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We prove a holomorphic extension theorem for CR mappings between real algebraic submanifolds of $\mathbb{C}^N$. The source and the target manifolds are assumed to be generic submanifolds of equal dimension with source being connected, holomorphically nondegenerate, and having at least one point at which it is of finite type. The mapping H is assumed to be a smooth mapping which is a local diffeomorphism near at least one point of the source. In proving our main result, we construct a CR function which is nonzero at a given point of the source if and only if H is a local diffeomorphism near that same point.

Introduction

We present an extension theorem for CR mappings between generic algebraic submanifolds of $\mathbb{C}^N$ (definitions follow). Let M and M' be two such submanifolds, of equal dimension, with M connected and holomorphically nondegenerate. It is well known that every holomorphically nondegenerate hypersurface must have points at which it is of finite type. However, this need not be the case for generic submanifolds of higher codimension. Therefore, we also assume that the set of points at which M is of finite type is nonempty. With M and M' as described, we demonstrate that any smooth CR mapping $H : M \to M'$ whose Jacobian is not identically zero extends holomorphically to a neighborhood of M; see Theorem 1.

If M is a real manifold, its tangent bundle TM may be complexified to form $\mathbb{C}TM = \bigcup_{p\in M} \mathbb{C}\otimes T_pM$. Because the complex space $\mathbb{C}^N$ is itself a real manifold of dimension $2N$, the bundle $\mathbb{C}T\mathbb{C}^N$ is defined the same way. Given $p \in \mathbb{C}^N$, an element of $\mathbb{C}T_p\mathbb{C}^N$ is a (0, 1) *vector* at p if it is of the form

$$\sum_{j=1}^{N} a_j \frac{\partial}{\partial \bar{Z}_j}\Big|_p,$$

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where the a_j are complex numbers. If M is a submanifold of $\mathbb{C}^N$, then for each $p \in M$ we may define $\mathcal{V}_p$ to be the complex vector space consisting of all (0,1) vectors of $\mathbb{C}^N$ at p which are also in $\mathbb{C}T_pM$. A smooth submanifold M of $\mathbb{C}^N$ is a *CR submanifold* if $\dim_{\mathbb{C}} \mathcal{V}_p$ does not vary with the base point p. Moreover, M is *generic* if $\dim_{\mathbb{C}} \mathcal{V}_p = \dim_{\mathbb{R}} M - N$. The *complex bundle* of a CR submanifold M is the bundle whose fiber at $p \in M$ consists of the span formed by the real and the imaginary parts of all vectors in $\mathcal{V}_p$. A CR submanifold M is of *finite type* at $p \in M$ (in the sense of Kohn; see [Bloom and Graham 1977]) if the sections of the complex bundle together with their commutators of all lengths span the tangent space of M at p. Further, M is of *infinite type* at p if M is not of finite type at p. We say that X is a *germ of a holomorphic vector field* at $p \in \mathbb{C}^N$ if X is of the form

$$X = \sum_{j=1}^{N} a_j(Z) \frac{\partial}{\partial Z_j},$$

where the a_j are germs of holomorphic functions at p. A real-analytic generic submanifold M is *holomorphically nondegenerate* at $p \in M$ if there does not exist a nonzero germ of a holomorphic vector field at p tangent to M. We say that M is *holomorphically nondegenerate* if it is holomorphically nondegenerate at all $p \in M$. A real-analytic submanifold M of $\mathbb{C}^N$ is an *algebraic submanifold* if the local defining functions for M can be taken to be real-algebraic. A smooth function defined on a CR submanifold is said to be a *CR function* if its differential vanishes identically on $\mathcal{V}_p$ for all $p \in M$. A smooth mapping between CR submanifolds is a *CR map* if its components, in any holomorphic coordinates on $\mathbb{C}^N$, are themselves CR functions.

If M and M' are manifolds of equal dimension, we use $\operatorname{Jac} H$ to denote the determinant of the Jacobian matrix of the map H in specified coordinates on M and M'. Wherever the coordinates are not specified, the use of $\operatorname{Jac} H$ is restricted to expressing whether a map H is a local diffeomorphism.

Theorem 1. *Let M and M' be generic real algebraic submanifolds in $\mathbb{C}^N$ of the same dimension. Suppose that M is connected, holomorphically nondegenerate, and that M has a point at which it is of finite type. If $H : M \to M'$ is a CR map of class C^∞ with* $\operatorname{Jac} H \not\equiv 0$, *then H extends holomorphically to an open neighborhood of M in $\mathbb{C}^N$.*

This is a generalization of an earlier result for algebraic hypersurfaces proved by Baouendi, Huang, and Rothschild [Baouendi et al. 1996b]. Early extension results for mappings between real-analytic hypersurfaces appear in [Lewy 1977] and [Pinčuk 1974]. More general results for hypersurfaces were proved in [Diederich and Webster 1980; Baouendi et al. 1988; Baouendi and Rothschild 1988; 1990; 1991; Diederich and Fornæss 1988]. The survey [Forstnerič 1993] contains many other useful references. For generic submanifolds of $\mathbb{C}^N$, including hypersurfaces, there are results by Baouendi, Jacobowitz, and Treves [Baouendi et al. 1985]. Recent results by Coupet, Pinčuk, and Sukhov [Coupet et al. 1999a; 1999b] pertain to cases where M and M' are embedded in complex spaces of different dimensions.

The conditions of Theorem 1 are optimal in the sense that no assumptions may be dropped. From [Baouendi et al. 1996b, Theorem 4], it is clear that the hypothesis that M be holomorphically nondegenerate cannot be removed. Ebenfelt [1996] gave an example demonstrating that the algebraicity of both M and M' is an essential part of the hypothesis. Examples also exist showing that the conclusion of Theorem 1 can be false if Jac $H \equiv 0$ or if M has no points at which it is of finite type. An example of the former type is easily constructed.

To construct an example of the latter type, suppose (z, w_1, w_2) are global coordinates on $\mathbb{C}^3$, and let M be defined by

$$\operatorname{Im} w_1 = |z|^2, \quad \operatorname{Im} w_2 = 0.$$

Clearly M is a generic algebraic submanifold of $\mathbb{C}^3$ parametrized by z, $s_1 := \operatorname{Re} w_1$, and $s_2 := \operatorname{Re} w_2$. Using this parametrization we can define a map $H : M \to M$ by setting

$$H(z, s_1, s_2) = \bigl(z, s_1, s_2 + \exp(-1/s_2^2)\bigr).$$

Because the CR bundle of M has dimension one, all CR vector fields on M are multiples of the vector field

$$\Lambda := \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s_1}.$$

Thus, knowing that

$$H_\star(\Lambda_p) = \Lambda_{H(p)}$$

is sufficient to conclude that H is a CR map. Further, it is clear that H is invertible near the origin. Although one easily checks that M is holomorphically nondegenerate, H is not a real-analytic map, and cannot be extended to a holomorphic function on an open neighborhood of M in $\mathbb{C}^N$. Note that M has no points at which it is of finite type.

The paper is organized as follows. In Section 1, we introduce the flattening of a real-analytic generic submanifold of $\mathbb{C}^N$. The flattening is then used to give an explicit formula that can be used to locally define smooth extensions of CR functions. The essential properties of such extensions are established in Proposition 1.1. In Section 2, we use these results to show that near every point of M there exists a locally defined CR function whose zero set coincides with that of Jac H. In order to accomplish this, we introduce a notion of a holomorphic Jacobian of a smooth extension. The properties of holomorphic Jacobians of extensions satisfying an additional condition are examined in Lemmas 2.1 and 2.2. The main result of the section is Proposition 2.3. In Section 3, we use the results of Section 2 to construct a globally defined CR function whose zero set again coincides with that of Jac H; this is Proposition 3.1. In Section 4, we use Proposition 3.1 to show that the set of points at which Jac H is nonzero must be dense in M. This is the content of Proposition 4.4, which is followed by the proof of Theorem 1.

