TIGHT CONTACT STRUCTURES OF CERTAIN SEIFERT FIBERED 3-MANIFOLDS WITH $e_0 = -1$

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We classify, up to contact isotopy, all tight contact structures on a family of Seifert fibered three-manifolds $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ satisfying $0 < \frac{\beta}{\alpha} < \frac{1}{6}$. We show that, if $[r_0, r_1, \ldots, r_l]$ is the continued fraction expansion of $-\frac{\alpha}{\beta}$, there are exactly $|r_0+5|\,|r_1+1|\cdots|r_l+1|$ tight contact structures on such Seifert fibered three-manifolds $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ as above, so all the tight contact structures are holomorphically fillable.

1. Introduction and results

An oriented 2-plane field distribution ξ on an oriented 3-manifold is called a *positive contact structure* if $\xi = \ker \eta$ for some global 1-form η satisfying $\eta \wedge d\eta > 0$. In this paper, a contact structure always means a positive contact structure. All contact structures in 3-manifolds fall largely into two categories: tight and overtwisted (see [Honda 2000] for details).

Let $M(r_1, r_2, r_3)$ denote a (small) Seifert fibered 3-manifold over S^2 with three singular fibers. The integer

$$e_0(M(r_1, r_2, r_3)) = \sum_{i=1}^{3} [r_i]$$

is an invariant of the Seifert fibered 3-manifold $M(r_1, r_2, r_3)$, called the *truncated Euler number*. H. Wu [2004] has classified tight contact structures on small Seifert fibered 3-manifolds with $e_0 \neq -2, -1, 0$; subsequently P. Ghiggini, P. Lisca, and A. Stipsicz extended the classification results of Wu to the case $e_0 = 0$ [Ghiggini et al. 2004].

Here we report the classification of tight contact structures, up to contact isotopy, on a family of the Seifert fibered 3-manifolds $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ satisfying $0 < \frac{\beta}{\alpha} < \frac{1}{6}$. Note that the Seifert fibered 3-manifolds of the present paper have the truncated Euler number $e_0 = -1$ which was considered as kind of an exceptional case in [Wu 2004] and [Ghiggini et al. 2004].

To state the main result, we write $-\frac{\alpha}{\beta}$ as a continued fraction $[r_0, r_1, \ldots, r_l]$.

Theorem 1.1. *Let M be the Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ with $0 < \frac{\beta}{\alpha} < \frac{1}{6}$. Then there exist exactly $|r_0 + 5| |r_1 + 1| \cdots |r_l + 1|$ tight contact structures on M up to contact isotopy. So all the tight contact structures on M are holomorphically fillable.*

As an immediate consequence, we have the following

Corollary 1.2. *On the Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ with $k \geq 2$, there exist exactly $3k - 5$ tight contact structures up to contact isotopy. So all the tight contact structures on $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ with $k \geq 2$ are holomorphically fillable.*

This corollary answers a question of P. Ghiggini and S. Schönenberger [2003, Section 5]. Note also that $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{11}\right)$ is just the Brieskorn homology 3-sphere $-\Sigma(2, 3, 11)$ equipped with orientation opposite to the one as a boundary of the Milnor fiber. The special case of Theorem 1.1 for $k = 2$ is proved in [Ghiggini and Schönenberger 2003]. It would also appear to be interesting to classify tight contact structures on the Brieskorn homology 3-spheres $-\Sigma(2, 3, 6k - 1)$ $(k > 2)$ equipped with orientation opposite to the one as a boundary of the Milnor fiber. Lisca and Matić [1997] showed that there exist at least $k - 1$ tight contact structures on $-\Sigma(2, 3, 6k - 1)$ which are holomorphically fillable. A preliminary analysis analogous to the present paper shows that there seem to exist exactly $\frac{k(k-1)}{2}$ tight contact structures on the Brieskorn homology 3-sphere $-\Sigma(2, 3, 6k - 1)$; this work will appear elsewhere. So the result of Theorem 1.1 seems to be sharp in the sense that $-\Sigma(2, 3, 6k - 1)$ is the Seifert 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{k}{6k-1}\right)$ in our notation and that $\frac{k}{6k-1}$ is greater than $\frac{1}{6}$. We hope to come back to the classification of tight contact structures on $-\Sigma(2, 3, 6k - 1)$ in the future work.

As a consequence of Theorem 1.1, each Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ has exactly $|r_0 + 5| |r_1 + 1| \cdots |r_l + 1|$ nonisotopic tight (in fact, holomorphically fillable) contact structures satisfying $0 < \frac{\beta}{\alpha} < \frac{1}{6}$. Hence these same manifolds are atoroidal (have no incompressible tori); this follows from a recent theorem of Colin, Giroux, and Honda [Colin et al. 2003, Theorem 0.5], to the effect that a closed, oriented, irreducible 3-manifold carries infinitely many tight contact structures up to isotopy if and only if it is toroidal.

We prove Theorem 1.1 by essentially employing the techniques developed in [Honda 2000; Etnyre and Honda 2001] and later implemented in [Ghiggini and Schönenberger 2003]. This paper is a result of the author's attempt to understand those techniques. We use the same definitions, terminology, and theorems as in the three references just cited. The reader is expected to be familiar with those papers.

