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REPRESENTATIONS OF THE BRAID GROUP BY AUTOMORPHISMS OF GROUPS, INVARIANTS OF LINKS, AND GARSIDE GROUPS

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From a group H and $h \in H$, we define a representation $\rho : B_n \to \mathrm{Aut}(H^{*n})$, where B_n denotes the braid group on n strands, and H^{*n} denotes the free product of n copies of H. We call ρ the Artin type representation associated to the pair (H, h). Here we study various aspects of such representations.

Firstly, we associate to each braid β a group $\Gamma_{(H,h)}(\beta)$ and prove that the operator $\Gamma_{(H,h)}$ determines a group invariant of oriented links. We then give a topological construction of the Artin type representations and of the link invariant $\Gamma_{(H,h)}$, and we prove that the Artin type representations are faithful if and only if h is nontrivial. The last part of the paper is devoted to the study of some semidirect products $H^{*n} \rtimes_\rho B_n$, where $\rho : B_n \to \mathrm{Aut}(H^{*n})$ is an Artin type representation. In particular, we show that $H^{*n} \rtimes_\rho B_n$ is a Garside group if H is a Garside group and h is a Garside element of H.

1. Introduction

Throughout the paper, we shall denote by B_n the braid group on n strands, and by $\sigma_1, \dots, \sigma_{n-1}$ the standard generators of B_n.

Let H be a group and fix $h \in H$. Take n copies $H_1, \dots, H_n$ of H and consider the group $H^{*n} = H_1 * \cdots * H_n$. We denote by $\phi_i : H \to H_i$ the natural isomorphism and we write $h_i = \phi_i(h) \in H_i$, for all $i = 1, \dots, n$. For $k = 1, \dots, n-1$, let $\tau_k : H^{*n} \to H^{*n}$ be the automorphism determined by

$$\tau_k : \begin{cases} \phi_k(y) \mapsto h_k^{-1}\, \phi_{k+1}(y)\, h_k, \\ \phi_{k+1}(y) \mapsto h_k\, \phi_k(y)\, h_k^{-1}, \\ \phi_j(y) \mapsto \phi_j(y) \qquad \text{if } j \neq k,\ k+1 \end{cases}$$

for $y \in H$. One can easily show the following.

Proposition 1.1. *The mapping $\sigma_k \mapsto \tau_k$, $k = 1, \dots, n-1$, determines a representation $\rho : B_n \to \mathrm{Aut}(H^{*n})$.*

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Proof. This involves checking, case by case, that the usual braid group relations are satisfied by the automorphisms τ_k. For example, both $\tau_k\tau_{k+1}\tau_k$ and $\tau_{k+1}\tau_k\tau_{k+1}$ map $\phi_k(y)$ to $h_k^{-1}h_{k+1}^{-1}\phi_{k+2}(y)h_{k+1}h_k$, $\phi_{k+1}(y)$ to $h_k^{-1}h_{k+1}\phi_{k+1}(y)h_{k+1}^{-1}h_k$, etc. Similarly, one checks that $\tau_k\tau_j = \tau_j\tau_k$ if $k < j-1$. We leave the details to the reader. □

Definition 1.2. The representation of Proposition 1.1 shall be called the *Artin type representation of B_n associated to the pair (H, h).*

The special case where h is taken to be the identity, $h = \mathrm{Id}_H$, gives a representation of B_n by permutations of the free factors of H^{*n}. This representation has image the full symmetric group S_n and kernel the pure braid group. All other Artin type representations will be shown to be faithful (see Proposition 4.1).

If $H = \mathbb{Z}$ and $h = 1$ (a generator of $\mathbb{Z}$ in the additive notation), then $H^{*n} = F_n$ is the free group of rank n and ρ is the classical representation introduced by Artin [1925; 1947]. Another example which appears in the literature is the case where $H = \mathbb{Z}$ and h is an arbitrary nonzero integer. This case was introduced by Wada [1992] in his construction of group invariants of links. Sections 2 and 3 of the present paper are inspired by [Wada 1992].

Our purpose in this paper is to study different aspects of the Artin type representations.

Definition 1.3. Let $\rho : B_n \to \mathrm{Aut}(H^{*n})$ be the Artin type representation associated to a pair (H, h). Let $\beta \in B_n$. Then we denote by $\Gamma(\beta) = \Gamma_{(H,h)}(\beta)$ the quotient of H^{*n} by the relations

$$g = \rho(\beta)g, \quad g \in H^{*n}.$$

For a braid β, we denote by $\hat{\beta}$ the oriented link (or more precisely the equivalence class of oriented links) represented by the closed braid of β as defined in [Birman 1974]. Given two braids β_1 and β_2 (not necessarily with the same number of strands), we prove in Section 2 that $\Gamma(\beta_1) \simeq \Gamma(\beta_2)$ if $\hat{\beta}_1 = \hat{\beta}_2$. This allows us to define a group invariant of oriented links, $\Gamma_{(H,h)}$, by setting $\Gamma_{(H,h)}(L)$ to be the group $\Gamma_{(H,h)}(\beta)$ for any braid β such that $L = \hat{\beta}$. Note that, in the case $H = \mathbb{Z}$ and $h = 1$, the invariant $\Gamma_{(\mathbb{Z},1)}$ computes the link group, namely $\Gamma_{(\mathbb{Z},1)}(L) \cong \pi_1(\boldsymbol{S}^3 \setminus L)$ for any link L in $\boldsymbol{S}^3$.

The goal of Section 3 is to give topological constructions of the Artin type representations and of the groups $\Gamma_{(H,h)}(\beta)$, for $\beta \in B_n$. If $H = \mathbb{Z}$ and h is a nonzero integer, then our constructions coincide with Wada's constructions [1992, Section 3]. In fact, our constructions are straightforward extensions of Wada's constructions to all Artin type representations.

In Section 4, we prove that Artin type representations are faithful whenever h is chosen nontrivial (Proposition 4.1). If h has infinite order, then the Artin type representation $\rho : B_n \to \mathrm{Aut}(H^{*n})$ contains the classical Artin representation and, therefore, is faithful by [Artin 1925; 1947]. So, Proposition 4.1 is mostly of interest in the case where h has finite order. In fact the proof may be easily reduced to the case $H = \mathbb{Z}/k\mathbb{Z}$ and $h = 1$, however we will not need to use any such reduction, as our method applies just as easily in all cases. We note also that the case where H is cyclic of order 2 follows (by somewhat different methods) from [Crisp and Paris 2005, Section 2.3]. The proof of Proposition 4.1 is inspired by the proof of [Shpilrain 2001, Theorem A], and it is based on Dehornoy's work [1994; 1997a] on orderings of the braid group.

The remaining sections (Sections 5 and 6) are dedicated to the study of semidirect products $H^{*n} \rtimes_\rho B_n$, where $\rho : B_n \to \mathrm{Aut}(H^{*n})$ is the Artin type representation associated to a pair (H, h).

If $H = \mathbb{Z}$ and $h = 1$, then $H^{*n} \rtimes_\rho B_n$ is the Artin group $A(B_n)$ associated to the Coxeter graph B_n (not to be confused with the braid group B_n, which is itself an Artin group, of type A_{n-1}). This result is implicit in [Lambropoulou 1994; Crisp 1999], and is described explicitly in [Crisp and Paris 2005]. The group $A(B_n)$ is well-understood. In particular, solutions to the word and conjugacy problems in this group are known [Deligne 1972; Brieskorn and Saito 1972], it is torsion free [Brieskorn 1973; Deligne 1972], its center is an infinite cyclic group [Deligne 1972; Brieskorn and Saito 1972], it is biautomatic [Charney 1992; 1995], and it has an explicit finite dimensional classifying space [Deligne 1972; Bestvina 1999].

A natural next step is to understand the groups $H^{*n} \rtimes_\rho B_n$ in the case where ρ is a Wada representation (of type 4), namely, when $H = \mathbb{Z}$ and $h \in \mathbb{Z} \setminus \{0\}$. One can readily establish that, for these representations, the group $H^{*n} \rtimes_\rho B_n$ fails to be an Artin group unless $h = \pm 1$. It turns out, however, that these groups do have quite a lot in common with Artin groups: like the Artin groups, they belong to a family of groups known as *Garside groups*.

Briefly, a *Garside group* is a group G which admits a left invariant lattice order and contains a so-called *Garside element*, a positive element Δ whose positive divisors generate G and such that conjugation by Δ leaves the lattice structure invariant (there are also conditions placed on the positive cone of G, that it be a finitely generated atomic monoid; see Section 5 for details). The notion of a Garside group was introduced in [Dehornoy and Paris 1999] in a slightly restricted sense, and in [Dehornoy 2002] in the larger sense in which it is now generally used. Their theory is largely inspired by [Garside 1969], which treated the case of braid groups, and [Brieskorn and Saito 1972], which generalized Garside's work to Artin groups. The Artin groups of spherical (or finite) type which include, notably, the braid groups as well as the groups $A(B_n)$ mentioned above, are motivating examples. Other interesting examples of Garside groups include all torus link groups

[Picantin 2003] and some generalized braid groups associated to finite complex reflection groups [Bessis and Corran 2004].

Garside groups have many attractive properties. Solutions to the word and conjugacy problems in these groups are known [Dehornoy 2002; Picantin 2001b; Franco and González-Meneses 2003], they are torsion free [Dehornoy 1998], they admit canonical decompositions as iterated direct products of "irreducible" components, and the center of each component is an infinite cyclic group [Picantin 2001a], they are biautomatic [Dehornoy 2002], and they admit finite dimensional classifying spaces [Dehornoy and Lafont 2003; Charney et al. 2004]. Another important property of the Garside groups is that there exist criteria in terms of presentations to detect them [Dehornoy and Paris 1999; Dehornoy 2002].

In Section 6, we prove that, if H is a Garside group, h a Garside element of H, and ρ the Artin type representation associated to (H, h), then $H^{*n} \rtimes_\rho B_n$ is also a Garside group (Theorem 6.1). This result applies in particular to the case $H = \mathbb{Z}$ and $h \in \mathbb{Z} \setminus \{0\}$, but also applies, for example, to the case where H is another braid group, say $H = B_l$, and $h = \Delta^k$ is a nontrivial power of the fundamental element of B_l.

The proof of Theorem 6.1 is based on a necessary and sufficient criterion, explained in Section 5, for a group to be Garside. This criterion rests largely on the "coherence" condition of [Dehornoy and Paris 1999] and is essentially a variation on [Dehornoy 2002, Proposition 6.14]. Our version differs from Dehornoy's [2002] in that it is not algorithmic. In particular, we do not give any method for finding a Garside element. However, our Criterion 5.9 is relatively easy to apply once one has an appropriate presentation and an expression for a Garside element to hand.

