UNIQUENESS AND A PRIORI ESTIMATES FOR SOME NONLINEAR ELLIPTIC NEUMANN EQUATIONS IN $\mathbb{R}^3$

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Under some conditions on $f(u)$, we show that for λ small and $\Omega \subset \mathbb{R}^3$ convex, the only solution to the elliptic equation $\Delta u - \lambda u + f(u) = 0$ in Ω, with $u > 0$ in Ω and $\partial u / \partial \nu = 0$ on $\partial \Omega$, is constant.

1. Introduction

We consider the nonlinear elliptic equation

$$(1.1) \qquad \begin{cases} \Delta u - \lambda u + f(u) = 0 & \text{in } \Omega, \\ u > 0 \ \text{in } \Omega, \ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \end{cases}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$, the function f lies in $C^{1+\theta}$ for some $0 < \theta < 1$, and ν is the unit outer normal vector field at $\partial \Omega$. We suppose that $f(u)$ can be written as $f(u) = u^5(1 + \rho(u))$, with $\rho'(u) \leq 0$, $0 \leq u^5 \rho(u) \leq C \sum_{i=1}^{K} u^{p_i}$ for some $1 < p_i \leq \frac{13}{3}$ and some constant $C > 0$. A typical example for $f(u)$ is the function

$$f(u) = u^5 + \sum_{i=1}^{K} a_i u^{p_i}, \quad \text{with } 1 < p_i \leq \tfrac{13}{3} \text{ and } a_i \geq 0 \text{ for } i = 1, \dots, K.$$

Our main result is the following.

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^3$ is convex. Then there exist $\lambda_0 > 0$ and $C > 0$ such that for $\lambda < \lambda_0$ we have*

$$u \leq C\lambda^{1/4}$$

for all solutions u of (1.1), *where C is independent of λ.*

As a consequence, $u \equiv C_\lambda$ for some constant C_λ.

Remark. The restriction that $u^5 \rho(u) \leq C \sum_{i=1}^{K} u^{p_i}$ for $p_i \leq \frac{13}{3}$ just reflects a technical difficulty. We believe that Theorem 1 also holds under the weaker assumption $0 \leq u^5 \rho(u) \leq C \sum_{i=1}^{K} u^{p_i}$ for $1 < p_i < 5$.

The proof of Theorem 1 is a direct consequence of integration by parts. It also gives a new and rather simple derivation of the following theorem.

Theorem 2. *Suppose that $\Omega \subset \mathbb{R}^3$ is convex and $f(u) = u^5$. Then for λ small, $u = \lambda^{1/4}$ is the unique solution of* (1.1).

Theorem 2 is a special case of Lin and Ni's conjecture [Lin and Ni 1988]. The conjecture says that for λ small and $f(u) = u^p$, $p > 1$, the problem (1.1) admits only constant solutions. (Problem (1.1) with $f(u) = u^p$ arises in the study of steady-state solutions of the model of Keller and Segel [1970] in chemotaxis and the Gierer–Meinhardt system [1972] in pattern formation. For more background on (1.1), see [Ni 1998].)

Lin and Ni proved their conjecture in the case when p is subcritical. Zhu [1999] proved the conjecture for three-dimensional convex domains (the same class of domains as Theorem 2) by using a very complicated blow-up analysis. Zhu's proof is by contradiction and thus indirect. It is unclear if Zhu's proof can be generalized to other functions $f(u)$. Our proof is much simpler and elementary. In fact since our proof is direct, it can yield an explicit value for the number λ_0 in Theorem 1. We remark that when $\Omega = B_R(0)$ and u is radial, similar results were proved by Adimurthi and Yadava [1993].

In higher dimensions, the Lin–Ni conjecture may be wrong. Counterexamples in the radial case are given in [Adimurthi and Yadava 1991; 1997]. For the nonradial case, see [Gui and Wei 2005].

We use an idea developed by Chang, Gursky and Yang in the study of three-dimensional prescribed scalar curvature problem [Chang et al. 1993]. Our starting point is to write equation (1.1) as

$$\Delta u - \lambda u + f(u) = \Delta u + R(u)u^5,$$

where $R(u) = 1 + \rho(u) - \lambda u^{-4}$. Thus if we introduce the conformal transformation $g = u^4 g_0$, where g_0 is the usual Euclid metric, R becomes the scalar curvature in the new metric.

