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In memory of my father

We study the classification of linear Hamiltonian systems satisfying Bolza boundary conditions and its applications to nonlinear Hamiltonian systems.

1. Introduction

Ivar Ekeland [1990] discussed the classification of convex Hamiltonian systems of the form

$$\dot{x} = JB(t)x, \quad x(0) = x(\tau), \tag{1–1}$$

where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

I_n being the $n \times n$ identity matrix. That is, for any continuous symmetric positive definite matrix map $B : [0, \tau] \to \mathrm{GL}(2n)$ with $B(0) = B(\tau)$, he defined a pair of numbers $(i^E(B), \nu^E(B)) \in \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots, 2n\}$ by making use of dual variational methods. This pair of integers is called the Ekeland index of B.

We define, as usual,

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbb{R}) \mid M^T JM = J\},$$
$$\mathcal{P}_\tau(2n) = \{\gamma : C([0, \tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\}.$$

For any $\gamma \in \mathcal{P}_\tau(2n)$, the Maslov-type index of γ is defined to be a pair of integers $(i^L(\gamma), \nu^L(\gamma)) \in \mathbb{Z} \times \{0, 1, 2, \ldots, 2n\}$; see [Conley and Zehnder 1984; Long and

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Zehnder 1990; Long 1990; Long 1997]. Denote by $\gamma_B(t)$ the matrizant of (1–1), that is, the solution of

$$\dot{\gamma}_B(t) = JB(t)\gamma_B(t), \quad \gamma_B(0) = I_{2n}.$$

Then $\gamma_B \in \mathcal{P}_\tau(2n)$ and

$$i^L(\gamma_B) = i^E(B) + n, \quad \nu^L(\gamma_B) = \nu^E(B),$$

whenever B is positive definite. So the Maslov-type index is more general than the Ekeland index. These index theories (about the properties of the index) and their iteration theories have important applications in the study of nonlinear Hamiltonian systems. See [Fei 1995; Conley and Zehnder 1984; Long and Zehnder 1990; Fei and Qiu 1997; Chang et al. 1997; Su 1998; Li and Liu 1989] for multiple periodic solutions of asymptotically linear Hamiltonian systems, [Ekeland and Hofer 1985; Dong and Long 1997] for Rabinowitz's minimal periodic problem, and [Ekeland and Hofer 1987; Long and Zhu 2002; Liu et al. 2002] for multiple closed characteristics on compact convex hyper-surfaces in $\mathbb{R}^{2n}$. For a systematic treatment and other applications, one can refer to the excellent books [Ekeland 1990; Long 2002].

Let $\mathcal{L}_{2n}^\infty := \{B : (0, 1) \to \mathrm{GL}(2n) \mid B(t) = (b_{ij}(t))_{2n\times 2n}, b_{ij} \in L^\infty(0, 1)$ and $b_{ij} = b_{ji}$ for all $i, j = 1, 2, \dots, 2n\}$. Throughout this paper, for any two symmetric matrices A_1 and A_2, we write $A_1 \le A_2$ if $A_2 - A_1$ is positive semidefinite, and write $A_1 < A_2$ if $A_2 - A_1$ is positive definite. For any $B_1, B_2 \in \mathcal{L}_{2n}^\infty$, we write $B_1 \le B_2$ if $B_1(t) \le B_2(t)$ for a.e. $t \in (0, 1)$; write $B_1 < B_2$ if $B_1 \le B_2$ and $B_1(t) < B_2(t)$ on a subset of $(0, 1)$ with positive measure. In this paper we consider the classification of systems of the form

$$\dot{x} = JB(t)x, \quad Px(0) = 0 = Px(1), \tag{1–2}$$

where $B \in \mathcal{L}_{2n}^\infty$ and $Px = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ for any $x = (x_1, \dots, x_{2n})^T \in \mathbb{R}^{2n}$. Specifically, to any $B \in \mathcal{L}_{2n}^\infty$ we associate a pair of numbers $(i(B), \nu(B)) \in \mathbb{Z} \times \{0, 1, \cdots, n\}$. The nullity $\nu(B)$ is the dimension of the solution space of (1–2). To define the index $i(B)$ we proceed in two steps. At first, we give the definition of $i(\lambda I_{2n})$ for any $\lambda \in \mathbb{R}$. Second, for any two $B_1, B_2 \in \mathcal{L}_{2n}^\infty$ with $B_1 < B_2$ we consider the nullity $\nu((1-\lambda)B_1 + \lambda B_2)$. We prove that the number of $\lambda \in [0, 1)$ with $\nu((1-\lambda)B_1 + \lambda B_2) \ne 0$ is finite and define the relative Morse index as $I(B_1, B_2) = \sum_{\lambda\in[0,1)} \nu((1-\lambda)B_1 + \lambda B_2)$. We prove the relative Morse index satisfies additivity properties and that $i(\lambda I_{2n}) - I(B, \lambda I_{2n})$ is independent of $\lambda \in \mathbb{R}$ with $\lambda I_{2n} > B$; we define $i(B)$ to be this number. We define an Ekeland type index $i_\mu(B)$, called the μ-index of B and satisfying $I(B_1, B_2) = i_\mu(B_2) - i_\mu(B_1)$. In this way we can prove additivity for the relative Morse index. This is the content of Section 2. In the process we only use the spectral theory of self-adjoint compact operators.

In [Dong 2005] we discussed the classification of the second-order Hamiltonian system

$$x'' + A(t)x = 0, \quad x(0) = 0 = x(1). \tag{1–3}$$

Namely, to every $A \in \mathscr{L}_n^\infty$ we associated a pair of numbers $(m^0(A), m^-(A))$. By making the change $z_1 = x$, $z_2 = \dot{x}$, $z = (z_1, z_2)$, we see that (1–3) is a special case of (1–2) with B the block-diagonal matrix $\operatorname{diag}\{A, I_n\}$. At the end of Section 2, we prove that $m^0(A) = \nu(\operatorname{diag}\{A, I_n\})$ and that $m^-(A) = i(\operatorname{diag}\{A, I_n\})$.

In Section 3 we discuss multiple solutions for the system

$$\dot{x} = JH'(t, x), \quad Px(0) = 0 = Px(1). \tag{1–4}$$

We first use the Leray–Schauder degree theory to obtain the existence of solutions and of nontrivial solutions. Then we use the dual variational method and Morse theory to discuss the multiple solutions of (1–4). The index and μ-index play an important role in the discussion. We stress that by making use of the μ-index, dual variational methods can be used instead of the saddle point reduction method whenever $H''(t, x)$ is bounded. Ekeland [1990] discussed the problem (1–4) and called it the Bolza problem. The method used here is also suitable for Hamiltonian systems with periodic boundary value conditions, for which we will write another paper.

2. Maslov type index theory for linear Hamiltonian systems satisfying Bolza boundary value conditions

As in the introduction we set $Px = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ for any $x = (x_1, \dots, x_{2n})^T \in \mathbb{R}^{2n}$. We shall give a classification for the system

$$\dot{x} = JB(t)x \text{ for } t \in (0, 1), \quad Px(0) = 0 = Px(1); \tag{2–1}$$

that is, for any $B \in \mathscr{L}_{2n}^\infty$, we shall define $i(B) \in \mathbb{Z}$ and $\nu(B) \in \{0, 1, \dots, n\}$, called the index and nullity of B respectively. We begin with the nullity. Set

$$W := \{x : [0, 1] \to \mathbb{R}^{2n} \mid x \text{ continuous on } [0, 1],\ Px(0) = 0 = Px(1),\ \dot{x} \in L^2\},$$

(where $L^2 := L^2((0, 1); \mathbb{R}^{2n})$), with the norm

$$\|x\|_W := \left(\int_0^1 (|\dot{x}(t)|^2 + |x(t)|^2)\, dt \right)^{1/2},$$

where $|x| = \left(\sum_{i=1}^{2n} |x_i|\right)^{1/2}$ for $x = (x_1, \dots, x_{2n})^T \in \mathbb{R}^{2n}$. Also define $\Lambda_1 : W \subset L^2 \to L^2$ by $(\Lambda_1 x)(t) := J\dot{x}(t)$ and $\bar{B} : L^2 \to L^2$ by $(\bar{B}x)(t) = B(t)x(t)$. Then Λ_1 and $\bar{B}$ are self-adjoint operators and $\bar{B}$ is bounded.

Definition 2.1. For any $B \in \mathcal{L}^\infty_{2n}$, the nullity of B is defined as

$$\nu(B) = \dim \ker(\Lambda_1 + \bar{B}).$$

By definition, $\nu(B) \geq 0$. Moreover, $\nu(B) = 0$ if and only if $\ker(\Lambda_1 + \bar{B}) = \{\theta\}$, and if and only if the problem (2–1) has no nontrivial solutions $x \neq \theta$.

Let $\gamma(t)$ be the matrizant of (2–1), i.e., the solution of

$$\dot{\gamma}(t) = JB(t)\gamma(t) \text{ for a.e. } t \in (0,1), \quad \gamma(0) = I_{2n}. \tag{2–2}$$

Then

$$\ker(\Lambda_1 + \bar{B}) = \left\{x = \gamma(t)\binom{0}{c} \mid c \in \mathbb{R}^n,\ Px(1) = 0\right\} \cong \left\{c \in \mathbb{R}^n \mid P\gamma(1)\binom{0}{c} = 0\right\} \subset \mathbb{R}^n.$$

Thus $\dim \ker(\Lambda_1 + \bar{B}) \leq n$, and $\nu(B) = \dim \ker(\Lambda_1 + \bar{B}) \in \{0, 1, \dots, n\}$. In particular, if $B \equiv \lambda I_{2n}$, we have

$$\gamma(t) = \exp J\lambda t = \begin{pmatrix} I_n \cos\lambda t & -I_n \sin\lambda t \\ I_n \sin\lambda t & I_n \cos\lambda t \end{pmatrix}, \quad P\gamma(1)\begin{pmatrix} 0 \\ c \end{pmatrix} = P\begin{pmatrix} -c\sin\lambda \\ c\cos\lambda \end{pmatrix} = -c\sin\lambda,$$

so $\ker(\Lambda_1 + \bar{B}) = \{c \in \mathbb{R}^n | c\sin\lambda = 0\}$. Therefore, $\nu(\lambda I_{2n}) = 0$ if and only if $\lambda/\pi \notin \mathbb{Z}$, and $\nu(\lambda I_{2n}) = n$ if $\lambda/\pi \in \mathbb{Z}$.

Still in the case $B \equiv \lambda I_{2n}$, the system (2–1) is equivalent to

$$y'' + \lambda^2 y = 0 \text{ for } t \in (0,1), \quad y(0) = 0 = y(1), \tag{2–3}$$

via the change $y = Px$. For the general second-order Hamiltonian system

$$x'' + A(t)x = 0 \text{ for } t \in (0,1), \quad x(0) = 0 = x(1), \tag{2–4}$$

with any $A \in \mathcal{L}^\infty_n$, we define as in [Dong 2005]

$$\psi_A(x,x) := \int_0^1 \left(|\dot{x}(t)|^2 - (A(t)x(t), x(t))\right) dt,$$

where $x \in F := \{x \in W^{1,2}([0,1]; \mathbb{R}^n) \mid x(0) = 0 = x(1)\}$. Then we have a ψ_A-orthogonal decomposition $F = F^+(A) \oplus F^0(A) \oplus F^-(A)$, and we define

$$m^-(A) := \dim F^-(A), \quad m^0(A) := \dim F^0(A).$$

In particular, $m^0(\lambda^2 I_n) = 0$ if $\lambda/\pi \notin \mathbb{Z}$, and $m^0(\lambda^2 I_n) = n$ if $\lambda/\pi \in \mathbb{Z}$. Thus $m^0(\lambda^2 I_n) = \nu(\lambda I_{2n})$ for $\lambda > 0$. Letting $x_1 = y$, $x_2 = -\dot{y}$, (2–3) is also equivalent to (2–1) with $B = \operatorname{diag}\{\lambda^2 I_n, I_n\}$. By Definition 2.1, we obtain $\nu(\operatorname{diag}\{\lambda^2 I_n, I_n\}) = m^0(\lambda^2 I_n)$.

