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The universal enveloping algebra of a semisimple Lie algebra is FCR. Complete reducibility for finite-dimensional modules is generalized to encompass the representations of reductive Lie algebras. A quantum example is presented as a nontrivial illustration of these ideas.

Since interest in FCR algebras was rekindled by [Farkas 1987], the literature has been growing [Kraft and Small 1994; Kraft et al. 1999; Kirkman and Small 2002]. Recall that an algebra R is FCR ("finite-dimensional representations are completely reducible") if it has enough finite-dimensional representations, in the sense that every nonzero element of R survives in some finite-dimensional image, and every finite-dimensional R-module is semisimple. The premier example is the enveloping algebra of a finite-dimensional semisimple Lie algebra in characteristic zero. However, classical representation theory of Lie groups frequently requires the examination of a slightly larger class of Lie algebras — the reductive ones. Intuitively, a reductive Lie algebra is "semisimple up to a central subalgebra": finite-dimensional modules on which the central subalgebra acts like a character will be completely reducible.

For example, if $\mathfrak{g} = \mathfrak{sl}(3)$ and θ is the involution of $\mathfrak{g}$ defined by

$$\theta(e_1) = f_2, \quad \theta(e_2) = f_1, \quad \theta(h_1) = -h_2$$

(using the generators presented in [Humphreys 1980]), then $\mathfrak{g}^\theta \simeq \mathfrak{sl}(2) \oplus \mathbb{C}$ is reductive. The intent of this paper is to formulate the appropriate definition of "reductive over a central subalgebra" and exhibit a nontrivial quantum cousin of this example.

The second author has developed a uniform theory of quantum symmetric pairs. There is a one-parameter family of quantum analogues to the enveloping algebra of $\mathfrak{g}^\theta$ described in [Letzter 2003]. Both that reference and [Letzter 2002] consider only nonzero values of the parameter. The algebra $\boldsymbol{B}$ we discuss in this paper is a degenerate member of the family with parameter set to zero. Dijkhuizen, Noumi,

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and Sugitani [Dijkhuizen 1996; Dijkhuizen and Noumi 1998; Noumi et al. 1997] study a closely related family of two-sided coideals in type A. In that context, the degenerate case occurs when the relevant parameter takes on the value 0 after applying a Hopf algebra automorphism.

We begin by discussing notions related to separability of ring extensions, but with conditions restricted to finite-dimensional modules. Section 2 is devoted to a reductive version of Weil's Lemma, which bounds the number of simple modules for a fixed dimension. The bulk of the paper studies the particular example $\boldsymbol{B}$. We show by a delicate specialization argument that finite-dimensional modules, on which a distinguished central element acts as a scalar, are completely reducible. The last section discusses residually finite-dimensional algebras.

The reader may ask whether the quantum symmetric pairs with standard parameters produce reductive algebras (in our sense) as well. The status of these quantum fixed algebras is unclear. They appear to have some misbehaving central characters, suggesting that complete reducibility will only hold "almost everywhere".

1. Semisimple extensions

For the purposes of this paper, k will always denote an algebraically closed field. By the unadorned term *algebra*, we will mean an algebra over k. From now on, assume that R and S are algebras. We say that $R|S$ is an extension if there is an algebra homomorphism from S to R; it is central when S is commutative and it maps into the center of R. We will look at well known representation-theoretic conditions on extensions but restrict their reach to finite-dimensional modules. A traditional treatment of these conditions can be found in [Kadison 1999].

There are two rival definitions to consider for reductivity. The extension $R|S$ is *finitarily semisimple* provided that every short exact sequence of finite-dimensional left R modules which splits as S-modules must also split as R-modules. We say $R|S$ is *finitarily Wedderburn* when every finite-dimensional left R-module which is semisimple as an S-module is also semisimple as an R-module. Clearly, a finitarily semisimple extension is Wedderburn.

Example 1.1. We now consider a finite-dimensional Wedderburn extension which is not semisimple. Let R be the $\mathbb{C}$-algebra of block upper triangular 4×4 matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where a, b and c range over all 2×2 complex matrices. Choose any nonzero 2×2 matrix t with $t^2 = 0$ and set

$$\sigma = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

The span of 1 and σ is a two-dimensional subalgebra S with radical $\mathbb{C}\sigma$. Suppose M is an R-module that is also a semisimple S-module. Then $(R\sigma R)M = 0$. But $R\sigma R$ contains 1 because the ring of all 2×2 matrices over $\mathbb{C}$ is simple. Hence $M = 0$. By default, M is a semisimple R-module. The collection J of all matrices

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

is a left ideal of R. Since the space of all block diagonal matrices is an S-module, we see that J splits as an S-submodule of R. On the other hand, any left ideal J' of R complementary to J must contain a matrix of the form

$$\begin{pmatrix} u & v \\ 0 & w \end{pmatrix}$$

with $w \neq 0$. Thus

$$\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in J',$$

which is a contradiction, and so the extension is not semisimple.

The discrepancy between the two definitions will not be decisive because S will be central in R for all of our reductive candidates. For the next four lemmas, it will be convenient to assume that S is a subalgebra of R.

Lemma 1.2. *Let R be a finite-dimensional algebra and suppose that $e \in R$ is a central idempotent. If $R|S$ is a finitarily Wedderburn extension then so is $eR|eS$.*

Proof. Let M be a finite-dimensional eR-module which is semisimple as an eS-module. Certainly M is an R-module, and $e\,\mathrm{Jac}(S) \subseteq \mathrm{Jac}(eS)$ so $e\,\mathrm{Jac}(S)M = 0$. But $(1-e)M = 0$, so $\mathrm{Jac}(S)M = 0$, and hence M is a semisimple S-module. The hypothesis of the lemma now tells us that M is a semisimple R-module, and so M is a semisimple eR-module. □

Lemma 1.3. *Let R be a finite-dimensional algebra and suppose $e \in R$ is a central idempotent. If both $eR|eS$ and $(1-e)R|(1-e)S$ are finitarily semisimple extensions then so is $R|S$.*

Proof. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of finite-dimensional R-modules that splits in the category of S-modules. Patch together back-maps for

$$0 \to eA \to eB \to eC \to 0$$

and

$$0 \to (1-e)A \to (1-e)B \to (1-e)C \to 0.$$ □

Lemma 1.4. *Let $R|S$ be a finitarily Wedderburn extension and suppose that R is a finite-dimensional algebra with Wedderburn Principal Decomposition*

$$R = \operatorname{Jac}(R) \oplus \Lambda.$$

If S is central in R then $R = S\Lambda$.