1. Smooth Extensions of CR Functions

Let M be a generic real-analytic submanifold of $\mathbb{C}^N$ of codimension d, and let $n := N - d$. Given $p_0 \in M$, we may choose coordinates $(z_1, \dots, z_n, w_1, \dots, w_d)$ on some neighborhood $\mathbb{O}$ of p_0 in $\mathbb{C}^N$, centered at p_0, such that

$$\mathbb{O} \cap M = \big\{ p \in \mathbb{O} \mid \operatorname{Im} w(p) = \phi(z(p), \bar{z}(p), \operatorname{Re} w(p)) \big\}, \tag{1–1}$$

where $z := (z_1, \dots, z_n)$, $w := (w_1, \dots, w_d)$, and where ϕ is a real-analytic function defined near 0 in $\mathbb{R}^{2n+d}$, valued in $\mathbb{R}^d$, such that $\phi(0) = 0$ and $d\phi(0) = 0$. These are *regular coordinates* for M at p_0. In addition to coordinates on $\mathbb{O}$, we also use coordinates $(z, \bar{z}, s_1, \dots, s_d, t_1, \dots, t_d)$ on $\mathbb{R}^{2n+2d}$. For brevity, set $s := (s_1, \dots, s_d)$ and $t := (t_1, \dots, t_d)$.

Because ϕ is a real-analytic function, there exists a neighborhood $\tilde{\mathbb{O}}$ of 0 in $\mathbb{R}^{2n+2d}$ on which $\phi(z, \bar{z}, s+it)$ is defined. By making sure that $\tilde{\mathbb{O}}$ is sufficiently small, we may, using coordinates specified above, define a mapping $\tilde{\Psi} : \tilde{\mathbb{O}} \to \mathbb{O}$ by

$$\tilde{\Psi}(z, \bar{z}, s, t) := \big(z,\ s + it + i\phi(z, \bar{z}, s+it)\big),$$

where we have used vector notation. Notice that $s_j = \operatorname{Re} w_j$ whenever $t = 0$. Since $\tilde{\Psi}(0) = 0$ and because $d\tilde{\Psi}(0)$ is clearly invertible, we may use the inverse function theorem to conclude that $\tilde{\Psi}$ is a real-analytic diffeomorphism provided that $\tilde{\mathbb{O}}$ and $\mathbb{O}$ are taken sufficiently small. Further, it follows from (1–1) that

$$\tilde{\Psi}^{-1}(\mathbb{O} \cap M) = \{(z, \bar{z}, s, t) \in \tilde{\mathbb{O}} \mid t = 0\}$$

and that $\mathbb{O} \cap M$ is parametrized by $\Psi := \tilde{\Psi}|_{t=0}$. We say that $\tilde{\Psi}$ is a real-analytic *flattening* for M near p_0.

Given a smooth function f on M, we may define its smooth extension F on $\mathbb{O}$ by setting

$$(F \circ \tilde{\Psi}) := (f \circ \Psi) + \sum_{l=1}^{d} i t_l \frac{\partial (f \circ \Psi)}{\partial s_l} - \frac{1}{2} \sum_{l,k=1}^{d} t_l t_k \frac{\partial^2 (f \circ \Psi)}{\partial s_l \partial s_k}, \tag{1–2}$$

where we have suppressed the arguments $(z, \bar{z}, s, t)$ of $\tilde{\Psi}$ and $(z, \bar{z}, s)$ of Ψ. Note that F is of second order in t.

Proposition 1.1. *Let M be a generic real-analytic submanifold of $\mathbb{C}^N$ and suppose that $f : M \to \mathbb{C}$ is a CR function. Given $p_0 \in M$, there exists a neighborhood $\mathbb{O}$ of p_0 in $\mathbb{C}^N$ and a smooth function $F : \mathbb{O} \to \mathbb{C}$ which extends f and for which, in any given set of holomorphic coordinates $(Z_1, \dots, Z_N)$ on $\mathbb{O}$,*

(i) $\displaystyle \frac{\partial}{\partial \bar{Z}_j} F\Big|_{M\cap\mathbb{O}} \equiv 0$ *for $j = 1, \dots, N$, and*

(ii) $\displaystyle \frac{\partial}{\partial Z_k}\frac{\partial}{\partial \bar{Z}_j} F\Big|_{M\cap\mathbb{O}} \equiv 0 \equiv \frac{\partial}{\partial \bar{Z}_k}\frac{\partial}{\partial \bar{Z}_j} F\Big|_{M\cap\mathbb{O}}$ *for $j, k = 1, \dots, N$.*

Proof. Clearly statements (i) and (ii) are independent of the choice of holomorphic coordinates. Thus, we may assume that $Z := (z, w)$, with z and w as in the beginning of this section. Let F be as in (1–2). An elementary calculation shows that

$$\begin{pmatrix} \tilde{\Psi}_\star^{-1}(\partial/\partial \bar{w}_1) \\ \vdots \\ \tilde{\Psi}_\star^{-1}(\partial/\partial \bar{w}_d) \end{pmatrix} = {}^\tau\big(I - i\phi_s(z, \bar{z}, s - it)\big)^{-1} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix},$$

where

$$\lambda_j = \frac{1}{2}\Big(\frac{\partial}{\partial s_j} - \frac{1}{i}\frac{\partial}{\partial t_j}\Big) \quad \text{for } j = 1, \dots, d. \tag{1–3}$$

Then

$$\Big(\frac{\partial}{\partial \bar{w}_j} F\Big) \circ \tilde{\Psi} = \tilde{\Psi}_\star^{-1}\Big(\frac{\partial}{\partial \bar{w}_j}\Big)(F \circ \tilde{\Psi}) = \sum_{m=1}^{d} \alpha_{jm}\lambda_m(F \circ \tilde{\Psi}),$$

where α_{jm} are some real-analytic functions. From (1–2) and (1–3) we get

$$\Big(\frac{\partial}{\partial \bar{w}_j} F\Big) \circ \tilde{\Psi} = -\sum_{m=1}^{d} \alpha_{jm}\Big(\sum_{l,k=1}^{d} \frac{t_l t_k}{4}\frac{\partial^3(f \circ \Psi)}{\partial s_m \partial s_l \partial s_k}\Big). \tag{1–4}$$

Immediately, we may observe that

$$\frac{\partial}{\partial \bar{w}_j} F\Big|_M \equiv 0 \quad \text{for } j = 1, \dots, d. \tag{1–5}$$

Let $\rho(z, \bar{z}, w, \bar{w}) := \operatorname{Im} w - \phi(z, \bar{z}, \operatorname{Re} w)$ be a defining function for M as in (1–1). Because $d\phi(0) = 0$, the matrix $\rho_{\bar{w}}$ can be seen to be invertible. In vector notation,

$$L := \frac{\partial}{\partial \bar{z}} - {}^\tau(\rho_{\bar{z}})\ {}^\tau(\rho_{\bar{w}})^{-1}\frac{\partial}{\partial \bar{w}} \tag{1–6}$$

is a column of antiholomorphic vector fields tangent to M. Using (1–5), it is then clear that

$$\frac{\partial}{\partial \bar{z}_j} F\Big|_M = L_j F\Big|_M \quad \text{for } j = 1, \dots, n.$$

Because $f = F|_M$ is a CR function, the right-hand side of this equality reduces to zero and we have

$$\frac{\partial}{\partial \bar{z}_j} F \Big|_M \equiv 0 \quad \text{for } j = 1, \ldots, n. \tag{1–7}$$

This, together with (1–5), implies statement (i) in Proposition 1.1.