Much of this article is devoted to the classification results of the Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ for $k \geq 3$, in detail; we think that the analysis of

these special cases will help the reader understand the proof of the general case. Specifically, in Section 2 we give an upper bound of the number of Corollary 1.2, and in Section 3 we construct $3k - 5$ holomorphically fillable contact structures using the Legendrian surgery description as in [Gompf 1991; 1998; [Eliashberg 1990]]. Section 4 gives a proof of the general case of a Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ satisfying the property $0 < \frac{\beta}{\alpha} < \frac{1}{6}$.

2. Proof of Corollary 1.2: upper bound

In this section M denotes the Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ over S^2 with three singular fibers f_i with invariants $\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$. Let V_i ($i = 1, 2, 3$) be tubular neighborhoods of the singular fibers f_i, and we identify ∂V_i with $\mathbb{R}^2/\mathbb{Z}^2$ by choosing $(1, 0)^T$ as the meridional direction and $(0, 1)^T$ as a longitudinal direction. Let Σ denote a pair of pants. We identify each component of $-\partial(S^1 \times \Sigma)$ with $\mathbb{R}^2/\mathbb{Z}^2$ in a similar way as above. Then $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ can be obtained by attaching the solid tori V_i to $S^1 \times \Sigma$ via the attaching maps

$$A_i : \partial V_i \to -\partial(S^1 \times \Sigma)$$

defined by

$$A_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 6k-1 & 3k \\ -2 & -1 \end{pmatrix}.$$

Now we recall the definition of the twisting number of a Legendrian curve used in this paper. A Legendrian knot in the manifold M which is smoothly isotopic to a regular fiber admits two framings, the fibration framing and the contact framing. The difference between the contact framing and the fibration framing is called the *twisting number* of the Legendrian curve.

Theorem 2.1. *On the Seifert fibered 3-manifold* $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ *with* $k \geq 3$, *there exist*, *up to contact isotopy*, *at most* $3k - 5$ *tight contact structures*.

Proof. The proof of the theorem consists of several steps. We first assume without loss of generality that the singular fibers f_i are simultaneously isotoped to Legendrian curves with negative twisting numbers n_i for $i = 1, 2, 3$. Then the slopes of ∂V_i of a standard neighborhood V_i of f_i with convex boundary become $1/n_i$. Furthermore, we assume that the Legendrian ruling slope on $-\partial(M \setminus V_i)$ is infinite, due to the flexibility of the Legendrian rulings [Honda 2000, Theorem 3.4; Ghiggini and Schönenberger 2003, Theorem 3].

Lemma 2.2. *We can increase the twisting numbers* n_1 *and* n_2 *up to* -2, *and the twisting number* n_3 *up to* -1.

Proof. By assumption, the boundary slopes of $-\partial(M \setminus V_i)$ under A_i are

$$s(-\partial(M \setminus V_1)) = \frac{n_1}{2n_1 - 1}, \qquad s(-\partial(M \setminus V_2)) = \frac{-n_2}{3n_2 + 1},$$

$$s(-\partial(M \setminus V_3)) = -\frac{2n_3 + 1}{(6k - 1)n_3 + 3k}.$$

Let A be a convex annulus whose boundary consists of Legendrian rulings of $-\partial(M \setminus V_1)$ and $-\partial(M \setminus V_2)$. We have three cases to consider:

Case 1. $2n_1 - 1 \neq 3n_2 + 1$. Assume first that $2n_1 - 1 < 3n_2 + 1 \leq 0$. Then it follows from the Imbalance Principle [Honda 2000, Proposition 3.17] that there exists a bypass along a Legendrian ruling on $-\partial(M \setminus V_1)$. By the Twisting Number Lemma [Honda 2000, Lemma 4.4] with Legendrian ruling slope $r_1 = 2$, we can increase the twisting number n_1 by 1 as long as $n_1 \leq -1$. Next, if $3n_2 + 1 < 2n_1 - 1 \leq 0$, then similarly with $r_2 = -\frac{1}{3}$ we can increase the twisting number n_2 by 1 as long as $n_2 \leq -2$.

To sum up, we can increase the twisting numbers n_1 and n_2 by one at least as long as $n_1 \leq -1$ and $n_2 \leq -2$.

Case 2. $2n_1 - 1 = 3n_2 + 1$ and there exists a bypass on the convex annulus. In this case we apply the Twisting Number Lemma as in Case 1 to increase the twisting numbers.

Case 3. $2n_1 - 1 = 3n_2 + 1$ and there exists no bypass on the convex annulus. In this case we cannot apply the Twisting Number Lemma. So we apply the cut-and-round procedure [Ghiggini and Schönenberger 2003, Proposition 6; Honda 2000, Lemma 3.11] along a convex annulus A to get a convex torus isotopic to $\partial(M \setminus V_3)$ with boundary slope

$$\frac{n_1}{2n_1 - 1} - \frac{n_2}{3n_2 + 1} - \frac{1}{2n_1 - 1} = \frac{n_2}{6n_2 + 2} = \frac{p}{6p + 1},$$

where $n_2 = 2p$ for $p \leq -1$. Since

$$\begin{pmatrix} -1 & -3k \\ 2 & 6k - 1 \end{pmatrix} \begin{pmatrix} 6n_2 + 2 \\ -n_2 \end{pmatrix} = \begin{pmatrix} 3(k - 2)n_2 - 2 \\ -(6k - 13)n_2 + 4 \end{pmatrix},$$

we obtain a boundary slope $-2 + n_2/(3(k-2)n_2 - 2)$ on ∂V_3, which is negative.