Finally, in the Appendix we answer a question posed by Shpilrain [2001] in his study of Wada's representations.

Definition 1.4. Let G be a group. Two representations $\rho, \rho' : B_n \to \mathrm{Aut}(G)$ are called *equivalent* if there exist automorphisms $\phi : G \to G$ and $\mu : B_n \to B_n$ such that $\rho'(\mu(\beta)) = \phi^{-1} \circ \rho(\beta) \circ \phi$ for all $\beta \in B_n$.

Remark. If two representations $\rho, \rho' : B_n \to \mathrm{Aut}(G)$ are equivalent, then the groups $G \rtimes_\rho B_n$ and $G \rtimes_{\rho'} B_n$ are isomorphic.

Shpilrain's question was simply to give a classification of Wada's representations up to equivalence. This classification is given in Proposition A.1.

2. Link invariants

Let H be a group, $h \in H$, and $\rho : B_n \to \mathrm{Aut}(H^{*n})$ be the Artin type representation associated to (H, h). Recall that the group H^{*n} is defined as $H^{*n} = H_1 * \cdots * H_n$, where group isomorphisms $\phi_i : H_i \to H$ are given for $i = 1, 2, \ldots, n$. The goal of this section is to prove the following.

Proposition 2.1. *Let $n, m \in \mathbb{N}$, and let $\beta_1 \in B_n$ and $\beta_2 \in B_m$. If $\hat{\beta}_1 = \hat{\beta}_2$, then $\Gamma_{(H,h)}(\beta_1) \simeq \Gamma_{(H,h)}(\beta_2)$.*

Definition 2.2 (Link invariant). Let L be an oriented link. We set $\Gamma_{(H,h)}(L) := \Gamma_{(H,h)}(\beta)$, where β is any braid (on any number of strings) such that $L = \hat{\beta}$. By Proposition 2.1, $\Gamma_{(H,h)}$ is a well-defined group invariant of oriented links.

Proof of Proposition 2.1. Let $n \in \mathbb{N}$ and let $\beta \in B_n$. We write Γ for $\Gamma_{(H,h)}$. By Markov's theorem [Birman 1974, Theorem 2.3], it suffices to show that

(1) $\Gamma(\alpha^{-1}\beta\alpha) \simeq \Gamma(\beta)$ for all $\alpha \in B_n$,

(2) $\Gamma(\beta\sigma_n) \simeq \Gamma(\beta)$, and

(3) $\Gamma(\beta\sigma_n^{-1}) \simeq \Gamma(\beta)$,

where $\beta\sigma_n$ and $\beta\sigma_n^{-1}$ are viewed as braids on $n+1$ strands.

Note that, if $\beta \in B_n$ and $n \leq m$, then the action of β via ρ on H^{*m} agrees with the action via ρ on $H^{*n} < H^{*m}$, and is trivial on the free factors $H_{n+1}, \ldots, H_m$. We suppress ρ from our notation, writing simply $\beta(g)$ to mean $\rho(\beta)g$, for any $\beta \in B_n$ and $g \in H^{*m}$. This also amounts to writing σ_k instead of τ_k.

We now prove conditions (1), (2) and (3) above.

(1) For $\beta \in B_n$, the group $\Gamma(\beta)$ is defined as the quotient of H^{*n} by the relations $g = \beta(g)$ for all $g \in H^{*n}$. Since, for $\alpha \in B_n$, the relation $g = \alpha^{-1}\beta\alpha(g)$ is equivalent to the relation $\alpha(g) = \beta(\alpha(g))$, and α is an automorphism of H^{*n}, it is clear that $\Gamma(\alpha^{-1}\beta\alpha)$ is defined by the same set of relations as $\Gamma(\beta)$.

(2) The group $\Gamma(\beta\sigma_n)$ may be defined as the quotient of $H^{*(n+1)}$ by the family of relations $R(i, x) : \phi_i(x) = \beta\sigma_n(\phi_i(x))$ for $i = 1, 2, \ldots, n+1$ and $x \in H$. Note that $\sigma_n(\phi_{n+1}(x)) = h_n\phi_n(x)h_n^{-1}$. Therefore the relation $R(n+1, x)$ is equivalent to the relation $R'(n+1, x) : \phi_{n+1}(x) = \beta(h_n\phi_n(x)h_n^{-1})$, where the right hand side is actually an element of H^{*n}. In particular $\Gamma(\beta\sigma_n)$ is generated by the image of H^{*n}. Also,

$$\beta\sigma_n(\phi_n(x)) = \beta(h_n^{-1}\phi_{n+1}(x)h_n) = \beta(h_n^{-1})\phi_{n+1}(x)\beta(h_n).$$

So, in view of $R'(n+1, x)$, the relation $R(n, x)$ is now equivalent to the relation $R'(n, x) : \phi_n(x) = \beta(\phi_n(x))$. Finally, since $\sigma_n(\phi_i(x)) = \phi_i(x)$ for all $i < n$, the remaining relations $R(i, x)$ are equivalent to $R'(i, x) : \phi_i(x) = \beta(\phi_i(x))$ for all $i = 1, 2, \ldots, n-1$, and all $x \in H$. It now follows that $\Gamma(\beta\sigma_n) \simeq \Gamma(\beta)$.

(3) Observe that $\Gamma(\beta^{-1}) \simeq \Gamma(\beta)$, since the relation $g = \beta(g)$ is equivalent to $\beta^{-1}(g) = g$, for all $g \in H^{*n}$. Then

$$\begin{aligned} \Gamma(\beta\sigma_n^{-1}) &\simeq \Gamma(\sigma_n\beta^{-1}) \\ &\simeq \Gamma(\beta^{-1}\sigma_n) \quad \text{by the proof of (1),} \\ &\simeq \Gamma(\beta^{-1}) \quad \text{by the proof of (2),} \\ &\simeq \Gamma(\beta). \end{aligned}$$

□

3. Topological construction of the link invariants

Let X be a CW-complex, let $P_0 \in X$ be a basepoint, and let $\alpha : [0, 1] \to X$ be a loop based at P_0. In this section we give a topological realization of the Artin type representation of B_n associated to the pair $(H, h) = (\pi_1(X, P_0), [\alpha])$, and we deduce a topological construction of the link invariant $\Gamma_{(H,h)}$ of the previous section.

Let $\boldsymbol{D} = \boldsymbol{D}(\frac{n+1}{2}, \frac{n+1}{2})$ denote the disk in $\mathbb{C}$ centered at $\frac{n+1}{2}$ of radius $\frac{n+1}{2}$. Now, we construct a space Y obtained from $\boldsymbol{D}$ by making n holes in $\boldsymbol{D}$ and gluing a copy of X into each hole by identifying the circular boundary of the hole to the loop α in X. Choose some small $\varepsilon > 0$ (we require only that $\varepsilon < \frac{1}{8}$). Let

$$Y' = \boldsymbol{D} \setminus \left(\bigcup_{k=1}^{n} \mathring{\boldsymbol{D}}(k, \varepsilon) \right),$$

where $\mathring{\boldsymbol{D}}(k, \varepsilon)$ denotes the open disk centered at k of radius ε. Take n copies $X_1, \dots, X_n$ of X, denote by $f_k : X \to X_k$ the natural homeomorphism, and write $\alpha_k = f_k \circ \alpha$ for all $k = 1, \dots, n$. Then

$$Y = \left(Y' \sqcup \left(\bigsqcup_{k=1}^{n} X_k \right) \right) / \sim,$$

where $\sim$ is the identification defined by

$$\alpha_k(t) \sim k + \varepsilon e^{2\sqrt{-1}\pi t}, \quad k = 1, \dots, n, \; t \in [0, 1].$$

Finally, choose a basepoint $Q_0 \in \partial \boldsymbol{D}$ for Y. The following result is a direct consequence of the above construction.

Lemma 3.1. *Let $H = \pi_1(X, P_0)$, and let $H_1, \dots, H_n$ be n copies of H. Then $\pi_1(Y, Q_0) \simeq H_1 * \cdots * H_n$.*

We now show that the braid group B_n acts on Y up to isotopy relative to the boundary of $\boldsymbol{D}$ in such a way that the induced action on $\pi_1(Y)$ is the Artin type representation associated to (H, h), where h is the element of $H = \pi_1(X, P_0)$ represented by α.

Let $\xi \in \mathbb{C}$ and $0 < r < R$. Define the *half Dehn twist* $T = T(\xi, r, R)$ by

$$T(\xi + \rho e^{\sqrt{-1}\theta}) = \begin{cases} \xi + \rho e^{\sqrt{-1}(\theta - \pi)} & \text{if } 0 \le \rho \le r, \\ \xi + \rho e^{\sqrt{-1}(\theta - t\pi)} & \text{if } r \le \rho \le R \text{ and } t = \dfrac{R-\rho}{R-r}, \\ \xi + \rho e^{\sqrt{-1}\theta} & \text{if } \rho \ge R \end{cases}$$

(see Figure 1).

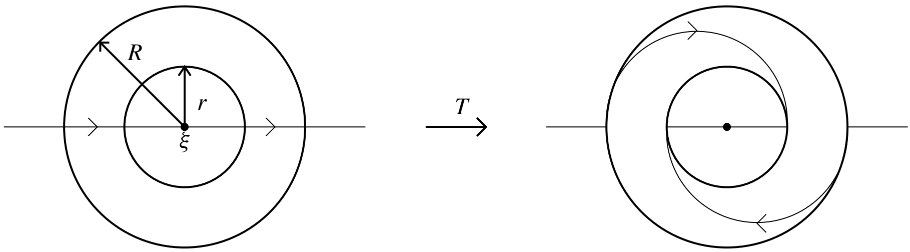

Figure 1. A half Dehn twist.

Let $T_k^{\boldsymbol{D}} : \boldsymbol{D} \to \boldsymbol{D}$ be the homeomorphism defined by

$$T_k^{\boldsymbol{D}} = T(k, \varepsilon, 2\varepsilon)^{-3} \circ T(k+1, \varepsilon, 2\varepsilon)^{-1} \circ T\big(k + \tfrac{1}{2}, \tfrac{1}{2} + \varepsilon, \tfrac{1}{2} + 2\varepsilon\big).$$

Note that $T_k^{\boldsymbol{D}}$ leaves invariant the set $\bigcup_{j=1}^{n} \boldsymbol{D}(j, \varepsilon)$, and therefore restricts to a homeomorphism $T_k' : Y' \to Y'$. See Figure 2.