Proposition 2.2. *For any $A \in \mathcal{L}^\infty_n$,*

$$\nu(\operatorname{diag}\{A, I_n\}) = m^0(A).$$

Proof. Making the change $x_1 = y$, $x_2 = -\dot{y}$, we see that (2–4) is equivalent to (2–1) with $B = \operatorname{diag}\{A, I_n\}$. □

In the following we consider the definition of $i(B)$ for any $B \in \mathscr{L}_{2n}^{\infty}$. Since (2–4) is a special case of (2–1), we begin with $B = \lambda I_{2n}$. Following [Ekeland 1990], we define $E[\alpha]$ for α real as the integer $a \in \mathbb{Z}$ such that $a < \alpha \le a+1$; in particular $E[\alpha] = \alpha - 1$ for integer α. Since (2–3) is equivalent to (2–1) when $B = \lambda I_{2n}$, and $m^-(\lambda^2 I_n) = nE[\lambda/\pi]$ as $\lambda > 0$, we have:

Definition 2.3. For any $\lambda \in \mathbb{R}$, we define

$$i(\lambda I_{2n}) = nE[\lambda/\pi].$$

In order to define $i(B)$ for any $B \in \mathscr{L}_{2n}^{\infty}$, we need to compare B with λI_{2n} for any $\lambda > 0$ with $\lambda I_{2n} > B$.

Definition 2.4. For any $B_1, B_2 \in \mathscr{L}_{2n}^{\infty}$ with $B_1 < B_2$, set

$$I(B_1, B_2) = \sum_{\lambda \in [0,1)} \nu((1-\lambda)B_1 + \lambda B_2).$$

We call $I(B_1, B_2)$ the relative Morse index, following [Fei 1995; Zhu and Long 1999; Long and Zhu 2000; Long 2002].

In Proposition 2.10 we will show that the relative Morse index is finite.

Proposition 2.5. *Assume $B, C \in \mathscr{L}_{2n}^{\infty}$ with $\nu(B) = 0$, $C \ge \epsilon I_{2n}$ for some positive constant $\epsilon \in \mathbb{R}$.*

1. *The operator $\Lambda_1 + \bar{B} : W \to L^2$ is invertible and the inverse $(\Lambda_1 + \bar{B})^{-1} : L^2 \to L^2$ is compact and self-adjoint.*
2. *There exists a sequence $\{\lambda_j\}_{j=-\infty}^{+\infty} \subset \mathbb{R}$ with $\lambda_j \to \pm\infty$ as $j \to \pm\infty$, such that the problem*

$$J\dot{x}(t) + B(t)x(t) = \lambda_j C(t)x(t), \qquad Px(0) = 0 = Px(1)$$

has a nontrivial solution subspace E_j with $\dim E_j \le n$ and $L^2 = \bigoplus_{j=-\infty}^{+\infty} E_j$.

Proof. (1) Let $\gamma(t)$ be the matrizant of (2–1) (see (2–2)). To prove that $\Lambda_1 + \bar{B}$ is invertible for any $u \in L^2$, we have to solve the problem

$$J\dot{x} + B(t)x = u, \qquad Px(0) = 0 = Px(1).$$

The first equation has a general solution

$$x(t) = \gamma(t)x(0) - \gamma(t)\int_0^t \gamma(s)^{-1} J u(s)\, ds. \tag{2–5}$$

The second condition gives

$$x(0)=\begin{pmatrix}0\\ c\end{pmatrix},\ Px(1)=P\gamma(1)\begin{pmatrix}0\\ c\end{pmatrix}-P\gamma(1)\int_0^1\gamma(s)^{-1}Ju(s)\,ds=0. \tag{2–6}$$

Since $\nu(B)=0$, setting $\gamma(1)=\begin{pmatrix}\gamma_{11}(1) & \gamma_{12}(1)\\ \gamma_{21}(1) & \gamma_{22}(1)\end{pmatrix}$, we see that the equation

$$0=P\gamma(1)\begin{pmatrix}0\\ c\end{pmatrix}=\gamma_{12}(1)c$$

has no nontrivial solutions, so $\gamma_{12}(1)$ is invertible. From (2–6) we have $\Lambda_1+\bar{B}$ is invertible and

$$((\Lambda_1+\bar{B})^{-1}u)(0)=\gamma_{12}(1)^{-1}P\gamma(1)\int_0^1\gamma(s)^{-1}Ju(s)\,ds.$$

From (2–5) and the Ascoli–Arzelà theorem, the operator $(\Lambda_1+\bar{B})^{-1}:L^2\to C([0,1],\mathbb{R}^{2n})$ is compact, and so is the operator $(\Lambda_1+\bar{B})^{-1}:L^2\to L^2$.

(2) Let $(\Pi u)(t)=(\Lambda_1+\bar{B})^{-1}C(t)u(t)$ for any $u\in L^2$. With $(x,y)=\sum_{i=1}^{2n}x_iy_i$, the inner product $(u,v)_1:=\int_0^1(C(t)u(t),v(t))\,dt$ defines a Hilbert space structure, and

$$\begin{aligned}(\Pi u,v)_1&=\int_0^1(C(t)(\Lambda_1+\bar{B})^{-1}C(t)u(t),v(t))\,dt\\ &=\int_0^1(C(t)u(t),(\Lambda_1+\bar{B})^{-1}C(t)v(t))\,dt=(u,\Pi v)_1\end{aligned}$$

for every $u,v\in L^2$. Thus $\Pi:L^2\to L^2$ is self-adjoint and compact. By spectral theory, there are $\{\mu_j\}_{j=1}^{\infty}\subset\mathbb{R}$ with $\mu_j\to 0$ as $j\to\infty$ and a basis $\{u_j\}_{j=1}^{\infty}$ of L^2 such that

$$(u_j,u_i)_1=\delta_{ij},\ \Pi u_j=\mu_ju_j,\ u_j\neq\theta.$$

By the definition of Π, we have $(\Lambda_1+\bar{B})\mu_ju_j=C(t)u_j$. So $\mu_j\neq 0$ and

$$J\dot{u}_j(t)+B(t)u_j(t)=\frac{1}{\mu_j}C(t)u_j(t),\ Pu_j(0)=0=Pu_j(1).$$

Hence, setting $E_j:=\ker(\Pi-\mu_j)$, we get

$$\dim E_j=\nu\Big(B-\frac{1}{\mu_j}C\Big)\le n.$$

Letting $\lambda_j:=1/\mu_j$, we only need to show that λ_j is bounded above and below. In fact, assume $\lambda_j\ge\bar{\lambda}$ for some $\bar{\lambda}\in\mathbb{R}$. For any $x\in W$, set $C(t)u(t):=J\dot{x}(t)+$

$B(t)x(t)$; then $u(t)=\sum_{j=1}^{+\infty}u_j(t)$ with $u_j\in E_j$, $x(t)=(\Pi u)(t)=\sum_{j=1}^{+\infty}\mu_j u_j(t)$ and

$$\begin{aligned}\int_0^1 (J\dot{x}(t)+B(t)x(t),x(t))\,dt &= \int_0^1 (C(t)u(t),x(t))\,dt\\ &= \int_0^1 \sum_{i=1}^{\infty}\mu_i(C(t)u_i(t),u_i(t))\,dt\\ &\geq \bar{\lambda}\int_0^1 (C(t)x(t),x(t))\,dt.\end{aligned}$$

Thus there exists a constant c_1 depending only on B, C and $\bar{\lambda}$ such that

$$\int_0^1 (J\dot{x}(t),x(t))\,dt \geq c_1\|x\|_{L^2}^2 \quad \text{for all } x\in W. \tag{2–7}$$

As in [Ekeland 1990], set $x_p(t):=\exp(p\pi(-J)(t))\begin{pmatrix}0\\ \xi\end{pmatrix}$ with $p\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$ satisfying $|\xi|=1$. Then $x_p(t)=\begin{pmatrix}\xi\sin p\pi t\\ \xi\cos p\pi t\end{pmatrix}\in W$, and

$$\|x_p\|_{L^2}^2=1,\qquad \int_0^1 (J\dot{x}_p(t),x_p(t))\,dt=p.$$

This contradicts (2–7) when $p<c_1$. □

Remark. From this Proposition, when $\nu(B_1)=0$, $B_2-B_1\geq\epsilon I_{2n}$, the number of $\lambda\in[0,1)$ such that $\nu((1-\lambda)B_1+\lambda B_2)\neq 0$ is finite, and $I(B_1,B_2)<+\infty$. According to Proposition 2.10 below, the relative Morse index is finite in general case.

Let card denote the cardinality of a set. When $\lambda_1<\lambda_2$, by the definition, we have $I(\lambda_1 I_{2n},\lambda_2 I_{2n})=n\ \mathrm{card}\{j\in\mathbb{Z}\mid j\in[\lambda_1/\pi,\lambda_2/\pi)\}$, and $I(\lambda_1 I_{2n},\lambda_3 I_{2n})=I(\lambda_1 I_{2n},\lambda_2 I_{2n})+I(\lambda_2 I_{2n},\lambda_3 I_{2n})$ if $\lambda_3>\lambda_2>\lambda_1$. More generally:

Proposition 2.6. *Assume B_i, $i=1,2,3$, $\in\mathscr{L}_{2n}^{\infty}$ with $B_1<B_2<B_3$. Then*

$$I(B_1,B_3)=I(B_1,B_2)+I(B_2,B_3).$$

We postpone the proof and first define $i(B)$. From Definition 2.3, for any $\lambda<\lambda_1$ we have $i(\lambda_1 I_{2n})=i(\lambda I_{2n})+I(\lambda I_{2n},\lambda_1 I_{2n})$, so

$$\begin{aligned}i(\lambda_1 I_{2n})-I(B,\lambda_1 I_{2n}) &= i(\lambda_1 I_{2n})-(I(\lambda I_{2n},\\ &\quad \lambda_1 I_{2n})+I(B,\lambda I_{2n}))=i(\lambda I_{2n})-I(B,\lambda I_{2n});\end{aligned}$$

that is, the number $i(\lambda I_{2n})-I(B,\lambda I_{2n})$ is independent of $\lambda\in\mathbb{R}$ with $\lambda I_{2n}>B$. Hence:

Definition 2.7. For any $B \in \mathscr{L}_{2n}^{\infty}$, we define

$$i(B) = i(\lambda I_{2n}) - I(B, \lambda I_{2n}),$$

where $\lambda \in \mathbb{R}$ satisfies $B < \lambda I_{2n}$.

This index $i(B)$ is monotonously nondecreasing with respect to B. That is:

Proposition 2.8. *Assume $B_1, B_2 \in \mathscr{L}_{2n}^{\infty}$ with $B_1 < B_2$. Then $i(B_1) + \nu(B_1) \leq i(B_2)$.*

Proof. By definition,

$$\begin{aligned} i(B_2) &= i(\lambda I_{2n}) - I(B_2, \lambda I_{2n}) = i(\lambda I_{2n}) - \big(I(B_1, \lambda I_{2n}) - I(B_1, B_2)\big) \\ &= i(\lambda I_{2n}) - I(B_1, \lambda I_{2n}) + I(B_1, B_2) = i(B_1) + I(B_1, B_2) \\ &\geq i(B_1) + \nu(B_1). \end{aligned}$$

□

We next prove that the relative Morse index defined in Definition 2.4 is finite for any B_1, B_2 with $B_1 < B_2$; we also prove Proposition 2.6. To do this, following [Ekeland 1990], we define a Morse type index $i_\mu(B)$ for any $B \in \mathscr{L}_{2n}^{\infty}$, and prove that $I(B_1, B_2) = i_\mu(B_2) - i_\mu(B_1)$. More precisely, for any $B \in \mathscr{L}_{2n}^{\infty}$, let $\mu \in \mathbb{R} \setminus \pi\mathbb{Z}$ with $B + \mu I_{2n} \geq I_{2n}$; then $\nu(-\mu I_{2n}) = 0$. The operator $\Lambda x := J\dot{x}(t) - \mu x(t)$ is invertible and its inverse $\Lambda^{-1} : L^2 \to L^2$ is self-adjoint and compact, by Proposition 2.5. Define a quadratic form by setting, for $u \in L^2$,

$$q_{\mu,B}(u, u) = \frac{1}{2}\int_0^1 \big(((\Lambda^{-1}u)(t), u(t)) + (C_\mu(t)u(t), u(t))\big)\, dt,$$

$$(\bar{C}_\mu u, u) := \int_0^1 (C_\mu(t)u(t), u(t))\, dt,$$

where $C_\mu(t) := (\mu I_{2n} + B(t))^{-1}$. Then $(\bar{C}_\mu u, u)$ defines a Hilbert space structure on L^2. $C_\mu^{-1}\Lambda^{-1}$ is a self-adjoint and compact operator under this interior product. By spectral theory there is a basis $\{e_j\}_{j\in\mathbb{N}}$ of L^2 and a sequence $\lambda_j \to 0$ in $\mathbb{R}$ such that

$$(\bar{C}_\mu e_i, e_j) = \delta_{ij}, \quad (\Lambda^{-1}e_j, u) = (\bar{C}_\mu \lambda_j e_j, u) \text{ for all} u \in L^2. \tag{2–8}$$