We now work with a convex annulus A between $-\partial(M \setminus V_2)$ and $-\partial(M \setminus V_3)$. Since $6n_2 + 2 < 3n_2 + 1 \leq 0$, we can apply the Imbalance Principle to A to obtain a bypass on ∂V_3. Then by the Twisting Number Lemma with the Legendrian ruling slope

$$r_3 = \frac{-(6k - 13)n_2 + 4}{(3k - 6)n_2 - 2}$$

we can increase the twisting number n_3 by 1 as long as $n_3 \leq -2$. In fact, since

$$-2 \leq -1 + \frac{1}{r_3} = \frac{(3k-6)n_2 - 2}{-(6k-13)n_2 + 4} \leq -\frac{3}{2}\left(1 + \frac{1}{18k - 27}\right) < -\frac{3}{2},$$

we should have

$$n_3 \leq -1 + \frac{1}{r_3}$$

as long as $n_3 \leq -2$. This implies that we can increase the twisting number n_3 up to -1.

Now in order to increase the twisting numbers n_1 and n_2, notice that as above the possible boundary slopes on $\partial(M \setminus V_3)$ are $p/(6p+1)$ for $p \leq -1$. Since $2/(6k-1) < \frac{1}{6} < p/(6p+1)$ for $k \geq 3$, inside V_3 there must exist a standard torus T with boundary slope $\frac{1}{6}$. Now, if we compare a Legendrian curve on T isotopic to the regular fiber with another Legendrian curve isotopic to the regular fiber on $\partial(M \setminus V_3)$, then clearly we obtain a bypass for $\partial(M \setminus V_3)$ except when $p/(6p+1) = \frac{1}{5}$ (i.e., $p = -1$). Thus we can increase the twisting number n_2 (and so n_1) up to -2.

From these three cases, we see that the twisting numbers n_1, n_2, and n_3 can be increased up to -2, -2, and -1, respectively. This completes the proof.   □

Remark 2.3. According to the referee, there exists a much simpler argument to deal with Case 3 in the proof of Lemma 2.2 as follows. Namely, let T' be the convex torus isotopic to $\partial(M \setminus V_3)$ obtained by the cut-and-round procedure with boundary slope $p/(6p+1)$, and let T be the convex torus with boundary slope $\frac{1}{6}$ ($< p/(6p+1)$) as above. Then take a vertical annulus from T to T' and use the Imbalance Principle to obtain a bypass for T along a ruling curve of slope ∞. This then can be used to thicken V_1 and V_2.

From now on, we assume that the twisting numbers n_i are $n_1 = n_2 = -2$ and $n_3 = -1$. Let A denote a convex vertical annulus whose boundary consists of Legendrian rulings of $\partial(M \setminus V_1)$ and $\partial(M \setminus V_2)$. Then we divide our proof into two cases, depending on whether or not A has a boundary-parallel dividing curve.

Lemma 2.4. *Assume that A has a boundary-parallel dividing curve. Then we can further increase the twisting numbers n_1, n_2, and n_3 up to 0, -1, and -1, respectively. Furthermore, there exists a regular fiber with twisting number 0.*

Proof. By taking the configurations of dividing curves on the annulus A into account, we see that there must exist a bypass on each side of $-\partial(M \setminus V_1)$ and $-\partial(M \setminus V_2)$. By the Twisting Number Lemma, we can increase n_1 and n_2 up to -1. But then there must be one more bypass on $-\partial(M \setminus V_1)$ by the Imbalance principle, since $t(-\partial(M \setminus V_1)) = -3 < t(-\partial(M \setminus V_2)) = -2 < 0$. Hence we can increase the

twisting number n_1 up to 0. With these new twisting numbers $n_1 = 0$ and $n_2 = -1$, we have the boundary slopes $s(-\partial(M \setminus V_1)) = 0$ and $s(-\partial(M \setminus V_2)) = -\frac{1}{2}$.

Now we apply the cut-and-round procedure to the convex annulus A. Then we obtain a convex torus with boundary slope $-\frac{1}{2} + 0 + \frac{1}{2} = 0$ on a torus isotopic to $\partial(M \setminus V_3)$. This implies that the slope of ∂V_3 can be assumed to be -2. Hence by Proposition 18 in [Ghiggini and Schönenberger 2003] we can find a convex torus parallel to the boundary ∂V_3 with boundary slope $-\frac{6k-1}{3k}$, since $-2 < -2 + \frac{1}{3k} = -\frac{6k-1}{3k} < 0$. But this slope is equivalent to infinity on $-\partial(M \setminus V_3)$. Hence we have a regular fiber with twisting number 0. This completes the proof. $\square$

Proposition 2.5. *Under the assumptions of Lemma 2.4, there does not exist, up to contact isotopy, any tight contact structure on $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ having twisting numbers $n_1 = 0$, $n_2 = -1$, and $n_3 = -1$.*

Proof. We use a slight modification of the proof in [Ghiggini and Schönenberger 2003, Section 4.2.2]. For the reader's sake, however, we give it here in some detail.

