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REPRESENTATIONS OF LOCALLY COMPACT GROUPS ON QSL$_p$-SPACES AND A p-ANALOG OF THE FOURIER–STIELTJES ALGEBRA

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For a locally compact group G and $p \in (1, \infty)$, we define $B_p(G)$ to be the space of all coefficient functions of isometric representations of G on quotients of subspaces of L_p spaces. For $p = 2$, this is the usual Fourier–Stieltjes algebra. We show that $B_p(G)$ is a commutative Banach algebra that contractively (isometrically, if G is amenable) contains the Figà-Talamanca–Herz algebra $A_p(G)$. If $2 \le q \le p$ or $p \le q \le 2$, we have a contractive inclusion $B_q(G) \subset B_p(G)$. We also show that $B_p(G)$ embeds contractively into the multiplier algebra of $A_p(G)$ and is a dual space. For amenable G, this multiplier algebra and $B_p(G)$ are isometrically isomorphic.

Introduction

P. Eymard [1964] introduced the *Fourier algebra* $A(G)$ of a locally compact group G. If G is abelian with dual group Γ, the Fourier transform yields an isometric isomorphism of $L_1(\Gamma)$ and $A(G)$: this motivates (and justifies) the name.

For any $p \in (1, \infty)$, as usual, we define $p' \in (1, \infty)$ to be such that $1/p + 1/p' = 1$; we say that p' is dual to p. The *Figà-Talamanca–Herz algebra* $A_p(G)$ is defined as the collection of those functions $f : G \to \mathbb{C}$ such that there are sequences $(\xi_n)_{n=1}^\infty$ in $L_{p'}(G)$ and $(\phi_n)_{n=1}^\infty$ in $L_p(G)$ such that

$$f(x) = \sum_{n=1}^{\infty} \langle \lambda_{p'}(x)\xi_n, \phi_n \rangle \qquad (x \in G), \tag{0–1}$$

where $\lambda_{p'}$ denotes the regular left representation of G on $L_{p'}(G)$, and

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty. \tag{0–2}$$

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The norm of $f \in A_p(G)$ is the infimum over all expressions of the form (0–2) satisfying (0–1). These Banach algebras were first considered by C. Herz [1971; 1973]; their study has been an active area of research ever since (see [Cowling 1979; Forrest 1993; 1994; Lambert et al. 2004; Miao 1996], and many more). For $p = 2$, the algebra $A_p(G)$ is nothing but the Fourier algebra $A(G)$.

Another algebra introduced in [Eymard 1964] is the *Fourier–Stieltjes algebra* $B(G)$. For abelian G, it is isometrically isomorphic to $M(\Gamma)$ via the Fourier–Stieltjes transform. It consists of all coefficient functions of unitary representations of G on some Hilbert space and contains $A(G)$ as a closed ideal.

Is there, for general $p \in (1, \infty)$, an analog of $B(G)$ in a p-setting that relates to $A_p(G)$ as does $B(G)$ to $A(G)$?

In the literature (see [Cowling 1979; Forrest 1994; Miao 1996; Pier 1984], for instance), sometimes an algebra $B_p(G)$ is considered: it is defined as the multiplier algebra of $A_p(G)$. If $p = 2$ and if G is amenable, we do have $B(G) = B_p(G)$; for nonamenable G, however, $B(G) \subsetneq B_2(G)$ holds. Hence, the value of $B_p(G)$ as the appropriate replacement for $B(G)$ when dealing with $A_p(G)$ is *a priori* limited to the amenable case.

In the present paper, we pursue a novel approach. We define $B_p(G)$ to consist of the coefficient functions of all representations of G on quotients of subspaces of $L_{p'}$-spaces, so-called $\mathrm{QSL}_{p'}$-spaces. This class of spaces is identical with the p'-spaces considered in [Herz 1973] and turns out to be appropriate for our purpose (such representations were considered only recently, in a completely different context, in [Jaming and Moran 2000]).

We list some properties of our $B_p(G)$:

- Under pointwise multiplication, $B_p(G)$ is a commutative Banach algebra with identity.
- $A_p(G)$ is an ideal of $B_p(G)$, into which it contractively embeds (isometrically if G is amenable).
- If $2 \le q \le p$ or $p \le q \le 2$, we have a contractive inclusion of $B_q(G)$ in $B_p(G)$.
- $B_p(G)$ is a dual Banach space.
- $B_p(G)$ embeds contractively into the multiplier algebra of $A_p(G)$ and is isometrically isomorphic to it if G is amenable.

This list shows that our $B_p(G)$ relates to $A_p(G)$ in a fashion similar to how $B(G)$ relates to $A(G)$ and therefore may be the right substitute for $B(G)$ when working with Figà-Talamanca–Herz algebras.

The main challenge when defining $B_p(G)$ and trying to establish its properties is that the powerful methods from C^*- and von Neumann algebras are no longer at one's disposal for $p \neq 2$, so that one has to look for appropriate substitutes.

1. Group representations and QSL$_p$-spaces

We begin with defining what we mean by a representation of a locally compact group on a Banach space:

Definition 1.1. A *representation* of a locally compact group G (on a Banach space) is a pair (π, E) where E is a Banach space and π is a group homomorphism from G into the invertible isometries on E which is continuous with respect to the given topology on G and the strong operator topology on $\mathcal{B}(E)$.

Remarks. 1. Our definition is more restrictive than the usual definition of a representation, which does not require the range of π to consist of isometries. Since we will not encounter any other representations, however, we feel justified to use the general term "representation" in the sense just defined.

2. Any representation (π, E) of a locally compact group G induces a representation of the group algebra $L^1(G)$ on E, i.e. a contractive algebra homomorphism $L_1(G)$ to $\mathcal{B}(E)$ — which we shall denote likewise by π — through

$$\pi(f) := \int_G f(x)\pi(x)\,dx \qquad (f \in L^1(G)), \tag{1–1}$$

where the integral (1–1) converges with respect to the strong operator topology.

3. Instead of requiring π to be continuous with respect to the strong operator topology on $\mathcal{B}(E)$, we could have demanded that π be continuous with respect to the weak operator topology on $\mathcal{B}(E)$: both definitions are equivalent by [de Leeuw and Glicksberg 1965].

Definition 1.2. Let G be a locally compact group, and let (π, E) and (ρ, F) be representations of G. Then:

(a) (π, E) and (ρ, F) are said to be *equivalent* if there is an invertible isometry $V : E \to F$ such that

$$V\pi(x)V^{-1} = \rho(x) \qquad (x \in G).$$

(b) (ρ, F) is called a subrepresentation of (π, E) if F is a closed subspace of E such that

$$\rho(x) = \pi(x)|_F \qquad (x \in G).$$

(c) (ρ, F) is said to be *contained* in (π, E) — in symbols: $(\rho, F) \subset (\pi, E)$ — if (ρ, F) is equivalent to a subrepresentation of (π, E).