One can verify (with a little effort) that $T_k' T_{k+1}' T_k'$ is isotopic to $T_{k+1}' T_k' T_{k+1}'$ relative to $\partial Y'$ for $k = 1, \dots, n-2$, and that $T_k' T_l'$ is isotopic to $T_l' T_k'$ relative to

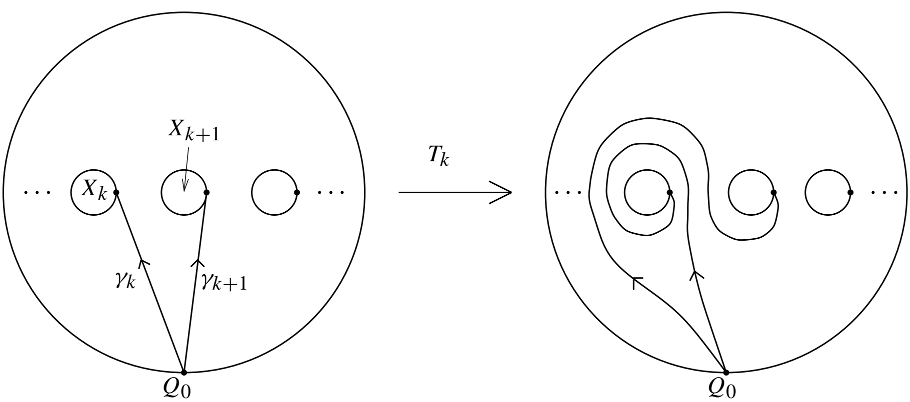

Figure 2. The homeomorphism $T_k' : Y' \to Y'$.

$\partial Y'$ for $|k-l| \geq 2$. Moreover, T'_k fixes $\partial \boldsymbol{D}$ and transforms the rest of $\partial Y'$ as follows:

$$T'_k(j+\varepsilon e^{\sqrt{-1}\theta}) = \begin{cases} j+\varepsilon e^{\sqrt{-1}\theta} & \text{if } j \neq k,\ k+1, \\ k+1+\varepsilon e^{\sqrt{-1}\theta} & \text{if } j = k, \\ k+\varepsilon e^{\sqrt{-1}\theta} & \text{if } j = k+1. \end{cases}$$

Therefore, T'_k extends to a homeomorphism $T_k : Y \to Y$ by setting, for all $x \in X$,

$$T_k(f_j(x)) = \begin{cases} f_j(x) & \text{if } j \neq k,\ k+1, \\ f_{k+1}(x) & \text{if } j = k, \\ f_k(x) & \text{if } j = k+1. \end{cases}$$

The homeomorphism T_k is the identity on $\partial \boldsymbol{D}$, $T_k T_{k+1} T_k$ is isotopic to $T_{k+1} T_k T_{k+1}$ relatively to $\partial \boldsymbol{D}$ for $k = 1, \dots, n-2$, and $T_k T_l$ is isotopic to $T_l T_k$ relatively to $\partial \boldsymbol{D}$ for $|k-l| \geq 2$.

These observations show that T_k determines an automorphism $\tau_k : \pi_1(Y, Q_0) \to \pi_1(Y, Q_0)$. Moreover,

$$\begin{aligned} \tau_k \tau_{k+1} \tau_k &= \tau_{k+1} \tau_k \tau_{k+1} && \text{for } k = 1, \dots, n-2, \\ \tau_k \tau_l &= \tau_l \tau_k && \text{for } |k-l| \geq 2. \end{aligned}$$

Thus the mapping $\sigma_k \to \tau_k$ determines a representation $\rho : B_n \to \mathrm{Aut}(\pi_1(Y, Q_0))$.

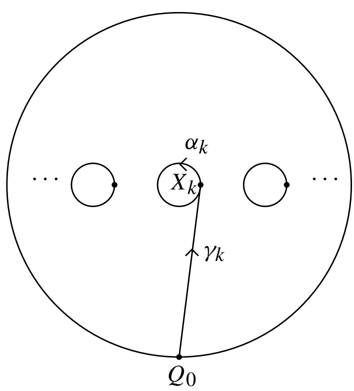

Figure 3. The path γ_k.

Set $Q_0 = \frac{n+1}{2} - \sqrt{-1}\,\frac{n+1}{2}$. Let $\gamma_k : [0, 1] \to Y$ be the path from Q_0 to $f_k(P_0)$ shown in Figure 3. We identify $\pi_1(Y, Q_0)$ with $H^{*n} = H_1 * \cdots * H_n$ in such a way that the k-th embedding $\phi_k : H = \pi_1(X, P_0) \to H_k \subset H^{*n}$ is defined by

$$\phi_k([\beta]) = [\gamma_k f_k(\beta) \gamma_k^{-1}].$$

With this assumption, one can easily show the following.

Proposition 3.2. *The representation $\rho : B_n \to \mathrm{Aut}(\pi_1(Y, Q_0))$ described above coincides with the Artin type representation of B_n associated to (H, h), where $H = \pi_1(X, P_0)$ and h is the element of H represented by α.*

Proof. It suffices to observe, with the aid of Figure 2, that, for all $k = 1, \ldots, n-1$ and all loops β at P_0 in X,

(i) $T_k(\gamma_k f_k(\beta)\gamma_k^{-1})$ is homotopic to $\gamma_k\alpha_k^{-1}\gamma_k^{-1}\gamma_{k+1} f_{k+1}(\beta)\gamma_{k+1}^{-1}\gamma_k\alpha_k\gamma_k^{-1}$;

(ii) $T_k(\gamma_{k+1} f_{k+1}(\beta)\gamma_{k+1}^{-1})$ is homotopic to $\gamma_k\alpha_k f_k(\beta)\alpha_k^{-1}\gamma_k^{-1}$; and

(iii) $T_k(\gamma_j f_j(\beta)\gamma_j^{-1})$ is homotopic to $\gamma_j f_j(\beta)\gamma_j^{-1}$, for all $j \neq k, k+1$. □

We now introduce some standard notions and facts concerning framings of links and linking numbers. We refer the reader to [Rolfsen 1990], or any similar introductory text on knot theory, for further details.

Consider an oriented m-component link $L = K_1 \cup \cdots \cup K_m$ in $\boldsymbol{S}^3$. The knot K_i is an embedding $K_i : \boldsymbol{S}^1 \to \boldsymbol{S}^3$, and $K_i(\boldsymbol{S}^1) \cap K_j(\boldsymbol{S}^1) = \varnothing$ for $i \neq j$. Define a *tubular neighborhood* of K_i to be an embedding $T_i : \boldsymbol{D}^2 \times \boldsymbol{S}^1 \to \boldsymbol{S}^3$ such that $T_i(0, \xi) = K_i(\xi)$ for all $\xi \in \boldsymbol{S}^1$. Here, $\boldsymbol{D}^2$ denotes the disk centered at 0 of radius 1 in $\mathbb{C}$. A *framing* of L is a collection $\{T_i : \boldsymbol{D}^2 \times \boldsymbol{S}^1 \to \boldsymbol{S}^3\}_{i=1}^m$ of embeddings such that T_i is a tubular neighborhood of K_i, for $i = 1, \ldots, m$, and $T_i(\boldsymbol{D}^2 \times \boldsymbol{S}^1) \cap T_j(\boldsymbol{D}^2 \times \boldsymbol{S}^1) = \varnothing$ for $i \neq j$. The *longitude* of the component K_i is the (oriented) embedding $\lambda_i : \boldsymbol{S}^1 \to \boldsymbol{S}^3$ such that $\lambda_i(\xi) = T_i(1, \xi)$ for all $\xi \in \boldsymbol{S}^1$. The framing of each component K_i is determined up to isotopy by the homology class of its longitude λ_i in the knot complement $\boldsymbol{S}^3 \setminus K_i$.

Given an oriented knot K, we identify $H_1(K) := H_1(\boldsymbol{S}^3 \setminus K)$ with $\mathbb{Z}$ in such a way that $1 \in \mathbb{Z}$ is represented by the 1-cycle depicted in Figure 4(a). Let K_1, K_2 denote disjoint oriented knots in $\boldsymbol{S}^3$. One defines the *linking number* $\mathrm{lk}(K_1, K_2) \in \mathbb{Z}$ to be the class $[K_1] \in H_1(K_2) = \mathbb{Z}$. The linking number $\mathrm{lk}(K_1, K_2)$ may be measured from any regular projection of the link $K_1 \cup K_2$ by counting with sign the crossings where K_1 passes over K_2, as indicated in Figure 4(b). (Equally one

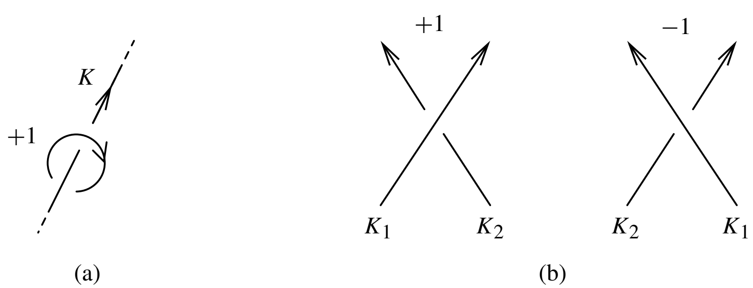

Figure 4. Sign conventions.

may choose to count undercrossings with the appropriate sign, and one quickly sees that $\mathrm{lk}(K_1, K_2) = \mathrm{lk}(K_2, K_1)$).

Notation (Preferred framing). Let $L = K_1 \cup \cdots \cup K_m$ be an m-component oriented link in $\boldsymbol{S}^3$. Up to isotopy, there is a unique framing in which the longitude λ_i for each component K_i satisfies the condition

$$\sum_{j=1}^{m} \mathrm{lk}(\lambda_i, K_j) = 0.$$

Note that, for $j \neq i$, $\mathrm{lk}(\lambda_i, K_j) = \mathrm{lk}(K_i, K_j)$ and is determined by the oriented link L. We shall refer to the above framing as the *preferred framing* of L.

We now wish to associate to an oriented link L the space $\Omega(L, X)$ obtained by performing a 'generalized' surgery on the link L according to the preferred framing just described. More precisely, let $L = K_1 \cup \cdots \cup K_m$ and let $\{T_i : \boldsymbol{D}^2 \times \boldsymbol{S}^1 \to \boldsymbol{S}^3\}_{i=1}^m$ be the preferred framing. Let $\mathring{T}_i$ denote the interior of $T_i(\boldsymbol{D}^2 \times \boldsymbol{S}^1)$ for $i = 1, \ldots, m$, and set

$$\Omega'(L) = \boldsymbol{S}^3 \setminus \left(\bigcup_{i=1}^{m} \mathring{T}_i \right).$$

Take m copies $X_1, \ldots, X_m$ of X, denote by $f_i : X \to X_i$ the natural homeomorphism, and write $\alpha_i = f_i \circ \alpha$. Then

$$\Omega(L, X) = \left(\Omega'(L) \sqcup \left(\bigsqcup_{i=1}^{m} (X_i \times \boldsymbol{S}^1) \right) \right) / \sim,$$

where $\sim$ is the identification defined by putting

$$(\alpha_i(t), \eta) \sim T_i(e^{2\sqrt{-1}\pi t}, \eta), \quad i = 1, \ldots, m, \; t \in [0, 1], \; \eta \in \boldsymbol{S}^1.$$

The following proposition yields a second proof of the fact that $\Gamma_{(H,h)}$ is a link invariant for any finitely generated group H and any element $h \in H$.