By Lemma 2.4, $M \setminus \bigcup_{i=1}^{3} V_i$ is diffeomorphic to $S^1 \times \Sigma$ with boundary slopes $0, -\frac{1}{2}$, and $-\frac{1}{3k-1}$, and further there exists another pair of pants Σ' inside Σ so that $S^1 \times \Sigma'$ is diffeomorphic to $M \setminus \bigcup_{i=1}^{3} V_i'$ with all boundary slopes equal to ∞, where V_i' is a neighborhood of the singular fiber f_i containing V_i for $i = 1, 2, 3$. Hence $S^1 \times (\Sigma - \Sigma')$ is diffeomorphic to the union of three thickened tori $T_i \times I$ so that $T_i \times \{0\} = -\partial(M \setminus V_i)$ and $T_i \times \{1\} = -\partial(M \setminus V_i')$.

Since $T_1 \times I$ has boundary slopes 0 and ∞, and $\frac{0}{1}$ and $\frac{1}{0}$ are consecutive in the Farey tessellation, $T_1 \times I$ is a basic slice. Similarly, since $-\frac{1}{2}, -1$, and $\frac{1}{0}$ are consecutive, $T_2 \times I$ is the union of two basic slices $T_2 \times [0, \frac{1}{2}]$ with boundary slopes $-\frac{1}{2}$ and -1 and $T_2 \times \left[\frac{1}{2}, 1\right]$ with boundary slopes -1 and ∞, and $T_3 \times I$ with boundary slopes $-\frac{1}{3k-1}$ and ∞ is the union of $3k-1$ basic slices $T_3 \times \left[\frac{i}{3k-1}, \frac{i+1}{3k-1}\right]$ with boundary slopes $-1/(3k-1-i)$ and $-1/(3k-2-i)$ for $i = 0, 1, \ldots, 3k-2$.

Let p_i denote the number of positive basic slices in $T_i \times I$. Then we have the following possibilities; $p_1 = 0, 1$, $p_2 = 0, 1, 2$, and $p_3 = 0, 1, \ldots, 3k-1$. (As the referee pointed out, we can let p_3 be the number of positive basic slices from slope $-\frac{1}{6}$ to ∞. Then $p_3 = 0, 1, \ldots, 6$. So we can simplify the argument below. See also Section 4.) Thus there exist at most $2 \times 3 \times 3k = 18k$ possible tight contact structures on $M \setminus \bigcup_{i=1}^{3} V_i$ by Lemma 31 in [Ghiggini and Schönenberger 2003].

First, we need the following definition:

Definition 2.6. Let $\tilde{V}_i$ be thickenings of the V_i so that the boundary slopes are zero. Their equivalent slopes

$$s(-\partial(M \setminus \tilde{V}_1)) = \frac{1}{2}, \quad s(-\partial(M \setminus \tilde{V}_2)) = -\frac{1}{3}, \quad \text{and} \quad s(-\partial(M \setminus \tilde{V}_3)) = -\frac{2}{6k-1}$$

are called *critical*.

In fact, they are slopes for which we can find an overtwisted contact structure on the 3-manifold M.

In order to prove the proposition, we need to consider following four cases;

Case 1. We first assume that $T_3 \times \left[\frac{3k-3}{3k-1}, 1\right]$ is contact isomorphic to $T_2 \times [0, 1]$. Note that this assumption covers $p_2 = 0$, 1, or 2. Then the manifold $M \setminus (V_1' \cup V_2 \cup V_3 \cup T_3 \times [0, \frac{3k-3}{3k-1}])$ has boundary slopes ∞, $-\frac{1}{2}$, and $-\frac{1}{2}$. Then it follows from Lemma 36 in [Ghiggini and Schönenberger 2003] that we can find a vertical annulus A between $T_2 \times \{0\}$ and $T_3 \times \{\frac{3k-3}{3k-1}\}$ without boundary parallel dividing arcs. Now apply the cut-and-round procedure to the annulus A so that we get a slope $-\frac{1}{2} - \frac{1}{2} + \frac{1}{2} = -\frac{1}{2}$ on a torus isotopic to $\partial(M \setminus V_1)$. But it is equivalent to the slope $\frac{1}{2}$ on a torus isotopic to $-\partial(M \setminus V_1)$ which is critical. Hence these cases do not occur. Thus we are left with the following cases $p_2 = 0$ and $p_3 = 3k - 2$ or $3k - 1$, $p_2 = 1$ and $p_3 = 0$ or $3k - 1$, or $p_2 = 2$ and $p_3 = 0$ or 1.