Proof. The finite-dimensional R-module $R/\operatorname{Jac}(S)R$ is a semisimple S-module. Hence $\operatorname{Jac}(R) \subseteq \operatorname{Jac}(S)R$. But S central implies $\operatorname{Jac}(S)R \subseteq \operatorname{Jac}(R)$. Thus

$$\operatorname{Jac}(R) = \operatorname{Jac}(S)R.$$

The lemma follows once we improve this equality to $\operatorname{Jac}(R) = \operatorname{Jac}(S)\Lambda$. We show by induction on $d \geq 2$ that $\operatorname{Jac}(R) \subseteq \operatorname{Jac}(R)^d + \operatorname{Jac}(S)\Lambda$. To begin,

$$\operatorname{Jac}(R) \subseteq \operatorname{Jac}(S)(\operatorname{Jac}(R) + \Lambda) \subseteq \operatorname{Jac}(R)^2 + \operatorname{Jac}(S)\Lambda.$$

Now assume the desired inclusion for d. Then

$$\begin{aligned} \operatorname{Jac}(R) &\subseteq \operatorname{Jac}(R)^d + \operatorname{Jac}(S)\Lambda \\ &\subseteq \left(\operatorname{Jac}(R)^2 + \operatorname{Jac}(S)\Lambda\right)^d + \operatorname{Jac}(S)\Lambda \\ &\subseteq \operatorname{Jac}(R)^{d+1} + \operatorname{Jac}(S)\Lambda. \end{aligned}$$

Since $\operatorname{Jac}(R)$ is nilpotent, we conclude that $\operatorname{Jac}(R) = \operatorname{Jac}(S)\Lambda$ □

Lemma 1.5. *If $R|S$ is a finitarily Wedderburn extension and S is central in R then $R|S$ is a finitarily semisimple extension.*

Proof. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of finite-dimensional R-modules which splits when all modules are restricted to be S-modules. To show that the sequence has an R-module splitting there is no harm in replacing R with its image modulo the annihilator of B. Thus we may assume that R is finite-dimensional in the statement of the theorem.

By the first two lemmas, we may also assume that R has no nontrivial central idempotents. Using the notation of Lemma 1.4, we have $R = S\Lambda$ where Λ is the semisimple portion of R. The lack of central idempotents and the centrality of S imply that $\Lambda \simeq \operatorname{Mat}_n(k)$ for some n. Also, $S\Lambda$ is a homomorphic image of

$$S \otimes_k \operatorname{Mat}_n(k) \simeq \operatorname{Mat}_n(S).$$

Hence $S\Lambda \simeq \operatorname{Mat}_n(S/I)$ for some ideal I of S. But S is a subalgebra of $S\Lambda$, so $I = 0$. In other words, $R = \operatorname{Mat}_n(S)$. Now classical separability theory (see [DeMeyer and Ingraham 1971]) says that $\operatorname{Mat}_n(S)|S$ is a semisimple extension. □

This proof also establishes that, *if S is a central subalgebra of R and Γ is a subalgebra of R such that $R = S\Gamma$ and $\Gamma|k$ is a finitarily semisimple extension, then $R|S$ is a finitarily semisimple extension.*

Theorem 1.6. *Let $R|S$ be a central extension. Then the following are equivalent:*

(1) *$R|S$ is finitarily semisimple.*

(2) *$R|S$ is finitarily Wedderburn.*

(3) *For each codimension-one maximal ideal $\mathcal{M}$ of S, finite-dimensional $R/\mathcal{M}R$-modules are semisimple.*

Proof. We have already established that (1) and (2) are equivalent. It is obvious that (1) implies (3). So we shall assume that condition (3) holds and prove (2). Let V be a finite-dimensional R-module which is semisimple as an S-module. Then V is a direct sum of isotypical components for S. The centrality of S forces each component to be an R-submodule. Thus it suffices to show that each component is semisimple over R. But an isotypical component is annihilated by a codimension-one maximal ideal of S, so (3) applies. □

We can use statement (3) of the theorem to produce a test for finitary semisimplicity. Recall the well known criterion for *all* finite-dimensional R-modules to be semisimple: there are no finite-dimensional indecomposable R-modules of length two. Now assume that $R|S$ is a central extension and apply the criterion to $R/\mathcal{M}R$ for a maximal ideal $\mathcal{M}$ of S. We see that $R|S$ is finitarily semisimple if and only if there are no finite-dimensional indecomposable R-modules of length two on which elements of S act like scalars.

Corollary 1.7. *If $R|S$ is a finitarily semisimple central extension and T is a commutative S-algebra then $R \otimes_S T|T$ is a finitarily semisimple extension.*

Proof. Let $\mathcal{N}$ be a codimension-one maximal ideal of T and let W be a nonzero finite-dimensional $R \otimes_S T$-module annihilated by $R \otimes_S \mathcal{N}$. (If there are no such modules, testing with condition (3) is vacuous.) Consider W as an R-module via the embedding $R \to R \otimes 1$. Let $\mathcal{M}$ be the annihilator of W in S. Since 1 is not in the annihilator and elements of T act like scalars on W, the ideal $\mathcal{M}$ must be maximal and of codimension one in S. By hypothesis, W is a semisimple R-module. Moreover, each simple R-summand of W is also a simple $R \otimes_S T$-summand, again because elements of T act like scalars. It follows that W is a semisimple $R \otimes_S T$-module. □

The corollary describes the enveloping algebra of a finite-dimensional reductive Lie algebra. In this case, the algebra is the tensor product over $\mathbb{C}$ of the FCR enveloping algebra of a semisimple Lie algebra and the enveloping algebra of the (Lie algebra) center.

2. Weil's Lemma

Weil's Lemma for semisimple representations states that if R is affine and FCR over k then there are only finitely many isomorphism classes of n-dimensional R-modules for each positive integer n. The rendition in [Farkas 1987] states that any affine polynomial identity algebra which is a finitarily semisimple extension of the scalar field must be finite-dimensional. We now address the analogue for central extensions.

Theorem 2.1. *Let $R|S$ be a central finitarily semisimple extension. Suppose also that R satisfies a polynomial identity and that both R and S are affine. Then there is a bound $B > 0$ such that for all maximal ideals $\mathcal{M}$ of S, the number of nonisomorphic simple R-modules annihilated by $\mathcal{M}$ is bounded by B.*

(Since R is an affine PI algebra, all simple R-modules are finite-dimensional.)

Proof. We shall assume that S is a subalgebra of R and argue by induction on the Krull dimension of S. For each maximal ideal $\mathcal{N}$ of S, Weil's Lemma tells us that there are only finitely many simple R-modules annihilated by $\mathcal{N}$. If S has Krull dimension 0, it has only finitely many maximal ideals so there is clearly a uniform bound.

Because we are only interested in counting simple R-modules, we may assume that R is semiprime. According to a theorem of Procesi [1967] about affine algebras, R satisfies the ascending chain condition on semiprime ideals. It follows (see, for example, [Rowen 1988, page 213]) that there are finitely many (minimal) prime ideals $P_1, \ldots, P_t$ with $P_1 \cap P_2 \cap \cdots \cap P_t = 0$. Every maximal ideal of S lies above some $S \cap P_j$; thus we may assume that R itself is prime.