We first make some preliminary notes. (Throughout this paper, C will denote different constants independent of λ.)

From equation (1.1), we see that

$$\int_{\Omega} u^5 \leq \int_{\Omega} f(u) = \lambda \int_{\Omega} u \leq C\lambda \left(\int_{\Omega} u^5 \right)^{1/5},$$

which implies that

$$(1.2) \qquad \int_\Omega u^5 \le C\lambda^{5/4}, \quad \int_\Omega u \le \lambda^{1/4}$$

and thus

$$(1.3) \qquad u_{\min} := \min_{x\in\bar\Omega} u(x) \le C\lambda^{1/4}.$$

From (1.2), we also obtain that

$$(1.4) \qquad \int_\Omega u^q \le C\lambda^{q/4} \quad \text{for all } 0 < q \le 5.$$

Next we need a well-known fact found in [Lions 1982, Appendix 1, Lemma 5], for example: if Ω is convex and $\partial u/\partial \nu = 0$ on $\partial\Omega$, then

$$(1.5) \qquad \frac{\partial}{\partial \nu}|\nabla u|^2 \le 0 \quad \text{on } \partial\Omega.$$

2. Proof of Theorem 1

Let g_0 be the usual Euclidean metric and $g = u^4 g_0$. In the new metric, we consider the trace-free Ricci tensor B. In a local coordinate system, we may write

$$B_{ij} = -u^{-2}\big((u^2)_{g,ij} - \tfrac{1}{3}(\Delta_g u^2)g_{ij}\big).$$

(Here the covariant derivatives are taken with respect to g, not g_0.)

We first obtain an integral estimate for $\int_\Omega |B|^{3/2}\, dV_g$. (Here dV_g denotes integration with respect to the new metric g. It is easy to see that $dV_g = u^6\, dx$.) To this end, we consider

$$\int_\Omega |B|^2 u^2\, dV_g = \int_\Omega g^{ik} g^{jl} B_{ij} B_{kl} u^2\, dV_g$$
$$= -\int_\Omega (u^2)_{g,ij} B_{kl} g^{ik} g^{jl}\, dV_g + \frac{1}{3}\int_\Omega (\Delta_g u^2) g_{ij} g^{ik} g^{jl} B_{kl}\, dV_g,$$

where the second term vanishes because B is trace-free.

Now using integration by parts, together with the Neumann boundary condition, we have

$$\int_\Omega |B|^2 u^2\, dV_g = \int_\Omega (u^2)_{g,i} B_{kl;j} g^{ik} g^{jl}\, dV_g + \int_{\partial\Omega} \frac{\partial}{\partial \nu}|\nabla u|^2 d\sigma_g,$$

where $d\sigma_g$ is the surface element in the new metric g.

Using the contracted second Bianchi identity and (1.5) we obtain

$$\int_{\Omega} |B|^2 u^2 \, dV_g \leq \frac{1}{6} \int_{\Omega} (u^2)_{g,i} R_{,k} g^{ijk} \, dV_g$$

$$= \frac{1}{6} \int_{\Omega} \langle \nabla_g (u^2), \nabla_g R \rangle \, dV_g = \frac{1}{12} \int_{\Omega} \langle \nabla (u^4), \nabla R \rangle \, dx$$

$$= \frac{1}{12} \int_{\Omega} \nabla u^4 \nabla (1 + \rho(u) - \lambda u^{-4})$$

$$\leq \frac{4}{3} \lambda \int_{\Omega} u^{-2} |\nabla u|^2 \, dx \quad \text{(since } \rho'(u) \leq 0\text{)}$$

$$= \frac{4}{3} \lambda \int_{\Omega} u^{-1} \Delta u \, dx = \frac{4}{3} \lambda \int_{\Omega} u^{-1} (\lambda u - f(u)) \leq C \lambda^2.$$

Thus we obtain the following key estimate (here we need $\Omega \subset \mathbb{R}^3$)

$$(2.6) \qquad \int_{\Omega} |B|^{3/2} \, dV_g \leq \left(\int_{\Omega} |B|^2 u^2 \, dV_g \right)^{3/4} \left(\int_{\Omega} u^{-6} dV_g \right)^{1/4} \leq C \lambda^{3/2}.$$