Case 2. Next assume that three basic slices in $T_3 \times \left[\frac{3k-4}{3k-1}, 1\right]$ have the same sign as $T_1 \times I$. Let V_1'' be a standard neighborhood whose boundary slope $s(-\partial(M \setminus V_1'')) = T_1 \times \{-1\}$ is $\frac{1}{3}$. Then the manifold

$$M \setminus \left(V_1'' \cup V_2' \cup V_3 \cup T_3 \times \left[0, \frac{3k-4}{3k-1}\right]\right)$$

has boundary slopes $\frac{1}{3}$, $-\frac{1}{3}$, and ∞. Then, by [Ghiggini and Schönenberger 2003, Lemma 37], we can find a vertical annulus A between $T_1 \times \{-1\}$ and $T_3 \times \{\frac{3k-4}{3k-1}\}$ without boundary-parallel arcs. Now apply the cut-and-round procedure along A to get the boundary slope $\frac{1}{3} - \frac{1}{3} + \frac{1}{3} = \frac{1}{3}$ on a torus isotopic to $\partial(M \setminus V_2)$. Thus we have a critical boundary $-\frac{1}{3}$ on $-\partial(M \setminus V_2)$. Thus we are left with the following cases $p_1 = 0$ and $p_3 = 3k - 3$, $3k - 2$, or $3k - 1$ or $p_1 = 1$ and $p_3 = 0$, 1, or 2.

Summarizing the two preceding cases, we are left with six possibilities: $p_1 = 0$, $p_2 = 0$, and $p_3 = 3k - 2$ or $3k - 1$, $p_1 = 0$, $p_2 = 1$ and $p_3 = 3k - 1$ or $p_1 = 1$, $p_2 = 1$ and $p_3 = 0$ or $p_1 = 1$, $p_2 = 2$ and $p_3 = 0$ or 1.

Case 3. This time we assume that the basic slices $T_2 \times I$ and $T_1 \times I$ have the same sign. This covers $p_1 = 0$, $p_2 = 0$ and $p_1 = 1$, $p_2 = 2$. Since we can decrease the twisting numbers n_1 and n_2 up to -2, we can take standard neighborhood V_1'' and V_2'' for which the boundary slopes are $s(-\partial(M \setminus V_1'')) = \frac{2}{5}$ and $s(-\partial(M \setminus V_2'')) = -\frac{2}{5}$. Thus the manifold $M \setminus (V_1'' \cup V_2'' \cup V_3')$ has boundary slopes $\frac{2}{5}$, $-\frac{2}{5}$, ∞. A similar argument as above yields a convex torus isotopic to $-\partial(M \setminus V_3)$ with slope $-\frac{1}{5}$. The thickened torus between $-\partial(M \setminus V_3)$ and $-\partial(M \setminus V_3'')$ has boundary slopes $-\frac{1}{3k-1}$ and $-\frac{1}{5}$. Since the thickened torus contains a convex torus with infinite boundary slope by assumption there exists a convex torus with boundary slope $-\frac{2}{6k-1}$, which is critical. Thus we are left with two cases: $p_1 = 0$, $p_2 = 1$, $p_3 = 3k - 1$, or $p_1 = 1$, $p_2 = 1$, $p_3 = 0$.

Case 4. We finally assume that the basic slice $T_2 \times \left[\frac{1}{2}, 1\right]$ has the same sign as $T_3 \times I$. This covers the last two cases. Using a convex annulus between $T_2 \times \left\{\frac{1}{2}\right\}$ and $T_3 \times \left\{\frac{3k-2}{3k-1}\right\}$ that does not have a boundary parallel curves, we obtain a convex torus isotopic to $-\partial(M \setminus V_1)$ with boundary slope 1. By the Twisting Number Lemma with Legendrian slope $r_1 = 1$, we can increase the twisting number n_1 up to 1. So by decreasing the twisting number n_1 we may assume that $T_1 \times I$ and $T_2 \times \left[\frac{1}{2}, 1\right]$ have the same sign. This assumption reduce this case to the previous case such as $p_1 = p_2 = 0$ or $p_1 = p_2 = 1$ which was proved to be overtwisted. $\square$

Now we are in a position to deal with the case where A has no boundary-parallel dividing curve.

Proposition 2.7. *If A has no boundary-parallel dividing curve, there exist, up to contact isotopy, at most $3k - 5$ tight contact structures on $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$.*

Proof. Recall that the twisting numbers n_i are $n_1 = n_2 = -2$ and $n_3 = -1$. Thus the boundary slopes are

$$s(-\partial(M \setminus V_1)) = \frac{-2}{-5}, \quad s(-\partial(M \setminus V_2)) = \frac{2}{-5}, \quad \text{and} \quad s(-\partial(M \setminus V_3)) = \frac{1}{-3k+1}.$$

Applying the cut-and-round procedure to a convex annulus A between $-\partial(M \setminus V_1)$ and $-\partial(M \setminus V_2)$, we obtain a torus with boundary slope $\frac{2}{5} - \frac{2}{5} + \frac{1}{5} = \frac{1}{5}$ on a torus T isotopic to $\partial(M \setminus V_3)$. Since we have

$$\begin{pmatrix} -1 & -3k \\ 2 & 6k - 1 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 - 3k \\ 6k - 11 \end{pmatrix},$$

the convex torus isotopic to ∂V_3 has boundary slope

$$-\frac{6k - 11}{3k - 5} = -2 - \frac{1}{-(3k - 5)}.$$

Thus it follows from the classification of the tight contact structures on a solid torus [Honda 2000, Theorem 2.3] that there exist, up to contact isotopy, exactly $|(-2 + 1)(3k - 5)| = 3k - 5$ tight contact structures on a solid torus V_3'' of a singular fiber f_3 with boundary T. Since the boundary slopes of ∂V_1 and ∂V_2 are all $\frac{1}{-2}$, clearly there exists a unique tight contact structure on each V_1 and V_2. Moreover, the thickened torus obtained by the cut-and-round procedure above should be nonrotative for the manifold M to be tight. $\square$