Throughout, we shall often not tell a particular representation apart from its equivalence class. This should, however, not be a source of confusions.

In this paper, we are interested in representations of locally compact groups on rather particular Banach spaces:

Definition 1.3. Let $p \in (1, \infty)$.

(a) A Banach space is called an L_p-space if it is of the form $L_p(X)$ for some measure space X.

(b) A Banach space is called a QSL_p-space if it is isometrically isomorphic to a quotient of a subspace of an L_p-space.

Remarks. 1. Equivalently, a Banach space is a QSL_p-space if and only if it is a subspace of a quotient of an L_p-space.

2. Trivially, the class of QSL_p-spaces is closed under taking subspaces and quotients.

3. If $(E_\alpha)_\alpha$ is a family of QSL_p-spaces, its ℓ_p-direct sum $\ell_p\text{-}\bigoplus_\alpha E_\alpha$ is again a QSL_p-space.

4. If E is a QSL_p-space and if $p' \in (1, \infty)$ is dual to p, the dual space E^* is an $\mathrm{QSL}_{p'}$-space. In particular, every QSL_p-space is reflexive.

5. By [Kwapień 1972, §4, Theorem 2], the QSL_p-spaces are precisely the p-spaces in the sense of [Herz 1971], i.e. those Banach spaces E such that for any two measure spaces X and Y the amplification map

$$\mathcal{B}(L_p(X), L_p(Y)) \to \mathcal{B}(L_p(X, E), L_p(Y, E)), \quad T \mapsto T \otimes \mathrm{id}_E$$

is an isometry. In particular, an L_q-space is a QSL_p-space if and only if $2 \le q \le p$ or $p \le q \le 2$. Consequently, if $2 \le q \le p$ or $p \le q \le 2$, then every QSL_q-space is a QSL_p-space.

6. All $\mathcal{L}_{p,1}$-spaces in the sense of [Lindenstrauss and Rosenthal 1969] — and, more generally, all $\mathfrak{L}^g_{p,1}$-spaces in the sense of [Defant and Floret 1993] — are QSL_p-spaces.

7. Since the class of L_p-space is stable under forming ultrapowers ([Heinrich 1980]), so is the class of QSL_p-spaces (this immediately yields that QSL_p-spaces are not only reflexive, but actually superreflexive). In the case where $X = Y = \mathbb{C}$, the QSL_p-spaces are therefore precisely those that occur in [Le Merdy 1996, Theorem 4.1] and play the rôle played by Hilbert spaces in Ruan's representation theorem for operator spaces ([Effros and Ruan 2000, Theorem 2.3.5]).

2. The linear space $B_p(G)$

We shall not so much be concerned with representations themselves, but rather with certain functions associated with them:

Definition 2.1. Let G be a locally compact group, and let (π, E) be a representation of G. A *coefficient function* of (π, E) is a function $f : G \to \mathbb{C}$ of the form

$$f(x) = \langle \pi(x)\xi, \phi \rangle \qquad (x \in G), \tag{2–1}$$

where $\xi \in E$ and $\phi \in E^*$.

Remark. It is clear that every coefficient function of the form (2–1) must be both bounded — by $\|\xi\|\|\phi\|$ — and continuous.

For any locally compact group G and $p \in (1, \infty)$, we denote by $\mathrm{Rep}_p(G)$ the collection of all (equivalence classes) of representations of G on a QSL_p-space.

Examples. 1. The *left regular representation* $(\lambda_p, L_p(G))$ of G with

$$\lambda_p(x)\xi(y) := \xi(x^{-1}y) \qquad (x, y \in G,\ \xi \in L_p(G))$$

belongs to $\mathrm{Rep}_p(G)$.

2. For any QSL_p-space E, the *trivial representation* (id_E, E) lies in $\mathrm{Rep}_p(G)$.
3. For $2 \le q \le p$ or $p \le q \le 2$, we have $\mathrm{Rep}_q(G) \subset \mathrm{Rep}_p(G)$, so that, in particular, every unitary representation of G on a Hilbert space belongs to $\mathrm{Rep}_p(G)$.

We can now define the main object of study in this article:

Definition 2.2. Let G be a locally compact and let $p, p' \in (1, \infty)$ be dual to each other. Let

$$B_p(G) := \left\{ f\colon G \to \mathbb{C} : f \text{ is a coefficient of some } (\pi, E) \in \mathrm{Rep}_{p'}(G) \right\}.$$

Remarks. 1. In the literature (see, for instance, [Pier 1984]), the symbol $B_p(G)$ is usually used to denote the *multiplier algebra* of $A_p(G)$, i.e. the set of those continuous functions f on G such that $f A_p(G) \subset A_p(G)$.

2. Since subspaces and quotients of Hilbert spaces are again Hilbert spaces, $B_2(G)$ is just the usual Fourier–Stieltjes algebra $B(G)$ introduced in [Eymard 1964]. For amenable G, this is consistent with the usage in [Pier 1984]. In the nonamenable case, however, $B_2(G) = B(G)$ as defined in Definition 2.2 and $B_2(G)$ in the sense of [Pier 1984] denote different objects.

We conclude this section with proving a few, very basic properties of $B_p(G)$:

Lemma 2.3. *Let G be a locally compact group, let $p, p' \in (1, \infty)$ be dual to each other, and let $f : G \to \mathbb{C}$ be a function such that the following holds: There are sequences $((\pi_n, E_n))_{n=1}^\infty$, $(\xi_n)_{n=1}^\infty$, and $(\phi_n)_{n=1}^\infty$ with $(\pi_n, E_n) \in \mathrm{Rep}_{p'}(G)$, $\xi_n \in E_n$, and $\phi_n \in E_n^*$ for $n \in \mathbb{N}$ such that*

$$\sum_{n=1}^\infty \|\xi_n\|\|\phi_n\| < \infty$$

and

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle \qquad (x \in G).$$

Then f lies in $B_p(G)$.

