above assumption for W_P, w_0 contains δ_{2m-3} at least once. By relation (1) we may assume that w_0 begins with δ_{2m-3} and ends with δ_{2m-3}. If $w_0 = \delta_{2m-3}$, we have $W_P = \alpha_0\delta_{2m-3}\delta_{2m-2}\delta_{2m-3}\beta_0$ by relation (2). Then the number of occurrences of the symbol δ_{2m-2} in W_P decreases.

Suppose that $w_0 \neq \delta_{2m-3}$. Then w_0 is of the form $\delta_{2m-3}w_1\rho_0$, where $w_1 \in \mathbf{D}_{2m-3}$ and $\rho_0 \in \mathbf{D}_{2m-2}$ begins and ends with δ_{2m-3}. We continue this process iteratively as long as possible, writing $w_i = \delta_{2m-3-i}w_{i+1}\rho_i$, where $w_{i+1} \in \mathbf{D}_{2m-3-i}$, and where ρ_i is either empty (in which case so is w_{i+1}) or begins and ends with δ_{2m-3-i}. Then we have

$$W_P = \alpha_0\delta_{2m-2}\delta_{2m-3}\dots\delta_{2m-2-i}\delta_{2m-3-i}\rho_{i-1}\dots\rho_0\delta_{2m-2}\beta_0$$

for some i, where ρ_{i-1} is not empty. Let $\rho_{i-1} = \delta_{2m-2-i}\rho'_{i-1}$. Then by relations (1) and (2) we have

$$W_P = \alpha_0\delta_{2m-3-i}\delta_{2m-2}\delta_{2m-3}\dots\delta_{2m-2-i}\delta_{2m-3-i}\rho'_{i-1}\rho_{i-2}\dots\rho_0\delta_{2m-2}\beta_0.$$

Hence we can decrease the length of w_0. We may thus assume that the length of w_0 is one; it contracted with $w_0 \neq \delta_{2m-3}$.

Therefore, if W_P does not include δ_{2m-1}, we may immediately assume that W_P contains δ_{2m-2}, that is $W_P = \alpha_0\delta_{2m-2}\beta_0$ for some $\alpha_0, \beta_0 \in \mathbf{D}_{2m-2}$. The ordered Morse circle divide represented as the word $\alpha_0\beta_0 \in \mathbf{D}_{2m-2}$ is obtained from P by removing an outermost kink adjacent to the double point represented by δ_{2m-2}. □

Lemma 5.3. *Let P be a connected, ordered Morse circle divide, and $W_P \in \mathbf{D}_{2m}$ be the word which represents P, where m is the number of x_1-maximal points of P. Suppose that $m > 1$, and that W_P does not include two consecutive letters equal under relations (1) and (2). If W_P includes δ_{2m-1} at least twice, then the number of occurrences of δ_{2m-1} in W_P is decreased by the relations (1) and (2) in $\mathbf{D}_{2m}$.*

Proof. The argument of the proof is same as that used for decreasing the number of occurrences of δ_{2m-2} in the proof of Lemma 5.2. □

Lemma 5.4. *Let P be a connected, ordered Morse circle divide, and $W_P \in \mathbf{D}_{2m}$ be the word representing P, where m is the number of x_1-maximal points of P. We suppose $m > 1$ and that W_P does not include two consecutive letters equal under relations (1) and (2). If W_P includes δ_{2m-1} once, then P can be transformed into a divide which has one of the parts of divides illustrated in Figure 8 by the relations (1) and (2) in $\mathbf{D}_{2m}$.*

Proof. For the given divide P, the word W_P has the form $\alpha\delta_{2m-1}\beta$ where α and β are words in $\mathbf{D}_{2m-1}$. By the same argument as that for decreasing the number of δ_{2m-2} in the proof of Lemma 5.2, we can show that if α or β includes δ_{2m-2} at least twice, then the number of occurrences of δ_{2m-2} decreases by relations (1) and (2) in $\mathbf{D}_{2m-1}$. If each of α and β includes δ_{2m-2} at most once, then P contains one of the parts of divides illustrated in Figure 8. □

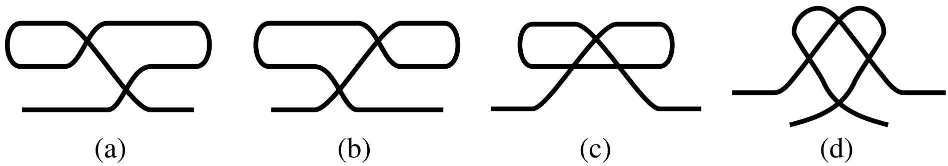

Figure 8. Parts of ordered Morse divides whose words include δ_{2m-1} once.