This completes the proof of Theorem 2.1. $\square$
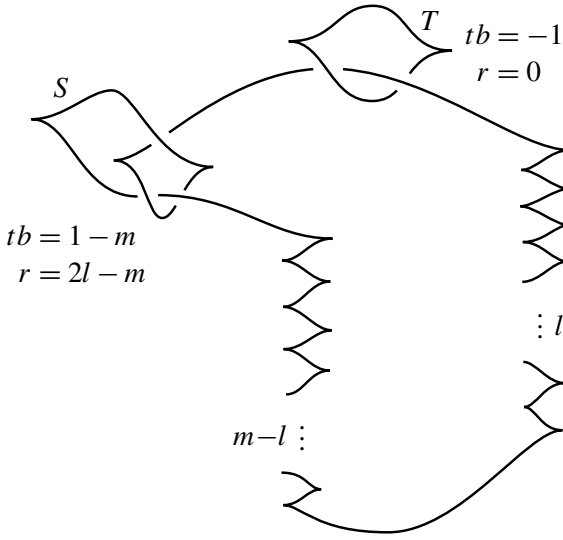
3. Proof of Corollary 1.2: lower bound

We now construct $3k - 5$ holomorphically fillable contact structures on the Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$.
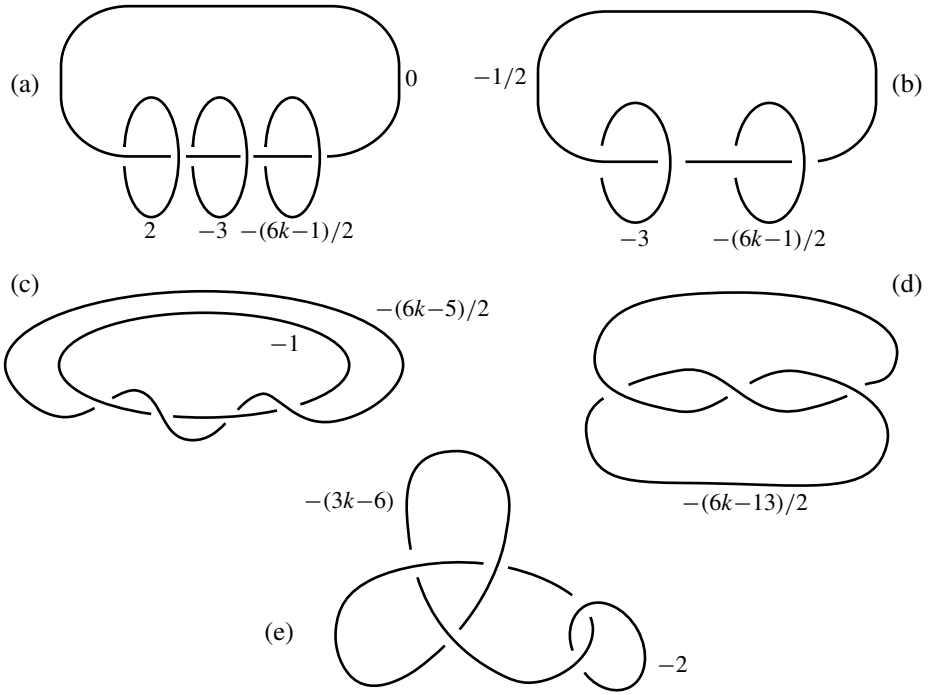
Figure 1. Surgery diagrams for $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$.

We use Kirby calculus to show that the manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ can be represented as Legendrian surgery on the framed link as follows. Indeed, the surgery description of the Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ is given in part (a) of Figure 1. Then we slam-dunk along the knot with framing coefficient 2 to obtain part (b). Then we perform the Rolfsen $(+2)$-twist along the knot with framing coefficient $-\frac{1}{2}$ to obtain (c). Next once more we do the Rolfsen $(+1)$-twist along the knot with framing coefficient -1 to obtain (d). Finally we do the inverse slam-dunk to get (e). The last framed link can be realized as Legendrian links in several ways as shown in Figure 2 (l downward zig-zags plus one zig on the right and $m-l$ upward zig-zags on the left).

For any $k \geq 2$, we let $m = 3k - 6$ and denote by X_k^l $(0 \leq l \leq m)$ the Stein manifolds with boundary constructed by attaching handles along these framed links. By applying [Lisca and Matić 1997, Theorem 2.1] we have $\langle \mathrm{PD}(c_1(X_k^l)), T \rangle = 0$ and $\langle \mathrm{PD}(c_1(X_k^l)), S \rangle = 2l - m$. Thus by the same theorem or [Kronheimer and Mrowka 1997], we have at least $m + 1 = 3k - 5$ nonisotopic holomorphically fillable contact structures on the Seifert fibered 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$. This complete the proof of Theorem 1.1.