Proposition 3.3. *Let β be a braid, and let $\hat{\beta}$ denote the closed braid of β. Let X be a CW-complex with basepoint P_0 and let α be a loop in X. Then $\pi_1(\Omega(\hat{\beta}, X))$ is isomorphic to $\Gamma_{(H,h)}(\beta)$, where $H = \pi_1(X, P_0)$ and h is the element of H represented by α.*

Proof. We first remind the reader of the standard construction of the closed braid $\hat{\beta}$ from a braid β [Birman 1974]. The notation used to describe this construction will be needed for the completion of the proof. Firstly, decompose $\boldsymbol{S}^3$ as follows: let T_1, T_2 be two copies of the solid torus $\boldsymbol{D} \times \boldsymbol{S}^1$ and write

$$\boldsymbol{S}^3 = T_1 \bigcup_{\kappa : \partial T_1 \to \partial T_2} T_2,$$

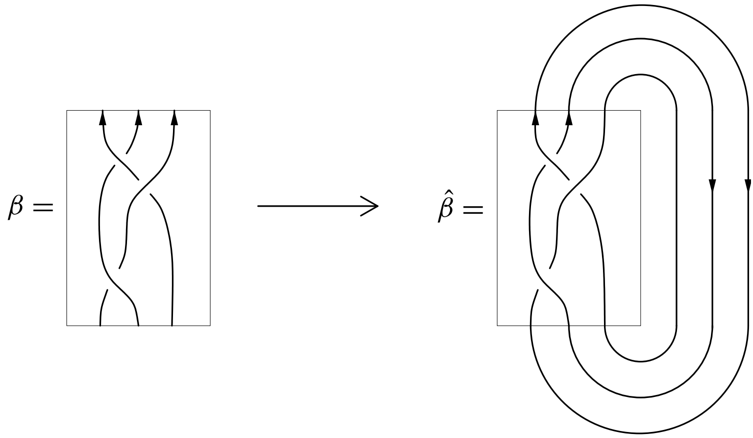

Figure 5. Braid closure.

where the identifying map κ is a homeomorphism carrying $\partial \boldsymbol{D}$ to $\boldsymbol{S}^1$ and $\boldsymbol{S}^1$ to $\partial \boldsymbol{D}$. Let g denote the inclusion of T_1 in $\boldsymbol{S}^3$, and let $f : \boldsymbol{D} \times [0, 1] \to T_1 = \boldsymbol{D} \times \boldsymbol{S}^1$ be the identification map defined by

$$f(p, t) = (p, e^{2\sqrt{-1}\pi t}), \quad \text{for } p \in \boldsymbol{D} \text{ and } t \in [0, 1].$$

The closed braid $\hat{\beta}$ is the oriented link which is induced by composing the braid $\beta : \{1, \dots, n\} \times [0, 1] \to \boldsymbol{D} \times [0, 1]$ with the map $g \circ f : \boldsymbol{D} \times [0, 1] \to \boldsymbol{S}^3$. The orientation on $\hat{\beta}$ is naturally induced from a choice of orientation of the interval $[0, 1]$.

Given a standard projection of a braid β we may describe a projection of the closed braid $\hat{\beta}$ with the same number of crossings, as indicated in Figure 5. We now produce a framing Λ of $\hat{\beta}$ by choosing a longitude λ_i for each component K_i of $\hat{\beta}$ whose projections are as indicated in Figure 6 in the vicinity of a crossing, and otherwise parallel to the link projection. It is easily enough verified, by counting overcrossings, that this framing is exactly the preferred framing of $\hat{\beta}$.

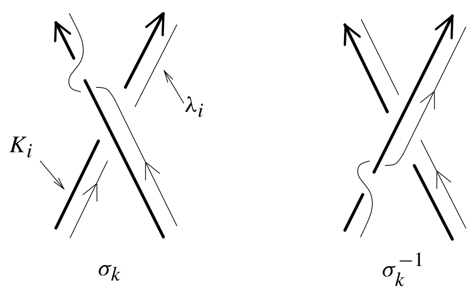

Figure 6. Choosing a framing for $\hat{\beta} = K_1 \cup \cdots \cup K_m$.

Write $\beta = \sigma_{k_1}^{\varepsilon_1}\sigma_{k_2}^{\varepsilon_2}\dots\sigma_{k_r}^{\varepsilon_r}$ and define $T_\beta^{\boldsymbol{D}} : \boldsymbol{D} \to \boldsymbol{D}$ as the composition of the homeomorphisms $(T_{k_j}^{\boldsymbol{D}})^{\varepsilon_j}$ for $j = 1, \dots, r$. Similarly, define $T_\beta = T_{k_1}^{\varepsilon_1} T_{k_2}^{\varepsilon_2} \dots T_{k_r}^{\varepsilon_r} : Y \to Y$. For $j = 1, \dots, n$, denote by b_j the point $j + \varepsilon$ on $\partial \boldsymbol{D}(j, \varepsilon)$. This is the point on $\partial Y'$ to which the basepoint of X_j is attached when forming Y. Since $T_\beta^{\boldsymbol{D}}$ is isotopic to $\mathrm{Id}_{\boldsymbol{D}}$ relative to $\partial \boldsymbol{D}$, there is a homeomorphism $U : \boldsymbol{D} \times [0, 1] \to \boldsymbol{D} \times [0, 1]$ such that $U(x, 0) = (x, 0)$, $U(x, 1) = (T_\beta^{\boldsymbol{D}}(x), 1)$, for all $x \in \boldsymbol{D}$, and U fixes $\partial \boldsymbol{D} \times [0, 1]$ pointwise. Moreover, by construction, U carries

$$\left(\bigsqcup_{j=1}^{n} \boldsymbol{D}(j, \varepsilon) \right) \times [0, 1]$$

to a tubular neighborhood of (a representative of) the braid β, and $g \circ f \circ U$ carries the arcs $\{b_j \times [0, 1] : j = 1, \dots, n\}$ to a framing of $\hat{\beta}$ equivalent to that described in Figure 6, namely the preferred framing. Consequently the space $\Omega(\hat{\beta}, X)$ is homeomorphic to $T_1' \cup T_2$ where

$$T_1' = Y \times [0, 1] / \big((y, 0) \sim (T_\beta(y), 1) \big).$$

We therefore have $\pi_1(T_1') \cong H^{*n} * \langle t \rangle \big/ \big(t x t^{-1} = \rho(\beta) x \text{ for } x \in H^{*n} \big)$, an HNN-extension. Attaching T_2 to T_1' has the effect of simply killing the stable letter t. Consequently

$$\pi_1(\Omega(\hat{\beta}, X)) \cong H^{*n} \big/ \big(x = \rho(\beta) x \text{ for } x \in H^{*n} \big) = \Gamma_{(H,h)}(\beta). \qquad \square$$

4. Faithfulness

Recall that, for any group H and for $n \in \mathbb{N}$, we write H^{*n} for the free product $H_1 * \dots * H_n$, where each free factor H_i is isomorphic to H by an isomorphism $\phi_i : H \to H_i$. The aim of this section is to prove the following.

Proposition 4.1. *Let $\rho : B_n \to \mathrm{Aut}(H^{*n})$ be the Artin type representation of B_n associated to the pair (H, h) where H is a group and $h \in H$.*

(i) *If $h \neq \mathrm{Id}_H$, then ρ is faithful.*

(ii) *If $h = \mathrm{Id}_H$, then $\ker(\rho)$ is the pure braid group and $B_n / \ker(\rho) \cong S_n$, the symmetric group, acts by permutations of the free factors of H^{*n} (respecting the isomorphisms $\{\phi_1, \dots, \phi_n\}$).*

Remark. Part (ii) of this proposition requires no proof but is included here for completeness. We concern ourselves below with the case of h nontrivial.

As pointed out in the introduction, the proof of Proposition 4.1(i) is strongly inspired by the proof of [Shpilrain 2001, Theorem A], and its main ingredient is the following:

Proposition 4.2 [Dehornoy 1994; 1997a]. *Let $B_{[2,n]}$ denote the subgroup of B_n generated by $\sigma_2, \dots, \sigma_{n-1}$ (namely the braid group on the second through nth strings). Let $\beta \in B_n$. Then either*

(1) *$\beta \in B_{[2,n]}$, or*

(2) *one of β or β^{-1} can be written as $\alpha_0\sigma_1\alpha_1\sigma_1\alpha_2\dots\sigma_1\alpha_l$, where $l \geq 1$ and $\alpha_0, \dots, \alpha_l \in B_{[2,n]}$.*

The following lemma is preliminary to the proof of Proposition 4.1. We first fix a nontrivial $h \in H$, and write $h_i = \phi_i(h)$ for $i = 1, \dots, n$.

Lemma 4.3. *Let $K = H_2 * \dots * H_n$. Let $u \in H^{*n}$ such that the normal form of u with respect to the decomposition $H^{*n} = H_1 * K$ starts with h_1^{-1} and ends with h_1.*

(1) *The normal form of $\rho(\sigma_1)(u)$ with respect to the decomposition $H^{*n} = H_1 * K$ also starts with h_1^{-1} and ends with h_1.*

(2) *Let $k \in \{2, \dots, n-1\}$ and $\varepsilon \in \{\pm 1\}$. The normal form of $\rho(\sigma_k^\varepsilon)(u)$ with respect to the decomposition $H^{*n} = H_1 * K$ also starts with h_1^{-1} and ends with h_1.*

Proof. Let $v \in H_1 * H_2$. Suppose that the normal form of v is

$$v = \phi_1(x_1)\,\phi_2(y_1)\ \dots\ \phi_1(x_l)\,\phi_2(y_l),$$

where $x_1, \dots, x_l, y_1, \dots, y_{l-1} \in H \setminus \{\mathrm{Id}\}$, and $y_l \in H$. Then

$$\rho(\sigma_1)(v) = h_1^{-1} \cdot \phi_2(x_1) \cdot h_1^2\phi_1(y_1)h_1^{-2} \cdot \dots \cdot \phi_2(x_l) \cdot h_1^2\phi_1(y_l)\ h_1^{-1};$$

thus the normal form of $\rho(\sigma_1)(v)$ starts with h_1^{-1}.