Next we estimate $\int_{\Omega} |\nabla u^{-2}|^6 \, dx$. Let $v = 1/u$. Recall the Bochner identity:

$$(2.7) \qquad \Delta_g |\nabla_g v|^2 = 2 |\nabla_g^2 v|^2 + 2 \langle \nabla_g v, \nabla_g (\Delta_g v) \rangle + 2 \operatorname{Ric}_g (\nabla_g v, \nabla_g v).$$

If Ω is convex, $\int_{\Omega} \Delta_g |\nabla_g v|^2 \, dV_g \leq 0$ by (1.5). Integrating both sides of (2.7) and using the divergence theorem, we get

$$\int_{\Omega} |\nabla_g^2 v|^2 \, dV_g \leq \int_{\Omega} (\Delta_g v)^2 \, dV_g + \int_{\Omega} |\operatorname{Ric}| |\nabla_g v|^2 dV_g.$$

Recall the Sobolev inequality in $\mathbb{R}^3$

$$\left(\int_{\Omega} \psi^6 \, dV_g \right)^{1/3} \leq C \int_{\Omega} |\nabla_g \psi|^2 \, dV_g + C \int_{\Omega} \psi^2 \, dV_g.$$

Take $\psi = |\nabla_g v|$. Then we obtain

$$(2.8) \quad \left(\int_{\Omega} |\nabla_g v|^6 \, dV_g \right)^{1/3}$$

$$\leq C \int_{\Omega} |\nabla_g |\nabla_g v|^2| + C \int_{\Omega} |\nabla_g v|^2$$

$$\leq C \int_{\Omega} |\nabla_g^2 v|^2 \, dV_g + C \int_{\Omega} |\nabla_g v|^2 \, dV_g \quad \text{(by Kato's inequality)}$$

$$\leq C \int_{\Omega} |\Delta_g v|^2 \, dV_g + C \int_{\Omega} |\operatorname{Ric}| |\nabla_g v|^2 \, dV_g + C \int_{\Omega} |\nabla_g v|^2 \, dV_g$$

$$\leq C \int_{\Omega} |\Delta_g v|^2 \, dV_g + C \int_{\Omega} |B| |\nabla_g v|^2 \, dV_g$$

$$+ C \int_{\Omega} |R| |\nabla_g v|^2 \, dV_g + C \int_{\Omega} |\nabla_g v|^2 \, dV_g,$$

where we have used the fact that $|\operatorname{Ric}| \leq |B| + \frac{1}{3}|R|$.

We estimate each term on the last two lines of (2.8).

Recall that $g_0 = u^{-4}g = v^4 g$. The scalar curvature R_0 with respect to g_0 is 0. So

$$\Delta_g v = (\Delta_g + \tfrac{1}{8}R)v - \tfrac{1}{8}Rv = -\tfrac{1}{8}R_0 v^5 - \tfrac{1}{8}Rv$$
$$= -\tfrac{1}{8}Rv = -\tfrac{1}{8}\big(1 + \rho(u) - \lambda u^{-4}\big)u^{-1}.$$

Therefore

$$(\Delta_g v)^2 \leq C\big(u^{-2} + u^{-2}\rho^2(u) + \lambda^2 u^{-10}\big) \leq C\bigg(u^{-2} + \sum_{i=1}^{K} u^{2p_i - 12} + \lambda^2 u^{-10}\bigg)$$

and

$$\int_{\Omega} (\Delta_g v)^2 \, dV_g \leq C + C\sum_{i=1}^{K} \int_{\Omega} u^{2p_i - 6}\, dx + C\lambda^2 \int_{\Omega} u^{-4}\, dx \leq C\big(1 + u_{\min}^{-2\tau} + \lambda^2 u_{\min}^{-4}\big),$$

where $\tau = \min(3 - p_1, \dots, 3 - p_K, 0) < 2$ since $1 < p_i < 5$.