Figure 2. Legendrian surgery diagrams for $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$.

Remark. By [Lisca and Matić 1997, Proposition 2.2], the $3k - 5$ nonisotopic holomorphically fillable contact structures on the 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{2}{6k-1}\right)$ constructed in the proof of Theorem 1.1 are not homotopic as fields of 2-planes.

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is closely modeled on that of Corollary 1.2, developed in Sections 2 and 3. Thus we just highlight the differences between the two proofs.

Throughout this section, M with denote the Seifert 3-manifold $M\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ fibered over S^2 with three singular fibers f_i with invariants $\left(-\frac{1}{2}, \frac{1}{3}, \frac{\beta}{\alpha}\right)$ satisfying $0 < \frac{\beta}{\alpha} < \frac{1}{6}$, and we will keep the notations of the preceding two sections, unless stated otherwise. In this general case, the attaching maps A_1 and A_2 are the same as in Section 2, but now the attaching map $A_3 : \partial V_3 \to -\partial(S^1 \times \Sigma)$ is given by

$$A_3 = \begin{pmatrix} \alpha & \alpha' \\ -\beta & -\beta' \end{pmatrix},$$

where $\alpha'\beta - \alpha\beta' = 1$ with $0 < \alpha' < \alpha$ and $\alpha, \beta > 0$. Note that $\beta' > 0$ and $\beta - \beta' > 0$. Then the boundary slope of $-\partial(M \setminus V_3)$ is

$$(4\text{–}1) \qquad\qquad -\frac{(\alpha - 6\beta)n_2 - 2\beta}{(\alpha' - 6\beta')n_2 - 2\beta'}.$$

Since $\frac{\beta}{\alpha} = \frac{\beta'}{\alpha'} + \frac{1}{\alpha\alpha'}$ and $0 < \frac{\beta}{\alpha} < \frac{1}{6}$, we see that $0 < \frac{\beta'}{\alpha'} < \frac{1}{6}$. Hence the number (4–1) is negative.

Now, we work on a convex annulus between $-\partial(M \setminus V_2)$ and $-\partial(M \setminus V_3)$. Since $6n_2 + 2 < 3n_2 + 1 \leq 0$, we can apply the Imbalance Principle with r_3 equal to (4–1). Indeed, we first claim that we have

$$(4\text{–}2) \qquad\qquad 0 < \frac{\alpha' - 6\beta'}{\alpha - 6\beta} < 1.$$

To see this, if the inequality (4–2) does not hold, we easily obtain using the relation $\alpha'\beta - \alpha\beta' = 1$ that

$$1 \geq (\alpha - 6\beta)(\beta - \beta'),$$

which implies that $\alpha - 6\beta = 1$ and $\beta - \beta' = 1$. Thus

$$1 = \alpha'\beta - \alpha\beta' = \beta(\alpha' - 6\beta + 5) + 1,$$

which implies that $\beta = 0$. This is a contradiction. Hence

$$(4\text{–}3) \qquad\qquad -2 \leq -1 + \frac{1}{r_3} \leq -1 - \frac{\alpha' - 4\beta'}{\alpha - 4\beta}.$$

Note also that the fraction on the right lies between 0 and 1. Thus by the Twisting Number Lemma as in Lemma 2.2 with Legendrian ruling slope r_3 equal to the equation (4–1), we have

$$n_3 \leq -1 + \frac{1}{r_3} < -1$$

as long as $n_2 \leq -2$. This implies that we increase the twisting number by one as long as the twisting number n_3 is less than or equal to -2. Now it is easy to see that the rest of the proof in Lemma 2.2 works without any modifications. Hence we conclude that we can increase the twisting numbers n_1, n_2, and n_3 up to -2, -2, and -1, respectively.

Now assume that the convex annulus A as in Lemma 2.4 has a boundary-parallel dividing curve. Since we do not have any change for the gluing maps A_1 and A_2, we can increase the twisting numbers n_1 and n_2 up to 0 and -1 as in Lemma 2.4. Furthermore, since the slope of ∂V_3 corresponding to the slope 0 (resp. ∞) on $\partial(M \setminus V_3)$ is $-\frac{\beta}{\beta'}$ (resp. $-\frac{\alpha}{\alpha'}$) and $-\frac{\beta}{\beta'} < -\frac{\alpha}{\alpha'} < 0$ using $\alpha'\beta - \alpha\beta' = 1$, we have a Legendrian regular fiber with twisting number 0.