For any $u \in L^2$, expressible as $u = \sum_{j=1}^{\infty} \xi_j e_j$, we have

$$q_{(\mu,B)}(u, u) = \frac{1}{2}\int_0^1 \big((\Lambda^{-1}u, u) + (C_\mu(t)u, u)\big)\, dt = \frac{1}{2}\sum_{j=1}^{\infty}(1 + \lambda_j)\xi_j^2.$$

Define

$$E_\mu^-(B) := \left\{ \sum_{j=1}^{\infty} \xi_j e_j \;\middle|\; \xi_j = 0 \text{ if } 1+\lambda_j \geq 0 \right\},$$

$$E_\mu^0(B) := \left\{ \sum_{j=1}^{\infty} \xi_j e_j \;\middle|\; \xi_j = 0 \text{ if } 1+\lambda_j \neq 0 \right\},$$

$$E_\mu^+(B) := \left\{ \sum_{j=1}^{\infty} \xi_j e_j \;\middle|\; \xi_j = 0 \text{ if } 1+\lambda_j \leq 0 \right\}.$$

Obviously, $E_\mu^-(B)$, $E_\mu^0(B)$ and $E_\mu^+(B)$ are $q_{(\mu,B)}$-orthogonal and

$$E_\mu^-(B) \oplus E_\mu^0(B) \oplus E_\mu^+(B) = L^2.$$

Since $\lambda_j \to 0$ as $j \to \infty$, the spaces $E_\mu^0(B)$ and $E_\mu^-(B)$ are finite-dimensional.

Definition 2.9. For any $B \in \mathscr{L}_{2n}^\infty$, $\mu \in \mathbb{R}$ with $\mu I_{2n} + B \geq I_{2n}$, we define

$$\nu_\mu(B) := \dim E_\mu^0(B),\, i_\mu(B) := \dim E_\mu^-(B).$$

We call $\nu_\mu(B)$ and $i_\mu(B)$ the μ-nullity and μ-index of B respectively.

Proposition 2.10. *For any B_1, B_2, $B \in \mathscr{L}_{2n}^\infty$ with $B_1 < B_2$, we have*

$$\nu_\mu(B) = \nu(B), \quad I(B_1, B_2) = i_\mu(B_2) - i_\mu(B_1).$$

Proof of the first equality. By the definitions, $E_\mu^-(B)$, $E_\mu^0(B)$ and $E_\mu^+(B)$ are $q_{(\mu,B)}$-orthogonal and satisfy $L^2 = E_\mu^-(B) \oplus E_\mu^0(B) \oplus E_\mu^+(B)$. For every $u \in E_\mu^0(B)$, we have

$$q_{(\mu,B)}(u, v) = 0 \text{ for any } v \in L^2.$$

So

$$\Lambda^{-1}u + C_\mu(t)u = 0.$$

Set $x := \Lambda^{-1}u$. Applying $C_\mu(t) = (B(t) + \mu I_{2n})^{-1}$ to both sides and using the equalities $\Lambda = J\frac{d}{dt} - \mu$ and $u = \Lambda x$, we get

$$(B(t) + \mu I_{2n})x(t) + J\dot{x}(t) - \mu x(t) = 0.$$

That is,

$$B(t)x(t) + J\dot{x}(t) = 0.$$

Hence, $\ker(\Lambda_1 + \bar{B}) \cong E_\mu^0(B)$ and $\nu(B) = \nu_\mu(B)$. □

Proof of the second equality. Step 1. We show that if X is a subspace of L^2 such that $q_{(\mu,B)}(u, u) < 0$ for every $u \in X \setminus \{\theta\}$, then $\dim X \leq i_\mu(B)$.

In fact, let $e_1, \ldots, e_k$ be a basis of X, we have the decomposition $e_i = e_i^- + e_i^*$ with $e_i^- \in E_\mu^-(B)$, $e_i^* \in E_\mu^0(B) \oplus E_\mu^+(B)$. Suppose there exist numbers $\alpha_i \in \mathbb{R}$, not all zero, such that $\sum_{i=1}^k \alpha_i e_i^- = \theta$. Write $e := \sum_{i=1}^k \alpha_i e_i$; then $e \in X \setminus \{\theta\}$ and $q_{(\mu,B)}(e, e) < 0$; at the same time, $e = \sum_{i=1}^k \alpha_i e_i^* \in E_\mu^0(B) \oplus E_\mu^+(B)$, and $q_{(\mu,B)}(e, e) \geq 0$, a contradiction. So $\{e_i^-\}_{i=1}^k$ is linearly independent and $i_\mu(B) \geq k = \dim X$.

Step 2. For $B_1 < B_2 \in \mathscr{L}_{2n}^\infty$, set $i(\lambda) := i_\mu((1-\lambda)B_1 + \lambda B_2)$ for $\lambda \in [0, 1]$. Then $i(\lambda_2) \geq i(\lambda_1) + \nu(\lambda_1)$ for any $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 < \lambda_2$.

In fact, write $A_i = (1-\lambda_i)B_1 + \lambda_i B_2$ for $i = 1, 2$; we only need to prove that

$$q_{(\mu,A_2)}(u, u) < 0 \quad \text{for all } u \in E_\mu^-(A_1) \oplus E_\mu^0(A_1) \setminus \{\theta\}.$$

Take any $u = u^0 + u^-$ with $u^0 \in E_\mu^0(A_1)$, $u^- \in E_\mu^-(A_1)$. If $u^- \neq \theta$ we have

$$\begin{aligned} q_{(\mu,A_2)}(u, u) \leq q_{(\mu,A_1)}(u, u) &= q_{(\mu,A_1)}(u^-, u^-) + q_{(\mu,A_1)}(u^0, u^0) \\ &= q_{(\mu,A_1)}(u^-, u^-) < 0. \end{aligned}$$

If $u^- = \theta$, write $x^0 = \Lambda^{-1}u^0$; then $u^0 = \Lambda x^0$ and x^0 is a nontrivial solution of

$$J\dot{x}(t) + A_1(t)x(t) = 0, \qquad Px(0) = Px(1) = 0.$$

So $x^0(t) \neq 0$ for every $t \in [0, 1]$, and $u^0(t) = -(A_1(t) + \mu I_{2n})x^0(t) \neq 0$ for a.e. $t \in (0, 1)$. Hence

$$\begin{aligned} &\frac{1}{\lambda_1-\lambda_2} q_{(\mu,A_2)}(u, u) \\ &\quad = \frac{1}{\lambda_1-\lambda_2}\big(q_{(\mu,A_2)}(u^0, u^0) - q_{(\mu,A_1)}(u^0, u^0)\big) \\ &\quad = \int_0^1 \big((\mu I_{2n} + A_2(t))^{-1}(B_1(t) - B_2(t))(\mu I_{2n} + A_1(t))^{-1}u^0(t),\ u^0(t)\big)\, dt \\ &\quad = \int_0^1 \big((B_1(t) - B_2(t))x^0(t),\ (\mu I_{2n} + A_2(t))^{-1}(\mu I_{2n} + A_1(t))x^0(t)\big)\, dt. \end{aligned}$$

If $\lambda_1 = \lambda_2$, we have $A_2(t) = A_1(t)$, and the last integral is

$$\int_0^1 \big((B_2(t) - B_1(t))x^0(t),\ x^0(t)\big)\, dt > 0.$$

Hence, if λ_2 is close to λ_1 and $\lambda_2 > \lambda_1$, we have $q_{(\mu,A_2)}(u, u) < 0$. So for $\lambda_2 < \lambda_1$ and λ_2 close to λ_1, we have $i(\lambda_2) \geq i(\lambda_1) + \nu(\lambda_1)$.

Step 3. For any $\lambda \in [0, 1)$, we have $i(\lambda + 0) = i(\lambda) + \nu(\lambda)$.

In fact, from Step 2, $i(\lambda) + \nu(\lambda) \leq i(\lambda + 0)$. So we only need to show that $i(\lambda) + \nu(\lambda) \geq i(\lambda + 0)$. Write $k := i(\lambda + 0)$. There exists $\lambda' > \lambda$ such that $i(s) = k$

and $\nu(s) = 0$ for $s \in (\lambda, \lambda')$. Set $C(s) := (\mu I_{2n} + (1-s)B_1(t) + sB_2(t))^{-1}$. By (2–8) we have

$$(C(s)e_j^s, e_i^s) = \delta_{ij} \quad \text{and} \quad \Lambda^{-1}e_j^s = C(s)\lambda_j^s e_j^s \quad \text{for all } u \in L^2. \tag{2–9}$$

Since $C(s) \geq (\mu I_{2n} + B_2)^{-1}$ for $s \in [0, 1]$, the sequence $\{e_j^s\}$ is bounded in L^2, and $\lambda_j^s = (\Lambda^{-1}e_j^s, e_j^s)$ is bounded in $\mathbb{R}$ for $j = 1, \ldots, k$ and $s \in [0, 1]$. So there exist $s_l \in (\lambda, \lambda')$ such that $s_l \to \lambda+0$, $e_j^{s_l} \rightharpoonup e_j$ in L^2, $\lambda_j^{s_l} \to \lambda_j$ in $\mathbb{R}$, and $\Lambda^{-1}e_j^{s_l} \to \Lambda^{-1}e_j$ in L^2. Taking the limit in (2–9) we obtain $(C(\lambda)e_j, e_i) = \delta_{ij}$ and $\Lambda^{-1}e_j = C(\lambda)\lambda_j e_j$ for $j = 1, \ldots, k$. Again for $j = 1, \ldots, k$, since $i(s) = k$ for every $s \in (\lambda, \lambda')$, we have by the definitions $1 + \lambda_j^{s_l} < 0$ and $\{1/\lambda_j^{s_l}\}$ bounded in $\mathbb{R}$. So

$$e_j^{s_l} = \frac{1}{\lambda_j^{s_l}} C(s_l)^{-1}\Lambda^{-1}e_j^{s_l} \to \frac{1}{\lambda_j} C(\lambda)^{-1}\Lambda^{-1}e_j = e_j$$

in L^2. It follows that $\{e_i\}_{i=1}^k$ is linearly independent and for every $u = \sum_{j=1}^k \alpha_j e_j$, since $\sum_{j=1}^k \alpha_j e_j^{s_l} \to u$ in L^2 and

$$q_{(\mu,(1-s_l)B_1+s_l B_2)}\left(\sum_{j=1}^k \alpha_j e_j^{s_l}, \sum_{j=1}^k \alpha_j e_j^{s_l}\right) < 0,$$

taking the limit as $s_l \to \lambda + 0$ we have $q_{(\mu,(1-\lambda)B_1+\lambda B_2)}(u, u) \leq 0$. In a way similar to the proof of Step 1, this implies $i(\lambda) + \nu(\lambda) \geq k := i(\lambda + 0)$.

Step 4. The function $i(\lambda)$ is left continuous for $\lambda \in (0, 1]$ and continuous for $\lambda \in (0, 1)$ with $\nu(\lambda) = 0$.