Let F be the field of fractions of S. By Corollary 1.7, the extension $R \otimes_S F | F$ is finitarily semisimple. Moreover, $R \otimes_S F$ is affine over F and satisfies a polynomial identity. By Weil's Lemma, it is finite-dimensional over F. A standard argument for affine algebras produces a nonzero $c \in S$ such that $R[c^{-1}]$ is a finitely generated $S[c^{-1}]$-module.

The collection of maximal ideals of S which contain c can be identified with the maximal ideals of S/cS. Now $R/cR \mid S/cS$ is finitarily semisimple and the Krull dimension of S/cS is less than that of S. Thus, by induction, there is a uniform bound B_1 on the number of isomorphism classes of simple R-modules annihilated by a maximal ideal of S containing c.

On the other hand, any remaining maximal ideal $\mathcal{M}$ of S does not contain c, so $\mathcal{M}' = \mathcal{M}S[c^{-1}]$ is a maximal ideal of $S[c^{-1}]$ and $R[c^{-1}]/\mathcal{M}'R[c^{-1}] \simeq R/\mathcal{M}R$. Since $R[c^{-1}]$ is a finitely generated module over $S[c^{-1}]$, we have

$$\dim_k R[c^{-1}]/\mathcal{M}'R[c^{-1}] \leq \dim_F RF.$$

Hence $\dim_k R/\mathcal{M}R \le \dim_F RF$. Again, $R/\mathcal{M}R$ is a semisimple k-algebra, so the number of nonisomorphic simple $R/\mathcal{M}R$-modules is at most $\dim_F FR$. This dimension may be taken as the bound B_2 for all remaining maximal ideals of S. $\square$

Notice that in the proof above, R "became" a finitely generated S-module after localizing at a single central element. When $R|S$ is module-finite, the extension is separable by [DeMeyer and Ingraham 1971, Theorem 2.7.1]. It is then known that S is an S-module direct summand of R.

Corollary 2.2. *Assume that $\mathfrak{R}$ and $\mathfrak{S}$ are affine algebras and that $\mathfrak{R}|\mathfrak{S}$ is a finitarily semisimple central extension. For each non-negative integer n there is a uniform bound $B(n)$ on the number of n-dimensional simple $\mathfrak{R}$-modules that afford a given character $\mathfrak{S} \to k$.*

Proof. Fix n and let J be the intersection of the annihilators of all simple n-dimensional $\mathfrak{R}$-modules. If we set $R = \mathfrak{R}/J$ and let S be the image of $\mathfrak{S}$ in R then R satisfies the polynomial identities of $n \times n$ matrices and each n-dimensional simple $\mathfrak{R}$-module is an R-module. Now apply Theorem 2.1. $\square$

3. A particular algebra

The algebra we describe here is one small example related to the theory of quantum symmetric pairs developed by the second author. It is a degenerate quantum analogue for the enveloping algebra of the fixed subalgebra of a simple Lie algebra under an involution. Specialization to a classical object will become clear in the next section.

Let q be an indeterminate. Then let $\boldsymbol{B}$ be the $\mathbb{C}(q)$-algebra generated by B_1, B_2, B_3 and $K^{\pm 1}$ subject to the relations

(1) $KK^{-1} = 1 = K^{-1}K$,

(2) $KB_1K^{-1} = q^{-1}B_1$ and $KB_2K^{-1} = qB_2$,

(3) $B_3 = B_1B_2 - qB_2B_1$,

(4) $B_1B_3 - q^{-1}B_3B_1 = -(q+q^{-1})q^2B_1K^{-3}$,

(5) $B_2B_3 - qB_3B_2 = (q+q^{-1})B_2K^{-3}$.

The reader will undoubtedly notice that B_3 is an extraneous generator. However, with its presence, the relations take a nice enough form that the theory of noncommutative Groebner bases (or, equivalently, the Diamond Lemma) easily establishes that

$$\{B_1^m B_2^n B_3^s K^{\pm t} \mid m, n, t, s \in \mathbb{N}\}$$

is a basis for $\boldsymbol{B}$ over $\mathbb{C}(q)$.

Finally, write
$$Z = \left(\frac{q^3+q}{q^2-1}\right) K^{-2} - B_3 K.$$
Notice that B_3 can be recovered from B_1, B_2, K, and Z because K is invertible. Thus $\boldsymbol{B}$ is generated as an algebra by B_1, B_2, K, and Z.

The polynomial ring over Z will be our central subalgebra in an eventual demonstration of the finitary semisimplicity of $\boldsymbol{B}$. Hence the first order of business is to show that Z is central in $\boldsymbol{B}$. Since K commutes with B_3, it suffices to check that Z commutes with both B_1 and B_2. We verify the first of these and leave the second for the reader. For notational simplicity, set $\xi = (q^3+q)/(q^2-1)$.

$$\begin{aligned} B_1 Z &= \xi B_1 K^{-2} - B_1 B_3 K \\ &= \xi q^{-2} K^{-2} B_1 - \left(q^{-1} B_3 B_1 - (q+q^{-1}) q^2 B_1 K^{-3}\right) K \\ &= \xi q^{-2} K^{-2} B_1 - B_3 K B_1 + (q+q^{-1}) K^{-2} B_1 \\ &= \left(\xi q^{-2} + (q+q^{-1})\right) K^{-2} B_1 - B_3 K B_1 \end{aligned}$$

Now simply check that $\xi q^{-2} + (q+q^{-1}) = \xi$.

Let $\overline{\mathbb{C}(q)}$ denote the algebraic closure of $\mathbb{C}(q)$ and write $\bar{\boldsymbol{B}} = \overline{\mathbb{C}(q)} \otimes_{\mathbb{C}(q)} \boldsymbol{B}$. The goal is to show that $\bar{\boldsymbol{B}}$ is a finitarily semisimple extension of $\overline{\mathbb{C}(q)}[Z]$. We begin by examining the structure of a finite-dimensional simple $\bar{\boldsymbol{B}}$-module V. The central element Z acts like some scalar c. Consider any eigenvector $w \in V$ for K with eigenvalue μ. Since K commutes with B_3,

$$K(B_1^m B_2^n B_3^s K^{\pm t} w) = q^{-m+n} \mu (B_1^m B_2^n B_3^s K^{\pm t} w).$$

In particular, V has a basis of eigenvectors for K, each of which corresponds to an eigenvalue of the form $q^e \mu$ with $e \in \mathbb{Z}$. Since V is finite-dimensional, there is a largest e which can actually appear; it corresponds to an eigenvector v.