Proof. Without loss of generality, we may suppose that

$$\sum_{n=1}^{\infty} \|\xi_n\|^{p'} < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \|\phi_n\|^{p} < \infty.$$

Define $(\pi, E) \in \mathrm{Rep}_{p'}(G)$ by letting $E := \ell_{p'}\text{-}\bigoplus_{n=1}^{\infty} E_n$ and, for $\eta = (\eta_1, \eta_2, \ldots)$ in E,

$$\pi(x)\eta := (\pi_1(x)\eta_1, \pi_2(x), \eta_2, \ldots) \qquad (x \in G).$$

It follows that $\xi := (\xi_1, \xi_2, \ldots) \in E$, that $\phi := (\phi_1, \phi_2, \ldots) \in E^*$, and that f is a coefficient function of (π, E) — therefore belonging to $B_p(G)$. □

For any topological space Ω, we use $\mathscr{C}_b(\Omega)$ to denote the bounded continuous functions on it.

Proposition 2.4. *Let G be a locally compact group, and let $p \in (1, \infty)$. Then $B_p(G)$ is a linear subspace of $\mathscr{C}_b(G)$ containing $A_p(G)$. Moreover, if $2 \le q \le p$ or $p \le q \le 2$, we have $B_q(G) \subset B_p(G)$.*

Proof. We have already seen that $B_p(G) \subset \mathscr{C}_b(G)$.

Let $p' \in (1, \infty)$ be dual to p, and let $f_1, f_2 \in B_p(G)$. By the definition of $B_p(G)$, there are $(\pi_1, E_1), (\pi_2, E_2) \in \mathrm{Rep}_{p'}(G)$ such that f_j is a coefficient function of (π_j, E_j) for $j = 1, 2$. It is clear that the pointwise sum $f_1 + f_2$ is then of the form considered in Lemma 2.3 (take $\xi_3 = \xi_4 = \cdots = 0$) and thus contained in $B_p(G)$.

To see that $A_p(G) \subset B_p(G)$, apply Lemma 2.3 again with

$$(\pi_n, E_n) = (\lambda_{p'}, L_{p'}(G)) \quad \text{for } n \in \mathbb{N}.$$

Suppose $2 \le q \le p$ or $p \le q \le 2$, and let $q' \in (1, \infty)$ be dual to q. Since every $\mathrm{QSL}_{q'}$ space is a is a $\mathrm{QSL}_{p'}$-space, the inclusion $B_q(G) \subset B_p(G)$ holds. □

3. Tensor products of QSL_p-spaces

Let G be a locally compact group. In $B(G) = B_2(G)$, the pointwise product of functions corresponds to the tensor product of representations, which, in turn, relies on the existence of the Hilbert space tensor product. In order to turn $B_p(G)$ into an algebra, we will therefore equip, in this section, the algebraic tensor product of two $\mathrm{QSL}_{p'}$-spaces (where p' is dual to p), with a suitable norm.

The main result is the following:

Theorem 3.1. *Let E and F be QSL_p-spaces. Then there is a norm $\|\cdot\|_p$ on the algebraic tensor product $E \otimes F$ such that*:

(i) *$\|\cdot\|_p$ dominates the injective norm*;

(ii) *$\|\cdot\|_p$ is a cross norm*;

(iii) *the completion $E\tilde{\otimes}_p F$ of $E \otimes F$ with respect to $\|\cdot\|_p$ is a QSL_p-space.*

Moreover, if G is a locally compact group with $(\pi, E), (\rho, F) \in \mathrm{Rep}_p(G)$, then $(\pi \otimes \rho, E\tilde{\otimes}_p F) \in \mathrm{Rep}_p(G)$ is well defined through

$$(\pi(x) \otimes \rho(x))(\xi \otimes \eta) := \pi(x)\xi \otimes \rho(x)\eta \qquad (x \in G,\ \xi \in E,\ \eta \in F).$$

Proof. Let X be a measure space, let E_1 and F_1 be closed subspaces of $L_p(X)$, and let E_2 and F_2 be closed subspaces of E_1 and F_1, respectively, such that $E = E_1/E_2$ and $F = F_1/F_2$.

We may embed the algebraic tensor product $L_p(X) \otimes L_p(X)$ into the vector valued L_p-space $L_p(X, L_p(X))$ and thus equip it with a norm, denoted by $|||\cdot|||_p$, which dominates the injective norm on $L_p(X) \otimes L_p(X)$ [Defant and Floret 1993, 7.1, Proposition]. Of course, we may restrict $|||\cdot|||_p$ to $E_1 \otimes E_2$. We denote the (uncompleted) injective tensor product by $\otimes_\epsilon$. Since $\otimes_\epsilon$ respects passage to subspaces, we see that the identity on $E_1 \otimes F_1$ induces a contraction from $(E_1 \otimes E_2, |||\cdot|||_p)$ to $E_1 \otimes_\epsilon F_1$. Let $\pi_E : E_1 \to E$ and $\pi_F : F_1 \to F$ denote the canonical quotient maps. The mapping property of the injective tensor product then yields that

$$\pi_E \otimes \pi_F : (E_1 \otimes F_1, |||\cdot|||_p) \to E_1 \otimes_\epsilon F_1 \to E \otimes_\epsilon F$$

is a surjective contraction, so that, in particular, $\ker(\pi_E \otimes \pi_F)$ is closed in

$$(E_1 \otimes F_1, |||\cdot|||_p).$$

Let $\|\cdot\|_p$ denote the induced quotient norm on $E \otimes F = (E_1 \otimes F_1)/\ker(\pi_E \otimes \pi_F)$. It is immediate that $\|\cdot\|_p$ dominates the injective tensor norm on $E \otimes F$, so that (i) holds. Moreover, since $|||\cdot|||_p$ is a cross norm on $E_1 \otimes E_2$, it is clear that $\|\cdot\|_p$ is at least subcross on $E \otimes F$. Since $\|\cdot\|_p$, however, dominates the injective norm — which is a cross norm — on $E \otimes F$, we conclude that $\|\cdot\|_p$ is indeed a cross norm on $E \otimes F$. This proves (ii).

For notational convenience, we write $L_p(X) \otimes_p L_p(X) := (L_p(X) \otimes L_p(X), |||\cdot|||_p)$, and let $E \otimes_p F := (E \otimes F, \|\cdot\|_p)$. Let Y and Z be any measure spaces. In view of [Defant and Floret 1993, 7.2 and 7.3], it is clear that the amplification map

$$\mathcal{B}(L_p(Y), L_p(Z)) \to \mathcal{B}\big(L_p(Y, L_p(X) \otimes_p L_p(X)), L_p(Z, L_p(X) \otimes_p L_p(X))\big),$$
$$T \mapsto T \otimes \mathrm{id}$$

is an isometry, and from [Defant and Floret 1993, 7.4, Proposition] we conclude that the same is true for

$$\mathcal{B}(L_p(Y), L_p(Z)) \to \mathcal{B}(L_p(Y, E \otimes_p F), L_p(Z, E \otimes_p F)), \quad T \mapsto T \otimes \mathrm{id}. \tag{3–1}$$

However, if we replace $E \otimes_p F$ in (3–1) by its completion $E \tilde{\otimes}_p F$, the map obviously remains an isometry. Hence, $E \tilde{\otimes}_p F$ is a p-space in the sense of [Herz 1971] and thus a QSL_p-space by [Kwapień 1972, §4, Theorem 2].