In fact, from Steps 2 and 3 we only need to show $i(\lambda) \leq i(\lambda - 0)$. Let $e_1, \ldots, e_k$ be a basis of $E^-(\lambda) := E_\mu^-((1-\lambda)B_1 + \lambda B_2)$, and set

$$S_1 := \left\{(\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k \;\middle|\; \sum_{i=1}^k \alpha_i^2 = 1\right\}.$$

Then

$$\begin{aligned}
f(s, &\alpha_1, \ldots, \alpha_k) \\
&:= q_{(\mu,(1-s)B_1+sB_2)}\left(\sum_{i=1}^k \alpha_i e_i, \sum_{i=1}^k \alpha_i e_i\right) \\
&= \frac{1}{2}\int_0^1 \Bigg(\left(\Lambda^{-1}\sum_{i=1}^k \alpha_i e_i(t), \sum_{i=1}^k \alpha_i e_i(t)\right) \\
&\qquad\qquad + (\mu I_{2n} + (1-s)B_1(t) + sB_2(t))^{-1}\sum_{i=1}^k \alpha_i e_i(t), \sum_{i=1}^k \alpha_i e_i(t)\Bigg)\, dt
\end{aligned}$$

is continuous on $[0, 1] \times S_1$. Since $f(\lambda, \alpha_1, \dots, \alpha_k) < 0$ for $(\alpha_1, \cdots, \alpha_k) \in S_1$ we have $f(s, \alpha_1, \dots, \alpha_k) < 0$ for $(\alpha_1, \dots, \alpha_k) \in S_1$ and s close enough to λ. From Step 1, we have $i(\lambda) \le i(s)$ for s close to λ. Hence $i(\lambda) \le i(\lambda - 0)$. In conclusion:

$$\begin{aligned} i_\mu(B_2) &= i_\mu(B_1) + \sum_{0\le\lambda<1} \nu_\mu((1-\lambda)B_1 + \lambda B_2) \\ &= i_\mu(B_1) + \sum_{0\le\lambda<1} \nu((1-\lambda)B_1 + \lambda B_2) = i_\mu(B_1) + I(B_1, B_2). \end{aligned}$$ □

Remark. The method of the proof comes from [Ekeland 1990, Theorem I.4.6], with some modifications.

Proof of Proposition 2.6. From Proposition 2.10, fix $\mu \in \mathbb{R}$ with $\mu I_{2n} + B_1 \ge I_{2n}$. Then

$$\begin{aligned} I(B_1, B_2) + I(B_2, B_3) &= i_\mu(B_2) - i_\mu(B_1) + i_\mu(B_3) - i_\mu(B_2) \\ &= i_\mu(B_3) - i_\mu(B_1) = I(B_1, B_3). \end{aligned}$$ □

Proposition 2.11. *For any $A \in \mathscr{L}_n^\infty$, we have*

$$i(\mathrm{diag}\{A, I_n\}) = m^-(A).$$

Proof. Fix $\epsilon > 0$ and $c > 0$ with $c^2 I_n > (1+\epsilon)A$. Since $0 \le \lambda_1 < \lambda_2 < 1$, we have

$$\begin{aligned} &((1-\lambda_2)A + \lambda_2 c^2 I_n)(1+\epsilon\lambda_2)^{-1} - ((1-\lambda_1)A + \lambda_1 c^2 I_n)(1+\epsilon\lambda_1)^{-1} \\ &\qquad = (\lambda_2 - \lambda_1)(1+\epsilon\lambda_1)^{-1}(1+\epsilon\lambda_2)^{-1}(c^2 I_n - (1+\epsilon)A) > 0 \end{aligned}$$

and

$$\begin{aligned} &\mathrm{diag}\big\{(\lambda_1 c^2 I_n + (1-\lambda_1)A)(1+\epsilon\lambda_1)^{-1},\ (1+\epsilon\lambda_1)I_n\big\} \\ &\qquad < \mathrm{diag}\big\{(\lambda_2 c^2 I_n + (1-\lambda_2)A)(1+\epsilon\lambda_2)^{-1},\ (1+\epsilon\lambda_2)I_n\big\}. \end{aligned}$$

As in the proof of Proposition 2.10, we obtain

$$\begin{aligned} \text{(2–10)}\quad &\sum_{0\le\lambda<1} \nu\, \mathrm{diag}\big\{(\lambda c^2 I_n + (1-\lambda)A)(1+\epsilon\lambda)^{-1}, (1+\epsilon\lambda)I_n\big\} \\ &\qquad = i_\mu\, \mathrm{diag}\{c^2 I_n(1+\epsilon)^{-1}, (1+\epsilon)I_n\} - i_\mu\, \mathrm{diag}\{A, I_n\}, \end{aligned}$$

where we have dropped the parentheses enclosing the argument of i_μ to lighten the notation. By Proposition 2.10 itself, we have

$$\begin{aligned} \text{(2–11)}\quad &i_\mu\, \mathrm{diag}\big\{c^2 I_n(1+\epsilon)^{-1}, (1+\epsilon)I_n\big\} - i_\mu\, \mathrm{diag}\{A, I_n\} \\ &\qquad = I\Big(\mathrm{diag}\{A, I_n\}, \mathrm{diag}\Big\{\frac{c^2}{1+\epsilon} I_n, (1+\epsilon)I_n\Big\}\Big). \end{aligned}$$

For any $A \in \mathscr{L}_n^\infty$, consider the system

$$x'' + A(t)x = 0, \quad x(0) = 0 = x(1).$$

Via the change $z_1 = x$, $z_2 = -(1+\epsilon)^{-1}\dot{z}_1$, this system is equivalent to

$$\dot{z} = JB(t)z, \quad Pz(0) = Pz(1) = 0,$$

where $B := \operatorname{diag}\{A(1+\epsilon)^{-1}, (1+\epsilon)I_n\}$. So

$$\nu\big(\operatorname{diag}\{A(1+\epsilon)^{-1}, (1+\epsilon)I_n\}\big) = m^0(A). \tag{2–12}$$

It follows that

$$\sum_{0\le\lambda<1} \nu \operatorname{diag}\big\{(\lambda c^2 I_n + (1-\lambda)A)(1+\epsilon\lambda)^{-1}, (1+\epsilon\lambda)I_n\big\} = \sum_{0\le\lambda<1} m^0(\lambda c^2 I_n + (1-\lambda)A).$$

Denote the right-hand side by $I_1(A, c^2 I_n)$. Again as in Proposition 2.10, we can show that $I_1(A, B) = m^-(B) - m^-(A)$ for any $A, B \in \mathscr{L}_n^\infty$ with $A < B$. Combining (2–10) and (2–11) with part (1) of Proposition 2.13 below we obtain

$$i \operatorname{diag}\Big\{\frac{c^2}{1+\epsilon} I_n, (1+\epsilon)I_n\Big\} - i \operatorname{diag}\{A, I_n\} = m^-(c^2 I_n) - m^-(A). \tag{2–13}$$

From Definitions 2.4 and 2.7 and equation (2–12), for any $c_1 \in \mathbb{R}$ with $c_1 > 1+\epsilon$ and $c_1 > c^2/(1+\epsilon)$ we have

$$\begin{aligned}
&i \operatorname{diag}\Big\{\frac{c^2}{1+\epsilon} I_n, (1+\epsilon)I_n\Big\}\\
&\quad = i(c_1 I_{2n}) - I\Big(\operatorname{diag}\Big\{\frac{c^2}{1+\epsilon} I_n, (1+\epsilon)I_n\Big\}, c_1 I_{2n}\Big)\\
&\quad = i(c_1 I_{2n}) - \sum_{0\le\lambda<1} \nu \operatorname{diag}\Big\{\Big(\lambda c_1 + (1-\lambda)\frac{c^2}{1+\epsilon}\Big) I_n, ((1+\epsilon)(1-\lambda) + c_1\lambda)I_n\Big\}\\
&\quad = i(c_1 I_{2n}) - \sum_{0\le\lambda<1} m^0\Big(\Big((1-\lambda)\frac{c^2}{1+\epsilon} + \lambda c_1\Big)((1+\epsilon)(1-\lambda) + c_1\lambda)I_n\Big)\\
&\quad = i(c_1 I_{2n}) - n \operatorname{card}\big\{j \in \mathbb{N} \mid j \in [c/\pi, c_1/\pi)\big\}\\
&\quad = i(c_1 I_{2n}) - I(cI_{2n}, c_1 I_{2n}) = i(cI_{2n}) = m^-(c^2 I_{2n}).
\end{aligned}$$

By (2–13) this proves the result. □

Definition 2.12. For any $B_1, B_2 \in \mathscr{L}_{2n}^\infty$, define

$$I(B_1, B_2) = I(B_1, \mu I_{2n}) - I(B_2, \mu I_{2n})$$

where $\mu \in \mathbb{R}$ satisfies $\mu I_{2n} > B_1$, $\mu I_{2n} > B_2$.

From Proposition 2.6, $I(B_1, B_2)$ is independent of μ and coincides with the object of Definition 2.4 when $B_1 < B_2$, so it is well defined.

Proposition 2.13. *The index defined by Definitions 2.3 and 2.7, the relative Morse index defined by Definitions 2.4 and 2.12, and the μ-index defined by Definition 2.9 have the following properties:*

(1) *For any $B_1, B_2 \in \mathscr{L}_{2n}^\infty$, we have*

$$I(B_1, B_2) = i(B_2) - i(B_1).$$

(2) *For any $B_1, B_2, B_3 \in \mathscr{L}_{2n}^\infty$, we have*

$$I(B_1, B_2) + I(B_2, B_3) = I(B_1, B_3).$$

(3) *For any $B \in \mathscr{L}_{2n}^\infty$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ we have*

$$\begin{aligned} \nu(B+\epsilon I_{2n}) &= 0 = \nu(B-\epsilon I_{2n}),\\ i(B-\epsilon I_{2n}) &= i(B),\\ i(B+\epsilon I_{2n}) &= i(B)+\nu(B). \end{aligned}$$

In particular, if $\nu(B) = 0$, we have $i(B+\epsilon I_{2n}) = i(B)$ for $\epsilon \in (0, \epsilon_0]$.

(4) *$i_\mu(B) - i(B)$ is a constant for B satisfying $B + \mu I_{2n} \geq I_{2n}$, i.e., $i_\mu(B) - i(B) = i_\mu(B_1) - i(B_1)$ for any other $B_1 \in \mathscr{L}_{2n}^\infty$ with $B_1 + \mu I_{2n} \geq I_{2n}$. For any $\mu > 1$, we have*

$$i_\mu(0) = nE\left[\frac{\mu}{\pi}\right] \quad \text{and} \quad i_\mu(B) = nE\left[\frac{\mu}{\pi}\right] + n + i(B)$$

for any $B \in \mathscr{L}_{2n}^\infty$ with $B + \mu I_{2n} \geq I_{2n}$.

Proof. (1) follows directly from Definitions 2.7 and 2.12, and (2) follows from (1). We prove the other two parts.

(3) From part (1), we have $i(B+I_{2n}) = i(B) + I(B, B+I_{2n})$. From Definition 2.4 and Proposition 2.10, we see that $I(B, B+I_{2n}) = \sum_{0\leq\lambda<1} \nu(B+\lambda I_{2n})$ is finite. So there is some $\epsilon_0 > 0$ such that $\nu(B+\epsilon I_{2n}) = 0$ for $\epsilon \in (0, \epsilon_0]$, and

$$i(B+\epsilon I_{2n}) = i(B) + \sum_{0\leq\lambda<1} \nu(B+\lambda\epsilon I_{2n}) = i(B)+\nu(B).$$

(4) By Definition 2.9, $i_\mu(0) = \dim E_\mu^-(0) = \sum_{1+\lambda_j<0} \dim\ker(\Lambda^{-1} - \bar{C}_\mu\lambda_j)$, where $\bar{C}_\mu = \mu^{-1} I_{2n}$. If $\Lambda^{-1}e_j - \bar{C}_\mu\lambda_j e_j = 0$, letting $\Lambda^{-1}e_j = x_j$, we have

$$J\dot{x}_j - \mu\Big(1+\frac{1}{\lambda_j}\Big)x_j = 0.$$

So $k\pi = \mu\Big(1+\dfrac{1}{\lambda_j}\Big)$ for $k \in \mathbb{Z}$, and $1+\lambda_j = \dfrac{\mu}{k\pi-\mu} + 1$. Since

$$1+\lambda_j < 0 \iff \{k\pi - \mu < 0, k\pi > 0\} \iff 0 < k < \frac{\mu}{\pi},$$

we have $\ker(\Lambda^{-1} - \bar{C}_\mu \lambda_j) \cong \ker(\Lambda_1 + \bar{B})$, where $\bar{B}x = -k\pi x$ and $\Lambda_1 x = J\dot{x}$. Hence $\dim\ker(\Lambda^{-1} - \bar{C}_\mu\lambda_j) = n$, and $i_\mu(0) = nE[\frac{\mu}{\pi}]$. This is the first displayed equality in (4). From (1) and Proposition 2.10, we have

$$i_\mu(B_1) - i_\mu(B) = I(B, B_1) = i(B_1) - i(B).$$

So

$$i_\mu(B) - i(B) = i_\mu(0) - i(0).$$

From Definition 2.3, we have $i(0) = -n$, and the second desired equality follows. □

3. Existence and multiplicity of solutions for nonlinear Hamiltonian systems

Consider the problem

$$\dot{x} = JH'(t, x), \qquad Px(0) = 0 = Px(1), \tag{3–1}$$

where $H : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous and $H'(t, x)$ is its gradient with respect to x. (For the definition of P see immediately after (1–2).) A problem of the form (3–1) is called a Bolza problem by I. Ekeland. We will always assume that $H' : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is continuous as well.