It is clear that any first derivative of the right-hand side of (1–4) vanishes after it is restricted to $t = 0$. Therefore

$$\frac{\partial}{\partial Z_k} \frac{\partial}{\partial \bar{w}_j} F \Big|_M \equiv \frac{\partial}{\partial \bar{Z}_k} \frac{\partial}{\partial \bar{w}_j} F \Big|_M \equiv 0 \quad \text{for } j = 1, \ldots, d,\ k = 1, \ldots, N. \tag{1–8}$$

Using vector notation as in (1–6), let

$$R := \frac{\partial}{\partial Z} - {}^{\tau}(\rho_Z)\ {}^{\tau}(\rho_{\bar{w}})^{-1} \frac{\partial}{\partial \bar{w}}$$

be a column of vector fields. Just like the vector fields L_k, the vector fields R_k are also tangent to M. Therefore, (1–7) yields

$$\begin{aligned} &L_k \frac{\partial}{\partial \bar{z}_j} F \Big|_M \equiv 0 \quad \text{for } j, k = 1, \ldots, n, \\ &R_k \frac{\partial}{\partial \bar{z}_j} F \Big|_M \equiv 0 \quad \text{for } j = 1, \ldots, n,\ k = 1, \ldots, N. \end{aligned} \tag{1–9}$$

Since mixed partial derivatives commute, equality in (1–8) implies that

$$\frac{\partial}{\partial \bar{w}_k} \frac{\partial}{\partial \bar{z}_j} F \Big|_M \equiv 0 \quad \text{for } j = 1, \ldots, n,\ k = 1, \ldots, d.$$

From the definition of R and L it is then clear that

$$\begin{aligned} \frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j} F \Big|_M &\equiv L_k \frac{\partial}{\partial \bar{z}_j} F \Big|_M \quad \text{for } j, k = 1, \ldots, n, \\ \frac{\partial}{\partial Z_k} \frac{\partial}{\partial \bar{z}_j} F \Big|_M &\equiv R_k \frac{\partial}{\partial \bar{z}_j} F \Big|_M \quad \text{for } j = 1, \ldots, n,\ k = 1, \ldots, N. \end{aligned}$$

Finally, from (1–9) it is immediate that

$$\begin{aligned} &\frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j} F \Big|_M \equiv 0 \quad \text{for } j, k = 1, \ldots, n, \\ &\frac{\partial}{\partial Z_k} \frac{\partial}{\partial \bar{z}_j} F \Big|_M \equiv 0 \quad \text{for } j = 1, \ldots, n,\ k = 1, \ldots, N. \end{aligned} \tag{1–10}$$

Statement (ii) then follows from (1–8) and (1–10). □

Proposition 1.1(i) states that the antiholomorphic derivatives of F vanish on M. By part (ii), these derivatives vanish to first order. Had we required that F satisfy (i) but not necessarily (ii), we could have omitted the terms of order 2, in t, from the right-hand side of (1–2). Conversely, by adding terms of higher order, we could have obtained an extension whose antiholomorphic derivatives vanish to any given order along M. Extensions whose antiholomorphic derivatives are completely flat along M, called *almost holomorphic extensions*, exist as well; see [Nirenberg 1971]. However, the conclusion of Proposition 1.1, as stated, is sufficient for our purposes. Part (ii) is explicitly used only in Lemma 2.2.

2. The CR Version of the Jacobian

Let M and M' be generic submanifolds of $\mathbb{C}^N$ and assume that $H : M \to M'$ is a CR map. Suppose $\mathcal{H}$ is a given smooth extension of H to $\mathbb{C}^N$ near $p_0 \in M$. Assume that $Z = (Z_1, \ldots, Z_N)$ are coordinates on $\mathbb{C}^N$ near p_0 and $Z' = (Z'_1, \ldots, Z'_N)$ are coordinates on $\mathbb{C}^N$ near $H(p_0)$. We say that $\mathcal{H}$ is *holomorphic at points* of M if

$$\frac{\partial}{\partial \bar{Z}_k}(Z'_j \circ \mathcal{H})\Big|_M \equiv 0 \quad \text{for } j, k = 1, \ldots, N. \tag{2–1}$$

Because the components of H in the coordinates Z' are CR functions defined near p_0, they can be extended using Proposition 1.1. Such local extensions of H not only satisfy condition (2–1), as is clear from Proposition 1.1(i), but also satisfy

$$\frac{\partial}{\partial Z_l}\frac{\partial}{\partial \bar{Z}_k}(Z'_j \circ \mathcal{H})\Big|_M \equiv 0 \quad \text{for } j, k, l = 1, \ldots, N, \tag{2–2}$$

which follows from Proposition 1.1(ii).

Given coordinates Z and Z' as above, we may locally define the *holomorphic Jacobian* of $\mathcal{H}$ by

$$\mathrm{HolJac}_{(Z,Z')}(\mathcal{H}) := \begin{vmatrix} \frac{\partial}{\partial Z_1}(Z'_1 \circ \mathcal{H}) & \ldots & \frac{\partial}{\partial Z_N}(Z'_1 \circ \mathcal{H}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial Z_1}(Z'_N \circ \mathcal{H}) & \ldots & \frac{\partial}{\partial Z_N}(Z'_N \circ \mathcal{H}) \end{vmatrix}. \tag{2–3}$$

This makes sense for any smooth extension $\mathcal{H}$ of H, whether or not it satisfies any of the earlier conditions. Although different extensions satisfying (2–1) need not have the same holomorphic Jacobian, we show in the next two lemmas that these Jacobians agree when restricted to M, and that they become CR functions when restricted to M.

Lemma 2.1. *Suppose $\mathcal{H}^1$ and $\mathcal{H}^2$ are C^∞ extensions of H near p_0 which are holomorphic at points of M. Then near p_0 one has*

$$\mathrm{HolJac}_{(Z,Z')}(\mathcal{H}^1)|_M = \mathrm{HolJac}_{(Z,Z')}(\mathcal{H}^2)|_M.$$

Proof. Because M is generic, we can pick regular coordinates $Z = (z, w)$ for M near p_0. As in Section 1, let ρ be the defining function for M near p_0 valued in $\mathbb{R}^d$. Recall that

$$R = \frac{\partial}{\partial Z} - {}^{\tau}(\rho_Z)\, {}^{\tau}(\rho_{\bar{w}})^{-1} \frac{\partial}{\partial \bar{w}}$$

is a column of vector fields tangent to M. Then

$$\frac{\partial}{\partial Z_j}(Z'_k \circ \mathcal{H}^l)\Big|_M = R_j(Z'_k \circ \mathcal{H}^l)\Big|_M = R_j(Z'_k \circ H)\Big|_M \quad \text{for } l = 1, 2,\ j, k = 1, \ldots, N,$$

where the first equality holds because $\mathcal{H}^1$ and $\mathcal{H}^2$ are holomorphic at points of M, and the second because $\mathcal{H}^1|_M = \mathcal{H}^2|_M = H$ and because R_j are tangent to M. This shows that the holomorphic derivatives of $\mathcal{H}^1$ and $\mathcal{H}^2$ actually agree on M. □

Lemma 2.2. *If $\mathcal{H}$ is a C^∞ extension of H which is holomorphic at points of M, then* $\mathrm{HolJac}_{(Z,Z')}(\mathcal{H})|_M$ *is a CR function on M.*

Proof. Let $\mathcal{H}^A$ be an extension of H, defined near a given point p_0 in M, satisfying (2–2) as well as (2–1). As remarked above, such extensions always exist. From (2–2) we see immediately that

$$\frac{\partial}{\partial \bar{Z}_l} \frac{\partial}{\partial Z_j}(Z'_k \circ \mathcal{H}^A)\Big|_M \equiv 0 \quad \text{for } l, j, k = 1, \ldots, N. \tag{2–4}$$

Suppose L is a CR vector field on M. Let $\mathcal{L}$ be a C^∞ extension of L to a neighborhood of p_0 in $\mathbb{C}^N$ such that

$$\mathcal{L} = \sum_{l=1}^{N} \sigma_l(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}_l},$$

where the σ_l are C^∞ functions defined on a neighborhood of p_0 in $\mathbb{C}^N$. From (2–4) we observe that

$$\mathcal{L} \frac{\partial}{\partial Z_j}(Z'_k \circ \mathcal{H}^A)\Big|_M \equiv 0 \quad \text{for } j, k = 1, \ldots, N.$$