For the moreover part of the theorem, it is sufficient to show that, for $S \in \mathcal{B}(E)$ and $T \in \mathcal{B}(F)$, their tensor product $S \otimes T$ is continuous on $E \otimes_p F$ and has operator norm at most $\|S\| \|T\|$. We first treat the case where $S = \mathrm{id}_E$. Let $E_1 \otimes_p F$ stand for $E_1 \otimes F$ equipped with the norm obtained by factoring $E_1 \otimes F_2$ out of $(E_1 \otimes F_1, |||\cdot|||_p)$. From [Defant and Floret 1993, 7.3], it follows that $\mathrm{id}_{E_1} \otimes T \in \mathcal{B}(E_1 \otimes F)$ and has operator norm such that

$$\|\mathrm{id}_{E_1} \otimes T\|_{\mathcal{B}(E_1 \otimes_p F)} = \|T\|_{\mathcal{B}(F)}.$$

It is easy to see that $E \otimes F$ is, in fact, the quotient space of $E_1 \otimes_p F$ modulo $E_2 \otimes F$, and it follows that

$$\|\mathrm{id}_E \otimes T\|_{\mathcal{B}(E \otimes_p F)} \leq \|\mathrm{id}_{E_1} \otimes_p T\|_{\mathcal{B}(E_1 \otimes F)} = \|T\|_{\mathcal{B}(F)}.$$

By symmetry, we obtain that

$$\|S \otimes \mathrm{id}_F\|_{\mathcal{B}(E \otimes_p F)} \leq \|S\|_{\mathcal{B}(E)}$$

as well. Consequently,

$$\|S \otimes T\|_{\mathcal{B}(E \otimes_p F)} \leq \|S \otimes \mathrm{id}_F\|_{\mathcal{B}(E \otimes_p F)} \|\mathrm{id}_E \otimes T\|_{\mathcal{B}(E \otimes_p F)} \leq \|S\|_{\mathcal{B}(E)} \|T\|_{\mathcal{B}(F)}. \quad \square$$

Remarks.

1. For a measure space X and for a QSL_p-space E, the tensor product $L_p(X) \tilde{\otimes}_p E$ constructed in the proof of Theorem 3.1 is nothing but the vector valued L_p-space $L_p(X, E)$.
2. We suspect, but have been unable to prove, that $\|\cdot\|_p$ is the Chevet–Saphar tensor norm d_p on $E \otimes F$ (see [Defant and Floret 1993, 12.7]). This is indeed the case when both E and F are $\mathfrak{L}^g_{p,1}$-spaces; see [Jaming and Moran 2000].

We conclude this section with two simple corollaries of Theorem 3.1:

Corollary 3.2. *Let G be a locally compact group, let $p \in (1, \infty)$, and let $f, g : G \to \mathbb{C}$ be the coefficient functions of (π, E) and (ρ, F) in* $\mathrm{Rep}_p(G)$, *respectively*:

$$f(x) = \langle \pi(x)\xi, \phi \rangle \quad \text{and} \quad g(x) = \langle \rho(x)\eta, \psi \rangle \qquad (x \in G)$$

where $\xi \in E$, $\phi \in E^*$, $\eta \in F$, and $\psi \in F^*$. Then $\phi \otimes \psi : E \otimes F \to \mathbb{C}$ is continuous with respect to $\|\cdot\|_p$ with norm at most $\|\phi\| \|\psi\|$, so that the pointwise product of

f and g is a coefficient function of $(\pi \otimes \rho, E\tilde{\otimes}_p F)$, namely

$$f(x)g(x) = \langle(\pi(x) \otimes \rho(x))(\xi \otimes \eta), \phi \otimes \psi\rangle \qquad (x \in G).$$

Proof. In view of the definition of $(\pi \otimes \rho, E\tilde{\otimes}_p F)$, only the claim about $\phi \otimes \psi$ needs some consideration: it is, however, an immediate consequence of parts (i) and (ii) of Theorem 3.1. □

Corollary 3.3. *Let G be a locally compact group, and let $p \in (1, \infty)$. Then $B_p(G)$ is a unital subalgebra of $\mathcal{C}_b(G)$.*

Proof. By Proposition 2.4, $B_p(G)$ is a linear subspace of $\mathcal{C}_b(G)$, and by Corollary 3.2, it is a subalgebra. The constant function 1 is a coefficient function of any trivial representation of G on an QSL$_p$-space. □

4. The Banach algebra $B_p(G)$

Our next goal is to equip the algebra $B_p(G)$ with a norm turning it into a Banach algebra.

Definition 4.1. Let G be a locally compact group, and let (π, E) be a representation of G. Then (π, E) is called *cyclic* if there is $x \in E$ such that $\pi(L_1(G))x$ is dense in E. For $p \in (1, \infty)$, we let

$$\mathrm{Cyc}_p(G) := \{(\pi, E) \in \mathrm{Rep}_p(G) : (\pi, E) \text{ is cyclic}\}.$$

Remark. Let $f \in B_p(G)$ be a coefficient function of $(\pi, E) \in \mathrm{Rep}_p(G)$, i.e.

$$f(x) = \langle f(x)\xi, \phi\rangle \qquad (x \in G)$$

with $\xi \in E$ and $\phi \in E^*$. Let $F := \overline{\pi(L_1(G))\xi}$, and define $\rho : G \to \mathcal{B}(F)$ by restriction of $\pi(x)$ to F for each $x \in G$. Then (ρ, F) is cyclic with f as a coefficient function.

Definition 4.2. Let G be a locally compact group, let $p, p' \in (1, \infty)$ be dual to each other, and let $f \in B_p(G)$. We define $\|f\|_{B_p(G)}$ as the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in \mathrm{Cyc}_{p'}(E)$ with $\xi_n \in E_n$ and $\phi_n \in E_n^*$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty \qquad \text{and} \qquad f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n\rangle \quad (x \in G).$$

Remarks. 1. In view of the remark after Definition 4.1, it is clear that $\|\cdot\|_{B_p(G)}$ is well defined, and it is easily checked that $\|\cdot\|_{B_p(G)}$ is indeed a norm on $B_p(G)$.