Theorem 3.1. *Assume that H and H' are continuous and that*

1. *$H'(t, x) = B(t, x)x + o(|x|)$ as $|x| \to +\infty$, where $B(t, x)$ is a symmetric $2n \times 2n$matrix and continuous with respect to $(t, x) \in [0, 1] \times \mathbb{R}^{2n}$;*
2. *there exist $B_1(t)$ and $B_2(t)$ such that $B_1(t) \le B(t, x) \le B_2(t)$, $i(B_2) = i(B_1)$ and $\nu(B_2) = 0$.*

Then (3–1) has at least one solution.

Proof. Consider

$$J\dot{x} + \lambda B_1(t)x + (1-\lambda)H'(t, x) = 0, \qquad Px(0) = 0 = Px(1), \tag{3–2}$$

where $\lambda \in (0, 1)$. Set

$$C_0([0, 1]; \mathbb{R}^{2n}) := \Big\{x : [0, 1] \to \mathbb{R}^{2n} \;\Big|\; Px(0) = 0 = Px(1) \text{ and } x(t) \text{ is continuous for } t \in [0, 1]\Big\},$$

with the norm $\|x\|_C = \max_{t\in[0,1]} |x(t)|$. We only need to show that there exists $r > 0$ such that $\|x_\lambda\|_C < r$ for any solution x_λ of (3–2). In fact, let $(\Lambda x)(t) = J\dot{x}(t) - \mu x(t)$, $(\bar{C}x)(t) = (\mu I_{2n} + B_1(t))x(t)$, $(Nx)(t) = \mu x(t) + H'(t, x(t))$. Then $\Lambda^{-1}\bar{C} : C_0([0, 1]; \mathbb{R}^{2n}) \to C_0([0, 1]; \mathbb{R}^{2n})$ is linear and compact, $\Lambda^{-1}N : C_0([0, 1]; \mathbb{R}^{2n}) \to C_0([0, 1]; \mathbb{R}^{2n})$ is compact, and $\ker(\Lambda^{-1}\bar{C} + \mathrm{id}) = \{\theta\}$. We have the Leray–Schauder degree $\deg(\mathrm{id} + \Lambda^{-1}\bar{C}, B_r, \theta) \ne 0$. By homotopy invariance we have $\deg(\mathrm{id}+\Lambda^{-1}N, B_r, \theta)=\deg(\mathrm{id}+\Lambda^{-1}\bar{C}, B_r, \theta)\neq 0$, and $\Lambda^{-1}Nx+x=0$, so (3–1) has a solution.

Now we prove that the solution x_λ of (3–2) is bounded. Assume to the contrary that there exists a sequence $\{x_k\}$ in $C([0,1];\mathbb{R}^{2n})$ such that $\|x_k\|_C\to+\infty$, $\{\lambda_k\}\subset(0,1)$, and

$$J\dot{x}_k+\lambda_k B_1(t)x_k(t)+(1-\lambda_k)H'(t,x_k(t))=0,$$
$$Px_k(0)=0=Px_k(1).$$

Set $y_k=x_k/\|x_k\|_C$ and $A_k(t)=\lambda_k B_1(t)+(1-\lambda_k)B(t,x_k(t))$; also define $e_k(t)=(1-\lambda_k)(H'(t,x_k(t))-B(t,x_k(t))x_k(t))\|x_k\|_C^{-1}$. Then

$$J\dot{y}_k(t)+A_k(t)y_k(t)+e_k(t)=0,\qquad Py_k(0)=0=Py_k(1). \tag{3–3}$$

From assumption (2), $\{\dot{y}_k\}$ is bounded in $C_0([0,1],\mathbb{R}^{2n})$. By the Ascoli–Arzelà theorem we may assume that $y_k\to y_0$ in $C_0([0,1];\mathbb{R}^{2n})$. From assumption (1), we have $e_k\to 0$ in $C([0,1];\mathbb{R}^{2n})$. From assumption (2) again we have $B_1\le A_k\le B_2$. Now write $A_k(t)=(a_{ij}^{(k)}(t))_{2n\times 2n}$ and $B_k(t)=(b_{ij}^{(k)}(t))_{2n\times 2n}$ for $k=1,2,\dots$. Then

$$b_{ii}^{(1)}\le a_{ii}^{(k)}\le b_{ii}^{(2)}\quad\text{and}\quad 2b_{ij}^{(1)}+b_{ii}^{(1)}+b_{jj}^{(1)}\le 2a_{ij}^{(k)}+a_{ii}^{(k)}+a_{jj}^{(k)}\le 2b_{ij}^{(2)}+b_{ii}^{(2)}+a_{jj}^{(2)}$$

for all distinct i and j. We may further assume $\lambda_k\to\lambda_0$ in $\mathbb{R}$ and $a_{ij}^{(k)}\rightharpoonup a_{ij}$ in $L^2(0,1)$ by going to subsequences if necessary. Write $A_0(t)=(a_{ij}(t))_{2n\times 2n}$; integrating (3–3) and taking the limit we have

$$J\dot{y}_0+A_0(t)y_0(t)=0,\qquad Py_0(0)=0=Py_0(1). \tag{3–4}$$

Since $B_1\le A_k\le B_2$, it follows that $B_1\le A_0\le B_2$, and $A_0\in\mathcal{L}_{2n}^\infty$. Because $i(B_2)=i(B_1)$ and $\nu(B_2)=0$, we see from Proposition 2.8 that $\nu(A_0)=0$. This contradicts the fact that y_0 is a nontrivial solution of (3–4). □

Theorem 3.2. *Assume that H and H' are continuous and that there exist $B_1, B_2\in\mathcal{L}_{2n}^\infty$ satisfying the following properties:*

1. *$B_1\le B_2$.*
2. *H is (B_1,B_2)-subquadratic at infinity, i.e., the difference*
$$N(t,x):=H(t,x)-\tfrac{1}{2}(B_1(t)x,x)$$
is strict convex with respect to x for all $t\in[0,1]$, and there exists $c\in\mathbb{R}$ such that $H(t,x)\le\frac{1}{2}(B_2(t)x,x)+c$ for all (t,x).
3. *$i(B_2)=i(B_1)+\nu(B_1)$, and $\nu(B_2)=0$ if $B_1<B_2$.*

Then (3–1) has one solution.

Proof. Step 1. Let $\Gamma : W \subset L^2 \to L^2$ be defined by $(\Gamma x)(t) := J\dot{x}(t) + B_1(t)x(t)$. We first prove that the range $R(\Gamma) := \Gamma(W)$ is closed in L^2, that

$$L^2 = \ker(\Gamma) \oplus R(\Gamma), \tag{3–5}$$

that the restriction $\Gamma_0 := \Gamma|_{R(\Gamma)}$ is invertible, and that $\Gamma_0^{-1} : R(\Gamma) \to R(\Gamma)$ is self-adjoint and compact.

In fact, if $\nu(B_1) = 0$, by Proposition 2.5, we have $\ker\Gamma = \{\theta\}$ and $R(\Gamma) = L^2$. Thus we need only consider the case $\nu(B_1) \neq 0$. From Proposition 2.13, there exists $\epsilon > 0$ such that $\nu(B_1+\epsilon I_{2n}) = 0$. From Proposition 2.5(2), there exist $\lambda_j \in \mathbb{R}$ with $\lambda_j \to \pm\infty$ as $\lambda_j \to \pm\infty$ such that

$$L^2 = \bigoplus_{j=-\infty}^{+\infty} E_j, \quad \text{where} E_j := \ker\left(J\frac{d}{dt} + B_1(\cdot) + \epsilon I_{2n} - \lambda_j I_{2n}\right).$$

Since $\nu(B_1) = \dim\ker(J(d/dt) + B_1(\cdot)) \neq 0$ and since $\Gamma u = (\lambda_j - \epsilon)u$ for all $u \in E_j$, there exists some $j_0 \in \mathbb{Z}$ such that $\epsilon - \lambda_{j_0} = 0$. Then $E_{j_0} = \ker\Gamma$, $R(\Gamma) = \bigoplus_{j=-\infty, j\neq j_0}^{+\infty} E_j$, and $L^2 = \ker(\Gamma) \oplus R(\Gamma)$. Now, given $u \in R(\Gamma)$, write

$$u = \sum_{j=-\infty, j\neq j_0}^{+\infty} u_j, \quad \text{with} \quad \sum_{j=-\infty, j\neq j_0}^{+\infty} \|u_j\|_{L^2}^2 < +\infty.$$

Since $\Gamma u_j = (\lambda_j - \epsilon)u_j$, we have $\Gamma_0^{-1} u_j = \dfrac{1}{\lambda_j - \epsilon} u_j$. So

$$\Gamma_0^{-1} u = \sum_{j=-\infty, j\neq j_0}^{+\infty} \frac{1}{\lambda_j - \epsilon} u_j.$$

Since $\lambda_j \to \infty$, $\Gamma_0^{-1} : R(\Gamma) \to R(\Gamma)$ is compact and self-adjoint.

Step 2. Consider the functional

$$\psi(u) := \int_0^1 \left(\tfrac{1}{2}(\Gamma_0^{-1}u, u) + N^*(t, -u)\right) dt, \tag{3–6}$$

where $N^*(t, u) = \sup_{v\in\mathbb{R}^{2n}}\{(u, v) - N(t, v)\}$ is the Fenchel conjugate of $N(t, u)$. From the continuity of H and H' as well as the definition and the strict convexity of N, we conclude that the gradient $N^{*\prime}(t, u)$ of $N^*(t, u)$ with respect to u exists and is continuous with respect to u, so $\psi \in C^1(L^2, \mathbb{R})$ and

$$(\psi'(u), v) = \int_0^1 \left((\Gamma_0^{-1}u, v) + (N^{*\prime}(t, -u), -v)\right) dt \tag{3–7}$$

for all $v \in R(\Gamma)$. In view of Proposition 2.13 and of $\nu(B_2) = 0$, there exists $\epsilon > 0$ such that $\nu(B_2 + \epsilon I_{2n}) = 0$ and $i(B_2 + \epsilon I_{2n}) = i(B_2)$. So without loss of generality we assume that, for some $\epsilon > 0$,

(3–8) $$B_2 - B_1 \geq \epsilon I_{2n}.$$

Write $C(t) := (B_2(t) - B_1(t))^{-1}$. Then $(\bar{C}u, v) := \int_0^1 (C(t)u(t), v(t))\,dt$ defines a Hilbert inner product structure on $R(\Gamma)$. By the spectral theory of self-adjoint compact operators we find a basis of $R(\Gamma)$, denoted by $\{e_j\}$, and a real sequence $\{\lambda_j\}$ with $\lambda_j \to 0$ satisfying

(3–9) $$(\bar{C}e_i, e_j) = \delta_{ij} \quad \text{and} \quad (\Gamma_0^{-1}e_j, u) = \lambda_j(\bar{C}e_j, u) \ \text{ for all } u \in R(\Gamma).$$

From (3–5), there exists $\xi_j \in \ker\Gamma$ such that

$$\lambda_j C(t)e_j - \Gamma_0^{-1}e_j = \xi_j.$$

Let $x_j = \Gamma_0^{-1}e_j + \xi_j$. We have $\Gamma x_j - \lambda_j^{-1}(B_2(t) - B_1(t))x_j = 0$. Since $i(B_2) = i(B_1) + \nu(B_1)$ and $\nu(B_2) = 0$, we have $\nu(B_1 + \lambda(B_2 - B_1)) = 0$ for any $\lambda \in (0, 1]$ by Definition 2.4 and Proposition 2.13(1). Hence, for $\lambda_j < 0$ we have $-\lambda_j^{-1} > 1$ and $\lambda_j + 1 > 0$. So for any $j \in \mathbb{N}$, we have $1 + \lambda_j > 0$. Set $\bar{\lambda} := \inf\{\lambda_j < 0 \mid j \in \mathbb{N}\}$. Since $\lambda_j \to 0$ as $j \to \infty$, we have