Because $L = \mathcal{L}|_M$ is tangent to M, this implies that

$$L\left(\frac{\partial}{\partial Z_j}(Z'_k \circ \mathcal{H}^A)\Big|_M\right) \equiv 0 \quad \text{for } j, k = 1, \ldots, N,$$

and we conclude that the holomorphic derivatives of $\mathcal{H}^A$, when restricted to M, are CR functions. Sums and products of CR functions are CR, so $\mathrm{HolJac}_{(Z,Z')}(\mathcal{H}^A)|_M$ is also a CR function. We finish the proof by applying Lemma 2.1. □

Recall that $\mathrm{HolJac}_{(Z,Z')}(\mathcal{H})|_M$ depends only on H and not on the choice of $\mathcal{H}$. Therefore, whenever the holomorphic coordinates are specified, $\mathrm{HolJac}(\mathcal{H})|_M$ is a complex-valued CR function naturally associated to H. Similarly, whenever the local coordinates are specified, Jac H is a real-valued function naturally associated to H. It turns out that the holomorphic Jacobian vanishes if and only if the real Jacobian vanishes:

Proposition 2.3. *Let M and M' be real-analytic generic submanifolds of $\mathbb{C}^N$ of equal dimension. Suppose $H : M \to M'$ is a smooth CR map. Given $p_0 \in M$, there exists a neighborhood $\mathcal{O}$ of p_0 in $\mathbb{C}^N$, a neighborhood $\mathcal{O}'$ of $H(p_0)$ in $\mathbb{C}^N$, and a smooth function $J^H : \mathcal{O} \to \mathbb{C}$ satisfying the following conditions*:

(i) *If $\mathcal{H}$ is a C^∞ extension of H to an open set $\mathcal{Y}$ in $\mathbb{C}^N$ which is holomorphic at points of M, there are local coordinates Z on $\mathcal{O}$ and Z' on $\mathcal{O}'$ such that*

$$\operatorname{HolJac}_{(Z,Z')}(\mathcal{H})|_{\mathcal{Y}\cap\mathcal{O}\cap M} \equiv J^H|_{\mathcal{Y}\cap\mathcal{O}\cap M}.$$

(ii) *$J^H|_{\mathcal{O}\cap M}$ is a CR function.*

(iii) *If $p \in \mathcal{O} \cap M$, then $J^H(p) = 0 \iff$* Jac $H(p) = 0$.

Proof. Let d be the codimension of M and M' in $\mathbb{C}^N$, and let $n := N - d$. As in Section 1, choose a neighborhood $\mathcal{O}$ of p_0 in $\mathbb{C}^N$ with regular coordinates (z, w) for M at p_0. Then

$$\mathcal{O} \cap M = \{p \in \mathcal{O} \mid \operatorname{Im} w(p) = \phi(z(p), \bar{z}(p), \operatorname{Re} w(p))\}, \tag{2–5}$$

where ϕ is a real-analytic function. Recall that there is a neighborhood $\tilde{\mathcal{O}}$ of 0 in $\mathbb{R}^{2n+2d}$ on which the flattening $\tilde{\Psi}$ is a diffeomorphism onto $\mathcal{O}$. Let $(z, \bar{z}, s, t)$ be coordinates on $\tilde{\mathcal{O}}$ as in Section 1.

Because M' is a real-analytic submanifold, Theorem 4.2.6 of [Baouendi et al. 1999] implies that there exists a neighborhood $\mathcal{O}'$ of $H(p_0)$ in $\mathbb{C}^N$ with regular coordinates (z', w') for M' at $H(p_0)$ such that

$$\begin{aligned}\mathcal{O}' \cap M' &= \{p \in \mathcal{O}' \mid \operatorname{Im} w'(p) = \phi'(z'(p), \bar{z}'(p), \operatorname{Re} w'(p))\}\\ &= \{p \in \mathcal{O}' \mid w'(p) = Q(z'(p), \bar{z}'(p), \bar{w}'(p))\}\\ &= \{p \in \mathcal{O}' \mid \bar{w}'(p) = \bar{Q}(\bar{z}'(p), z'(p), w'(p))\},\end{aligned}$$

where ϕ' is a real-analytic function defined near 0 in $\mathbb{R}^{2n+d}$, valued in $\mathbb{R}^d$, and Q is a holomorphic function defined in a neighborhood of 0 in $\mathbb{C}^{2n+d}$, valued in $\mathbb{C}^d$, such that $Q(z, 0, \tau) = \tau$ and $\bar{Q}(0, \chi, \tau) = \tau$. The coordinates (z', w'), as above, are referred to as *normal coordinates* for M' at $H(p_0)$.

Without loss of generality, assume that $\mathcal{O}$ is small enough that $H(\mathcal{O} \cap M) \subset \mathcal{O}'$. The components of H in the coordinates above are then

$$\begin{aligned} f_j &:= z'_j \circ H \qquad \text{for } j = 1, \dots, n,\\ g_j &:= w'_j \circ H \qquad \text{for } j = 1, \dots, d,\end{aligned}$$

and may be thought of as complex-valued functions of $(z, \bar{z}, s)$. If we let

$$F_j : \mathcal{O} \to \mathbb{C} \quad \text{for } j = 1, \dots, n,$$
$$G_j : \mathcal{O} \to \mathbb{C} \quad \text{for } j = 1, \dots, d$$

be extensions of f_j and g_j to all of $\mathcal{O}$ as in (1–2), we obtain a smooth extension $\mathcal{H}$ of H such that

$$z'_j \circ \mathcal{H} := F_j \qquad \text{for } j = 1, \dots, n,$$
$$w'_j \circ \mathcal{H} := G_j \qquad \text{for } j = 1, \dots, d.$$

By further shrinking the neighborhood $\mathcal{O}$, if necessary, we may assume that $\mathcal{H}$ is defined on all of $\mathcal{O}$.

The map $\mathcal{H}$ is constructed as in the beginning of this section, albeit with a more particular choice of coordinates, and is therefore holomorphic at points of M. Thus, it makes sense to define $J^H : \mathcal{O} \to \mathbb{C}$ by setting

$$J^H := \operatorname{HolJac}_{((z,w),(z',w'))}(\mathcal{H}).$$

With this definition, (i) and (ii) in the statement of the proposition follow immediately from Lemmas 2.1 and 2.2. It is therefore only left to establish part (iii). As in the proof of Proposition 1.1, we introduce the CR vector fields L_j defined near $p_0 \in M$. The pullbacks $\Lambda_j := \Psi_\star^{-1}(L_j)$ are then the rows of the column vector

$$\Lambda := \frac{\partial}{\partial \bar{z}} - i\ {}^\tau(\phi_{\bar{z}})\ {}^\tau(I + i\phi_s)^{-1} \frac{\partial}{\partial s}. \tag{2–6}$$

This is clear from (1–1), (1–6), and the explicit form of Ψ.

Claim 2.4. *With J^H defined as above,*

$$J^H|_M = \frac{1}{\det(I + i\phi_s)} \begin{vmatrix} \bar{\Lambda}_1 f_1 & \dots & \bar{\Lambda}_n f_1 \\ \vdots & & \vdots \\ \bar{\Lambda}_1 f_n & \dots & \bar{\Lambda}_n f_n \end{vmatrix} \det \mathcal{U}^H,$$

where $\mathcal{U}^H$ is a $d \times d$ matrix with entries

$$(\mathcal{U}^H)_{pq} := \frac{\partial}{\partial s_q} g_p - \sum_{k=1}^{n} Q_{p z'_k} \frac{\partial}{\partial s_q} f_k$$

and Λ_j are CR vector fields given by (2–6).