If we apply divide self-tangency moves to the parts illustrated in Figure 8 (d), the new divide may not be an ordered Morse divide. So, we define a new set of words which represent such divides. For an ordered Morse circle divide, we replace some of the double points represented by the symbol δ_{n-1} with the copies of the part illustrated on the left in Figure 9, which we represent by the symbol ε_{n-1}. Let $\mathbf{D}_n(\varepsilon)$ be the set of products of ε_{n-1} and words in $\mathbf{D}_n$ with relations

$$\varepsilon_{n-1}\delta_i = \delta_i\varepsilon_{n-1} \quad \text{for } 1 \le i \le n-3. \tag{3}$$

We shall call a divide represented by the word in $\mathbf{D}_n(\varepsilon)$ an *almost ordered Morse divide* with index n. For example, the circle divide illustrated on the right in Figure 9 is represented by the word $\delta_2\delta_4\delta_3\delta_1\varepsilon_5\delta_3\delta_2\delta_4\delta_3$ where $\delta_1, \dots, \delta_5 \in \mathbf{D}_6$. An almost ordered Morse circle divide with index 2 is isotopic to the disjoint sum of ordered Morse circle divides with one x_1-maximal point.

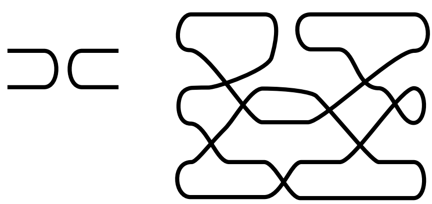

Figure 9. An almost ordered Morse circle divide and a part of it.

Lemma 5.5. *Let P be a connected, almost ordered Morse circle divide with index $2m$, and $W_P \in \mathbf{D}_{2m}(\varepsilon)$ the word that represents P. Suppose that $m > 1$, and that W_P does not contain two consecutive letters equal under relations (1)–(3) in $\mathbf{D}_{2m}(\varepsilon)$. Then we can obtain an almost ordered Morse circle divide with index $2(m-1)$ from P by a finite sequence of the transformations illustrated in Figure 2 and diffeomorphisms of the unit disk. Thus P can be transformed into a disjoint sum of ordered Morse circle divides with fewer than m x_1-maximal points, by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk.*

Proof. The word W_P is represented by the product $\alpha_1\varepsilon_{2m-1}\alpha_2 \dots \varepsilon_{2m-1}\alpha_k$ where $\alpha_1, \dots, \alpha_k \in \mathbf{D}_{2m}$. By the same argument as that in the proof of Lemma 5.2, we may assume that δ_{2m-1} appears at most once in each α_j. When α_j does not include δ_{2m-1}, by the same argument as that in the proof of Lemma 5.2, we may assume that δ_{2m-2} appears in α_j once, that is $\alpha_j = \beta_j\delta_{2m-2}\beta'_j$ where $\beta_j, \beta'_j \in \mathbf{D}_{2m-2}$.

Suppose that α_j includes δ_{2m-1} once, that is, $\alpha_j = \gamma_j\delta_{2m-1}\gamma'_j$ with γ_j, γ'_j in $\mathbf{D}_{2m-1}$. By the same argument as in the proof of Lemma 5.4, we may assume that each of γ_j and γ'_j includes δ_{2m-2} at most once. Then we may assume that $\alpha_j = \beta_j\delta'_j\beta'_j$, where β_j, β'_j lie in $\mathbf{D}_{2m-2}$ and δ'_j has the form $\delta_{2m-2}\delta_{2m-1}$ or $\delta_{2m-1}\delta_{2m-2}$ or $\delta_{2m-2}\delta_{2m-1}\delta_{2m-2}$ ($=\delta_{2m-1}\delta_{2m-2}\delta_{2m-1}$) or $\delta_{2m-2}\delta_{2m-1}\delta_{2m-3}\delta_{2m-2}$.