(3–10) $$1 + \bar{\lambda} > 0.$$

For any $u = \sum_{i=1}^{\infty} \xi_i e_i \in R(\Gamma)$, we get from (3–9)

(3–11) $$(\Gamma_0^{-1}u, u) = \sum_{j=1}^{\infty} \xi_j(\Gamma_0^{-1}u, e_j) = \sum_{i=1}^{\infty} \lambda_j|\xi_j|^2 \geq \bar{\lambda}(\bar{C}u, u).$$

On the other hand, we know that $N(t, u) \leq \frac{1}{2}((B_2(t) - B_1(t))u, u) + c$ for all (t, u), by assumption (2) of the theorem. Thus

$$N^*(t, u) \geq \tfrac{1}{2}(C(t)u, u) - c$$

and

$$\int_0^1 N^*(t, -u)\,dt \geq \tfrac{1}{2}(\bar{C}u, u) - c$$

for all $u \in R(\Gamma)$. Combining this with (3–10), (3–11) and (3–6), we obtain

$$\psi(u) \geq (\bar{\lambda} + 1)(\bar{C}u, u) - c \to +\infty$$

as $(\bar{C}u, u) \to +\infty$. Let $\{u_j\} \subset R(\Gamma)$ satisfy $\psi(u_n) \to \inf_{u \in R(\Gamma)} \psi(u)$. Then $\{u_j\}$ is bounded in $R(\Gamma)$, and there exists a subsequence, denoted again by $\{u_j\}$, converging weakly to u_0 in $R(\Gamma)$. As in [Ekeland 1990, Theorem II.2.1], ψ is weakly lower semicontinuous, so $\inf_{u \in R(\Gamma)} \psi(u) = \lim \psi(u_j) \geq \psi(u_0)$, and $\psi(u_0) = \inf_{u \in R(\Gamma)} \psi(u)$. Therefore, $\psi'(u_0) = 0$.

By (3–7), we have

$$\int_0^1 \big(\Gamma_0^{-1}u_0 - N^{*\prime}(t, -u_0), v\big)\, dt = 0$$

for all $v \in R(\Gamma)$. From (3–5) again, $\xi_0 := \Gamma_0^{-1}u_0 - N^{*\prime}(t, -u_0)$ lies in $\ker\Gamma$. Writing $\Gamma_0^{-1}u_0 - \xi_0 = x_0$, we get

$$J\dot{x}_0(t) + B_1(t)x_0(t) + N'(t, x_0(t)) = 0, \quad Px(0) = 0 = Px(1).$$

The proof is complete. □

Remark. As in [Ekeland 1990, Theorem III.2.1], it is not necessary to assume that $N(t, u)$ is strictly convex with respect to u: simple convexity suffices. We used the strict condition only in order to derive that $N^*(t, u)$ is differentiable with respect to u, to simplify the proof.

Corollary 3.3. *Assume $B_1, B_2 \in \mathscr{L}^\infty_{2n}$, $B_1(t)$, $B_2(t)$ are continuous with respect to t, and satisfy assumption (1) of Theorem 3.2. Let $\gamma_i(t)$ with $0 \le \gamma_1(t) \le \cdots \le \gamma_{2n}(t)$ be the eigenvalues of $B_2(t) - B_1(t)$, and let λ_j be the eigenvalues of Γ, and set:*

$$\gamma := \max\{\gamma_{2n}(t) | 0 \le t \le 1\}, \quad \lambda := \max\{\lambda_j < 0\}.$$

Then (3–1) has a solution if $|\lambda| > \gamma$.

Proof. We only need to prove that $\nu(B_2) = 0$ and $i(B_2) = i(B_1) + \nu(B_1)$. In fact, by the definition of γ we have $B_2(t) - B_1(t) \le \gamma I_{2n}$, and $B_2(t) \le B_1(t) + \gamma I_{2n}$. Since $-\lambda = |\lambda| > \gamma$, then $\Gamma x + \mu x = 0$ has no nontrivial solutions for any $\mu \in (0, \gamma]$. By Proposition 2.13 and Definition 2.4, we have $i(B_1 + \gamma I_{2n}) = i(B_1) + \nu(B_1)$. Since $B_1 < B_2 \le B_1 + \gamma I_{2n}$, we have $i(B_1) + \nu(B_1) \le i(B_2)$, $i(B_2) + \nu(B_2) \le i(B_1 + \gamma I_{2n})$. So $\nu(B_2) = 0$ and $i(B_2) = i(B_1) + \nu(B_1)$. □

Corollary 3.4. *Assume that*

$$((k-1)\pi + \epsilon)I_{2n} \le H''(t, x) \le (k\pi - \epsilon)I_{2n} \quad \textit{for all } (t, x),$$

where $\epsilon > 0$ is small and $k \in \mathbb{Z}$. Then (3–1) has at least one solution.

Proof. Take $B_1(t) := (k-1)\pi I_{2n}$ and $B_2(t) := (k\pi - \frac{1}{2}\epsilon)I_{2n}$. Then $N(t, x) := H(t, x) - \frac{1}{2}(B_1(t)x, x)$ is convex with respect to x, since $N''(t, x) \ge \epsilon I_{2n}$, and

$$\begin{aligned} H(t, x) &= \int_0^1 ds \bigg(\int_0^1 H''(t, \tau s x) x s\, d\tau, x \bigg) + (H'(t, \theta), x) + H(t, \theta) \\ &\le \tfrac{1}{2}(k\pi - \epsilon)|x|^2 + (H'(t, \theta), x) + H(t, \theta) \\ &\le \tfrac{1}{2} b|x|^2 + c, \end{aligned}$$

where $b := k\pi - \frac{1}{2}\epsilon$ and $c > 0$ is a constant. Since $i(bI_{2n}) = i((k-1)\pi I_{2n}) + \nu((k-1)\pi I_{2n})$, the hypotheses of Theorem 3.2 are satisfied. □

Example 3.5. Let

$$b(t)=\begin{cases}\dfrac{\pi}{2(1-t_1)} & \text{for } t\in(t_1,1),\\ \dfrac{\pi}{2t_1} & \text{for } t\in(0,t_1).\end{cases}$$

Then $i(bI_{2n})=\sum_{\lambda\in[0,1)}\nu(\lambda bI_{2n})-n=0$. Let $H(t,x)=(b(t)-\epsilon)|x|^2+|x|^{\epsilon+1}+h(t)$ and $\epsilon\in(0,1)$ is fixed. By Theorem 3.2, (3–1) has a solution. But $\gamma_i(t)=b(t)-\epsilon\to+\infty$ as $t_1\to 0^+$ for $t\in(0,t_1)$. So the assumptions in Corollary 3.3 are not satisfied.

Remark. Corollary 3.3 can be compared with [Ekeland 1990, Theorem II.2.1]. There Ekeland discusses periodic solutions of the Hamiltonian system consisting of the first equation in (3–1), and proves existence under similar conditions.

We now discuss the multiplicity for solutions of (3–1).

Theorem 3.6. *Let the assumptions of Theorem 3.1 be satisfied, and assume moreover that*

$$H'(t,x)=A(t,x)x+o(|x|)\quad \textit{for small } |x|,$$

where $A(t,x)$ is a symmetric $2n\times 2n$ matrix varying continuously with $(t,x)\in[0,1]\times\mathbb{R}^{2n}$ and satisfying $A_1(t)\le A(t,x)\le A_2(t)$, for all (t,x), where A_1 and A_2 are such that $i(A_1)=i(A_2)$ and $\nu(A_2)=0$.

Then (3–1) has a nontrivial solution if $i(B_1)-i(A_1)$ is odd.

We will use Leray–Schauder degree theory to prove the theorem.

Lemma 3.7 [Chang 1986, Chapter 1, Proposition 4.1′]. *Assume that $K: X\to X$ is a linear compact operator and $\sigma(K)$, the spectral set of K, does not contain -1. Then the Leray–Schauder degree* $\deg(\mathrm{id}+K,\Omega,\theta)$ *is $(-1)^\beta$, where $\beta=\sum_{\lambda_j+1<0,\lambda_j\in\sigma(K)}\beta_j$ and $\beta_j=\dim\bigcup_{k=1}^{\infty}\ker(K-\lambda_j\,\mathrm{id})^k$.* □

Proof of Theorem 3.6. We use the notations in Theorem 3.1. For $\mu\in\mathbb{R}$ with $\mu I_{2n}+B_1\ge I_{2n}$ and $\mu I_{2n}+A_1\ge I_{2n}$, we want to show that there exist $r_1,r\in\mathbb{R}$ with $r>r_1>0$ such that

$$\deg(\mathrm{id}+\Lambda^{-1}N,B_r,\theta)=(-1)^{i_\mu(B_1)}, \tag{3–12}$$

$$\deg(\mathrm{id}+\Lambda^{-1}N,B_{r_1},\theta)=(-1)^{i_\mu(A_1)}. \tag{3–13}$$

From Proposition 2.13, we have $i_\mu(B_1)-i_\mu(A_1)=i(B_1)-i(A_1)$, and hence $\deg(\mathrm{id}+\Lambda^{-1}N,B_r\setminus B_{r_1},\theta)=(-1)^{i_\mu(B_1)}-(-1)^{i_\mu(A_1)}\ne 0$, so the problem (3–1) has a nontrivial solution $x=x(t)$ with $r>\|x\|_C>r_1$. We only prove (3–12) since in a similar way we can get (3–13). From the proof of Theorem 3.1, we have

$$\deg(\mathrm{id}+\Lambda^{-1}N,B_r,\theta)=\deg(\mathrm{id}+\Lambda^{-1}\bar{C},B_r,\theta).$$

Set $K = \Lambda^{-1}\bar{C}$. If $\lambda \in \sigma(K)$ with $1+\lambda < 0$, we have $Kx = \lambda x$ for some $x \neq \theta$. Set $y(t) := (\mu I_{2n} + B_1(t))x(t)$; then $\Lambda^{-1} y = \bar{C}^{-1}\lambda y$, and $\ker(K - \lambda\,\mathrm{id}) \cong \ker(\Lambda^{-1} - \bar{C}^{-1}\lambda)$. From the definitions,

$$E_\mu^-(B_1) = \bigoplus_{\lambda+1<0} \ker(\Lambda^{-1} - \bar{C}^{-1}\lambda) \cong \bigoplus_{\lambda+1<0} \ker(K - \lambda\,\mathrm{id}),$$

$$i_\mu(B_1) = \dim E_\mu^-(B_1) = \dim \bigoplus_{\lambda+1<0} \ker(K - \lambda\,\mathrm{id}).$$

By Lemma 3.7, in order to prove (3–12) we only need to prove that

$$\ker(K - \lambda\,\mathrm{id})^2 = \ker(K - \lambda\,\mathrm{id}). \tag{3–14}$$

From (2–8), we have an orthogonal decomposition

$$L^2 = R(\Lambda^{-1} - \bar{C}^{-1}\lambda) \oplus \ker(\Lambda^{-1} - \bar{C}^{-1}\lambda). \tag{3–15}$$

If $(K - \lambda\,\mathrm{id})^2 x = \theta$, let $\bar{x} := (K - \lambda\,\mathrm{id})x = \Lambda^{-1}\bar{C}x - \lambda x$. Then

$$\bar{x} = (\Lambda^{-1} - \lambda\bar{C}^{-1})(\bar{C}x).$$

Since $(K - \lambda\,\mathrm{id})\bar{x} = 0$, we have $\bar{C}\bar{x} \in \ker(\Lambda^{-1} - \lambda\bar{C}^{-1})$. From (3–15) we get $(\bar{C}\bar{x}, \bar{x}) = 0$ and $\bar{x} = \theta$. This proves (3–14). □

To conclude we will use Morse theory to discuss the multiplicity of solutions. We will make the assumption that the second derivative $H'' : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ is continuous.