Proof. As in the proof of Proposition 1.1,

$$\begin{pmatrix} \tilde{\Psi}_\star^{-1}(\partial/\partial w_1) \\ \vdots \\ \tilde{\Psi}_\star^{-1}(\partial/\partial w_d) \end{pmatrix} = {}^\tau\big(I + i\phi_s(z, \bar{z}, s{+}it)\big)^{-1} \begin{pmatrix} \bar{\lambda}_1 \\ \vdots \\ \bar{\lambda}_d \end{pmatrix},$$

where

$$\bar{\lambda}_j = \frac{1}{2}\Big(\frac{\partial}{\partial s_j} + \frac{1}{i}\frac{\partial}{\partial t_j}\Big) \quad \text{for } j = 1, \dots, d.$$

Let k be one of f_j or g_j and suppose that K is its extension given by (1–2). If we let $\bar{\alpha} := {}^{\tau}\big(I + i\phi_s(z, \bar{z}, s+it)\big)^{-1}$, then

$$\frac{\partial}{\partial w_j} K\Big|_M = \tilde{\Psi}_\star^{-1}\Big(\frac{\partial}{\partial w_j}\Big)(K \circ \tilde{\Psi})\Big|_{t=0} = \sum_{m=1}^{d} \bar{\alpha}_{jm}\bar{\lambda}_m (K \circ \tilde{\Psi})\Big|_{t=0}$$
$$= \sum_{m=1}^{d} \frac{\bar{\alpha}_{jm}}{2}\Big(\frac{\partial}{\partial s_m} + \frac{1}{i}\frac{\partial}{\partial t_m}\Big)(K \circ \tilde{\Psi})\Big|_{t=0}.$$

By using (1–2) and subsequently differentiating, we get

$$\frac{\partial}{\partial w_j} K\Big|_M = \Big(\sum_{m=1}^{d} \bar{\alpha}_{jm}\frac{\partial}{\partial s_m}\Big)k\,\Big|_{t=0}. \tag{2–7}$$

An elementary calculation shows that there exist functions γ'_{jl} and γ''_{jl} such that

$$\tilde{\Psi}_\star^{-1}\Big(\frac{\partial}{\partial z_j}\Big) = \frac{\partial}{\partial z_j} - \sum_{l=1}^{d} \gamma'_{jl}\frac{\partial}{\partial s_l} - \sum_{l=1}^{d} \gamma''_{jl}\frac{\partial}{\partial t_l},$$

from which we conclude as above that there also exist functions γ_{jl} such that

$$\frac{\partial}{\partial z_j} K\Big|_M = \Big(\frac{\partial}{\partial z_j} - \sum_{l=1}^{d} \gamma_{jl}\frac{\partial}{\partial s_l}\Big)k\,\Big|_{t=0}. \tag{2–8}$$

The specific form of $\mathscr{H}$ therefore allows us to express the holomorphic derivatives of the functions F_j and G_j, at points of M, in terms of the derivatives of the functions f_j and g_j. By definition,

$$J^H = \begin{vmatrix} \frac{\partial}{\partial z_1}F_1 & \dots & \frac{\partial}{\partial z_n}F_1 & \frac{\partial}{\partial w_1}F_1 & \dots & \frac{\partial}{\partial w_d}F_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1}F_n & \dots & \frac{\partial}{\partial z_n}F_n & \frac{\partial}{\partial w_1}F_n & \dots & \frac{\partial}{\partial w_d}F_n \\ \frac{\partial}{\partial z_1}G_1 & \dots & \frac{\partial}{\partial z_n}G_1 & \frac{\partial}{\partial w_1}G_1 & \dots & \frac{\partial}{\partial w_d}G_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1}G_d & \dots & \frac{\partial}{\partial z_n}G_d & \frac{\partial}{\partial w_1}G_d & \dots & \frac{\partial}{\partial w_d}G_d \end{vmatrix}.$$

Using (2–7) we get

$$J^H|_M = \begin{vmatrix} \frac{\partial}{\partial z_1}F_1\,\big|_M & \cdots & \frac{\partial}{\partial z_n}F_1\,\big|_M & \frac{\partial}{\partial s_1}f_1 & \cdots & \frac{\partial}{\partial s_d}f_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1}F_n\,\big|_M & \cdots & \frac{\partial}{\partial z_n}F_n\,\big|_M & \frac{\partial}{\partial s_1}f_n & \cdots & \frac{\partial}{\partial s_d}f_n \\ \frac{\partial}{\partial z_1}G_1\big|_M & \cdots & \frac{\partial}{\partial z_n}G_1\big|_M & \frac{\partial}{\partial s_1}g_1 & \cdots & \frac{\partial}{\partial s_d}g_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1}G_d\big|_M & \cdots & \frac{\partial}{\partial z_n}G_d\big|_M & \frac{\partial}{\partial s_1}g_d & \cdots & \frac{\partial}{\partial s_d}g_d \end{vmatrix} \begin{vmatrix} I_{n\times n} & O \\ O & {}^{\tau}\bar{\alpha}|_{t=0} \end{vmatrix},$$

which after using (2–8) and performing elementary column operations becomes

$$J^H|_M = \det(\,{}^{\tau}\bar{\alpha}|_{t=0}) \begin{vmatrix} \frac{\partial}{\partial z_1}f_1 & \cdots & \frac{\partial}{\partial z_n}f_1 & \frac{\partial}{\partial s_1}f_1 & \cdots & \frac{\partial}{\partial s_d}f_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1}f_n & \cdots & \frac{\partial}{\partial z_n}f_n & \frac{\partial}{\partial s_1}f_n & \cdots & \frac{\partial}{\partial s_d}f_n \\ \frac{\partial}{\partial z_1}g_1 & \cdots & \frac{\partial}{\partial z_n}g_1 & \frac{\partial}{\partial s_1}g_1 & \cdots & \frac{\partial}{\partial s_d}g_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1}g_d & \cdots & \frac{\partial}{\partial z_n}g_d & \frac{\partial}{\partial s_1}g_d & \cdots & \frac{\partial}{\partial s_d}g_d \end{vmatrix}.$$

It is clear from (2–6) that there exist functions μ_{jl} such that

$$\Lambda_j = \frac{\partial}{\partial \bar{z}_j} - \sum_{l=1}^{d} \mu_{jl}\frac{\partial}{\partial s_l} \quad \text{and} \quad \bar{\Lambda}_j = \frac{\partial}{\partial z_j} - \sum_{l=1}^{d} \bar{\mu}_{jl}\frac{\partial}{\partial s_l}. \tag{2–9}$$

Applying elementary column operations to the determinant above then yields

$$J^H|_M = \det(\,{}^{\tau}\bar{\alpha}|_{t=0}) \begin{vmatrix} \bar{\Lambda}_1 f_1 & \ldots & \bar{\Lambda}_n f_1 & \frac{\partial}{\partial s_1}f_1 & \cdots & \frac{\partial}{\partial s_d}f_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \bar{\Lambda}_1 f_n & \ldots & \bar{\Lambda}_n f_n & \frac{\partial}{\partial s_1}f_n & \cdots & \frac{\partial}{\partial s_d}f_n \\ \bar{\Lambda}_1 g_1 & \ldots & \bar{\Lambda}_n g_1 & \frac{\partial}{\partial s_1}g_1 & \cdots & \frac{\partial}{\partial s_d}g_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \bar{\Lambda}_1 g_d & \ldots & \bar{\Lambda}_n g_d & \frac{\partial}{\partial s_1}g_d & \cdots & \frac{\partial}{\partial s_d}g_d \end{vmatrix}. \tag{2–10}$$

At this time we wish to eliminate the lower left block of the above determinant. Since $H(M) \subset M'$, we get

$$g_j = Q_j(f_1, \dots, \bar{f}_n, \bar{g}_1, \dots, \bar{g}_d) \quad \text{for } j = 1, \dots, d, \tag{2–11}$$

$$\bar{g}_j = \bar{Q}_j(\bar{f}_1, \dots, f_n, g_1, \dots, g_d) \quad \text{for } j = 1, \dots, d. \tag{2–12}$$