Write

$$Kv = \lambda v. \tag{3–1}$$

By the choice of highest q-power,

$$B_2 v = 0. \tag{3–2}$$

Now $B_1^m v$ is an eigenvector for K associated to the eigenvalue $q^{-m}\lambda$. We claim that the vector space spanned by these special vectors is all of V. Equivalently, we show that their span is a $\bar{\boldsymbol{B}}$-module and invoke simplicity. The span is obviously closed under the actions of B_1, K, and Z. The formula for Z tells us that it is stabilized by B_3. Finally,

$$B_2 B_1^m v = q^{-1}(B_1 B_2 - B_3) B_1^{m-1} v = q^{-1} B_1 (B_2 B_1^{m-1} v) - q^{-1} B_3 (B_1^{m-1} v),$$

so by induction (beginning with $B_2 v = 0$), we see that the span is closed under the action of B_2. We summarize:

- Any eigenvalue for K on V has the form $q^t\lambda$ for $t \le 0$.
- $B_1^r v, B_1^{r-1} v, \ldots, B_1 v, v$ is a $\overline{\mathbb{C}[q]}$-basis for V and $B_1^{r+1} v = 0$.

This is the beginning of an analysis of Verma modules for $\bar{\boldsymbol{B}}$; in this direction, one can ultimately prove that given a nonzero choice c for the central element Z, there are four simple modules in each dimension. Our purposes here are more modest. We will be satisfied to say that a finite-dimensional $\bar{\boldsymbol{B}}$-module is of type (λ, c) with highest weight vector v and highest weight λ provided that Z acts like the scalar c and the two bulleted conditions above hold, together with (3–1) and (3–2) on the previous page. Occasionally, we will be lazy and just say that we have a *highest weight* module.

For the remainder of the paper, we will focus on finite-dimensional $\bar{\boldsymbol{B}}$-modules of length two on which Z acts as a scalar c. Such a module N has a simple submodule V of type (β_1, c) and N/V is simple of type (β_2, c). We use a shorthand for this set-up: N has type (β_2, β_1, c).

Lemma 3.1. *Let N have type (β_2, β_1, c). If β_2/β_1 is not an integer power of q then N splits.*

Proof. The eigenvalues for K acting on N all have multiplicity one. Thus there is some $u \in N$ which is an eigenvector for K and which maps to a highest weight vector for N/N_1. The $\bar{\boldsymbol{B}}$-module generated by u is spanned by eigenvectors of K, none of whose eigenvalues can appear as eigenvalues for K acting on V. Thus N is the direct sum of V and the submodule generated by u. □

It is useful to recognize the role of u in the previous lemma. If a module N of length two (with distinguished simple submodule V) has a highest weight vector outside V then this vector maps to a highest weight vector for N/V. As observed in the lemma, with such a vector either N splits or the submodule generated by the highest weight vector is all of N. Of course, it need not be the case, a priori, that N has a highest weight vector like this. But sometimes it does.

Lemma 3.2. *If N has type (β_2, β_1, c) and if $\beta_1 = q^s\beta_2$ for some $s < 0$ then N splits or is a highest weight module.*

Proof. According to the previous lemma, we may assume that β_2/β_1 is a power of q. The eigenvalues for the action of K on the distinguished simple module V all have the form $q^d\beta_2$ for $d < 0$; it must be that β_2 is an eigenvalue of multiplicity one for the action of K on N. Thus there exists an eigenvector x in N for K which has associated eigenvalue β_2. By the "maximality" of β_2 we see that $B_2 x = 0$. Finally, N/V has a unique eigenvector corresponding to the eigenvalue β_2 up to scalar, so x maps to it. □

We can force the condition of the lemma by replacing N with its dual if necessary. There is an antiautomorphism of $\overline{\boldsymbol{B}}$ that sends

$$K \mapsto K, \quad B_1 \mapsto B_2, \quad B_2 \mapsto B_1.$$

One can check that B_3 maps to B_3, so Z is sent to itself. If M is any left $\overline{\boldsymbol{B}}$-module, write M^* for its dual regarded as a left module via this antiautomorphism.

Lemma 3.3. *Let V be any finite-dimensional simple $\overline{\boldsymbol{B}}$-module of type (λ, c), Then V^* is also simple of the same type.*

Proof. It is well known that V^* is simple. If v is a highest weight vector for V then $v, B_1 v, \ldots, B_1^r v$ is a basis for V. Let $v_0^*, \ldots, v_r^*$ be a dual basis, that is,

$$\langle v_i^*, B_1^j v \rangle = \delta_{ij}.$$

Since K is fixed by the dualizing antiautomorphism, it follows that

$$\langle K v_i^*, B_1^j v \rangle = \langle v_i^*, K B_1^j v \rangle = q^{-j} \lambda \delta_{ij}.$$

Thus $K v_i^* = q^{-i} \lambda v_i^*$. Consequently, v_0^* is a highest weight vector for V with weight λ. Similarly, Z is fixed, so it is forced to act like c on V^*. $\square$

A module splits as a direct sum of two submodules if and only if its dual does. So to prove that finite-dimensional modules of length two (on which Z acts like a scalar) split, there is no harm in replacing a module by its dual. However, dualizing “flips" the simple submodule to become a simple factor and vice versa. Putting together the last three lemmas, we are reduced to showing that two kinds of modules are semisimple: those that are highest weight modules and those of type (β, β, c). We handle these cases by finding an order in $\overline{\boldsymbol{B}}$ and a lattice inside the module so that the “residue” module splits for classical reasons and the splitting lifts.

4. Orders and lattices

In order to prove that $\overline{\boldsymbol{B}}$-modules of type (β_2, β_1, c) split, it suffices to prove splitting over a finite extension field L of $\mathbb{C}(q)$ containing β_1, β_2, and c. Write $\boldsymbol{B}_L = L \otimes_{\mathbb{C}(q)} \boldsymbol{B}$. Consider the discrete valuation ring $\mathbb{C}[q]_{(q-1)}$. It extends to a discrete valuation ring $\mathcal{R}$ in L with uniformizing parameter a. (The field of fractions for $\mathcal{R}$ is L.) The initial step in specializing to classical reductive extensions is to find an appropriate $\mathcal{R}$-order inside $\boldsymbol{B}_L$. First we remind the reader of a simple result about invariants.