Theorem 3.8. *Assume that H, H', H'' are all continuous, that $H(t, \theta) \equiv 0$, that $H'(t, \theta) \equiv \theta$ and that the following conditions are satisfied:*

(1) *There exist $B_1, B_2 \in \mathscr{L}_{2n}^\infty$ with $i(B_2) = i(B_1)$, $\nu(B_2) = 0$ such that*

$$B_1(t) \le H''(t, x) \le B_2(t) \quad \textit{for all } (t, x) \textit{ with } |x| \ge r > 0.$$

(2) *With $B_0(t) := H''(t, \theta)$, we have*

$$i(B_1) \notin [i(B_0), i(B_0) + \nu(B_0)].$$

Then (3–1) has one nontrivial solution. Moreover, under the further assumption that

(3) *$0 = \nu(B_0)$ and $|i(B_1) - i(B_0)| \ge n$, equation (3–1) has two nontrivial solutions.*

Proof. From assumption (1), $H''(t, x)$ is bounded and there exist $\mu_1, \mu > 0$ such that

$$\mu_1 I_{2n} \ge H''(t, x) + \mu I_{2n} \ge I_{2n} \quad \text{for all } (t, x). \tag{3–16}$$

Recall that $(\Lambda x)(t) = J\dot{x}(t) - \mu x(t)$. Define $N(t,x) = H(t,x) + \frac{1}{2}\mu|x|^2$ and $N^*(t,x) = \sup_{y\in\mathbb{R}^{2n}}\{(x,y) - N(t,y)\}$. From [Ekeland 1990, Proposition II.2.10] we have

$$N^{*\prime\prime}(t,u^*) = N''(t,u)^{-1} \tag{3–17}$$

if $u = N^{*\prime}(t,u^*)$, or equivalently if $u^* = N'(t,u)$ (by the Fenchel conjugate formula; see [Ekeland 1990, Proposition II.1.15]). By (3–16) we have

$$I_{2n} \le N^{*\prime\prime}(t,u^*) \le \mu_1^{-1} I_{2n} \quad \text{for all } (t,u) \in [0,1]\times\mathbb{R}^{2n}, \tag{3–18}$$

and hence $|u| \to +\infty$ if and only if $|u^*| \to +\infty$. From assumption (1) and (3–17) there exists $r_1 > 0$ such that

$$(B_2(t) + \mu I_{2n})^{-1} \le N^{*\prime\prime}(t,u^*) \le (B_1(t) + \mu I_{2n})^{-1} \tag{3–19}$$

for all $(t,u^*) \in [0,1]\times\mathbb{R}^{2n}$ such that $|u^*| \ge r_1$. Consider the functional defined by

$$\psi(u) = \int_0^1 \left(\tfrac{1}{2}(\Lambda^{-1}u(t), u(t)) + N^*(t,u(t))\right) dt \quad \text{for all } u \in L^2. \tag{3–20}$$

We prove that ψ satisfies the Palais–Smale condition. Assume that $\{u_j\}$ is a sequence in L^2 such that $\psi(u_j)$ is bounded and $\psi'(u_j) \to \theta$ in L^2. From $N'(t,\theta) \equiv \theta$, we have $N^{*\prime}(t,\theta) \equiv \theta$ and

$$(\psi'(u), v) = \int_0^1 \left((\Lambda^{-1}u(t), v(t)) + (N^{*\prime}(t,u(t)), v(t))\right) dt \tag{3–21}$$

for all $v \in L^2$. Noticing that $\int_0^1 N^{*\prime\prime}(t,\theta u_j(t))\,d\theta u_j(t) = N^{*\prime}(t,u_j(t))$, we have

$$\Lambda^{-1}u_j + \int_0^1 N^{*\prime\prime}(t,\theta u_j(t))\,d\theta u_j = \psi'(u_j) \to \theta, \text{ in } L^2. \tag{3–22}$$

If $\|u_j\|_{L^2} \to \infty$, we set $x_j = u_j/\|u_j\|_{L^2}$. Without loss of generality, we assume $x_j \rightharpoonup x_0$ in L^2, and hence $\Lambda^{-1}x_j \to \Lambda^{-1}x_0$ in L^2. For any $\delta \in (0,1)$ fixed, set

$$C_j(t) = \begin{cases} \int_0^1 N^{*\prime\prime}(t,\theta u_j(t))\,d\theta & \text{if } |u_j(t)| \ge r_1/\delta, \\ (B_1(t) + \mu I_{2n})^{-1} & \text{otherwise,} \end{cases}$$

$$\xi_j(t) = \int_0^1 N^{*\prime\prime}(t,\theta u_j(t))\,d\theta u_j(t) - C_j(t)u_j(t).$$

Then there exists a constant $M_1 > 0$ such that

$$|\xi_j(t)| \le M_1 \quad \text{for a.e. } t \in (0,1), \tag{3–23}$$

and

$$(1-\delta)(B_2(t)+\mu I_{2n})^{-1}+\delta I_{2n} \le C_j(t) \le (1-\delta)(B_1(t)+\mu I_{2n})^{-1}+\mu_1^{-1}\delta I_{2n}.$$

So for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$((B_2(t)+\epsilon I_{2n})+\mu I_{2n})^{-1} \le C_j(t) \le ((B_1(t)-\epsilon I_{2n})+\mu I_{2n})^{-1} \tag{3–24}$$

for all $t \in (0, 1)$. Now we may further assume $C_j^{-1}(t)u(t) \rightharpoonup B_0(t)u(t)$ in L^2 for every $u \in L^2$ with $\mu I_{2n}+B_1-\epsilon I_{2n} \le B_0 \le \mu I_{2n}+B_2+\epsilon I_{2n}$. Let $\Lambda^{-1}x_0(t) = y_0(t)$, from equations (3–22)–(3–24), we have

$$J\dot{y}_0(t)+(B_0(t)-\mu I_{2n})y_0(t) = 0,\ Py_0(0) = 0 = Py_0(1). \tag{3–25}$$

From assumption (1) and Proposition 2.13(3), for $\varepsilon > 0$ is small enough, we have $\nu(B_1-\varepsilon I_{2n}) = \nu(B_2+\varepsilon I_{2n}) = 0$ and $i(B_1-\varepsilon I_{2n}) = i(B_2+\varepsilon I_{2n})$. So $\nu(B_0-\mu I_{2n})$ vanishes. This is impossible since $\|y_0\|_{L^2} = 1$ and y_0 is a nontrivial solution of (3–25); thus $\|u_j\|_{L^2}$ is bounded. Assume $u_j \rightharpoonup u_0$ in L^2; then $\Lambda^{-1}u_j \to \Lambda^{-1}u_0$. Let $\zeta_j := \Lambda^{-1}u_j + N^{*\prime}(t, u_j)$; then $N^{*\prime}(t, u_j) = \zeta_j - \Lambda^{-1}u_j \to -\Lambda^{-1}u_0$ in L^2, from (3–22). The Fenchel conjugate formula gives $u_j = N'(\zeta_j - \Lambda^{-1}u_j) \to N'(-\Lambda^{-1}u_0)$ in L^2, by [Ekeland 1990, II, Theorem 4]. So ψ satisfies the PS condition.

In order to continue the proof we need a lemma. Let X be a Banach space and take $f \in C^2(X, R^1)$. Set $K = \{x \in X \mid f'(x) = \theta\}$ and $f_a = \{x \in X | f(x) \le a\}$. If $f'(p) = \theta$ and $c = f(p)$, we say that p is a critical point of f and c is a critical value. Otherwise, we say that $c \in \mathbb{R}$ is a regular value of f. For any $p \in K$, $f''(p)$ is a self-adjoint operator; the *Morse index* of p is defined as the dimension of the negative space corresponding to the spectral decomposing, and is denoted by $m^-(f''(p))$. We also set $m^0(f''(p_0)) = \dim\ker f''(p_0)$. If $f''(p)$ has a bounded inverse we say that p is nondegenerate.

From [Chang 1993, Chapter III, Theorem 3.1; Chapter II, Theorems 5.1, 5.2 and Corollary 5.2], one can prove:

Lemma 3.9. *Assume $f \in C^2(X, R)$ satisfies the PS condition, $f'(\theta) = \theta$, and there is a positive integer γ such that $\gamma \notin [m^-(f''(\theta)), m^0(f''(\theta))+m^-(f''(\theta))]$ and $H_q(X, f_a; \mathbb{R}) = \delta_{q\gamma}\mathbb{R}$ for some regular value $a < f(\theta)$. Then f has a critical point $p_0 \ne \theta$ with $C_\gamma(f, p_0) \ne 0$. Moreover, if θ is a nondegenerate critical point and $m^0(f''(p_0)) \le |\gamma - m^-(f''(\theta))|$, then f has another critical point $p_1 \ne p_0, \theta$.* □

We resume the proof of the theorem. By (3–17), we have

$$N^{*\prime\prime}(t,\theta) = (N''(t,\theta))^{-1} = (H''(t,\theta)+\mu I_{2n})^{-1}.$$

Thus, for any $u \in L^2$,

$$(\psi''(\theta)u, u) = \int_0^1 \big((\Lambda^{-1}u, u) + (N^{*\prime\prime}(t,\theta)u, u)\big)\,dt$$
$$= \int_0^1 \big((\Lambda^{-1}u, u) + ((B_0(t) + \mu I_{2n})^{-1}u(t), u(t))\big)\,dt.$$

By definition, $m^-(\psi''(\theta)) = i_\mu(B_0)$ and $m^0(\psi''(\theta)) = \nu_\mu(B_0)$. By Propositions 2.10 and 2.13, we have $i(B_1) \notin [i(B_0), i(B_0) + \nu(B_0)]$ if and only if $i_\mu(B_1) \in [i_\mu(B_0), i_\mu(B_0) + \nu_\mu(B_0)]$; and $\nu(B_0) = 0$, $|i(B_0) - i(B_0)| \geq n$ if and only if $\nu_\mu(B_0) = 0$, $|i_\mu(B_1) - i_\mu(B_0)| \geq n$. Hence, by Lemma 3.9, we only need to show

$$H_q(L^2, \psi_{-a}; \mathbb{R}) \cong \delta_{q\gamma}\mathbb{R} \quad \text{for } q = 0, 1, 2, \ldots, \tag{3–26}$$

for $a > 0$ is large enough, where $\gamma := i_\mu(B_1)$. We proceed in three steps.

Step 1. For any $B_1, B_2 \in \mathscr{L}_{2n}^\infty$ with $B_1 < B_2$, $i(B_1) = i(B_2)$ and $\nu(B_2) = 0$, we have $L^2 = E_\mu^-(B_1) \oplus E_\mu^+(B_2)$.

In fact, if $\theta \neq u \in E_\mu^-(B_1)$, then $\psi_{(\mu,B_1)}(u,u) < 0$,

$$\psi_{(\mu,B_2)}(u,u) \leq \psi_{(\mu,B_1)}(u,u) < 0,$$

and $u \notin E_\mu^+(B_2)$. So $E_\mu^-(B_1) \cap E_\mu^+(B_2) = \{\theta\}$. We only need to prove that $L^2 = E_\mu^-(B_1) + E_\mu^+(B_2)$. By definition, $L^2 = E_\mu^-(B_2) \oplus E_\mu^+(B_2)$, and $i_\mu(B_2) = \dim E_\mu^-(B_2) < \infty$. Let $\{e_j\}_{j=1}^\gamma$ be a basis of $E_\mu^-(B_1)$ where $\gamma := i_\mu(B_1)$. We have decompositions $e_j = e_j^- + e_j^+$ with $e_j^- \in E_\mu^-(B_2)$ and $e_j^+ \in E_\mu^+(B_2)$. If $\sum_{j=1}^\gamma \alpha_j e_j^- = 0$, then $\bar{x} := \sum_{j=1}^\gamma \alpha_j e_j = \sum_{j=1}^\gamma \alpha_j e_j^+ \in E_\mu^+(B_2)$, and $\bar{x} \in E_\mu^-(B_1)$, so $\bar{x} = \theta$ and $\alpha_j = 0$, $j = 1, 2, \ldots, \gamma$. Hence $\{e_j^-\}_{j=1}^\gamma$ is linear independent. Since $\dim E_\mu^-(B_2) = i_\mu(B_2) = i_\mu(B_1) = \gamma$, $\{e_j^-\}_{j=1}^\gamma$ is a basis of $E_\mu^-(B_2)$. For any $u \in L^2$ written as $u = u^- + u^+$ with $u^- \in E_\mu^-(B_2)$ and $u^+ \in E_\mu^+(B_2)$, we have $u^- = \sum_{j=1}^\gamma \beta_j e_j^-$. So $u = \sum_{j=1}^\gamma \beta_j e_j + \big(u^+ - \sum_{j=1}^\gamma \beta_j e_j^+\big)$; the first sum lies in $E_\mu^-(B_1)$ and the remainder is in $E_\mu^+(B_2)$.