For each α_j, we define $\alpha'_j \in \mathbf{D}_{2m-2}$ as follows:

$$\alpha'_j = \begin{cases} \beta_j\beta'_j & \text{if } \alpha_j \in \big\{\beta_j\delta_{2m-2}\beta'_j,\ \beta_j\delta_{2m-2}\delta_{2m-1}\beta'_j, \\ & \qquad\qquad\qquad \beta_j\delta_{2m-1}\delta_{2m-2}\beta'_j,\ \beta_j\delta_{2m-2}\delta_{2m-1}\delta_{2m-2}\beta'_j\big\}, \\ \beta_j\varepsilon_{2m-3}\beta'_j & \text{if } \alpha_j = \beta_j\delta_{2m-2}\delta_{2m-1}\delta_{2m-3}\delta_{2m-2}\beta'_j. \end{cases}$$

The product $W' = \alpha'_1\alpha'_2\dots\alpha'_k$ is a word in $\mathbf{D}_{2m-2}$, representing an almost ordered Morse circle divide with index $2(m-1)$ as follows.

Each α_j represents an outermost kink or one of the parts illustrated in Figure 8. The divide represented by W' is obtained from P by a finite sequence of the transformations illustrated in Figure 2 and diffeomorphisms of the unit disk. Actually, we obtain α'_j from α_j by the following transformations.

We remove the outermost kink adjacent to the double point represented by δ_{2m-2} for $\alpha_j = \beta_j\delta_{2m-2}\beta'_j$, and we apply the divide self-tangency move to reduce two double points δ_{2m-1} and δ_{2m-2} for $\alpha_j = \beta_j\delta_{2m-2}\delta_{2m-1}\beta'_j$ or $\alpha_j = \beta_j\delta_{2m-1}\delta_{2m-2}\beta'_j$. For $\alpha_j = \beta_j\delta_{2m-2}\delta_{2m-1}\delta_{2m-2}\beta'_j$, we apply the triangle move as illustrated on the left in Figure 10, remove two new outermost kinks, and finally remove a new outermost kink. For $\alpha_j = \beta_j\delta_{2m-2}\delta_{2m-1}\delta_{2m-3}\delta_{2m-2}\beta'_j$, we apply the transformation illustrated on the right in Figure 10, where the divide self-tangency move is applied twice. □

Proof of Proposition 5.1. Let $W_P \in \mathbf{D}_{2m}$ represent a given ordered Morse circle divide P with m x_1-maximal points. We may assume that P is connected.

If $m = 1$, then P is an immersed circle and it is obvious that P can be transformed into a trivial divide by a finite sequence of divide self-tangency moves and removing outermost kinks.

Next we consider the case $m > 1$. We assume that any ordered Morse circle divide with fewer than m x_1-maximal points can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk. If P is an immersion of at least two circles, then it can be transformed into a disjoint sum of ordered Morse divides by divide self-tangency moves and divide triangle moves, with at most $(m-1)$ x_1-maximal points.

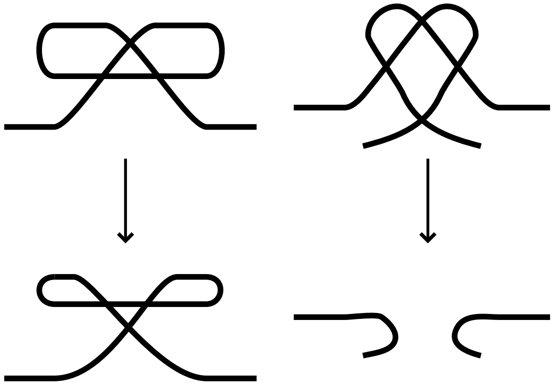

Figure 10. Transformations on certain (almost) ordered Morse divides.