Lemma 4.1. *Let $S[T, T^{-1}]$ be the group ring of the infinite cyclic group over the commutative coefficient ring S. Then $S[T, T^{-1}]$ is a free S-module on*

$$\left\{((T-1)(T^{-1}-1))^m (T-1)^{\epsilon} \mid m \in \mathbb{N} \textit{ and } \epsilon = 0, 1\right\}.$$

Proof. Let σ be the involution which sends T to T^{-1}. The fixed ring of $S[T, T^{-1}]$ under σ is the polynomial ring $S[T + T^{-1}]$ and the entire ring is a free module over the subring with basis $\{1, T\}$. Hence $S[T, T^{-1}]$ is a free module over $S[2-(T+T^{-1})]$ with basis $\{1, T-1\}$. $\square$

Theorem 4.2. *Let $\lambda \in L$ be nonzero. Set $\boldsymbol{A}_{\mathcal{R}}(\lambda)$ to be the $\mathcal{R}$-subalgebra of $\boldsymbol{B}_L$ generated by*

$$A_1 = B_1, \quad A_2 = \lambda^3 B_2, \quad \Delta = \frac{1}{q-1}(\lambda^{-1}K - 1), \quad \nabla = \frac{1}{q-1}(\lambda K^{-1} - 1).$$

Then $\boldsymbol{A}_{\mathcal{R}}(\lambda)/a\boldsymbol{A}_{\mathcal{R}}(\lambda) \simeq \mathbb{U}(\mathfrak{sl}(2) \oplus \mathbb{C})$.

Proof. As might be expected, we set

$$A_3 = A_1A_2 - qA_2A_1 = \lambda^3 B_3.$$

Then $\boldsymbol{A}_{\mathcal{R}}(\lambda)$ is the $\mathcal{R}$-algebra with generators A_1, A_2, A_3, Δ and ∇, subject to the additional relations

(1) $(q-1)\Delta\nabla + \Delta + \nabla = (q-1)\nabla\Delta + \Delta + \nabla = 0$,

(2) $\Delta A_1 - q^{-1}A_1\Delta = -q^{-1}A_1$,

(3) $\Delta A_2 - qA_2\Delta = A_2$,

(4) $A_1A_3 - q^{-1}A_3A_1 = -q^2(q+q^{-1})A_1((q-1)\nabla + 1)^3$,

(5) $A_2A_3 - qA_3A_2 = (q+q^{-1})A_2((q-1)\nabla + 1)^3$.

Moreover, by applying Lemma 4.1, we see $\boldsymbol{A}_{\mathcal{R}}(\lambda)$ has a basis over $\mathcal{R}$ consisting of

$$\left\{(A_1)^r(A_2)^s(A_3)^t(\Delta\nabla)^u(\Delta)^\epsilon \mid r, s, t, u \in \mathbb{N} \text{ and } \epsilon = 0, 1\right\}.$$

Let b_1, b_2, b_3 and d be, respectively, the images of A_1, A_2, A_3 and Δ in the quotient $\boldsymbol{A}_{\mathcal{R}}(\lambda)/a\boldsymbol{A}_{\mathcal{R}}(\lambda)$. Since $q \equiv 1 \pmod{a}$, relation (1) tells us that the image of ∇ is $-d$. Thus the factor ring is a $\mathbb{C}$-algebra on the generators above subject to the relations

(1) $[b_1, b_2] = b_3$,

(2) $[d, b_1] = -b_1$ and $[d, b_2] = b_2$,

(3) $[b_1, b_3] = -2b_1$ and $[b_2, b_3] = 2b_2$.

Moreoever, $\{b_1^r\, b_2^s\, b_3^t\, d^u \mid r, s, t, u \in \mathbb{N}\}$ is a vector space basis for the factor ring over $\mathbb{C}$. Notice that $2d + b_3$ is central and $\{b_1, b_2, b_3\}$ span a Lie algebra isomorphic to $\mathfrak{sl}(2)$ under the identification

$$b_2 \mapsto e, \quad b_1 \mapsto f, \quad b_3 \mapsto -h.$$

It follows that the factor ring is a homomorphic image of the desired enveloping algebra. But a comparison of the basis for this ring with the PBW basis for the enveloping algebra forces the identification to induce an isomorphism. □

Notice that $\boldsymbol{A}_{\mathfrak{R}}(\lambda)$ is actually independent of λ up to isomorphism because the parameter never appears in the relations. The scalar is, however, relevant since the λ indicates a particular imbedding of $\boldsymbol{A}_{\mathfrak{R}}$ in $\boldsymbol{B}_L$.

We have to modify Z slightly to get it inside $\boldsymbol{A}_{\mathfrak{R}}(\lambda)$. Recall that

$$Z = \xi K^{-2} - B_3 K \text{ where } \xi = \frac{q^3+q}{q^2-1}.$$

The difficulty is that $q-1$ appears as a factor of the denominator in ξ. We show how to compensate for this. Multiplying Z by λ^2,

$$\begin{aligned}\lambda^2 Z &= \xi(\lambda K^{-1})^2 - (\lambda^3 B_3)(\lambda^{-1}K)\\ &= (q-1)\xi\frac{1}{q-1}(\lambda K^{-1}-1)(\lambda K^{-1}+1) + \xi - A_3(\lambda^{-1}K)\\ &= (q-1)\xi\nabla(\lambda K^{-1}+1) + \xi - A_3(\lambda^{-1}K).\end{aligned}$$

Since $\lambda^{\pm 1}K^{\mp 1} \in \boldsymbol{A}_{\mathfrak{R}}(\lambda)$, we see that $\lambda^2 Z - \xi \in \boldsymbol{A}_{\mathfrak{R}}(\lambda)$. Thus we set

$$Z_{\boldsymbol{A}} = \lambda^2 Z - \xi = \frac{q^3+q}{q+1}\nabla((q-1)\nabla+2) - A_3((q-1)\Delta+1).$$

We use the notation from the previous theorem to compute $Z_{\boldsymbol{A}}$ modulo a. Since $q-1 \equiv 0 \pmod{a}$,

$$Z_{\boldsymbol{A}} \equiv 2\nabla - A_3 \pmod{a}.$$

In particular, $Z_{\boldsymbol{A}}$ is sent to $-2d-b_3$, a nonzero element of the center of $\mathfrak{sl}(2)\oplus\mathbb{C}$.

Let M be a finite-dimensional highest weight $\boldsymbol{B}_L$-module with highest weight vector v. Then M has a basis over L of the form

$$v,\ B_1 v,\ B_1^2 v,\ \dots,\ B_1^t v.$$

The *full highest weight $\mathfrak{R}$-lattice* in M generated by v is the free $\mathfrak{R}$-module with this same basis. The next step is to recognize this lattice as a module over some of our orders.

Theorem 4.3. *Assume that $\lambda \in L$ is nonzero and $m \in \mathbb{Z}$. If M is a finite-dimensional highest weight $\boldsymbol{B}_L$-module of type $(q^m\lambda, c)$ with highest weight vector v then the full highest weight $\mathfrak{R}$-lattice generated by v is an $\boldsymbol{A}_{\mathfrak{R}}(\lambda)$-module.*

Proof. Let Γ denote the full highest weight $\mathfrak{R}$-lattice in M, as described above. Then Γ is closed under the action of A_1 because $A_1 = B_1$. Recall that

$$K B_1^s v = q^{m-s}\lambda B_1^s v$$

for $s = 0, 1, \ldots, t$ and $\dim M = t+1$. Thus $B_1^s v$ is an eigenvector for $\lambda^{-1}K$ with eigenvalue an integral power of q. It follows that Γ is closed under the actions of Δ and ∇.