Applying Λ_l and $\bar{\Lambda}_l$ to these equalities and using that f_j and g_j are CR functions gives

$$\bar{\Lambda}_l g_j = \sum_{k=1}^{n} Q_{j z'_k} \bar{\Lambda}_l f_k \quad \text{and} \quad \Lambda_l \bar{g}_j = \sum_{k=1}^{n} \bar{Q}_{j \bar{z}'_k} \Lambda_l \bar{f}_k. \tag{2–13}$$

Using the first of these relations to perform the appropriate row operations on the determinant in (2–10) we finally get

$$J^H|_M = \det({}^{\tau}\bar{\alpha}|_{t=0}) \begin{vmatrix} \bar{\Lambda}_1 f_1 & \dots & \bar{\Lambda}_n f_1 \\ \vdots & & \vdots \\ \bar{\Lambda}_1 f_n & \dots & \bar{\Lambda}_n f_n \end{vmatrix} \det \mathcal{U}^H,$$

where $\mathcal{U}^H$ is a matrix defined in the statement of Claim 2.4. The final expression for $J^H|_M$ then follows directly from the definition of $\bar{\alpha}$. □

Recall that $s_j = \operatorname{Re} w_j$ whenever $t = 0$. If we let $z_j = x_j + i y_j$, it is clear from (2–5) that $(x_1, y_1, \dots, x_n, y_n, s_1, \dots, s_d)$ are coordinates on M near p_0. Similarly, by letting $z'_j = x'_j + i y'_j$ and $s'_j = \operatorname{Re} w'_j$, we get coordinates

$$(x'_1, y'_1, \dots, x'_n, y'_n, s'_1, \dots, s'_d)$$

on M' near $H(p_0)$. In these coordinates, $\operatorname{Jac} H$ is a real-valued function defined near p_0 on M.

Claim 2.5. *In the coordinates above*,

$$\operatorname{Jac} H = \left(\tfrac{1}{2}\right)^d \det(I + \bar{Q}_{w'}) \begin{vmatrix} \bar{\Lambda}_1 f_1 & \dots & \bar{\Lambda}_n f_1 \\ \vdots & & \vdots \\ \bar{\Lambda}_1 f_n & \dots & \bar{\Lambda}_n f_n \end{vmatrix} \begin{vmatrix} \Lambda_1 \bar{f}_1 & \dots & \Lambda_n \bar{f}_1 \\ \vdots & & \vdots \\ \Lambda_1 \bar{f}_n & \dots & \Lambda_n \bar{f}_n \end{vmatrix} \det \mathcal{U}^H,$$

where $\mathcal{U}^H$ is a $d \times d$ matrix as in the statement of Claim 2.4 and Λ_j are CR vector fields introduced above.

Proof. After performing elementary row and column operations on the determinant of the Jacobian matrix in above coordinates, it is easily seen that

$$\operatorname{Jac} H = \begin{vmatrix} \frac{\partial}{\partial z_1} f_1 & \dots & \frac{\partial}{\partial \bar{z}_n} f_1 & \frac{\partial}{\partial s_1} f_1 & \dots & \frac{\partial}{\partial s_d} f_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1} \bar{f}_n & \dots & \frac{\partial}{\partial \bar{z}_n} \bar{f}_n & \frac{\partial}{\partial s_1} \bar{f}_n & \dots & \frac{\partial}{\partial s_d} \bar{f}_n \\ \frac{\partial}{\partial z_1} \operatorname{Re} g_1 & \dots & \frac{\partial}{\partial \bar{z}_n} \operatorname{Re} g_1 & \frac{\partial}{\partial s_1} \operatorname{Re} g_1 & \dots & \frac{\partial}{\partial s_d} \operatorname{Re} g_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1} \operatorname{Re} g_d & \dots & \frac{\partial}{\partial \bar{z}_n} \operatorname{Re} g_d & \frac{\partial}{\partial s_1} \operatorname{Re} g_d & \dots & \frac{\partial}{\partial s_d} \operatorname{Re} g_d \end{vmatrix}.$$

From (2–9) we see that this equals

$$\begin{vmatrix} \bar{\Lambda}_1 f_1 & \dots & \Lambda_n f_1 & \frac{\partial}{\partial s_1} f_1 & \dots & \frac{\partial}{\partial s_d} f_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \bar{\Lambda}_1 \bar{f}_n & \dots & \Lambda_n \bar{f}_n & \frac{\partial}{\partial s_1} \bar{f}_n & \dots & \frac{\partial}{\partial s_d} \bar{f}_n \\ \bar{\Lambda}_1 \operatorname{Re} g_1 & \dots & \Lambda_n \operatorname{Re} g_1 & \frac{\partial}{\partial s_1} \operatorname{Re} g_1 & \dots & \frac{\partial}{\partial s_d} \operatorname{Re} g_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \bar{\Lambda}_1 \operatorname{Re} g_d & \dots & \Lambda_n \operatorname{Re} g_d & \frac{\partial}{\partial s_1} \operatorname{Re} g_d & \dots & \frac{\partial}{\partial s_d} \operatorname{Re} g_d \end{vmatrix}.$$

Because H is a CR map, this simplifies to

$$\left(\frac{1}{2}\right)^d \begin{vmatrix} \bar{\Lambda}_1 f_1 & \dots & \bar{\Lambda}_n f_1 & 0 & \dots & 0 & \frac{\partial}{\partial s_1} f_1 & \dots & \frac{\partial}{\partial s_d} f_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \bar{\Lambda}_1 f_n & \dots & \bar{\Lambda}_n f_n & 0 & \dots & 0 & \frac{\partial}{\partial s_1} f_n & \dots & \frac{\partial}{\partial s_d} f_n \\ 0 & \dots & 0 & \Lambda_1 \bar{f}_1 & \dots & \Lambda_n \bar{f}_1 & \frac{\partial}{\partial s_1} \bar{f}_1 & \dots & \frac{\partial}{\partial s_d} \bar{f}_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \Lambda_1 \bar{f}_n & \dots & \Lambda_n \bar{f}_n & \frac{\partial}{\partial s_1} \bar{f}_n & \dots & \frac{\partial}{\partial s_d} \bar{f}_n \\ \bar{\Lambda}_1 g_1 & \dots & \bar{\Lambda}_n g_1 & \Lambda_1 \bar{g}_1 & \dots & \Lambda_n \bar{g}_1 & \frac{\partial}{\partial s_1} (g_1 + \bar{g}_1) & \dots & \frac{\partial}{\partial s_d} (g_1 + \bar{g}_1) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \bar{\Lambda}_1 g_d & \dots & \bar{\Lambda}_n g_d & \Lambda_1 \bar{g}_d & \dots & \Lambda_n \bar{g}_d & \frac{\partial}{\partial s_1} (g_d + \bar{g}_d) & \dots & \frac{\partial}{\partial s_d} (g_d + \bar{g}_d) \end{vmatrix}.$$

As in the proof of Claim 2.4, we use the relations in (2–13) to eliminate the terms of the form $\Lambda_j \bar{g}_k$ and $\bar{\Lambda}_j g_k$. Factoring the resulting determinant then shows that

$$\operatorname{Jac} H = \left(\frac{1}{2}\right)^d \begin{vmatrix} \bar{\Lambda}_1 f_1 & \dots & \bar{\Lambda}_n f_1 \\ \vdots & & \vdots \\ \bar{\Lambda}_1 f_n & \dots & \bar{\Lambda}_n f_n \end{vmatrix} \begin{vmatrix} \Lambda_1 \bar{f}_1 & \dots & \Lambda_n \bar{f}_1 \\ \vdots & & \vdots \\ \Lambda_1 \bar{f}_n & \dots & \Lambda_n \bar{f}_n \end{vmatrix} \det \tilde{\mathcal{U}}^H,$$

where $\tilde{\mathcal{U}}^H$ is a $d \times d$ matrix with entries

$$(\tilde{\mathcal{U}}^H)_{pq} := \frac{\partial}{\partial s_q}(g_p + \bar{g}_p) - \sum_{k=1}^{n} \left(Q_{p z'_k} \frac{\partial}{\partial s_q} f_k + \bar{Q}_{p \bar{z}'_k} \frac{\partial}{\partial s_q} \bar{f}_k \right). \tag{2–14}$$