Removing two consecutive equal letters from W_P gives a divide self-tangency move on P and decreases the number of double points of P. We assume that W_P does not contain two equal consecutive letters by relations (1)–(3) for $\mathbf{D}_{2m}(\varepsilon)$. By Lemma 5.3, we may assume that δ_{2m-1} appears in W_P at most once. By Lemma 5.2, if W_P does not include δ_{2m-1}, P can be transformed into an ordered Morse circle divide which has $(m-1)$ x_1-maximal points, by removing an outermost kink.

If W_P includes δ_{2m-1} once, we may assume, by Lemma 5.4, that P has one of the parts of divides illustrated in Figure 8. The argument for this case is almost the same as that in the proof of Lemma 5.5. If P has one of the parts illustrated in Figure 8 (a) or (b), then we can smooth two double points represented by $\delta_{2m-1}\delta_{2m-2}$ or $\delta_{2m-2}\delta_{2m-1}$ in W_P by a divide self-tangency move. The new divide is an ordered Morse circle divide with $(m-1)$ x_1-maximal points.

We suppose that the ordered Morse circle divide P contains the part illustrated in Figure 8 (c). We make a new divide P' from P by a divide triangle move as illustrated on the left in Figure 10. Though the word $W_{P'}$ which represents P' in the above rule includes δ_{2m-1} twice, we can remove both double points represented as δ_{2m-1} by removing two outermost kinks.

Therefore, for the above cases, the divide P can be transformed into an ordered Morse circle divide with $(m-1)$ x_1-maximal points, by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk.

If P contains the part illustrated in Figure 8 (d), we obtain an almost ordered Morse circle divide P'' with index $2(m-1)$ if $m > 2$, or a disjoint sum of ordered Morse circle divides with one x_1-maximal point if $m = 2$, by applying two divide self-tangency moves as illustrated on the right in Figure 10. By Lemma 5.5, P'' can be transformed into a disjoint sum of ordered Morse circle divides with fewer than $(m-1)$ x_1-maximal points by a finite sequence of the moves illustrated in

Figure 2 and diffeomorphisms of the unit disk. This proves the proposition for $m > 1$. □

Proposition 5.1 implies the following result by means of Theorem 1.1.

Corollary 5.6. *For any ordered Morse circle divide, the gordian number and the 4-dimensional clasp number of its link are equal to the sum of the number of double points and the number of branches.* □

By Proposition 5.1 and Lemma 5.5, we have the following result, which gives other examples of the divides which can be applied Theorem 1.1.

Proposition 5.7. *Any almost ordered Morse circle divide can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk. Therefore the gordian number and the 4-dimensional clasp number of the link of such a divide are equal to the sum of the number of double points and the number of branches.* □

The argument in this section implies that Theorem 1.1 can be applied to a large subset of circle divide links.

6. Free divide links

Free divides and their links were defined by Gibson and Ishikawa [2002]. A *free divide* P is a generic nonrelative immersion in the unit disk D of a finite number of 1-manifolds. The image of the boundary of the copies of the unit interval might not lie in the boundary of the unit disk. We call a point of such a boundary image a *free endpoint* of P.

Let E_P be the set of all free endpoints of P. At each free endpoint $x \in E_P$, there are two limits $v = (r\cos\theta, r\sin\theta)$ and $v' = (r\cos\theta', r\sin\theta')$ of tangent vectors of P, where $r = \sqrt{1 - \|x\|^2}$ and $\theta' = \theta \pm \pi$. Let $[a, b]$ be a small interval with $a < b$, and let $\phi : [a, b] \to P$ be an embedding with $\phi(b) = x$. We define

$$v = \lim_{t\to b} \frac{\sqrt{1 - \|\phi(t)\|^2}}{\|\dot{\phi}(t)\|}\dot{\phi}(t)$$

and define two paths from v to v' in $\{u \in T_x D \mid \|x\|^2 + \|u\|^2 = 1\}$ as follows:

$$l(P, x, +) = \left\{(r\cos(\theta + t), r\sin(\theta + t)) \in T_x D \;\middle|\; r = \sqrt{1 - \|x\|^2}, t \in [0, \pi]\right\}$$
$$l(P, x, -) = \left\{(r\cos(\theta - t), r\sin(\theta - t)) \in T_x D \;\middle|\; r = \sqrt{1 - \|x\|^2}, t \in [0, \pi]\right\}$$