For the second term in the right-hand side of (2.8), we have, by (2.6),

$$\int_{\Omega} |B|\,|\nabla_g v|^2\, dV_g \leq C\bigg(\int_{\Omega} |B|^{3/2}\, dV_g\bigg)^{2/3}\bigg(\int_{\Omega} |\nabla_g v|^6\, dV_g\bigg)^{1/3}$$
$$\leq C\lambda\bigg(\int_{\Omega} |\nabla_g v|^6\, dV_g\bigg)^{1/3}.$$

The terms on the last line of (2.8) can be estimated as follows:

$$\int_{\Omega} |\nabla_g v|^2\, dV_g = \int_{\Omega} |\nabla_0 u|^2 u^{-2}\, dx \leq C\lambda,$$

$$\int_{\Omega} |R|\,|\nabla_g v|^2\, dV_g \leq \int_{\Omega} C\big(1 + \rho(u) + \lambda u^{-4}\big)|\nabla_g v|^2\, dV_g$$
$$\leq C\lambda + \bigg(\int_{\Omega} |\rho(u) + \lambda u^{-4}|^{3/2}\, dV_g\bigg)^{2/3}\bigg(\int_{\Omega} |\nabla_g v|^6\, dV_g\bigg)^{1/3}$$
$$\leq C\lambda + C\bigg(\int_{\Omega} \bigg(\sum_{i=1}^{K} u^{(3/2)(p_i - 5) + 6} + \lambda^{3/2}\bigg)\, dx\bigg)^{2/3}$$
$$\times \bigg(\int_{\Omega} |\nabla_g v|^6\, dV_g\bigg)^{1/3}$$
$$\leq C\lambda + C\sum_{i=1}^{K} \lambda^{(1/4)(p_i - 5) + 1}\bigg(\int_{\Omega} |\nabla_g v|^6\, dV_g\bigg)^{1/3}$$

by (1.4), since $0 < \frac{3}{2}(p_i - 5) + 6 \leq 5$. (This is where we need the assumption $p_i \leq \frac{13}{3}$.)

Combining all the previous estimates, we have

$$\left(\int_\Omega |\nabla_g v|^6 \, dV_g\right)^{1/3} \le C\left(1 + u_{\min}^{-2\tau} + \lambda^2 u_{\min}^{-4}\right).$$

Note that

$$\int_\Omega |\nabla_g v|^6 \, dV_g = \int_\Omega |\nabla_0 u|^6 u^{-18} \, dx.$$

So

$$\left(\int_\Omega |\nabla_g u^{-2}|^6 \, dx\right)^{1/3} \le C\left(1 + u_{\min}^{-2\tau} + \lambda^2 u_{\min}^{-4}\right).$$

By the Sobolev embedding theorem, for any $P, Q \in \bar\Omega$, we have

$$\left|u^{-2}(P) - u^{-2}(Q)\right| \le C \left\|\nabla(u^{-2})\right\|_{L^6(\Omega)} |P - Q|^{1/2}$$
$$\le C \left\|\nabla(u^{-2})\right\|_{L^6(\Omega)} \le C\left(1 + u_{\min}^{-\tau} + \lambda u_{\min}^{-2}\right).$$

Therefore, for any $P, Q \in \bar\Omega$,

$$u^{-2}(P) \ge u^{-2}(Q) - C\left(1 + u_{\min}^{-\tau} + \lambda u_{\min}^{-2}\right).$$

Choose Q so that $u(Q) = u_{\min} = \min_{\bar\Omega} u$. Since $u_{\min} \le C\lambda^{1/4}$ (by (1.3)) and $\tau < 2$, we see that

$$u^{-2}(Q) - C\left(1 + u_{\min}^{-\tau} + \lambda u_{\min}^{-2}\right) \ge \tfrac{1}{2} u_{\min}^{-2},$$

which implies that

$$u(P) \le C u_{\min} \le C\lambda^{1/4} \quad \text{for all } P \in \bar\Omega.$$

Now let $w = u - \bar u$, where $\bar u = (1/|\Omega|) \int_\Omega u$. Then w satisfies

$$\Delta w - \lambda w + f(w + \bar u) - f(\bar u) - \lambda \bar u + f(\bar u) = 0$$

Multiplying by w and integrating by parts, we get

(2.9) $$\int_\Omega \left(|\nabla w|^2 + \lambda w^2 + c(w) w^2\right) = 0,$$

where

$$c(w) = -\frac{f(w + \bar u) - f(\bar u)}{w} \to 0 \quad \text{as } \lambda \to 0.$$

Since $\int_\Omega w = 0$, (2.9) implies that $w \equiv 0$ and hence $u \equiv \bar u$. Theorem 1 is thus proved.

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