2. One might think that it would be more appropriate to define $\|\cdot\|_{B_p(G)}$ in such a way that the infimum is taken over general $(\pi_n, E_n) \in \mathrm{Rep}_{p'}(G)$ instead of only in $\mathrm{Cyc}_{p'}(G)$. The problem here, however, is that QSL_p-spaces can be of arbitrarily large cardinality, so that $\mathrm{Rep}_{p'}(G)$ is not a set, but only a class. Since, for $(\pi, E) \in \mathrm{Cyc}_{p'}(G)$, the space E has a cardinality not larger than $|L_1(G)|^{\aleph_0}$, it follows that $\mathrm{Cyc}_{p'}(G)$ — unlike all of $\mathrm{Rep}_{p'}(G)$ — is indeed a set, so that it makes sense to take an infimum over it.

In view of the last one of the two preceding remarks, the following lemma is comforting:

Lemma 4.3. *Let G be a locally compact group, let $p, p' \in (1, \infty)$ be dual to each other, and let $((\pi_n, E_n))_{n=1}^{\infty}$ be a sequence in $\mathrm{Rep}_{p'}(G)$ such that, with $\xi_n \in E_n$ and $\phi_n \in E_n^*$ for $n \in \mathbb{N}$, we have $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty$. Then, for each $n \in \mathbb{N}$, there are $(\rho_n, F_n) \in \mathrm{Cyc}_{p'}(G)$ with $(\rho_n, F_n) \subset (\pi_n, E_n)$, $\eta_n \in F_n$, and $\psi_n \in E^*$, such that*

$$\sum_{n=1}^{\infty} \|\eta_n\| \|\psi_n\| \leq \sum_{n=1}^{\infty} \|\xi_n\| \|\phi\|$$

and

$$\sum_{n=1}^{\infty} \langle \rho_n(x)\eta_n, \psi_n \rangle = \sum_{n=1}^{\infty} \langle \rho_n(x)\xi_n, \phi_n \rangle \quad (x \in G)$$

Proof. We proceed as in the remark immediately following Definition 4.1: For $n \in \mathbb{N}$, let $F_n := \overline{\pi_n(L_1(G))\xi_n}$, define ρ_n through restriction, let $\eta_n := \xi_n$, and let ψ_n be the restriction of ϕ_n to F_n. □

Lemma 4.4. *Let G be a locally compact group, let $p, p' \in (1, \infty)$ be dual to each other, and let $f \in A_p(G)$. Then $\|f\|_{A_p(G)}$ is the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in \mathrm{Cyc}_{p'}(E)$ contained in $(\lambda_{p'}, L_{p'}(G))$ with $\xi_n \in E_n$ and $\phi_n \in E_n^*$ such that*

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty \qquad \textit{and} \qquad f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle \quad (x \in G).$$

Proof. From Lemma 4.3, it follows that the infimum in the statement of Lemma 4.4 is less or equal to $\|f\|_{A_p(G)}$. Let this infimum be denoted by C_f. Let $\epsilon > 0$, and choose a sequence $((\pi_n, E_n))_{n=1}^{\infty}$ of cyclic subrepresentations of $(\lambda_{p'}, L_{p'}(G))$ and, for each $n \in \mathbb{N}$, $\xi_n \in E_n$ and $\phi_n \in E_n^*$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < C_f + \epsilon \qquad \text{and} \qquad f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle \quad (x \in G).$$

For each $n \in \mathbb{N}$, use the Hahn–Banach theorem to extend $\phi_n \in E_n^*$ to $\psi_n \in L_{p'}(G)^* = L_p(G)$ with $\|\psi_n\| = \|\phi_n\|$. It follows that

$$\|f\|_{A_p(G)} \le \sum_{n=1}^{\infty} \|\xi_n\| \|\psi_n\| = \sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < C_f + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\|f\|_{A_p(G)} \le C_f$. □

Definition 4.5. Let G be a locally compact group, and let $p \in (1, \infty)$. Then $(\pi, E) \in \mathrm{Rep}_p(G)$ is called *p-universal* if $(\rho, F) \subset (\pi, E)$ for all $(\rho, F) \in \mathrm{Cyc}_p(G)$.

Example. Let G be a locally compact group, and let $p \in (1, \infty)$. Since $\mathrm{Cyc}_p(G)$ is a set, we can form the ℓ_p-direct sum of all $(\rho, F) \in \mathrm{Cyc}_p(G)$. This representation is then obviously p-universal.

Lemma 4.6. *Let G be a locally compact group, let $p, p' \in (1, \infty)$ be dual to each other, and let $(\pi, E) \in \mathrm{Rep}_{p'}(G)$ be p'-universal. Then, for each $f \in B_p(G)$, the norm $\|f\|_{B_p(G)}$ is the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\|$ with $\xi_n \in E$ and $\phi_n \in E^*$ for each $n \in \mathbb{N}$ such that*

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty \qquad \textit{and} \qquad f(x) = \sum_{n=1}^{\infty} \langle \pi(x)\xi_n, \phi_n \rangle \quad (x \in G).$$

Proof. Obvious in the light of Definition 4.5. □

In the end, we obtain:

Theorem 4.7. *Let G be a locally compact group, let $p \in (1, \infty)$, and let $B_p(G)$ be equipped with $\|\cdot\|_{B_p(G)}$. Then:*

(i) *$B_p(G)$ is a commutative Banach algebra.*

(ii) *The inclusion $A_p(G) \subset B_p(G)$ is a contraction.*

(iii) *For $2 \le q \le p$ or $p \le q \le 2$, the inclusion $B_q(G) \subset B_p(G)$ is a contraction.*

Proof. Let $p' \in (1, \infty)$ be dual to p, and let $(\pi, E) \in \mathrm{Rep}_{p'}(G)$ be p'-universal. It follows that $B_p(G)$ is a quotient space of the complete projective tensor product $E \tilde{\otimes}_\pi E^*$ and thus complete. By Corollary 3.3, $B_p(G)$ is an algebra, so that all that remains to prove (i) is to show that $\|\cdot\|_{B_p(G)}$ is submultiplicative.