We shall complete the proof of the claim by showing that

$$\tilde{\mathcal{U}}^H = (I + \bar{Q}_{w'}) \mathcal{U}^H.$$

Taking partial derivatives of (2–12) with respect to s_q we see that

$$\frac{\partial}{\partial s_q} \bar{g}_p = \sum_{k=1}^{n} \left(\bar{Q}_{p \bar{z}'_k} \frac{\partial}{\partial s_q} \bar{f}_k + \bar{Q}_{p z'_k} \frac{\partial}{\partial s_q} f_k \right) + \sum_{k=1}^{d} \bar{Q}_{p w'_k} \frac{\partial}{\partial s_q} g_k.$$

Substituting this into (2–14) gives

$$(\tilde{\mathcal{U}}^H)_{pq} = \sum_{k=1}^{d} (\delta_{pk} + \bar{Q}_{p w'_k}) \frac{\partial}{\partial s_q} g_k + \sum_{k=1}^{n} (\bar{Q}_{p z'_k} - Q_{p z'_k}) \frac{\partial}{\partial s_q} f_k.$$

Using above,

$$(\tilde{\mathcal{U}}^H - (I + \bar{Q}_{w'}) \mathcal{U}^H)_{pq} = \sum_{k=1}^{n} \left((\bar{Q}_{p z'_k} - Q_{p z'_k}) + \sum_{l=1}^{d} (\delta_{pl} + \bar{Q}_{p w'_l}) Q_{l z'_k} \right) \frac{\partial}{\partial s_q} f_k.$$

It is therefore sufficient to show that

$$(\bar{Q}_{p z'_k} - Q_{p z'_k}) + \sum_{l=1}^{d} (\delta_{pl} + \bar{Q}_{p w'_l}) Q_{l z'_k} = 0,$$

which in matrix notation amounts to

$$(I + \bar{Q}_{w'}) Q_{z'} = Q_{z'} - \bar{Q}_{z'}. \tag{2–15}$$

The normality of coordinates implies that $Q\big(z', \bar{z}', \bar{Q}(\bar{z}', z', w')\big) \equiv w'$. Taking partial derivatives with respect to w' and z' we get, respectively,

$$Q_{\bar{w}'} \bar{Q}_{w'} \equiv I, \tag{2–16}$$

$$Q_{z'} + Q_{\bar{w}'} \bar{Q}_{z'} \equiv 0. \tag{2–17}$$

Then

$$\begin{aligned}
Q'_z &= (I + Q_{\bar{w}'})^{-1}(I + Q_{\bar{w}'})Q_{z'} \\
&= (Q_{\bar{w}'}\bar{Q}_{w'} + Q_{\bar{w}'})^{-1}(Q_{\bar{w}'}Q_{z'} + Q_{z'}) \qquad \text{(using (2–16))} \\
&= (Q_{\bar{w}'}(\bar{Q}_{w'} + I))^{-1}(Q_{\bar{w}'}Q_{z'} + Q_{z'}) \\
&= (Q_{\bar{w}'}(\bar{Q}_{w'} + I))^{-1}(Q_{\bar{w}'}Q_{z'} - Q_{\bar{w}'}\bar{Q}_{z'}) \quad \text{(using (2–17))} \\
&= (I + \bar{Q}_{w'})^{-1}(Q_{\bar{w}'})^{-1}(Q_{\bar{w}'})(Q_{z'} - \bar{Q}_{z'}) \\
&= (I + \bar{Q}_{w'})^{-1}(Q_{z'} - \bar{Q}_{z'}),
\end{aligned}$$

which clearly implies (2–15). All inverses used in the calculation above exist near $H(p_0)$. This is sufficient, as we are proving a local result. □

Although it might be necessary to shrink the neighborhoods one last time, we may assume that the factors

$$\left(\tfrac{1}{2}\right)^d \det(I + \bar{Q}_{w'}) \quad \text{and} \quad \frac{1}{\det(I + i\phi_s)}$$

do not vanish near p_0. Part (iii) of Proposition 2.3 then follows immediately from Claims 2.4 and 2.5. □

3. The Global CR Jacobian

The result of the previous section is strictly local. In this section, we use the global coordinates on $\mathbb{C}^N$ to obtain a more global result. Let $Z^C = (Z_1^C, \dots, Z_N^C)$ be some global holomorphic coordinates on $\mathbb{C}^N$.

Proposition 3.1. *Let M and M' be real-analytic generic submanifolds of $\mathbb{C}^N$ of equal dimension. Suppose $H : M \to M'$ is a smooth CR map. Then there exists a smooth function $J^C : M \to \mathbb{C}$ satisfying the following conditions*:

(i) *If $\mathcal{H}$ is a C^∞ extension of H to an open set $\mathcal{Y}$ in $\mathbb{C}^N$ which is holomorphic at points of M, then*

$$\operatorname{HolJac}_{(Z^C, Z^C)}(\mathcal{H})|_{\mathcal{Y}\cap M} \equiv J^C|_{\mathcal{Y}\cap M}.$$

(ii) *J^C is a CR function.*

(iii) *If $p \in M$, then $J^C(p) = 0 \iff \operatorname{Jac} H(p) = 0$.*

Proof. Suppose $\mathcal{H}$ is a C^∞ extension of H to an open set $\mathcal{X}$ in $\mathbb{C}^N$ which is holomorphic at points of M. It was argued in the previous section that such extensions exist near every point of M. Define J^C on $\mathcal{X} \cap M$ by setting

$$J^C(p) := \operatorname{HolJac}_{(Z^C, Z^C)}(\mathcal{H})(p)$$

for all $p \in \mathcal{X} \cap M$. By Lemma 2.1, such local definitions may be pieced together to define J^C on all of M. It is then clear that $J^C \in C^\infty(M)$. Notice that (i) and (ii) again follow immediately from Lemmas 2.1 and 2.2.

Let $p \in M$ and suppose $\mathcal{H}$ is a C^∞ extension of H to a neighborhood $\mathcal{Y}$ of p in $\mathbb{C}^N$ which is holomorphic at points of M. By Proposition 2.3, there exists a neighborhood $\mathbb{O}$ of p contained in $\mathcal{Y}$, a neighborhood $\mathbb{O}'$ of $H(p)$ containing $\mathcal{H}(\mathbb{O})$, coordinates Z on $\mathbb{O}$, and coordinates Z' on $\mathbb{O}'$ such that

$$\mathrm{HolJac}_{(Z,Z')}(\mathcal{H})(p) = 0 \iff \mathrm{Jac}\, H(p) = 0.$$

Because $\mathcal{H}$ is holomorphic at points of M, we must also have

$$\mathrm{HolJac}_{(Z,Z')}(\mathcal{H})(p) = 0 \iff \mathrm{HolJac}_{(Z^C,Z^C)}(\mathcal{H})(p) = 0.$$

From these two equivalences, the equivalence in (iii) follows immediately. □

4. The Proof of Theorem 1

Let M, M', and H be as in the statement of Theorem 1. The set

$$U := \{p \in M \mid M \text{ is of finite type at } p\}$$

is nonempty by assumption. Because M is a connected real-analytic manifold, and because finite type is an open condition, U is an open dense subset of M. Moreover, the complement of U in M,

$$A := \{p \in M \mid M \text{ is of infinite type at } p\}, \tag{4–1}$$

is a proper real-analytic variety. In other words, every point of M has a neighborhood V such that $A \cap V$ is the zero set of a real-analytic function; see, for example, [Baouendi et al. 1999, Theorem 1.5.10].