The central element $Z_{\boldsymbol{A}}$ acts like some scalar $\omega \in L$ on M. The formula for $Z_{\boldsymbol{A}}$ allows us to recover information about the action of A_3. Indeed,

$$A_3 = (q-1)\xi\nabla(\lambda K^{-1}+1)(\lambda K^{-1}) - Z_{\boldsymbol{A}}\lambda K^{-1}.$$

Thus

$$A_3(B_1^s v) = q^{m-s}(p-\omega)B_1^s v \tag{4–1}$$

for some $p \in \mathbb{C}[q]_{(q-1)}$. On the other hand, the relation $KB_2 = qB_2K$ implies that $B_2(B_1^s v)$ lies in the same eigenspace for K as $B_1^{s-1}v$. Thus we may write

$$A_2(B_1^s v) = \mu_s B_1^{s-1} v$$

for some $\mu_s \in L$. (Note that $\mu_0 = 0$ and, by default, $\mu_{t+1} = 0$.) Now $A_3 = A_1A_2 - qA_2A_1$ forces

$$A_3(B_1^s v) = (\mu_s - q\mu_{s+1})B_1^s v.$$

Putting together the two descriptions of the action of A_3, we obtain

$$\mu_s - q\mu_{s+1} = q^{m-s}(p-\omega). \tag{4–2}$$

There are two cases to consider. First suppose that $\omega \in \mathcal{R}$. Then one can "back-solve" equation (4–2) to see that each $\mu_s \in \mathcal{R}$. Thus Γ is closed under the action of A_2. This completes the proof that Γ is a $\boldsymbol{A}_{\mathcal{R}}(\lambda)$-module. So suppose that $\omega \notin \mathcal{R}$. We borrow a Weyl algebra trick to reach a contradiction. By the DVR assumption,

$$\omega^{-1} \in a\mathcal{R}.$$

It follows from (4–1) that

$$-\omega^{-1}A_3(B_1^s v) = B_1^s v + (q^{m-s}-1)(B_1^s v) - \omega^{-1}(q^{m-s}p)(B_1^s v).$$

The essence of the last equality is that

$$-\omega^{-1}A_3(B_1^s v) \in B_1^s v + a\Gamma \tag{4–3}$$

for $s = 0, 1, \ldots, t$. Therefore $\omega^{-1}A_3$ stabilizes Γ. Solving the linear system (4–2) as before, we conclude that $\omega^{-1}A_2$ also stabilizes Γ.

There is a powerful way to summarize the previous three sentences. The actions of A_1, $-\omega^{-1}A_2$ and $-\omega^{-1}A_3$ on Γ induce actions C_1, C_2 and C_3, respectively, on $\Gamma/a\Gamma$. (One can think of C_j as a matrix over $\mathbb{C}$.) Then (4–3) above simply states that

$$C_3 = I.$$

The defining relation for A_3 now yields

$$C_1C_2 - C_2C_1 = I.$$

Such an equality cannot hold for matrices over a field of characteristic zero. $\square$

The next result says that finite-dimensional highest weight $\boldsymbol{B}_L$-modules specialize to finite-dimensional simple $(\mathfrak{sl}(2) \oplus \mathbb{C})$-modules via our orders.

Theorem 4.4. *Assume that $\lambda \in L$ is nonzero and $m \in \mathbb{Z}$. Let M be a finite-dimensional highest weight $\boldsymbol{B}_L$-module of type $(q^m\lambda, c)$ with highest weight vector v and set Γ to be the full highest weight $\mathcal{R}$-lattice generated by v. Then $\Gamma/a\Gamma$ is a finite-dimensional simple $\boldsymbol{A}_{\mathcal{R}}(\lambda)/a\boldsymbol{A}_{\mathcal{R}}(\lambda)$-module.*

Proof. We freely use the identification of $\boldsymbol{A}_{\mathcal{R}}(\lambda)/a\boldsymbol{A}_{\mathcal{R}}(\lambda)$ with $\mathfrak{sl}(2) \oplus \mathbb{C}$ presented in the specialization theorem. In particular, if u is the image of v in $\Gamma/a\Gamma$ then

$$u,\ fu,\ f^2u,\ \ldots,\ f^tu$$

is a basis for $\Gamma/a\Gamma$ over $\mathbb{C}$ for $t+1 = \dim M$. As indicated in the proof of the previous theorem, v is an eigenvector for A_3; since Γ is a $\boldsymbol{A}_{\mathcal{R}}(\lambda)$-module, the corresponding eigenvector lies in $\mathcal{R}$. Thus u is an eigenvector for h. Finally, $A_2v = 0$ implies that $eu = 0$. It follows from classical highest weight considerations that $\Gamma/a\Gamma$ is a simple $\mathfrak{sl}(2)$-module. $\square$

5. Splitting by specialization

We can describe the end-game for decomposability in rather general terms. The goal is to help the reader navigate through the final argument unencumbered by all of our notation. Let $\mathcal{R}$ denote any discrete valuation ring with uniformizing element a and field of fractions L. Assume that $\mathfrak{B}$ is some L-algebra and $\mathfrak{A}$ is a finitely generated $\mathcal{R}$-algebra such that $\mathfrak{B} = L \cdot \mathfrak{A}$. We study a finite-dimensional $\mathfrak{B}$-module N with a submodule V having the following additional structure:

(1) There is a vector space decomposition $N = V \oplus W$.

(2) There are $\mathcal{R}$-lattices $C \subseteq V$ and $D \subseteq W$ such that $L \cdot C = V$ and $L \cdot D = W$.

(3) C is an $\mathfrak{A}$-module.

(4) $\mathfrak{A} \cdot D \subseteq V + D$.

Let $\pi : N \to V$ denote the projection map relative to the vector space direct sum above. For fixed $x \in \mathfrak{A}$ and $d \in D$ there exists $0 \neq r \in \mathcal{R}$ such that

$$\pi(xd) \in \frac{1}{r}C.$$

Since D has finite rank as an $\mathcal{R}$-module, we can simultaneously choose r with

$$\pi(xD) \subseteq \frac{1}{r}C.$$

Suppose $y \in \mathfrak{A}$ and $\pi(yD) \subseteq (1/s)\,C$ as well. A straightforward calculation shows that

$$\pi(xyD) \subseteq \frac{1}{r}C + \frac{1}{s}C.$$

It follows from the finite generation of $\mathfrak{A}$ that there exists a nonzero $t \in \mathcal{R}$ with $\pi(\mathfrak{A} \cdot D) \subseteq (1/t)\,C$.