Similarly, if the normal form of v is

$$v = \phi_2(y_1)\,\phi_1(x_1)\ \dots\ \phi_2(y_l)\,\phi_1(x_l),$$

where $x_1, \dots, x_l, y_2, \dots, y_l \in H \setminus \{\mathrm{Id}\}$ and $y_1 \in H$, then the normal form of $\rho(\sigma_1)(v)$ ends with h_1.

Now, write

$$u = v_0\, w_1\, v_1 \dots w_l\, v_l,$$

where $v_i \in (H_1 * H_2) \setminus \{\mathrm{Id}\}$ and $w_j \in (H_3 * \dots * H_n) \setminus \{\mathrm{Id}\}$, and $l \geq 0$. The hypothesis that u starts with h_1^{-1} implies that v_0 starts with h_1^{-1}, and the hypothesis that u ends with h_1 implies that v_l ends with h_1. Both groups, $H_1 * H_2$ and $H_3 * \dots * H_n$, are invariant by $\rho(\sigma_1)$, and $\rho(\sigma_1)$ is the identity on $H_3 * \dots * H_n$. So,

$$\rho(\sigma_1)(u) = \rho(\sigma_1)(v_0) \cdot w_1 \cdot \rho(\sigma_1)(v_1) \cdot \dots \cdot w_l \cdot \rho(\sigma_1)(v_l).$$

By the observations above, $\rho(\sigma_1)(v_0)$ starts with h_1^{-1} and $\rho(\sigma_1)(v_l)$ ends with h_1; thus $\rho(\sigma_1)(u)$ starts with h_1^{-1} and ends with h_1.

Let $k \in \{2, \ldots, n-1\}$ and $\varepsilon \in \{\pm 1\}$. Write

$$u = h_1^{-1}\, w_1\, v_1 \ldots v_{l-1}\, w_l\, h_1,$$

where $v_1, \ldots, v_{l-1} \in H_1 \setminus \{\mathrm{Id}\}$ and $w_1, \ldots, w_l \in K \setminus \{\mathrm{Id}\}$. Both groups, H_1 and K, are invariant by $\rho(\sigma_k^\varepsilon)$, and $\rho(\sigma_k^\varepsilon)$ is the identity on H_1. So

$$\rho(\sigma_k^\varepsilon)(u) = h_1^{-1} \cdot \rho(\sigma_k^\varepsilon)(w_1) \cdot v_1 \cdot \cdots \cdot v_{l-1} \cdot \rho(\sigma_k^\varepsilon)(w_l) \cdot h_1;$$

thus the normal form of $\rho(\sigma_k^\varepsilon)(u)$ starts with h_1^{-1} and ends with h_1. □

Proof of Proposition 4.1(i). We argue by induction on n. Assume $n = 2$. We have

$$\rho(\sigma_1^{2l})(h_1) = (h_2h_1)^{-l} h_1 (h_2h_1)^l \neq h_1, \quad \text{for } l \in \mathbb{Z} \setminus \{0\}$$
$$\rho(\sigma_1^{2l+1})(h_1) = (h_2h_1)^{-l} h_1^{-1} h_2 h_1 (h_2h_1)^l \neq h_1, \quad \text{for } l \in \mathbb{Z};$$

thus the representation $\rho : B_2 \to \mathrm{Aut}(H_1 * H_2)$ is faithful.

Now, assume $n \geq 3$. Let $\beta \in B_n \setminus \{\mathrm{Id}\}$. By Proposition 4.2, either $\beta \in B_{[2,n]}$, or one of β or β^{-1} is written $\alpha_0\sigma_1 \ldots \sigma_1\alpha_l$, where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{[2,n]}$.

Suppose $\beta \in B_{[2,n]}$. By induction, $\rho(\beta)$ acts nontrivially on $K = H_2 * \cdots * H_n$; thus $\rho(\beta)$ acts nontrivially on $H^{*n} = H_1 * K$.

Suppose $\beta = \alpha_0\sigma_1 \ldots \sigma_1\alpha_l$, where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{[2,n]}$. Let

$$u = \rho(\sigma_1\alpha_l)(h_1) = \rho(\sigma_1)(h_1) = h_1^{-1}h_2h_1.$$

By Lemma 4.3, the normal form of $\rho(\alpha_0\sigma_1 \ldots \sigma_1\alpha_{l-1})(u) = \rho(\beta)(h_1)$ starts with h_1^{-1} and ends with h_1. In particular, $\rho(\beta)(h_1) \neq h_1$; thus $\rho(\beta) \neq \mathrm{Id}$.

Finally, suppose $\beta^{-1} = \alpha_0\sigma_1 \ldots \sigma_1\alpha_l$, where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{[2,n]}$. By the previous case, $\rho(\beta^{-1}) \neq \mathrm{Id}$; thus $\rho(\beta) \neq \mathrm{Id}$. □

5. Garside groups

In this section we give a brief presentation of the definition and salient properties of a Garside group, and then establish the necessary and sufficient criteria for a group to be a Garside group which we shall use in the subsequent section. Our Criterion 5.9 is essentially a variation on [Dehornoy 2002, Proposition 2.1]. The theory of Garside groups, as developed in [Dehornoy and Paris 1999; Dehornoy 1997b; 2002], provides the most natural general setting for the combinatorial arguments contained in Garside's original treatment [1969] of the braid groups, and its generalization to Artin groups in [Brieskorn and Saito 1972].

Definition 5.1. Let M be an arbitrary monoid. We say that M is *atomic* if there exists a function $\nu : M \to \mathbb{N}$ such that

- $\nu(a) = 0$ if and only if $a = 1$;
- $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Such a function $\nu : M \to \mathbb{N}$ is called a *norm* on M. An element $a \in M$ is called an *atom* if it is indecomposable, namely, if $a = bc$ then either $b = 1$ or $c = 1$.

We note that any generating set of M contains the set of all atoms. In particular, M is finitely generated if and only if it has only finitely many atoms. For details see [Dehornoy and Paris 1999].

Given that a monoid M is atomic, we may define left and right invariant partial orders $\leq_L$ and $\leq_R$ on M as follows:

- set $a \leq_L b$ if there exists $c \in M$ such that $ac = b$;
- set $a \leq_R b$ if there exists $c \in M$ such that $ca = b$.

We shall call these the *left* and *right divisibility orders* on M.

Definition 5.2. A *Garside monoid* is a monoid M such that

(i) M is atomic and finitely generated;

(ii) M is (left and right) cancellative, i.e. $abc = ab'c$ implies $b = b'$;

(iii) $(M, \leq_L)$ and $(M, \leq_R)$ are lattices;

(iv) there exists an element $\Delta \in M$, which we call a *Garside element*, such that

  (a) the set $L(\Delta) := \{x \in M : x \leq_L \Delta\}$ generates M, and
  (b) the sets $L(\Delta)$ and $R(\Delta) := \{x \in M : x \leq_R \Delta\}$ are equal.

Definition 5.3. For any monoid M one can define the group $G(M)$ which is presented by the generating set M and relations $ab = c$ whenever $ab = c$ in M. There is an obvious canonical homomorphism $M \to G(M)$. This homomorphism is not injective in general. The group $G(M)$ is known as the *group of fractions* of M. Define a *Garside group* to be the group of fractions of a Garside monoid.

Remark. (1) A Garside monoid M satisfies Öre's conditions (left and right cancellativity and the existence of common upper bounds in $(M, \leq_L)$); thus the canonical homomorphism $M \to G(M)$ is injective. Moreover the partial orders $\leq_L$ and $\leq_R$ extend respectively to left- and right-invariant lattice orders on $G(M)$ with positive cone M.

(2) A Garside element is never unique. For example, if Δ is a Garside element, then Δ^k is also a Garside element for all $k \geq 1$ [Dehornoy 2002, Lemma 2.2].

(3) Elsewhere in the literature the condition that M is finitely generated is often incorporated into condition (iv) of the definition by saying that the set $L(\Delta)$ is finite. It seems more natural to state this condition separately. Note that, if M is finitely generated and atomic, then $L(a) = \{x \in M : x \leq_L a\}$ is finite for all $a \in M$.

We now introduce some terminology needed in order to state Criterion 5.9.

For a finite set S, we denote by S^* the free monoid on S. The elements of S^* are called *words* on S. The empty word is denoted by ϵ. Let $\equiv$ be a congruence relation on S^*, and let $M = (S^*/\equiv)$. For $w \in S^*$, we denote by $\overline{w}$ the element of M represented by w, and we call w an *expression* of $\overline{w}$.

Definition 5.4. A *complement* is a function $f : S \times S \to S^*$ such that $f(x, x) = \epsilon$ for all $x \in S$. To a complement $f : S \times S \to S^*$ we associate the two monoids

$$M_L^f = \left\langle S \mid xf(x, y) = yf(y, x) \text{ for } x, y \in S \right\rangle^+,$$
$$M_R^f = \left\langle S \mid f(y, x)x = f(x, y)y \text{ for } x, y \in S \right\rangle^+.$$

For $u, v \in S^*$, we write $u \equiv_L^f v$ if u and v are expressions of the same element of M_L^f, and we write $u \equiv_R^f v$ if u and v are expressions of the same element of M_R^f.

Definition 5.5. A word w in $(S \cup S^{-1})^*$ is *f-reversible on the left in one step* to a word w' if w' is obtained from w by replacing some subword $x^{-1}y$ (with $x, y \in S$) by the corresponding word $f(x, y)f(y, x)^{-1}$. Let $p \geq 0$. We say that w is *f-reversible on the left in p steps* to a word w' if there exists a sequence $w_0 = w, w_1, \dots, w_p = w'$ in $(S \cup S^{-1})^*$ such that w_{i-1} is f-reversible on the left in one step to w_i for all $i = 1, \dots, p$. The property "w is f-reversible on the left to w'" is denoted by $w \to_L^f w'$.

We define *f-reversibility on the right* in a similar way, replacing subwords yx^{-1} (with $x, y \in S$) by the corresponding words $f(x, y)^{-1}f(y, x)$. The property "w is f-reversible on the right to w'" is denoted by $w \to_R^f w'$.