We also denote by $l(P, x, \pm)$ the image of $l(P, x, \pm)$ under the map

$$T_x D \to TD; \quad u \mapsto (x, u).$$

A *link* of a free divide P is the set

$$L(P;\{\epsilon_x\}_{x\in E_P}) = \left\{(x,u)\in ST\mathbb{R}^2 \mid x\in P-E_P,\, u\in T_xP\right\} \cup \bigcup_{x\in E_P} l(P,x,\epsilon_x)$$

where ϵ_x is a sign $+$ or $-$ for the free endpoint x. The orientation of the link of a free divide is given by same argument as that of the link of a divide. We note that the link of the given free divide may not be defined uniquely and depends on the signs of the free endpoints, though Proposition 3.3 and Theorem 3.4 do not depend on signs. A *free divide link* is the oriented link ambient isotopic to the link of a free divide with some signs for the free endpoints. We note that replacing an endpoint in ∂D with a free endpoint accessible from ∂D does not change the link type. Then we may assume that all free endpoints are not accessible from ∂D.

If an interval branch has no free endpoint, then we shall call it a *relative branch*, otherwise it is said to be a *free branch*. A free branch is *even* if it has two free endpoints with same sign, and *odd* if not [Gibson and Ishikawa 2002]. We shall say that a branch is *trivial* if it includes no double points of a given divide.

By extending Hirasawa's visualization algorithm to a free divide, we obtain a diagram of its link. Except near free endpoints, we draw a diagram by the same algorithm as that for divides. Around each free endpoint, we modify the diagram as illustrated in Figure 11, where dashed curves represent ∂D (see [Kawamura 2002c]).

By this algorithm, we obtain the following lemma.

Lemma 6.1. *Let P be a free divide and I be a trivial interval branch of P. If I is odd or relative, then the link of I is a trivial component which is split from other components of the link of P.* □

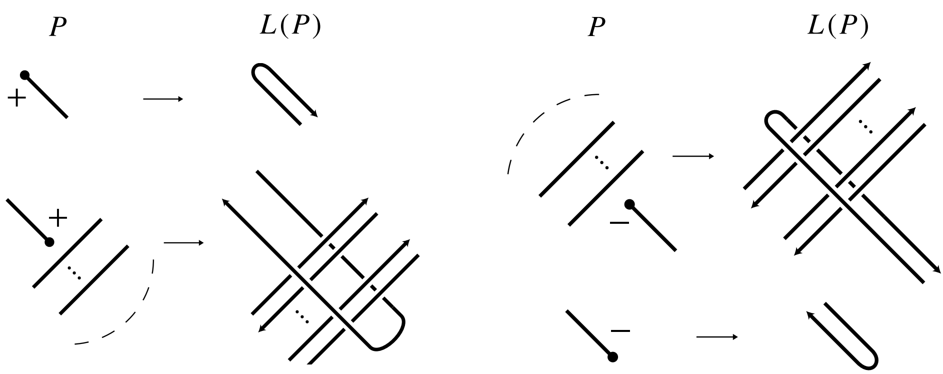

Figure 11. $l(P,x,\epsilon_x)$.

Gibson and Ishikawa [2002] showed that a *reduction*, that is the transformation on divides illustrated in Figure 12, is lifted to a crossing change on links as follows.

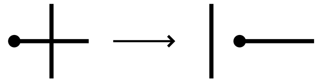

Figure 12. A reduction.

Lemma 6.2 [Gibson and Ishikawa 2002]. *Let P and P' be free divides. If P' is obtained from P by a reduction, and the signs of vertices of P' are equal to those of P, then $L(P'; \{\epsilon_x\}_{x \in E_{P'}})$ is obtained from $L(P; \{\epsilon_x\}_{x \in E_P})$ by changing one crossing.* □

The preceding lemmas yield the next theorem as an extension of Theorem 1.1.