Moreover, $(1/t)\,C \simeq C$ as $\mathfrak{A}$-modules. We refer to the replacement of C with $(1/t)\,C$ as the “first modification" of our given decomposition. After the modification, we may assume that $C + D$ is an $\mathfrak{A}$-submodule of N with C, D full sublattices in V, W respectively.

We go further. Set

$$\Phi = \{\pi(xd) \mid x \in \mathfrak{A} \text{ and } d \in D\}.$$

If $\Phi = 0$ then D is an $\mathfrak{A}$-module. Hence W is a $\mathfrak{B}$-module and $N = V \oplus W$ is a $\mathfrak{B}$-module splitting. If $\Phi \neq 0$ we shall lay out a strategy for reaching a contradiction when $\mathfrak{B}$ is our special algebra. Since

$$\bigcap_{m \geq 0} a^m C = 0,$$

there must be a largest $m \geq 0$ such that $\Phi \subseteq a^m C$. Our “second modification" is to replace C with $a^m C$. Equivalently, we may assume that $\Phi \nsubseteq aC$. Suppose it turns out that D/aD is an $\mathfrak{A}/a\mathfrak{A}$-module. (In other words, the induced

$$C/aC \oplus D/aD$$

is actually an $\mathfrak{A}/a\mathfrak{A}$-module splitting.) Then, for all $x \in \mathfrak{A}$ and $d \in D$,

$$\pi(xd) \in a(C + D) \cap C.$$

Thus $\Phi \subseteq aC$, which is impossible.

The splitting hypothesis modulo a will come from the following straightforward application of the representation theory for $\mathfrak{sl}(2)$ over $\mathbb{C}$.

Lemma 5.1. *Let M be a finite-dimensional $\mathbb{U}(\mathfrak{sl}(2))$-module. Suppose that x, $y \in M$ are linearly independent eigenvectors for h such that $ex = ey = 0$. Then $\mathbb{U}(\mathfrak{sl}(2))x$ and $\mathbb{U}(\mathfrak{sl}(2))y$ are simple submodules of M whose intersection is zero.*

We are finally ready to prove the main theorem for our special algebra.

Theorem 5.2. *$\bar{B}$ is a finitarily semisimple extension of $\overline{\mathbb{C}(q)}[Z]$.*

Proof. The goal is to show that if N is a finite-dimensional $\bar{\boldsymbol{B}}$-module on which Z acts like a scalar then N is a sum of simple modules. We briefly review the reductions made to this point. We may assume that N has length two. The $\overline{\mathbb{C}(q)}$-algebra $\bar{\boldsymbol{B}}$ may be replaced by the L-algebra $\boldsymbol{B}_L$ for some finite field extension $L|\mathbb{C}(q)$. Finally, N has one of two types: either it is a highest weight module or its highest weight "appears twice".

Case 1: N has type $(\beta, q^m\beta, c)$ with $m < 0$.

Assume N is a nonsplit $(r+1)$-dimensional module. It has a highest weight vector w with weight β. By Lemma 3.2 and Theorem 4.3, the $\mathcal{R}$-lattice Γ with basis

$$B_1^r w,\, B_1^{r-1} w,\, \ldots,\, B_1 w,\, w$$

is an $\boldsymbol{A}_{\mathcal{R}}(\beta)$-module. Recall that N has a simple submodule V with highest weight $q^m\beta$. Since our basis for N consists of eigenvectors of K, we may assume the highest weight vector for V is $B_1^m w$. Consider the $\mathcal{R}$-lattice direct sum

$$\Gamma = (\mathcal{R}B_1^r w + \cdots + \mathcal{R}B_1^m w) \oplus (\mathcal{R}B_1^{m-1} w + \cdots + \mathcal{R}B_1 w + \mathcal{R}w).$$

It satisfies the set-up described at the beginning of this section, primarily due to Theorem 4.3. If the right-most of the two direct summands is an $\boldsymbol{A}_{\mathcal{R}}(\beta)$-module then we have an $\boldsymbol{A}_{\mathcal{R}}(\beta)$-module direct sum, a contradiction. Make the second modification on Γ.

The specialization $\boldsymbol{A}_{\mathcal{R}}(\beta) \to \mathfrak{sl}(2) \oplus \mathbb{C}z$ sends

$$A_1 \mapsto f, \quad A_2 \mapsto e, \quad A_3 \mapsto -h, \quad \Delta \mapsto \tfrac{1}{2}(h-z), \quad Z_{\boldsymbol{A}} \mapsto z.$$

Moreover, z acts like a scalar on $\Gamma/a\Gamma$. Since w and $B_1^m w$ are annihilated by A_2, the images $w + a\Gamma$ and $B_1^m w + a\Gamma$ are annihilated by e. Similarly, w and $B_1^m w$ are eigenvectors for Δ with eigenvalues in $\mathcal{R}$, so their images in $\Gamma/a\Gamma$ are eigenvectors for h. (Each one is an eigenvector for z.) It follows from the previous lemma that our lattice splitting becomes a module splitting modulo a. This contradicts the punchline of the discussion preceding the lemma.

Case 2: N has type (β, β, c).

By assumption, β appears with multiplicity two as an eigenvalue for K. The generalized eigenspace corresponding to β has a basis $\{v, w\}$ where

$$Kv = \beta v \text{ and } Kw = \beta w + \epsilon v$$

for ϵ either 0 or 1. Thus we may assume that v is the highest weight vector for the simple submodule V and $w + V$ is the highest weight vector for N/V. We know $B_2 v = 0$. We claim that $B_2 w = 0$. Indeed, $KB_2 = qB_2K$ implies that

$$KB_2 w = qB_2Kw = qB_2(\beta w + \epsilon v) = (q\beta)B_2 w.$$

Since all eigenvalues for K acting on N have the form $q^t\beta$ for $t \le 0$, the claim follows.

Assume that $\dim V = l+1$ and $\dim N/V = m+1$. Consider the $\mathcal{R}$-lattice decomposition

$$\Gamma = (\mathcal{R}B_1^l v + \cdots + \mathcal{R}B_1 v + \mathcal{R}v) \oplus (\mathcal{R}B_1^m w + \cdots + \mathcal{R}B_1 w + \mathcal{R}w).$$

By Theorem 4.3, we may assume that the direct summand on the left is an $\boldsymbol{A}_{\mathcal{R}}(\beta)$-module and the summand on the right is a module after factoring out V. Thus we may perform the first modification on Γ so that it, too, becomes an $\boldsymbol{A}_{\mathcal{R}}(\beta)$-module.