It is shown in [Dehornoy 1997b] that a reversing process is confluent, namely:

Proposition 5.6 [Dehornoy 1997b, Lemma 1]. *Let $f : S \times S \to S^*$ be a complement, and let $w \in (S \cup S^{-1})^*$. Suppose that the word w is f-reversible on the left in p steps to a word uv^{-1}, with $u, v \in S^*$. Then any sequence of left f-reversing transformations starting from w leads in p steps to uv^{-1}.*

Definition 5.7. Let $f : S \times S \to S^*$ be a complement and let $u, v \in S^*$. Assume that there exist $u', v' \in S^*$ such that $u^{-1}v \to_L^f u'(v')^{-1}$. By Proposition 5.6, u' and v' are unique (if they exist). Then we write $u' = C_L^f(u, v)$ and $v' = C_L^f(v, u)$. One has, by [Dehornoy 1997b, Lemma 2],

$$uC_L^f(u, v) \equiv_L^f vC_L^f(v, u).$$

If no such words u', v' exist, we write $C_L^f(u, v) = C_L^f(v, u) = \infty$.

Similarly, define the words $C_R^f(u, v)$ and $C_R^f(v, u)$ to be the unique elements of S^* which satisfy $vu^{-1} \to_R^f C_R^f(u, v)^{-1}C_R^f(v, u)$, or write $C_R^f(u, v) = C_R^f(v, u) = \infty$ if no such words exist.

Definition 5.8 [Dehornoy 1997b, p. 120]. Let $f : S \times S \to S^*$ be a complement. We say that f is *coherent on the left* if, for all $x, y, z \in S$ such that $C_L^f(f(x, y), f(x, z)) \neq \infty$ we have

$$C_L^f(f(x, y), f(x, z)) \equiv_L^f C_L^f(f(y, x), f(y, z)).$$

Similarly, we say that f is *coherent on the right* if, for all $x, y, z \in S$ such that $C_R^f(f(z, x), f(y, x)) \neq \infty$ we have

$$C_R^f(f(z, x), f(y, x)) \equiv_R^f C_R^f(f(z, y), f(x, y)).$$

It can be shown [Dehornoy 1997b, Lemma 4] that if an atomic monoid M can be written $M = M_L^f$ where the complement f is coherent on the left, then M is left cancellative and $(M, \leq_L)$ is a *quasi-lattice*: every pair of elements $x, y \in M$ which has a common upper bound (z such that $x \leq_L z$ and $y \leq_L z$) has a least upper bound, written $x \vee_L y$. This argument is based on Garside's [1969] original argument (see also [Brieskorn and Saito 1972]), and forms the cornerstone of the theory of Garside groups. (The analogous statement when $M = M_R^g$ is atomic and g is coherent on the right obviously holds as well.)

We are now ready to state a criterion for a monoid M to be a Garside monoid:

Criterion 5.9. *Let M be a monoid. Then M is a Garside monoid if and only if it satisfies the following properties*:

(C1) *M is finitely generated and atomic.*

(C2) *There exist complements $f : S_1 \times S_1 \to S_1^*$, coherent on the left, and $g : S_2 \times S_2 \to S_2^*$, coherent on the right, such that $M \cong M_L^f$ and $M \cong M_R^g$.*

(C3) *M possesses a Garside element, namely an element $\Delta \in M$ such that the sets $L(\Delta) = \{x \in M : x \leq_L \Delta\}$ and $R(\Delta) = \{x \in M : x \leq_R \Delta\}$ are equal and generate M.*

Proof. Suppose first that M satisfies (C1), (C2), and (C3). It follows from [Dehornoy 1997b, Lemma 4] (see the remark above) that M is left and right cancellative and $(M, \leq_L)$ is a quasi-lattice. In this situation we may define an operation $\backslash_L : M \times M \to M \cup \{\infty\}$ such that $a(a \backslash_L b) = a \vee_L b$ if a and b have a common upper bound, and $a \backslash_L b = \infty$ otherwise. According to [Dehornoy 2002, Proposition 2.1], the above conditions together with the following condition (D) are sufficient to show that M is a Garside monoid:

(D) There exists a finite subset $P \subset M$ which generates M and which is closed under the operation $\backslash_L$ (namely, if $a, b \in P$ then $a \backslash_L b \in P$).

We show that M satisfies (D). Let $P = L(\Delta) = R(\Delta)$. Note that, by (C3), P generates M. Let $a, b \in P$. Since $a \leq_L \Delta$ and $b \leq_L \Delta$, we have $a \vee_L b \leq_L \Delta$.

Let $c \in M$ such that $\Delta = (a \vee_L b)c = a(a \backslash_L b)c$. Then $(a \backslash_L b)c \leq_R \Delta$; thus $(a \backslash_L b)c \leq_L \Delta$ (since, by (C3), $L(\Delta) = R(\Delta)$); therefore $(a \backslash_L b) \leq_L \Delta$, that is $(a \backslash_L b) \in P$.

Now suppose that M is a Garside monoid. Clearly, M satisfies (C1) and (C3). So, we just need to show that M satisfies (C2). Choose some finite generating set S for M, and consider complements $f : S \times S \to S^*$ and $g : S \times S \to S^*$ such that

$$x\overline{f(x, y)} = x \vee_L y, \quad \overline{g(x, y)}x = y \vee_R x,$$

for all $x, y \in S$. Then, by [Dehornoy and Paris 1999, Theorem 4.1], $M = M_L^f = M_R^g$, and, by [Dehornoy 2002, Lemma 5.2], f is coherent on the left and g is coherent on the right. □

It will be convenient, in Section 6, to have the following characterization of a Garside element.

Lemma 5.10 (Garside elements). *Let M be a* (*left and right*) *cancellative monoid. Then Δ is a Garside element* (*meaning that $L(\Delta)$ coincides with $R(\Delta)$ and generates M*) *if and only if the following condition holds*:

(C4) $L(\Delta) := \{x \in M : x \leq_L \Delta\}$ *generates M and there exists a* (*necessarily unique*) *monoid automorphism $\tau : M \to M$ such that $w\Delta = \Delta\tau(w)$ for all $w \in M$.*

Consequently, *we may replace condition* (C3) *in Criterion 5.9 with condition* (C4).

Proof. We first show sufficiency. Suppose that (C4) is satisfied. In particular, we have $\tau(\Delta) = \Delta$ and therefore $\tau(L(\Delta)) = L(\Delta)$ (since τ is a monoid automorphism). On the other hand, by using left and right cancellation one easily obtains from the equation $x\Delta = \Delta\tau(x)$ that $\tau(L(\Delta)) = R(\Delta)$. But then $L(\Delta) = \tau(L(\Delta)) = R(\Delta)$ and, by hypothesis (C4), $L(\Delta)$ also generates M. Thus Δ is a Garside element.

Now suppose that Δ is a Garside element. By cancellativity and the fact that $L(\Delta) = R(\Delta)$, one has a well-defined bijection $c : L(\Delta) \to L(\Delta)$ such that $x\,c(x) = \Delta$ for all $x \in L(\Delta)$. Note that, if $x \in L(\Delta)$ then so is $c(x)$ and Δ may be written either $x\,c(x)$ or $c(x)c^2(x)$. Thus $x\,\Delta = x\,c(x)c^2(x) = \Delta c^2(x)$, for all $x \in L(\Delta)$. Since $L(\Delta)$ generates M, it follows by cancellativity that the bijection c^2 extends uniquely to a monoid automorphism τ satisfying (C4). □

6. Semidirect products

We now turn back to the Artin type representations. Given an Artin type representation $\rho : B_n \to \mathrm{Aut}(H^{*n})$ associated to a group H and an element $h \in H$, we may form the semidirect product $H^{*n} \rtimes_\rho B_n$. The aim of this section is to prove the following.

Theorem 6.1. *Assume that H is the group of fractions of a Garside monoid M and that $h \in M$ is a Garside element. Let $G = H^{*n} \rtimes_\rho B_n$, where $\rho : B_n \to \mathrm{Aut}(H^{*n})$ denotes the Artin type representation associated to (H, h) (as defined in the Introduction), and let P be the submonoid of G generated by $M_1 = \phi_1(M)$ and the monoid B_n^+ of positive braids. Then P is a Garside monoid, $\Delta = (h_1\sigma_1\sigma_2 \dots \sigma_{n-1})^n$ is a Garside element of P, and G is the group of fractions of P.*

The first step in the proof is to find a presentation for $H^{*n} \rtimes_\rho B_n$:

Proposition 6.2. *Let $H = \langle S \mid \mathcal{R} \rangle$ be a presentation for H, and let $D \in S^*$ be an expression for h. Then $G = H^{*n} \rtimes_\rho B_n$ has a presentation with generators*

$$S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$$

and relations

$$\begin{array}{ll} r & \text{for } r \in \mathcal{R}, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \dots, n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2, \\ \sigma_i x = x\sigma_i & \text{for } x \in S \text{ and } i = 2, \dots, n-1, \\ x\sigma_1 D\sigma_1 = \sigma_1 D\sigma_1 D^{-1} x D & \text{for } x \in S. \end{array}$$

Proof of the proposition. Let G_0 denote the abstract group generated by the union $S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$, subject to the relations given in the statement of the proposition. Set $X = \left(\bigcup_{i=1}^n \phi_i(S)\right) \cup \{\sigma_1, \dots, \sigma_{n-1}\}$. With a little effort one can verify that the mapping $\varphi : X \to G_0$ defined by

$$\begin{aligned} \varphi(\phi_i(x)) &= \sigma_{i-1}^{-1} \dots \sigma_1^{-1} D^{i-1} x D^{1-i} \sigma_1 \dots \sigma_{i-1} && \text{for } i = 1, \dots, n \text{ and } x \in S, \\ \varphi(\sigma_i) &= \sigma_i && \text{for } i = 1, \dots, n-1 \end{aligned}$$

determines a homomorphism $\varphi : G \to G_0$, and somewhat more easily that the mapping $\psi : S \cup \{\sigma_1, \dots, \sigma_{n-1}\} \to G$ defined by

$$\begin{aligned} \psi(x) &= \phi_1(x) && \text{for } x \in S \\ \psi(\sigma_i) &= \sigma_i && \text{for } i = 1, \dots, n-1 \end{aligned}$$

determines a homomorphism $\psi : G_0 \to G$. One checks without much difficulty that $(\psi \circ \varphi)(a) = a$ for all $a \in X$, and $(\varphi \circ \psi)(b) = b$ for all $b \in S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$; thus $\psi \circ \varphi = \mathrm{Id}_G$ and $\varphi \circ \psi = \mathrm{Id}_{G_0}$. □

Proof of Theorem 6.1. Let $\tau : M \to M$ denote the automorphism of M induced by conjugation by h^{-1}, so that $xh = h\tau(x)$ for all $x \in M$ (see Lemma 5.10). Let S be a finite generating set for M. We may, and do, choose S so that $\tau(S) = S$ (for instance we may simply choose S to be the set of atoms of M). Define $f : S \times S \to S^*$ so that $\overline{xf(x,y)} = \overline{yf(y,x)} = x \vee_L y$ for all pairs $x, y \in S$. Similarly define $g : S \times S \to S^*$

so that $\overline{g(x,y)y} = \overline{g(y,x)x} = x \vee_R y$ for all pairs $x, y \in S$. As pointed out in the proof of Criterion 5.9, one has $M = M_L^f = M_R^g$, f is coherent on the left, and g is coherent on the right. We simply write $\sim$ for the congruence relation on S^* defined by the relations in M (namely, $\equiv_L^f$, or equally $\equiv_R^g$). Let $D \in S^*$ be an expression of h. Note that for $x \in S$ we have $xD \sim D\tau(x)$ and $\tau^{-1}(x)D \sim Dx$, where $\tau(x)$ and $\tau^{-1}(x)$ also denote elements of the generating set S. The last family of relations appearing in Proposition 6.2 may be replaced with $x\sigma_1 D\sigma_1 = \sigma_1 D\sigma_1\tau(x)$ for all $x \in S$, or equivalently with $\tau^{-1}(x)\sigma_1 D\sigma_1 = \sigma_1 D\sigma_1 x$ for all $x \in S$.