Step 2. For $\varepsilon > 0$ small enough, set $\mathcal{M}_R := (E_\mu^+(B_2 + \epsilon I_{2n}) \cap B_R) \oplus E_\mu^-(B_1 - \epsilon I_{2n})$. For $R, a > 0$ large enough, then,

$$H_q(L^2, \psi_{-a}; \mathbb{R}) = H_q(\mathcal{M}_R, \mathcal{M}_R \cap \psi_{-a}; \mathbb{R}) \quad \text{for } q = 0, 1, 2, \ldots. \tag{3–27}$$

In fact, from assumption (1) and Proposition 2.13, we have $\nu(B_2 + \varepsilon I_{2n}) = 0$ and $i(B_2 + \varepsilon I_{2n}) = i(B_1 - \varepsilon I_{2n})$. In addition, by Step 1, we have

$$L^2 = E_\mu^-(B_1 - \varepsilon I_{2n}) \oplus E_\mu^+(B_2 + \varepsilon I_{2n}).$$

For every $u = u_1 + u_2 \in L^2$ with $u_1 \in E^-_\mu(B_1 - \epsilon I_{2n})$ and $u_2 \in E^+_\mu(B_2 + \epsilon I_{2n})$, from (3–21) we have

$$\begin{aligned}
&(\psi'(u), u_2 - u_1)\\
&\quad= \int_0^1 \big((\Lambda^{-1}u, u_2 - u_1) + (N^{*\prime}(t, -u), u_1 - u_2)\big)\,dt\\
&\quad= -\int_0^1 \bigg((\Lambda^{-1}u_1, u_1) + \bigg(\int_0^1 N^{*\prime\prime}(t, -\theta u)\,d\theta u_1, u_1\bigg)\bigg)\,dt\\
&\qquad\qquad + \int_0^1 \bigg((\Lambda^{-1}u_2, u_2) - \bigg(\int_0^1 N^{*\prime\prime}(t, -\theta u)\,d\theta u_2, u_2\bigg)\bigg)\,dt\\
&\quad\geq -\int_0^1 \big((\Lambda^{-1}u_1, u_1) + ((\mu I_{2n} + B_1(t) - \varepsilon I_{2n})^{-1}u_1, u_1)\big)\,dt\\
&\qquad\qquad + \int_0^1 \big((\Lambda^{-1}u_2, u_2) + ((\mu I_{2n} + \varepsilon I_{2n} + B_2(t))^{-1}u_2, u_2)\big)\,dt - c_3,
\end{aligned}$$

where $c_3 > 0$ is a constant. Now we bound these last two integrals using the fact that in the subspace $E^-_\mu(B_1 - \varepsilon I_{2n})$ of L^2, the norm $\|\cdot\|_{L^2}$ is equivalent to $\|\cdot\|_1$ defined by

$$\|u\|_1 := \Big(-\int_0^1 [(\Lambda^{-1}u, u) + ((\mu I_{2n} + B_1(t) - \varepsilon I_{2n})^{-1}u, u)]\,dt\Big)^{1/2};$$

in this way we obtain

$$(\psi'(u), u_2 - u_1) \geq c_2\|u_2\|^2_{L^2} + c_1\|u_1\|^2_{L^2} - c_3, \tag{3–28}$$

with $c_1, c_2 > 0$. Thus when R is large enough we have $(\psi'(u), u_2 - u_1) > 1$ for every $u = u_1 + u_2$ with $u_1 \in E^-_\mu(B_1 - \epsilon I_{2n})$, $u_2 \in E^+_\mu(B_2 + \epsilon I_{2n})$ and $\|u_2\|_{L^2} \geq R$, or $\|u_1\|_{L^2} \geq R$. For any $u = u_2 + u_1 \notin \mathcal{M}_R$, let $\sigma(t, u) = e^{-t}u_2 + e^t u_1$, $T_u = \ln\|u_2\| - \ln R$, and

$$\eta(t, u_2 + u_1) = \begin{cases} u_2 + u_1 & \text{if } \|u_2\| \leq R,\\ \sigma(T_u t, u) & \text{if } \|u_2\| > R.\end{cases}$$

Then $\eta : [0, 1] \times L^2 \to L^2$ is continuous and satisfies $\eta(0, \cdot) = \mathrm{id}_{L^2}$, $\eta(1, L^2) \subset \mathcal{M}_R$, $\eta(1, \psi_a) \subset \mathcal{M}_R \cap \psi_a$,

$$\eta(t, \psi_a) \subset \psi_a, \quad \eta(t, \cdot)|_{\mathcal{M}_R} = \mathrm{id}_{\mathcal{M}_R} \quad \text{for all } t \in [0, 1].$$

So $(\mathcal{M}_R, \mathcal{M}_R \cap \psi_a)$ is a deformation retract of (L^2, ψ_a), yielding (3–27).

Step 3. For $R, -a > 0$ are large enough,

$$H_q(\mathcal{M}_R, \mathcal{M}_R \cap \psi_a; \mathbb{R}) \cong \delta_{q\gamma}\mathbb{R} \quad \text{for } q = 0, 1, \ldots.$$

In fact, we have from (3–18) and (3–19) that

$$
\begin{aligned}
&\int_0^1 N^*(t,u(t))\,dt\\
&= \int_0^1 \Big(\int_0^1 \theta d\theta \int_0^1 N^{*\prime\prime}(t,\theta s u(t))\,ds u(t), u(t)\Big)\,dt + \int_0^1 N^*(t,\theta)\,dt\\
&\qquad\qquad \le \int_{|u(t)|\ge kr_1} \Big(\int_0^1 \theta d\theta \int_0^1 N^{*\prime\prime}(t,\theta s u(t))\,ds u(t), u(t)\Big)\,dt + M_k\\
&= \int_{|u(t)|\ge kr_1} \Big(\iint_{|\theta s u(t)|\ge r_1,\theta,s\in[0,1]} \theta N^{*\prime\prime}(t,\theta s u(t))\,ds\,d\theta\, u(t), u(t)\Big)\,dt\\
&\quad + \int_{|u(t)|\ge kr_1} \Big(\iint_{|\theta s u(t)|\le r_1,\theta,s\in[0,1]} N^{*\prime\prime}(t,\theta s u(t))\,ds\,d\theta\, u(t), u(t)\Big)\,dt + M_k\\
&\le \frac{1}{2}\int_0^1 ((\mu I_{2n}+B_1(t))^{-1}u(t),u(t))\,dt\\
&\qquad\qquad\qquad + \mu_1^{-1}\int_0^1 (u(t),u(t))\,dt \iint_{|\theta s|\le \frac{1}{k},\theta,s\in[0,1]} \theta ds\,d\theta + M_k,
\end{aligned}
$$

where M_k is a constant depending only on k. Hence, for every $\varepsilon > 0$ there exists a constant M such that

$$\int_0^1 N^*(t,u(t))\,dt \le \frac{1}{2}\int_0^1 ((\mu I_{2n}+B_1(t)-\varepsilon I_{2n})^{-1}u(t),u(t))\,dt + M$$

for all $u \in L^2$. Together with (3–20), this yields, for any $u = u_1 + u_2$ with $u_1 \in E^-_\mu(B_1 - \epsilon I_{2n})$ and $u_2 \in E^+_\mu(B_2 + \epsilon I_{2n}) \cap B_R$, the bound

$$\psi(u) \le -c_1\|u_1\|^2_{L^2} + c_4\|u_1\|_{L^2} + c_5,$$

where $c_4, c_5 > 0$, and c_1 is the constant in (3–28). Hence $\psi(u) \to -\infty$ if and only if $\|u_1\| \to +\infty$ uniformly in $u_2 \in E^+_\mu(B_2 + \epsilon I_{2n}) \cap B_R$. Thus there exist $T > 0$, $a_1 < a_2 < -T$, and $R_0 > R_1 > R_2 > 0$ such that

$$
\begin{aligned}
&(E^+_\mu(B_2+\epsilon I_{2n})\cap B_{R_0}) \oplus (E^-_\mu(B_1-\epsilon I_{2n})\setminus B_{R_2}) \subset \psi_{a_1}\cap \mathcal{M}_{R_0}\\
&\qquad\qquad \subset (E^+_\mu(B_2+\epsilon I_{2n})\cap B_{R_0}) \oplus (E^-_\mu(B_1-\epsilon I_{2n})\setminus B_{R_1}) \subset \psi_{a_2}\cap \mathcal{M}_{R_0}.
\end{aligned}
$$

For any $u \in \mathcal{M}_{R_0} \cap (\psi_{a_2}\setminus\psi_{a_1})$, since $\sigma(t,u) = e^{-t}u_2 + e^t u_1$, the function $\psi(\sigma(t,x))$ is continuous in t and satisfies $\psi(\sigma(0,x)) = \psi(u) > a_1$ and $\psi(\sigma(t,u)) \to -\infty$ as $t \to +\infty$. Thus there exists a unique $t = T_1(u)$ such that $\psi(\sigma(t,u)) = a_1$. Since

$$
\begin{aligned}
\frac{d}{dt}\psi(\sigma(t,u)) &= \langle d\psi(\sigma(t,u)), \sigma'(t,u)\rangle\\
&= \langle d\psi(e^{-t}u_2 + e^t u_1), -e^{-t}u_2 + e^t u_1\rangle \le -1
\end{aligned}
$$

as $t > 0$, by the implicit function theorem, $t = T_1(u)$ is continuous. Define

$$\eta_1(t, u) = u, u \in \psi_{a_1} \cap \mathcal{M}_{R_0} = \sigma(T_1(u)t, u), u \in \mathcal{M} \cap (\psi_{a_2} \setminus \psi_{a_1});$$

then $\eta_1 : [0, 1] \times \psi_{a_2} \cap \mathcal{M}_{R_0} \to \psi_{a_2} \cap \mathcal{M}_{R_0}$ is a deformation from $\psi_{a_2} \cap \mathcal{M}_{R_0}$ to $\psi_{a_1} \cap \mathcal{M}_{R_0}$. Set $\tau_1 := \eta_1(1, \cdot) : \mathcal{M}_{R_0} \cap \psi_{a_2} \to \mathcal{M}_{R_0} \cap \psi_{a_1}$, and define

$$\tau_2(u) = \begin{cases} u & \text{if } \|u_1\| \geq R_1, \\ u_2 + (u_1/\|u_1\|)R_1 & \text{if } \|u_1\| < R_1. \end{cases}$$

Then $\tau = \tau_2 \circ \tau_1$ is a strong deformation retract:

$$\tau : \mathcal{M}_{R_0} \cap \psi_{a_2} \to (E_\mu^+(B_2 + \epsilon I_{2n}) \cap B_{R_0}) \oplus (E_\mu^-(B_1 - \epsilon I_{2n}) \setminus \text{int} B_{R_1}),$$

where int B_{R_1} is the interior of B_{R_1}. Hence, for $q = 0, 1, 2, \ldots,$

$$\begin{aligned} H(\mathcal{M}_{R_0}, \mathcal{M}_{R_0} \cap \psi_{a_2}; \mathbb{R}) &\cong H_q\big((E_\mu^+(B_2 + \epsilon I_{2n}) \cap B_{R_0}) \oplus E_\mu^-(B_1 - \epsilon I_{2n}), \\ &\qquad (E_\mu^+(B_2 + \epsilon I_{2n}) \cap B_{R_0}) \oplus (E_\mu^-(B_1 - \epsilon I_{2n}) \setminus \text{int } B_{R_1}); \mathbb{R}\big) \\ &\cong H_q(E_\mu^-(B_1 - \epsilon I_{2n}) \cap B_{R_1}, \partial(E_\mu^-(B_1 - \epsilon I_{2n}) \cap B_{R_1}); \mathbb{R}) \\ &\cong \delta_{q\gamma}\mathbb{R}. \end{aligned}$$ □

Remark. The method of the proof of (3–26) comes from [Chang 1993], but we have modified it to suit our case.

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