Theorem 6.3. *Let P be a divide or a free divide and P' the circle divide obtained from P by removing all interval branches. We suppose that each interval branch is odd or can be transformed into a relative trivial branch by reductions, and that P' can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk. Then the gordian number and the 4-dimensional clasp number of any link of P are equal to the sum of the number of circle branches and the number of double points of P.*

The supposition for interval branches cannot be omitted in Theorem 6.3. For example, consider a small trivial interval branch I surrounded by a trivial circle branch C. If I is odd, the link $L(I \cup C, \{\epsilon_1, \epsilon_2\})$ is the split sum of a trivial knot and a Hopf link. Then $L(I \cup C, \{\epsilon_1, \epsilon_2\})$ has gordian number and 4-dimensional clasp number 1. If I is even, $L(I \cup C, \{\epsilon_1, \epsilon_2\})$ is a $(3, -3)$-torus link with the opposite orientation on one component. Then $L(I \cup C, \{\epsilon_1, \epsilon_2\})$ has gordian number and 4-dimensional clasp number 3 (see [Kawamura 2002a]).

Proof of Theorem 6.3. Let B be an interval branch of the given divide P. By Lemmas 6.1 and 6.2, the component of the link of P associated to B is transformed into a trivial component split from the other components by crossing changes associated with reductions on B. By repeating such moves, the link $L(P)$ is transformed into the split sum of $L(P')$ and a trivial link with same many components as the interval branches. By Theorem 1.1, the gordian number of $L(P')$ is equal to the sum of the number of circle branches and the number of double points of P'. Hence the gordian number and the 4-dimensional clasp number of $L(P)$ are not greater than the sum of the number of circle branches and the number of double points of P. We complete the proof by applying Theorem 3.4. □

This result gives examples which include circle branches and where the equality holds for the inequality in Theorem 3.4. In particular, Theorem 6.3 implies the following result.

Corollary 6.4. *Let P be a divide and P' be the circle divide obtained from P by removing all interval branches. We suppose that P' can be transformed into a trivial circle divide by a finite sequence of the moves illustrated in Figure 2 and diffeomorphisms of the unit disk. Then the gordian number and the 4-dimensional clasp number of the link of P are equal to the sum of the number of circle branches and the number of double points of P.* □

Let P be a slalom divide or an ordered Morse divide. By means of this result, the gordian number and the 4-dimensional clasp number of the link of P are equal to the sum of the number of circle branches and the number of double points of P, even if P has both of interval branches and circle branches.

7. A limaçon and concentric circles

We now consider two types of circle divide to which Theorem 1.1 cannot be applied.

Example 7.1. The divide illustrated in Figure 13 is obtained as a limaçon and none of the transformations illustrated in Figure 2 can be applied to it. Its link consists of two trivial knots with linking number 4. The 4-dimensional clasp number is not less than the linking number [Kawamura 2002b], hence both the gordian number and the 4-dimensional clasp number of the link of this divide are 4. Therefore they are not equal to the sum of the number of circle branches and the number of double points. Ishikawa [2001a] also remarked on this fact for the gordian number.

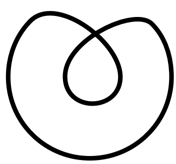

Figure 13. A limaçon.

Finally, we consider a nontrivial circle divide without double points.

Example 7.2. If a divide consists of n concentric circles, then its link consists of $2n$ trivial knots and the linking number of any two components is 1 or -1. Hence its gordian number and 4-dimensional clasp number are $2n^2 - n$, since the 4-dimensional clasp number is not less than the sum of the absolute values of the linking numbers for any two components [Kawamura 2002b]. Therefore they also differ from the sum of the number of circle branches and the number of double points if n is greater than 1.

Examples 7.1 and 7.2 show that Theorem 1.1 does not in general determine the gordian numbers and the 4-dimensional clasp numbers of circle divide links.

Acknowledgements

The author thanks Professor Masaharu Ishikawa for his helpful suggestions and useful comments.

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Received December 5, 2003. Revised September 10, 2004.

TOMOMI KAWAMURA
DEPARTMENT OF PHYSICS AND MATHEMATICS
AOYAMA GAKUIN UNIVERSITY
5-10-1 FUCHINOBE SAGAMIHARA
KANAGAWA 229-8558
JAPAN
tomomi@gem.aoyama.ac.jp