Now mimic the argument of the previous case. The vectors $v+a\Gamma$ and $w+a\Gamma$ in $\Gamma/a\Gamma$ are $\mathbb{C}$-linearly independent and annihilated by e (according to the claim). As in the first case, $v+a\Gamma$ is an eigenvector for h. It is also true that the span of $v+a\Gamma$ and $w+a\Gamma$ is stabilized by $\frac{1}{2}(h-z)$ (which is the image of Δ) and has a multiple eigenvalue. But $h-z$ acts semisimply on $\Gamma/a\Gamma$. Thus $w+a\Gamma$ is an eigenvector for h as well. The second case is now completed along the lines of Case 1. □

6. Reductive extensions

A k-algebra A is *residually finite-dimensional* if the zero ideal is the intersection of ideals with finite codimension. It is well known that the enveloping algebra of a finite-dimensional Lie algebra is residually finite-dimensional. The more well-known parallel notion appears in the theory of infinite groups. Indeed, the group algebra of a residually finite group is residually finite-dimensional. However, the concept for algebras has not been explored to the same extent as that for groups; one of the few general results is Mal'cev's proof that the Hopf property holds for such algebras.

Theorem 6.1 [Malcev 1943]. *Suppose that R is an affine, residually finite-dimensional algebra. Then a surjective algebra endomorphism of R is an automorphism.*

It is much easier to verify that a subalgebra of (or a full matrix ring over) a residually finite-dimensional algebra is residually finite-dimensional. In particular, if $R|S$ is a ring extension such that R is a finitely generated free S-module and S is residually finite-dimensional then so is R.

Theorem 6.2. *$\bar{\boldsymbol{B}}$ is a residually finite-dimensional $\overline{\mathbb{C}(q)}$-algebra.*

Proof. The short demonstration for experts is that $\bar{\boldsymbol{B}}$ imbeds in the simply connected quantized enveloping algebra for $\mathfrak{sl}(3)$ and the larger algebra is residually finite-dimensional. However, detailed proofs of these two assertions are not easily accessible. For that reason, we outline an argument.

Let $\mathbb{U}$ denote the quantized enveloping algebra for the Lie algebra $\mathfrak{sl}(3)$. As an algebra over $\overline{\mathbb{C}(q)}$ it is generated by the symbols x_i, y_i, $t_i^{\pm 1}$ (for $i = 1, 2$) subject to the relations

$$x_i y_j - y_j x_i = \delta_{ij}(t_i - t_i^{-1})/(q - q^{-1}),$$
$$t_i x_i = q^2 x_i t_i \quad \text{and} \quad t_i y_i = q^{-2} y_i t_i,$$
$$t_i x_{3-i} = q^{-1} x_{3-i} t_i \quad \text{and} \quad t_i y_{3-i} = q y_{3-i} t_i,$$
$$x_i^2 x_{3-i} - (q + q^{-1}) x_i x_{3-i} x_i + x_{3-i} x_i^2 = 0,$$
$$y_i^2 y_{3-i} - (q + q^{-1}) y_i y_{3-i} y_i + y_{3-i} y_i^2 = 0,$$

together with the added assumption that t_1 and t_2 generate a free abelian group.

Next, set

$$p_1 = y_1 t_1, \quad p_2 = y_2 t_2 + t_1^{-1} x_1 t_1, \quad \kappa = t_1 t_2^{-1}$$

and write $p_3 = p_1 p_2 - q p_2 p_1$. One can check directly that

$$\kappa p_1 \kappa^{-1} = q^{-3} p_1 \quad \text{and} \quad \kappa p_2 \kappa^{-1} = q^3 p_2,$$
$$p_1 p_3 - q^{-1} p_3 p_1 = -(q + q^{-1}) q^2 p_1 \kappa^{-1},$$
$$p_2 p_3 - q p_3 p_2 = (q + q^{-1}) p_2 \kappa^{-1}.$$

(Alternatively, the relations are derived more generally in [Letzter 2003, Theorem 7.1].) If we let $\widetilde{\boldsymbol{B}}$ denote the subalgebra of $\overline{\boldsymbol{B}}$ generated by B_1, B_2, B_3, and $K^{\pm 3}$ then $\overline{\boldsymbol{B}}$ is a finitely generated free right $\widetilde{\boldsymbol{B}}$-module and the algebra generated by p_1, p_2, p_3, and $\kappa^{\pm 1}$ is a homomorphic image of $\widetilde{\boldsymbol{B}}$. If the two algebras are isomorphic then the theorem is a consequence of the remarks preceding the proof and the fact that quantized enveloping algebras are residually finite dimensional (see [Joseph 1995, 7.19]).

Isomorphism will be an obvious corollary of the observation that the subalgebra of $\mathbb{U}$ generated by p_1, p_2, and $\kappa^{\pm 1}$ has a basis consisting of all $p_1^m p_2^n p_3^s \kappa^{\pm t}$ as m, n, s, t run over all natural numbers. Of course, this list clearly spans the subalgebra, so we need only examine linear independence. Temporarily set

$$y_3 = y_1 y_2 - q y_2 y_1.$$

It is known that $\{y_1^m y_2^n y_3^s \mid m, n, s \in \mathbb{N}\}$ is a basis for $\mathbb{U}^-$ (see, for example, [Klimyk and Schmüdgen 1997, pages 175–176]). By [Joseph 1995, 3.2.8], there is an injective algebra map from $\mathbb{U}^-$ into $\mathbb{U}$ that sends y_i to $y_i' = y_i t_i$. If we write y_3' for the image of y_3 then we see that

$$\{(y_1')^m (y_2')^n (y_3')^s \kappa^{\pm t} \mid m, n, s, t \in \mathbb{N}\}$$

is linearly independent.

The technical tool we need is an algebra grading of $\mathbb{U}$ by the root lattice $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ of $\mathfrak{sl}(3)$ (see [Jantzen 1996, 4.7]). In brief, $\mathbb{U}$ is spanned by weight vectors in such a way that y_1' has weight $-\alpha_1$, y_2' has weight $-\alpha_2$, y_3' has weight $-(\alpha_1+\alpha_2)$, and x_1 has weight α_1. If we regard the weight as a kind of degree under the pointwise order, then

$$\begin{aligned} p_1 &= y_1', \\ p_2 &= y_2' + \text{higher degree terms}, \\ p_3 &= y_3' + \text{higher degree terms}. \end{aligned}$$

It follows that

$$p_1^m p_2^n p_3^s \kappa^{\pm t} = (y_1')^m (y_2')^n (y_3')^s \kappa^{\pm t} + \text{higher degree terms}.$$

Therefore linear independence of the p-monomials is a consequence of linear independence of the y'-monomials. □

We close with the proposed definition of a reductive pair of algebras. Let $R|S$ be a central extension of algebras. It is a *reductive extension* provided it is finitarily semisimple and R is residually finite-dimensional.

Theorem 6.3. *$\bar{\boldsymbol{B}}$ is a reductive extension of* $\overline{\mathbb{C}(q)}[Z]$.

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