Lemma 4.1. *Let $\mathcal{M}$ be a connected real-analytic generic submanifold of $\mathbb{C}^N$ which we assume is of finite type at all of its points. If f is a continuous CR function on $\mathcal{M}$ which vanishes on some nonempty open subset of $\mathcal{M}$, then $f \equiv 0$.*

Proof. Let

$$S := \{Z \in \mathcal{M} \mid \text{there is an open neighborhood } V \text{ of } Z \text{ such that } f|_V \equiv 0\}.$$

The set S is nonempty by hypothesis. Because $\mathcal{M}$ is connected, and clearly open as well, it is sufficient to show that S is also closed. Assume x is an accumulation point of S. Since $\mathcal{M}$ is of finite type, we may apply a result from [Tumanov 1988] to conclude that there must exist an open connected neighborhood W of x which is an edge of a wedge $\mathcal{W}$ into which f extends holomorphically to F (also see [Baouendi et al. 1999, Theorem 8.1.1], for example). Because x is an accumulation point of S, the set $S \cap W$ must be nonempty. If we pick $y \in S \cap W$, then, similarly as above, there also exists an open neighborhood W' of y which is an edge of a wedge $\mathcal{W}'$ into which f extends holomorphically to F'. By shrinking $\mathcal{W}'$, if necessary, we may assume that $\mathcal{W}' \subset \mathcal{W}$. Because $y \in S$, we may also assume that $f \equiv 0$ on W'. By uniqueness, then, $F' \equiv 0$; see for example [Baouendi et al. 1999, Theorem 7.2.6]. Analytic continuation then implies that $F \equiv 0$ as well. Once again, by uniqueness we may conclude that $f \equiv 0$ on W. This shows that $x \in S$ and the lemma follows. □

Let $Z^C = (Z_1^C, \dots, Z_N^C)$ be global holomorphic coordinates on $\mathbb{C}^N$. Although these coordinates are assumed given, they are not unique. In fact, there are many global coordinates on $\mathbb{C}^N$ that preserve the algebraicity of both M and M'. Let J^C be a C^∞ function as given by Proposition 3.1. We now show that J^C must satisfy a nontrivial polynomial relation on each connected component of U. This is used in Lemma 4.3 to show that J^C satisfies such a relation near every point of M.

Lemma 4.2. *If U_0 is a connected component of U, then there exists a nonzero polynomial $P \in \mathbb{C}[Z_1, \dots, Z_N, X]$ such that $P(Z^C, J^C)|_{U_0} \equiv 0$.*

Proof. If $J^C|_{U_0} \equiv 0$, take $P(Z, X) = X$. Otherwise, there exists a point $p \in U_0$ such that $J^C(p) \neq 0$. Moreover, since J^C is continuous, there exists an open neighborhood W of p in U_0 on which J^C is never zero. Because M is holomorphically nondegenerate, Theorem 11.5.1 in [Baouendi et al. 1999] implies that there is a point $p_0 \in W$ at which M is finitely nondegenerate. Using a result from [Baouendi et al. 1985], we may conclude that H extends holomorphically to $\mathcal{H}$ on an open connected neighborhood $\mathcal{Y}$ of p_0 in $\mathbb{C}^N$. The algebraicity of $\mathcal{H}$ then follows from a result in [Baouendi et al. 1996a]. Because derivatives and products of algebraic functions are algebraic, (2–3) shows that the function $\mathcal{J}^C := \operatorname{HolJac}_{(Z^C, Z^C)}(\mathcal{H})$ is algebraic. Therefore, there exists $P \in \mathbb{C}[Z_1, \dots, Z_N, X]$ such that $P(Z^C, \mathcal{J}^C)|_{\mathcal{Y}} \equiv 0$. As in the conclusion of Proposition 3.1, we have $J^C|_{\mathcal{Y} \cap M} \equiv \mathcal{J}^C|_{\mathcal{Y} \cap M}$. It is then clear that $P(Z^C, J^C)|_{\mathcal{Y} \cap M} \equiv 0$. Because $Z_1^C|_M, \dots, Z_N^C|_M$, and J^C are CR functions, so is $P(Z^C, J^C)$. By applying Lemma 4.1, we may therefore conclude that $P(Z^C, J^C)|_{U_0} \equiv 0$. □

Lemma 4.3. *Given $p \in M$, there exists an open neighborhood W of p in M and a nonzero polynomial $P \in \mathbb{C}[Z_1, \dots, Z_N, X]$ such that $P(Z^C, J^C)|_W \equiv 0$.*

Proof. If $p \in U$, we are done by the previous lemma. Therefore, assume $p \in A$ with A defined by (4–1). By [Malgrange 1967, Lemma 1.5], there exists an open neighborhood W of p in M such that $W \setminus A$ has only finitely many connected components $W_1, \dots, W_m$. Because each W_j lies in some component U_j of U, Lemma 4.2 implies that there exist polynomials $P_1, \dots, P_m$ such that $P_j(Z^C, J^C)|_{W_j} \equiv 0$. Let $P := \prod_{j=1}^m P_j$. Because $P(Z^C, J^C)$ clearly vanishes on $W \setminus A$ and because $W \setminus A$ is dense in W, we conclude by continuity that $P(Z^C, J^C)|_W \equiv 0$. □

Since J^C is smooth, Lemma 4.3 shows that we may apply a known result (see [Baouendi et al. 1996b, Lemma 2.7], for instance) to conclude that the function J^C is actually real-analytic. Because $\operatorname{Jac} H \not\equiv 0$ by hypothesis, Proposition 3.1(iii) implies that $J^C \not\equiv 0$ as well. As M is connected and J^C is real-analytic everywhere, it follows immediately that J^C, and consequently $\operatorname{Jac} H$, cannot vanish on any nonempty open set in M. We have therefore proved:

Proposition 4.4. *Let M, M', and H be as in Theorem 1. Then* Jac H *does not vanish identically on any nonempty open subset of M.*

Proof of Theorem 1. Since H is a CR map, and because M is a generic submanifold in $\mathbb{C}^N$, it is sufficient to show that the components of H are real-analytic functions; see [Baouendi et al. 1999, Corollary 1.7.13]. Let U_0 be a connected component of U. Proposition 4.4 implies that $\operatorname{Jac} H|_{U_0} \not\equiv 0$. Proceeding as in the proof of Lemma 4.2, we see that there exists a point $p_0 \in U_0$ at which M is finitely nondegenerate and at which $\operatorname{Jac} H$ does not vanish. Further, H extends holomorphically to $\mathcal{H}$ on an open connected neighborhood $\mathcal{Y}$ of p_0 in $\mathbb{C}^N$. Suppose K is a component of $\mathcal{H}$ in coordinates Z^C. Algebraicity of K, once again, follows from a result in [Baouendi et al. 1996a]. Therefore, there exists a nonzero polynomial $P \in \mathbb{C}[Z_1, \dots, Z_N, X]$ such that $P(Z^C, K)|_{\mathcal{Y}} \equiv 0$. Moreover, Lemma 4.1 implies that $P(Z^C, K)|_{U_0} \equiv 0$.

To summarize, we have demonstrated that if k is a component of H in coordinates Z^C, and if U_0 is any connected component of U, there exists a nonzero polynomial $P \in \mathbb{C}[Z_1, \dots, Z_N, X]$ such that $P(Z^C, k)|_{U_0} \equiv 0$. Using this to argue as in the proof of Lemma 4.3, we conclude that the components of H satisfy such nontrivial polynomial relations near every point of M. Since the components of H are smooth, we may, once again, apply [Baouendi et al. 1996b, Lemma 2.7] to conclude that these components must be real-analytic. This completes the proof of Theorem 1. □

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