Let $f, g \in B_p(G)$, and let $\epsilon > 0$. Let $((\pi_n, E_n))_{n=1}^{\infty}$ and $((\rho_n, F_n))_{n=1}^{\infty}$ be sequences in $\mathrm{Cyc}_{p'}(G)$ and, for $n \in \mathbb{N}$, let $\xi_n \in E_n$, $\phi_n \in E_n^*$, $\eta_n \in F_n$, and $\psi_n \in F_n^*$ such that

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \langle \rho_n(x)\eta_n, \psi_n \rangle \qquad (x \in G)$$

and

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| \leq \|f\|_{B_p(G)} + \epsilon \qquad \text{and} \qquad \sum_{n=1}^{\infty} \|\eta_n\| \|\psi_n\| \leq \|g\|_{B_p(G)} + \epsilon.$$

By the "moreover" part of Theorem 3.1, we see that

$$(\pi_n \otimes \rho_m, E_n \tilde{\otimes}_p F_m) \in \mathrm{Rep}_{p'}(G)$$

for $n, m \in \mathbb{N}$, and Corollary 3.2 yields

$$f(x)g(x) = \sum_{n,m=1}^{\infty} \langle (\pi_n(x) \otimes \rho_m(x))(\xi_n \otimes \eta_m), \phi_n \otimes \psi_m \rangle \qquad (x \in G)$$

and that

$$\begin{aligned}\sum_{n,m=1}^{\infty} \|\xi_n \otimes \eta_m\|_{E_n \tilde{\otimes}_p F_n} \|\phi_n \otimes \psi_m\|_{(E_n \tilde{\otimes}_p F_n)^*} &\leq \sum_{n,m=1}^{\infty} \|\xi_n\| \|\eta_m\| \|\phi_n\| \psi_m\| \\ &\leq \left(\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| \right) \left(\sum_{m=1}^{\infty} \|\eta_m\| \|\psi_m\| \right) \\ &\leq (\|f\|_{B_p(G)} + \epsilon)(\|g\|_{B_p(G)} + \epsilon).\end{aligned}$$

From Lemma 4.3 and Definition 4.2, we conclude that

$$\|fg\|_{B_p(G)} \leq (\|f\|_{B_p(G)} + \epsilon)(\|g\|_{B_p(G)} + \epsilon).$$

Since $\epsilon > 0$ was arbitrary, this yields the submultiplicativity of $\|\cdot\|_{B_p(G)}$ and thus completes the proof of (i).

From Lemma 4.4 and Definition 4.2, (ii) is immediate.

Let $2 \leq q \leq p$ or $p \leq q \leq 2$, and let $q' \in (1, \infty)$ be dual to q. Since $\mathrm{Cyc}_{q'}(G) \subset \mathrm{Cyc}_{p'}(G)$, this proves (iii). □

5. $B_p(G)$ and $A_p(G)$

For any locally compact group G, the Fourier algebra $A(G)$ embeds isometrically into $B(G)$ and can be identified with the closed ideal of $B(G)$ generated by the functions in $B(G)$ with compact support [Eymard 1964].

For general $p \in (1, \infty)$, the only information we have so far about the relation between $B_p(G)$ and $A_p(G)$ is Theorem 4.7(ii). In the present section, we further explore the relation between those algebras.

Our first result is known for $p = 2$ as *Fell's absorption principle*:

Proposition 5.1. *Let G be a locally compact group, let $p \in (1, \infty)$, and let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then the representations $(\lambda_p \otimes \pi, L_p(G, E))$ and $(\lambda_p \otimes \mathrm{id}_E, L_p(G, E))$ are equivalent.*

Proof. The proof very much goes along the lines of the case $p = 2$.

Let $\mathscr{C}_{00}(G, E)$ denote the continuous E-valued functions on G with compact support (so that $\mathscr{C}_{00}(G, E)$ is a dense subspace of $L_p(G, E)$). Define

$$W_\pi : \mathscr{C}_{00}(G, E) \to \mathscr{C}_{00}(G, E)$$

by letting

$$(W_\pi \xi)(x) := \pi(x)\xi(x) \qquad (\xi \in \mathscr{C}_{00}(G, E),\ x \in G).$$

Since $\pi(G)$ consists of isometries, we have

$$\|W_\pi \xi\|^p_{L_p(G,E)} = \int_G \|\pi(x)\xi(x)\|^p\,dx = \int_G \|\xi(x)\|^p\,dx \qquad (\xi \in \mathscr{C}_{00}(G, E)),$$

so that W_p is an isometry with respect to the norm of $L_p(G, E)$ and thus extends to all of $L_p(G, E)$ as an isometry. Clearly, W_π is invertible with inverse given by

$$(W_\pi^{-1}\xi)(x) := \pi(x^{-1})\xi(x) \qquad (\xi \in \mathscr{C}_{00}(G, E),\ x \in G).$$

Let $\xi \in \mathscr{C}_{00}(G, E)$, and let $x \in G$. Then we have

$$((\lambda_p(x) \otimes \mathrm{id}_E)W_\pi^{-1}\xi)(y) = \pi(y^{-1}x)\xi(x^{-1}y) \qquad (y \in G)$$

and thus

$$\begin{aligned}(W_\pi(\lambda_p(x) \otimes \mathrm{id}_E)W_\pi^{-1}\xi)(y) &= \pi(y)\pi(y^{-1}x)\xi(x^{-1}y)\\ &= \pi(x)\xi(x^{-1}y)\\ &= ((\lambda_p(x) \otimes \pi(x))\xi)(y) \qquad (y \in G).\end{aligned}$$

Hence,

$$W_\pi(\lambda_p(x) \otimes \mathrm{id}_E)W_\pi^{-1} = \lambda_p(x) \otimes \pi(x) \qquad (x \in G)$$

holds, so that $(\lambda_p \otimes \pi, L_p(G, E))$ and $(\lambda_p \otimes \mathrm{id}_E, L_p(G, E))$ are equivalent as claimed. □

Corollary 5.2. *Let G be a locally compact group, let $p \in G$, let $f \in A_p(G)$, and let $g \in B_p(G)$. Then fg lies in $A_p(G)$ such that*

$$\|fg\|_{A_p(G)} \le \|f\|_{A_p(G)}\|g\|_{B_p(G)}.$$

Proof. Apply Proposition 5.1 (with p replaced by p' dual to p) to a p'-universal representation $(\pi, E) \in \mathrm{Rep}_{p'}(G)$. The norm estimate is proven as is the submultiplicativity assertion of Theorem 4.7. □

Let G be a locally compact group, and let $p \in (1, \infty)$. A *multiplier* of $A_p(G)$ is a function $f \in \mathscr{C}_b(G)$ such that $fA_p(G) \subset A_p(G)$. We denote the set of all multipliers of $A_p(G)$ by $\mathcal{M}(A_p(G))$. Clearly, $\mathcal{M}(A_p(G))$ is a subalgebra of $\mathscr{C}_b(G)$. From the closed graph theorem, it is immediate that multiplication with $f \in \mathcal{M}(A_p(G))$

is a bounded linear operator on $A_p(G)$, so that $\mathcal{M}(A_p(G))$ embeds canonically into $\mathcal{B}(A_p(G))$ turning it into a Banach algebra.