Let $X = S \cup \{\sigma_1, \dots, \sigma_{n-1}\}$. Let $\lambda : X \times X \to X^*$ be the complement defined by

$$\begin{array}{llll} \lambda(x,y) = f(x,y) & \text{for } x, y \in S, & \lambda(\sigma_i, x) = x & \text{for } x \in S \text{ and } i \geq 2, \\ \lambda(x,\sigma_1) = \sigma_1 D\sigma_1 & \text{for } x \in S, & \lambda(\sigma_i,\sigma_j) = \sigma_j\sigma_i & \text{for } |i-j| = 1, \\ \lambda(\sigma_1, x) = D\sigma_1\tau(x) & \text{for } x \in S, & \lambda(\sigma_i,\sigma_j) = \sigma_j & \text{for } |i-j| \geq 2, \\ \lambda(x,\sigma_i) = \sigma_i & \text{for } x \in S \text{ and } i \geq 2, & & \end{array}$$

and let $\delta : X \times X \to X^*$ be the complement defined by

$$\begin{array}{llll} \delta(x,y) = g(x,y) & \text{for } x, y \in S, & \delta(x, \sigma_i) = x & \text{for } x \in S \text{ and } i \geq 2, \\ \delta(\sigma_1, x) = \sigma_1 D\sigma_1 & \text{for } x \in S, & \delta(\sigma_j,\sigma_i) = \sigma_i\sigma_j & \text{for } |i-j| = 1, \\ \delta(x,\sigma_1) = \tau^{-1}(x)\sigma_1 D & \text{for } x \in S, & \delta(\sigma_j,\sigma_i) = \sigma_j & \text{for } |i-j| \geq 2, \\ \delta(\sigma_i, x) = \sigma_i & \text{for } x \in S \text{ and } i \geq 2. & & \end{array}$$

Let P_0 denote the monoid defined by the presentation with generators X and relations as laid out in Proposition 6.2. Then clearly $P_0 \cong M_L^\lambda \cong M_R^\delta$. We denote by $\approx$ the congruence relation on X^* defined by the relations of P_0. (So $\approx$ is the same congruence relation as $\equiv_L^\lambda$ and $\equiv_R^\delta$). We now show that P_0 satisfies Criterion 5.9 with complements λ and δ and Garside element $\Delta = (D\sigma_1\sigma_2 \dots \sigma_{n-1})^n$. It will follow that P_0 is a Garside monoid with group of fractions G and is canonically isomorphic to the submonoid $P \subset G$ in the statement of the Theorem.

Clearly P_0 is finitely generated. We check that P_0 is atomic. Let $\nu : M \to \mathbb{N}$ be a norm for M. Let $\Sigma = \{\sigma_1, \dots, \sigma_{n-1}\}$ and define the function $\ell : \Sigma^* \to \mathbb{N}$ by $\ell(\sigma_{i_1} \dots \sigma_{i_l}) = l$. We define a function $\nu_P : X^* \to \mathbb{N}$ as follows. Let $w \in X^*$. Write $w = u_1 v_1 \dots u_l v_l$, where $u_1 \in S^*$, $u_2, \dots, u_l \in S^* \setminus \{\epsilon\}$, $v_1, \dots, v_{l-1} \in \Sigma^* \setminus \{\epsilon\}$, and $v_l \in \Sigma^*$. Then

$$\nu_P(w) = \nu(u_1 u_2 \dots u_l) + \ell(v_1 v_2 \dots v_l).$$

One can easily verify that ν_P is invariant with respect to all of the relations given in Proposition 6.2, and therefore defines a function $\nu_P : P_0 \to \mathbb{N}$. Moreover, it is easily seen that ν_P is a norm, and therefore P_0 is atomic.

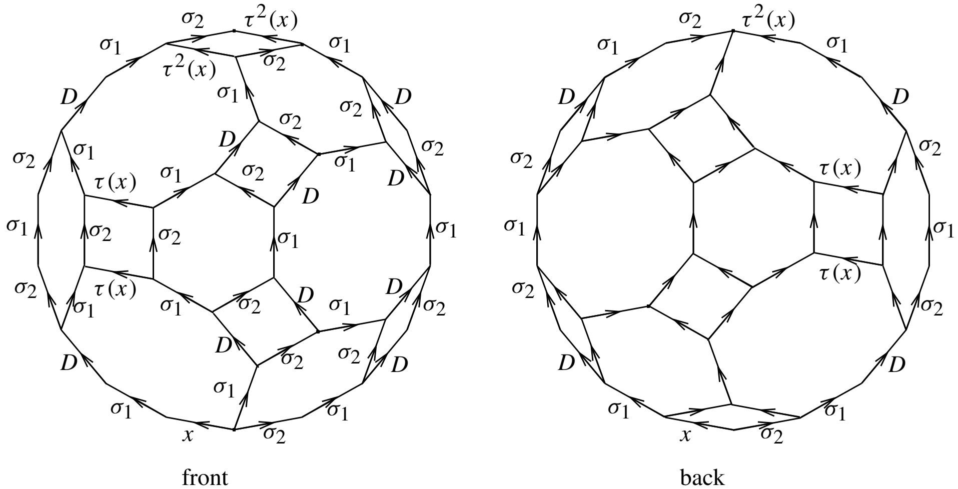

Figure 7. Left coherence of λ with respect to triple $\{\sigma_1, \sigma_2, x\}$. The labels which, for clarity, are missing from the back side may be easily inferred from the relations shown on the front side.

The proof that λ is coherent on the left may be deduced from the existence, for each triple $\alpha, \beta, \gamma \in X$, of a certain tiling of the 2-sphere by relations from M_L^λ (i.e., relations of the form $\alpha\lambda(\alpha, \beta) \approx \beta\lambda(\beta, \alpha)$ for $\alpha, \beta \in X$). We illustrate the two most difficult cases, namely when $\{\alpha, \beta, \gamma\} = \{\sigma_1, \sigma_2, x\}$ for some $x \in S$ (Figure 7), and when $\{\alpha, \beta, \gamma\} = \{\sigma_1, x, y\}$ for some $x \neq y \in S$ (Figure 8). In the latter case note that, if $f(x, y)$ is written $a_1 a_2 \dots a_k$ as a product of generators $a_i \in S$ then $\tau(f(x, y)) \approx \tau(a_1)\tau(a_2)\dots\tau(a_k)$ and the face containing $f(x, y)$ and $\tau(f(x, y))$ in Figure 8 decomposes into k faces corresponding to the relations

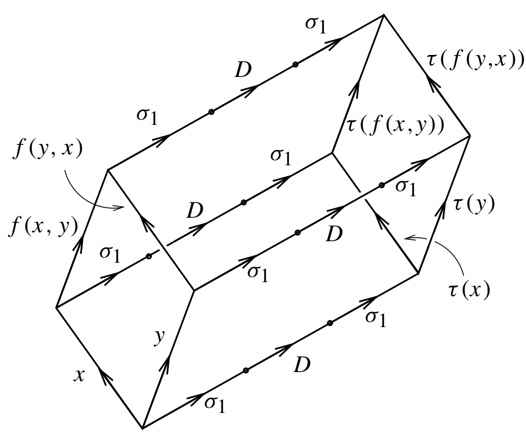

Figure 8. Left coherence of λ with respect to triple $\{\sigma_1, x, y\}$.

$a_i\sigma_1 D\sigma_1 \approx \sigma_1 D\sigma_1\tau(a_i)$. Similarly for $f(y,x)$. The remaining cases are easily handled since in these cases at least one of α, β, γ satisfies a commuting relation (explicit in the presentation M_L^λ) with each of the others.

The proof that δ is coherent on the right is similar.

Finally we show that the word $\Delta = (D\sigma_1\sigma_2 \dots \sigma_{n-1})^n$ represents a Garside element of P_0. We shall employ condition (C4) of Lemma 5.10. Consider the Artin monoid presentation

$$A^+(B_n) = \big\langle \beta_1, \beta_2, \dots, \beta_n \mid \beta_1\beta_2\beta_1\beta_2 = \beta_2\beta_1\beta_2\beta_1, \\ \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1} \ \text{ for } \ 2 \le i \le n-1, \\ \beta_i\beta_j = \beta_j\beta_i \ \text{ for } \ |i-j| \ge 2 \big\rangle^+ .$$

This monoid $A^+(B_n)$ is well-known as the Artin monoid of type B_n, and has Garside element $\Delta_B = (\beta_1\beta_2 \dots \beta_n)^n$. Clearly there exists a monoid homomorphism $A^+(B_n) \to P_0$ such that $\beta_1 \mapsto D$ and $\beta_i \mapsto \sigma_{i-1}$ for $i = 2, 3, \dots, n$. Thus any relation which is observed in $A^+(B_n)$ may be deduced in P_0. In particular, the fact that Δ_B is a Garside element in $A^+(B_n)$ implies that Δ is left divisible by $D, \sigma_1, \dots, \sigma_{n-1}$ and hence is left divisible by every element of X. It remains to verify that there exists an automorphism $\tau_P : P_0 \to P_0$ such that $w\Delta = \Delta\tau_P(w)$ for all $w \in P_0$.