Next as in Proposition 2.5, we show that there does not exist any tight contact structures on M, up to isotopy, which have the twisting numbers $n_1 = 0$, $n_2 = -1$, and $n_3 = -1$. But this can be shown in the similar way as in Section 2, since $M \setminus \bigcup_{i=1}^{3} V_i$ is diffeomorphic to $S^1 \times \Sigma$ with boundary slope 0, $-\frac{1}{2}$, and

$$(4\text{–}4) \qquad\qquad -\frac{1}{3} < -\frac{\beta - \beta'}{\alpha - \alpha'} = -\left(\frac{\beta}{\alpha} + \frac{1}{\alpha(\alpha - \alpha')}\right) < 0$$

To be precise, it follows from the inequality (4–4) that there exists a convex standard neighborhood V_3'' of the singular fiber f_3 containing V_3 inside V_3' whose boundary slope $s(-\partial(M \setminus V_3''))$ is $-\frac{1}{3}$. Let T_3'' denote the convex torus bounding V_3''. Thus we have a thickened torus $T_3'' \times I$ with $T_3'' \times \{0\} = -\partial(M \setminus V_3'')$ and $T_3'' \times \{1\} = -\partial(M \setminus V_3')$ whose boundary slopes are $-\frac{1}{3}$ and ∞. Note also that the thickened torus $T_3'' \times I$ is the union of three basic slices $T_3'' \times \left[\frac{i}{3}, \frac{i+1}{3}\right]$ with boundary slopes $-\frac{1}{3-i}$ and $-\frac{1}{2-i}$ for $i = 0, 1, 2$ which is a continued fraction block.

As before, let p_i denote the number of positive basic slices in $T_i \times I$ ($i = 1, 2$) and let p_3 be the number of positive basic slices in $T_3'' \times I$. Then there exist *at least* $2 \times 3 \times 3 = 18$ possible tight contact structures on $M \setminus (V_1 \cup V_2 \cup V_3'')$. But we will show that these cases do not occur under the assumption that there exists a Legendrian regular fiber with twisting number 0. This will clearly finish the proof that there does not exist any tight contact structures on M, up to isotopy, which have the twisting numbers $n_1 = 0$, $n_2 = -1$, and $n_3 = -1$. The proof is just a repetition of the arguments in Section 2. But in our case the critical slopes are respectively

$$s(-\partial(M \setminus \tilde{V}_1)) = \tfrac{1}{2}, \quad s(-\partial(M \setminus \tilde{V}_2)) = -\tfrac{1}{3}, \quad \text{and} \quad s(-\partial(M \setminus \tilde{V}_3)) = -\tfrac{\beta}{\alpha} > -\tfrac{1}{6}.$$

Using an argument similar to that of Case 1 of Proposition 2.5, we are left with the cases $p_2 = 0$ and $p_3 = 2$ or 3, $p_1 = 1$ and $p_3 = 0$ or 3, and $p_2 = 1$ and $p_3 = 0$ or 1. Next as in Case 2 we are left with the following six cases $p_1 = 0 = p_2$ and $p_3 = 1$, $p_1 = 0$, $p_2 = 1$ and $p_3 = 3$, $p_1 = 1$, $p_2 = 0$ and $p_3 = 2$, $p_1 = p_2 = 1$ and $p_3 = 0$, and $p_1 = 1$, $p_2 = 2$, and $p_3 = 0$ or 1. Applying the argument in Case 3, we can reduce the above six cases to three cases $p_0 = 0, p_2 = 1$ and $p_3 = 3$, $p_1 = 1$, $p_2 = 1$ and $p_3 = 0$, and $p_1 = 1$, $p_2 = 0$ and $p_3 = 2$. This is possible, since $-\frac{1}{5} < -\frac{\beta}{\alpha}$ and the thickened torus obtained by cut-and-round procedure along the convex annulus between $-\partial(M \setminus V_1'')$ and $-\partial(M \setminus V_2'')$ as in Case 3 in the proof of Proposition 2.5 contains a convex torus with infinite boundary slope. Finally, an argument as in Case 4 concludes that there does not exist any tight contact structures on M, up to isotopy, which have the twisting numbers $n_1 = 0$, $n_2 = -1$, and $n_3 = -1$.

We are thus left with the case that A has no boundary-parallel dividing curve. In this case, it follows as in Proposition 2.7 that we have the convex torus isotopic to ∂V_3 with boundary slope $-(\alpha - 5\beta)/(\alpha' - 5\beta')$ which is equal to

$$[r_l, r_{l-1}, \ldots, r_0 + 5].$$

In fact, we can see this as follows. Since $(\alpha' - 5\beta')\beta - (\alpha - 5\beta)\beta' = 1$, the fraction $-(\alpha - 5\beta)/\beta$ is written as $[5 + r_0, r_1, \ldots, r_l]$. Thus the fraction

$$-\frac{\alpha - 5\beta}{\alpha' - 5\beta'}$$

is equal to $[r_l, r_{l-1}, \ldots, r_0+5]$; see [Ghiggini et al. 2004, Lemma 2.5], for example. Hence we conclude that there exist at most $|r_0 + 5|\, |r_1 + 1| \cdots |r_l + 1|$ tight contact structures on M up to isotopy by the classification of Honda.

Finally we need to construct $|r_0 + 5|\, |r_1 + 1| \cdots |r_l + 1|$ tight contact structures on M by the Legendrian surgery argument. But M is the result of a surgery on the right-handed trefoil knot with surgery coefficient $-\frac{\alpha}{\beta} + 6 < 0$. Hence, as in Section 3 or as in [Gompf 1991], we can show that there exist at least

$$|r_0 + 5|\, |r_1 + 1| \cdots |r_l + 1|$$

tight contact structures on M. This completes the proof of Theorem 1.1.

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