We have the following (compare [Herz 1971, Lemma 0]):

Corollary 5.3. *Let G be a locally compact group, and let $p \in (1, \infty)$. Then $B_p(G)$ is contained in $\mathcal{M}(A_p(G))$ such that*

$$\|f\|_{\mathcal{M}(A_p(G))} \leq \|f\|_{B_p(G)} \qquad (f \in B_p(G)). \tag{5–1}$$

In particular,

$$\|f\|_{\mathcal{M}(A_p(G))} \leq \|f\|_{B_p(G)} \leq \|f\|_{A_p(G)} \qquad (f \in A_p(G)) \tag{5–2}$$

holds with equality throughout if G is amenable.

Proof. By Corollary 5.2, $B_p(G) \subset \mathcal{M}(A_p(G))$ holds as does (5–1). The first inequality of (5–2) follows from (5–1) and the second one from Theorem 4.7(ii). Finally, if G is amenable, $A_p(G)$ has an approximate identity bounded by one [Pier 1984, Theorem 4.10], so that $\|f\|_{\mathcal{M}(A_p(G))} = \|f\|_{A_p(G)}$ holds for all $f \in A_p(G)$. □

Remark. Let G be a locally compact group such that, for any $p \in (1, \infty)$, the embedding of $A_p(G)$ into $B_p(G)$ is an isometry. Since $A_p(G)$ is regular [Herz 1973], this means that $A_p(G)$ can be identified with the closed ideal of $B_p(G)$ generated by the functions in $B_p(G)$ with compact support. In view of Theorem 4.7(iii), this would yield a contractive inclusion $A_p(G) \subset A_q(G)$ whenever $2 \leq q \leq p$ or $p \leq q \leq 2$. Such in inclusion result is indeed true for amenable G [Herz 1971] — and also for certain nonamenable G (see [Herz and Rivière 1972]) — but is false for noncompact, semisimple Lie groups with finite center [Lohoué 1980], as was pointed out to me by Michael Cowling.

6. $B_p(G)$ as a dual space

The Fourier–Stieltjes algebra $B(G)$ of a locally compact group G can be identified with the dual space of the full group C^*-algebra $C^*(G)$ [Eymard 1964].

In this section, we show that $B_p(G)$ is a dual space in a canonical fashion for *arbitrary* $p \in (1, \infty)$. This, in turn, will enable us to further clarify the relation between $B_p(G)$ and $\mathcal{M}(A_p(G))$.

We begin with some more definitions:

Definition 6.1. Let G be a locally compact group, let $p \in (1, \infty)$, and let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then:

(a) $\|\cdot\|_\pi$ is the algebra seminorm on $L_1(G)$ defined through

$$\|f\|_\pi := \|\pi(f)\|_{\mathcal{B}(E)} \qquad (f \in L_1(G)).$$

(b) The algebra $\mathrm{PF}_{p,\pi}(G)$ of *p-pseudofunctions associated with* (π, E) is the closure of $\pi(L_1(G))$ in $\mathcal{B}(E)$.

(c) If $(\pi, E) = (\lambda_p, L_p(G))$, we simply speak of *p-pseudofunctions* and write $\mathrm{PF}_p(G)$ instead of $\mathrm{PF}_{p,\lambda_p}(G)$.

(d) If (π, E) is p-universal, we denote $\mathrm{PF}_{p,\pi}(G)$ by $\mathrm{UPF}_p(G)$ and call it the algebra of *universal p-pseudofunctions*.

Remarks. 1. The notion of p-pseudofunctions is well established in the literature; the other definitions seem to be new.

2. For $p = 2$, the algebra $\mathrm{PF}_p(G)$ is the reduced group C^*-algebra and $\mathrm{UPF}_p(G)$ is the full group C^*-algebra of G.

3. If $(\rho, F) \in \mathrm{Rep}_p(G)$ is such that (π, E) contains every cyclic subrepresentation of (ρ, F), then $\|\cdot\|_\rho \le \|\cdot\|_\pi$ holds. In particular, the definition of $\mathrm{UPF}_p(G)$ is independent of a particular p-universal representation.

4. With $\langle\cdot,\cdot\rangle$ denoting the $L_1(G)$-$L_\infty(G)$ duality and with (π, E) a p-universal representation of G, we have

$$\|f\|_\pi = \sup\{|\langle f, g\rangle| : f \in B_{p'}(G),\ \|g\|_{B_{p'}(G)} \le 1\} \qquad (f \in L_1(G)),$$

where $p' \in (1, \infty)$ is dual to p: this follows from Lemma 4.6.

We now turn to representations of Banach algebras.

Definition 6.2. A *representation* of a Banach algebra $\mathfrak{A}$ is a pair (π, E) where E is a Banach space and π is a contractive algebra homomorphism from $\mathfrak{A}$ to $\mathcal{B}(E)$. We call (π, E) *isometric* if π is an isometry and *essential* if the linear span of $\{\pi(a)\xi : a \in \mathfrak{A},\ \xi \in E\}$ is dense in E.

Remarks. 1. As with Definition 1.1, our definition of a representation of a Banach algebra is somewhat more restrictive than the one usually used in a literature. Our reasons for this are the same as given after Definition 1.1.

2. If G is a locally compact group and (π, E) is a representation of G in the sense of Definition 1.1, then (1–1) induces an essential representation of $L_1(G)$. Conversely, every essential representation of $L_1(G)$ arises in the fashion.

3. The notions introduced in Definition 1.2 for representations of locally compact groups carry over to representations of Banach algebras accordingly.

We require three lemmas:

Lemma 6.3. *Let $\mathfrak{A}$ be a Banach algebra with an approximate identity bounded by one, and let (π, E) be a representation of $\mathfrak{A}$. Let F be the closed linear span of $\{\pi(a)\xi : a \in \mathfrak{A},\ \xi \in E\}$, and define*

$$\rho : \mathfrak{A} \to \mathcal{B}(F), \quad a \mapsto \pi(a)|_F.$$

Then (ρ, F) is an essential subrepresentation of (π, E) which is isometric if (π, E) is. Moreover, if E is a reflexive Banach space — so that $\mathcal{B}(E)$ is a dual space — and π is weak-weak continuous, then so is ρ.*

Proof. Straightforward. □

For our next lemma, recall the notion of an ultrapower of a Banach space E with respect to a (free) ultrafilter $\mathcal{U}$ (see [Heinrich 1980]); we denote it by $E_{\mathcal{U}}$.