We already know that Δ_B is central in $A^+(B_n)$. Thus we have $\sigma_i\Delta = \Delta\sigma_i$ for all $i = 1, 2, \dots, n-1$. We may also check (by performing the calculation in $A^+(B_n)$) that

$$\Delta \approx D\,U^{n-1} \qquad \text{where } \ U := \sigma_1 D\sigma_1\sigma_2\sigma_3 \cdots \sigma_{n-1} .$$

Recall that τ denotes the automorphism of M such that, at the level of words, $xD \sim D\tau(x)$ for all $x \in S^*$. Observe also that $xU \approx U\tau(x)$ for all $x \in S^*$ (or more loosely speaking, for all $x \in M$). We now define $\tau_P : P_0 \to P_0$ such that

$$\tau_P(\sigma_i) = \sigma_i \qquad \text{for } \ i = 1, 2, \dots, n-1,$$
$$\tau_P(x) = \tau^n(x) \qquad \text{for all } \ x \in M .$$

It is easily seen that τ_P is a monoid isomorphism. Moreover, for all $x \in M$,

$$x\Delta \approx xDU^{n-1} \approx D\tau(x)U^{n-1} \approx DU^{n-1}\tau^n(x) \approx \Delta\tau_P(x),$$

and $\sigma_i\Delta \approx \Delta\sigma_i$ for all $i = 1, 2, \dots, n-1$. Thus condition (C4) of Lemma 5.10 is satisfied, and Δ is a Garside element. □

Remark. In closing, we remark that both the above proof and the formulation of Theorem 6.1 were strongly inspired by the example of the Artin group $A(B_n)$ which, as noted in the introduction, is isomorphic to the semidirect product $F_n \rtimes B_n$ associated to Artin's 1925 representation [Artin 1925; 1947], namely the Artin type representation associated to $(\mathbb{Z}, 1)$ (using additive notation). This is evident in both the description of the fundamental element, and the checking of coherence (see Figures 7 and 8) which follow closely the proof that $A(B_n)$ has a Garside structure. Note, in particular, that the diagram shown in Figure 7 depicts the Cayley graph for the Coxeter group of type B_3, once the labels D, x, $\tau(x)$ and $\tau^2(x)$ are replaced with a single generator.

In response to a question posed by the referee, we are not aware of any other general constructions of Garside groups obtained in a similar fashion by studying other Artin groups of finite type. We note however that the Artin group of type D_n is isomorphic to the index 2 torsion free subgroup of the semidirect product $(C_2)^{*n} \rtimes B_n$ associated to the Artin type representation determined by the nontrivial element of C_2. However, the group C_2 of order 2 is clearly not Garside (it has torsion!) so that while $A(D_n)$ admits a Garside structure, this does *not* arise by virtue of Theorem 6.1 just proved. The Artin groups of type B_n, $n \geq 2$, would appear to be the only Artin groups of irreducible finite type which are covered in this way by Theorem 6.1.

Appendix

We denote by F_n the free group of rank n, and fix a basis $x_1, \ldots, x_n$ for F_n.

Definition. According to Shpilrain's terminology [2001], a *Wada representation of type* (1) is an Artin type representation associated to $(\mathbb{Z}, h)$, where h is a nonzero integer. Such a representation will be denoted by $\rho_h^{(1)} : B_n \to \operatorname{Aut}(F_n)$. It is determined by

$$\rho_h^{(1)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k,\ k+1, \\ x_k^{-h} x_{k+1} x_k^h & \text{if } i = k, \\ x_k & \text{if } i = k+1. \end{cases}$$

The *Wada representation of type* (2) is the representation $\rho^{(2)} : B_n \to \operatorname{Aut}(F_n)$ determined by

$$\rho^{(2)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k,\ k+1, \\ x_k x_{k+1}^{-1} x_k & \text{if } i = k, \\ x_k & \text{if } i = k+1. \end{cases}$$

and the *Wada representation of type* (3) is the representation $\rho^{(3)} : B_n \to \operatorname{Aut}(F_n)$ determined by

$$\rho^{(3)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k,\ k+1, \\ x_k^2 x_{k+1} & \text{if } i = k, \\ x_{k+1}^{-1} x_k^{-1} x_{k+1} & \text{if } i = k+1. \end{cases}$$

Proposition A.1. (1) *Let $k, l \in \mathbb{Z} \setminus \{0\}$. Then $\rho_k^{(1)}$ and $\rho_l^{(1)}$ are equivalent if and only if $l = \pm k$.*

(2) *$\rho^{(2)}$ and $\rho^{(3)}$ are equivalent.*

(3) *Let $k \in \mathbb{Z} \setminus \{0\}$. Then $\rho^{(2)}$ and $\rho_k^{(1)}$ are not equivalent.*

The following lemmas are preliminary to the proof of this proposition.

Lemma A.2. *Consider the action of B_n on F_n via the representation $\rho_h^{(1)}$. For all $i = 1, \dots, n-1$, the subgroup of F_n left fixed by $\langle \sigma_i \rangle$, and written $F_n^{\langle \sigma_i \rangle}$, is freely generated by the elements*

$$x_1, \dots, x_{i-1}, x_{i+1}^h x_i^h, x_{i+2}, \dots, x_n.$$

Proof. Write $F_n = C * D$, where $C = \langle x_i, x_{i+1} \rangle$, $D = \langle x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_n \rangle$. Both groups, C and D, are invariant by the action of σ_i. Moreover, σ_i is the identity on D and acts on C by $x_i \mapsto x_i^{-h} x_{i+1} x_i^h$, $x_{i+1} \mapsto x_i$. In particular, $F_n^{\langle \sigma_i \rangle} = C^{\langle \sigma_i \rangle} * D$.

Let $u \in C^{\langle \sigma_i \rangle}$. Write

$$u = x_i^{n_1} x_{i+1}^{m_1} \dots x_i^{n_r} x_{i+1}^{m_r},$$

where $r \geq 1$, $m_1, \dots, m_{r-1}, n_2, \dots, n_r \in \mathbb{Z} \setminus \{0\}$, and $m_r, n_1 \in \mathbb{Z}$. First, suppose $n_1 \neq 0$. Then

$$\sigma_i(u) = x_i^{-h} x_{i+1}^{n_1} x_i^{m_1} \dots x_{i+1}^{n_r} x_i^{m_r+h} = u.$$

Thus

$$-h = n_1, \ n_1 = m_1, \ \dots, \ n_r = m_r, \ \text{and } m_r + h = 0,$$

hence $u = (x_{i+1}^h x_i^h)^{-r}$. Now, suppose $n_1 = 0$. Then

$$\sigma_i(u) = x_i^{m_1-h} x_{i+1}^{n_2} x_i^{m_2} \dots x_{i+1}^{n_r} x_i^{m_r+h}.$$

Thus

$$m_1 - h = 0, \ m_1 = n_2, \ n_2 = m_2, \ \dots, \ n_r = m_r + h, \ \text{and } m_r = 0,$$

hence $u = (x_{i+1}^h x_i^h)^{r-1}$. □

Lemma A.3. *Consider the action of B_n on F_n via $\rho_h^{(1)}$. Then the fixed subgroup $F_n^{B_n}$ is the cyclic subgroup of F_n generated by $x_n^h \dots x_2^h x_1^h$.*

Proof. Let $u \in F_n^{B_n}$. We have $u \in F_n^{\langle \sigma_i \rangle}$ for all $i = 1, \dots, n-1$. Thus, by Lemma A.2, the reduced form of u satisfies the following properties:

- All the exponents are either equal to h or equal to $-h$.
- If $i \neq 1$, then x_i^h is followed by x_{i-1}^h, and, if $i \neq n$, then x_i^h is preceded by x_{i+1}^h.
- If $i \neq n$, then x_i^{-h} is followed by x_{i+1}^{-h}, and, if $i \neq 1$, then x_i^{-h} is preceded by x_{i-1}^{-h}.

Clearly, these properties hold if and only if u is of the form $u = (x_n^h \dots x_2^h x_1^h)^r$, with $r \in \mathbb{Z}$. □

Proof of Proposition A.1. (1) Let $k \in \mathbb{Z} \setminus \{0\}$. Let $\phi : F_n \to F_n$ be the automorphism determined by $\phi(x_i) = x_i^{-1}$ for all $i = 1, \dots, n$. One can easily verify that

$$\phi^{-1} \circ \rho_k^{(1)}(\sigma_i) \circ \phi = \rho_{-k}^{(1)}(\sigma_i)$$

for all $i = 1, \dots, n-1$; thus ρ_k and ρ_{-k} are equivalent.

Let $k, l > 0$. For a group G, we denote by $H_1(G)$ the abelianization of G, and, for a subgroup H of G, we denote by $\langle\langle H \rangle\rangle$ the normal subgroup of G generated by H. By Lemma A.3, we have

$$F_n / \langle\langle F_n^{\rho_k^{(1)}(B_n)} \rangle\rangle \simeq \langle x_1, \dots, x_n \mid x_n^k \dots x_2^k x_1^k = 1 \rangle;$$

hence

$$H_1(F_n / \langle\langle F_n^{\rho_k^{(1)}(B_n)} \rangle\rangle) \simeq (\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}^{n-1}.$$

So, if $\rho_k^{(1)}$ and $\rho_l^{(1)}$ are equivalent, then $(\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}^{n-1} \simeq (\mathbb{Z}/l\mathbb{Z}) \times \mathbb{Z}^{n-1}$; thus $k = l$.

(2) Write

$$y_i = x_1^2 \dots x_{i-1}^2 x_i \quad \text{for } i = 1, \dots, n.$$

One can easily verify that

$$\rho^{(3)}(\sigma_k)(y_i) = \begin{cases} y_i & \text{if } i \neq k,\ k+1, \\ y_{k+1} & \text{if } i = k, \\ y_{k+1} y_k^{-1} y_{k+1} & \text{if } i = k+1. \end{cases}$$

Let $\phi : F_n \to F_n$ be the automorphism determined by $\phi(x_i) = y_i$ for $i = 1, \dots, n$, and let $\mu : B_n \to B_n$ be the automorphism determined by $\mu(\sigma_i) = \sigma_i^{-1}$ for $i = 1, \dots, n-1$. From the expression of $\rho^{(3)}(\sigma_k)(y_i)$ given above, there follows

$$\phi^{-1} \circ \rho^{(3)}(\sigma_i) \circ \phi = \rho^{(2)}(\mu(\sigma_i))$$

for all $i = 1, \dots, n-1$; thus $\rho^{(2)}$ and $\rho^{(3)}$ are equivalent.

(3) Let $k > 0$. For $u \in F_n$, we denote by $[u]$ the element of $H_1(F_n) \simeq \mathbb{Z}^n$ represented by u. We have

$$\rho^{(2)}(\sigma_1^t)[x_1] = (t+1)[x_1] - t[x_2]$$

for all $t \in \mathbb{N}$. On the other hand, $\rho_k^{(1)}(\beta)$ has finite order as an automorphism of $H_1(F_n)$, for all $\beta \in B_n$. This shows that $\rho^{(2)}$ and $\rho_k^{(1)}$ are not equivalent. □

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