The lemma is a straightforward consequence of [Daws 2004, Proposition 5]:

Lemma 6.4. *Let E be a superreflexive Banach space, and let $p \in (1, \infty)$. Then there is a free ultrafilter $\mathcal{U}$ such that the canonical representation of $\mathcal{B}(E)$ on $\ell_p(\mathbb{N}, E)_{\mathcal{U}}$ is weak-weak* continuous.*

Lemma 6.5. *Let G be a locally compact group, let $p, p' \in (1, \infty)$ be dual to each other, and let $(\pi, E) \in \mathrm{Rep}_{p'}(G)$. Then, for each $\phi \in \mathrm{PF}_{p',\pi}(G)$, there is a unique $g \in B_p(G)$ with $\|g\|_{B_p(G)} \leq \|\phi\|$ such that*

$$\langle \pi(f), \phi \rangle = \int_G f(x) g(x)\, dx \qquad (f \in L_1(G)). \tag{6–1}$$

Moreover, if (π, E) is p'-universal, we have $\|g\|_{B_p(G)} = \|\phi\|$.

Proof. By Lemma 6.4, there is a free ultrafilter such that the canonical representation of $\mathrm{PF}_{p',\pi}(G)$ on $\ell_{p'}(\mathbb{N}, E)_{\mathcal{U}}$ is weak-weak* continuous. Use Lemma 6.3 to obtain an isometric, essential, and still weak-weak* continuous subrepresentation (ρ, F) of it.

Since E is a $\mathrm{QSL}_{p'}$-space and since the class of all $\mathrm{QSL}_{p'}$-spaces is closed under the formation of $\ell_{p'}$-direct sums, of ultrapowers, and of subspaces, F is again a $\mathrm{QSL}_{p'}$-space. Since ρ is weak-weak* continuous and an isometry, it follows that ρ^* restricted to $F \tilde{\otimes}_\pi F^*$ is a quotient map onto $\mathrm{PF}_{p',\pi}(G)$. Let $\epsilon > 0$. Then there are sequences $(\xi_n)_{n=1}^\infty$ in F and $(\psi_n)_{n=1}^\infty$ in F^* such that, for $f \in L_1(G)$.

$$\|\phi\| \leq \sum_{n=1}^\infty \|\xi_n\| \|\psi_n\| < \|\phi\| + \epsilon \qquad \text{and} \qquad \langle \rho(\pi(f)), \phi \rangle = \sum_{n=1}^\infty \langle \rho(f)\xi_n, \psi_n \rangle.$$

Since $\pi(L_1(G))$ is dense in $\mathrm{PF}_{p,\pi}(G)$, it follows that $(\rho \circ \pi, F)$ is an essential representation of $L_1(G)$, which therefore can be identified via (1–1) with an element (σ, F) of $\mathrm{Rep}_{p'}(G)$. Letting

$$g(x) := \sum_{n=1}^\infty \langle \sigma(x)\xi_n, \psi_n \rangle \qquad (x \in G)$$

we obtain $g \in B_p(G)$ such that (6–1) holds. Moreover,

$$\|g\|_{B_p(G)} \le \sum_{n=1}^{\infty} \|\xi_n\| \|\psi_n\| < \|\phi\| + \epsilon;$$

holds, and since $\epsilon > 0$ was arbitrary, this means that even $\|g\|_{B_p(G)} \le \|\phi\|$.

Suppose now that (π, E) is p'-universal. Since the representation of $L_1(G)$ induced by (π, E) is essential, so is its infinite amplification $(\pi^\infty, \ell_{p'}(\mathbb{N}, E))$. With the appropriate identifications in place, we thus have

$$\ell_{p'}(\mathbb{N}, E) \subset F \subset \ell_{p'}(\mathbb{N}, E)_{\mathfrak{U}}.$$

Consequently, (σ, F) is also p'-universal. It then follows from Lemma 4.6 that $\|g\|_{B_p(G)} = \|\phi\|$. □

In view of Lemma 6.5, the following is now immediate:

Theorem 6.6. *Let G be a locally compact group, and let $p, p' \in (1, \infty)$ be dual to each other. Then:*

(i) *For any $(\pi, E) \in \mathrm{Rep}_{p'}(G)$, the dual space $\mathrm{PF}_{p',\pi}(G)^*$ embeds contractively into $B_p(G)$.*

(ii) *The embedding of $\mathrm{UPF}_{p'}(G)^*$ into $B_p(G)$ is an isometric isomorphism.*

Remarks. 1. For $p = 2$, the adverb "contractively" can be replaced by "isometrically". For $p \neq 2$, this is not true. To see this, assume otherwise, and let $2 \le q \le p$ or $p \le q \le p$. Since $(\lambda_{q'}, L_{q'}(G)) \in \mathrm{Rep}_{p'}(G)$, we would thus have an isometric embedding of $\mathrm{PF}_q(G)^*$ — and thus of $A_q(G)$ — into $B_p(G)$. For amenable G, this, in turn, would entail that $A_q(G) = A_p(G)$ holds isometrically. This is clearly impossible except in trivial cases.

2. As Michael Cowling pointed out to me, there is some overlap of this section with [Cowling and Fendler 1984]. In particular, it is an immediate consequence of Theorem 2 in that reference that $B_p(G)$ is a dual Banach space.

We conclude this section with a theorem that further clarifies the relation between $B_p(G)$ and $A_p(G)$:

Theorem 6.7. *Let G be an amenable, locally compact group, and let $p, p' \in (1, \infty)$ be dual to each other. Then $\mathrm{PF}_{p'}(G)^*$, $B_p(G)$, and $\mathcal{M}(A_p(G))$ are equal with identical norms.*

Proof. Since G is amenable, we have $\mathrm{PF}_{p'}(G)^* = \mathcal{M}(A_p(G))$ with identical norms by [Cowling 1979, Theorem 5], so that, by Theorem 6.6 and Corollary 5.3, we have a chain

$$\mathrm{PF}_{p'}(G)^* \subset B_p(G) \subset \mathcal{M}(A_p(G)) = \mathrm{PF}_{p'}(G)^*$$

of contractive inclusions. This proves the claim. □

Remark. By [Cowling 1979, Theorem 5], the equality $\mathrm{PF}_{p'}(G)^* = \mathcal{M}(A_p(G))$, even with merely equivalent and not necessarily identical norms, is also sufficient for the amenability of G. In view of the situation where $p=2$, we suspect that G is amenable if and only if $B_p(G)=\mathcal{M}(A_p(G))$ and if and only if $B_p(G)=\mathrm{PF}_{p'}(G)^*$.

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