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A GENERALIZATION OF THE CARTAN–HELGASON THEOREM FOR RIEMANNIAN SYMMETRIC SPACES OF RANK ONE

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Let U/K be a compact Riemannian symmetric space with U simply connected and K connected. Let G/K be the noncompact dual space, with G and U analytic subgroups of the simply connected complexification $G^{\mathbb{C}}$. Let $G = KAN$ be an Iwasawa decomposition of G, and let M be the centralizer of A in K. For $\delta \in \widehat{U}$, let μ be the highest restricted weight of δ, and let σ be the M-type acting in the highest restricted weight subspace of H_δ. Fix a K-type τ. Earlier we proved that if U/K has rank one, then $\delta|_K$ contains τ if and only if $\tau|_M$ contains σ and $\mu \in \mu_{\sigma,\tau} + \Lambda_{\text{sph}}$, where Λ_{sph} is the set of highest restricted spherical weights and $\mu_{\sigma,\tau}$ is a suitable element of $\mathfrak{a}^*$ uniquely determined by σ and τ. In this paper we obtain an explicit formula for this element in the case of $U/K = S^n$, $P^n(\mathbb{C})$, $P^n(\mathbb{H})$. This gives a generalization of the Cartan–Helgason theorem to arbitrary K-types on these rank one symmetric spaces.

1. Introduction

Let U be a compact semisimple simply connected Lie group, K the (necessarily connected) fixed point group of an involutive automorphism of U, and U/K the corresponding Riemannian symmetric space of the compact type.

Along with U/K consider the noncompact dual symmetric space G/K, where we assume that both G and U are analytic subgroups of the (complex semisimple) simply connected Lie group $G^{\mathbb{C}} = U^{\mathbb{C}}$ whose Lie algebra is the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ of G.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, and let $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ be the corresponding decomposition of the Lie algebra $\mathfrak{u}$ of U, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k} = \text{Lie}(K)$ in $\mathfrak{g}$ with respect to the Killing form.

Let $\mathfrak{a}$ be maximal abelian in $\mathfrak{p}$, let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and let A, M_e be the analytic subgroups of $G^{\mathbb{C}}$ with Lie algebras $\mathfrak{a}$ and $\mathfrak{m}$ respectively. The centralizer M of A in K is not connected, in general, and is the product $M = M_e\, F_M$

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of its identity component M_e and the finite abelian subgroup $F_M = \exp(i\mathfrak{a}) \cap K$; see [Kostant 2004, Lemma 2.4]. As is well known, F_M is generated by the (order-two) elements $\gamma_\alpha = \exp(2\pi i A_\alpha/|\alpha|^2)$, where $\alpha \in \Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is a restricted root, with $|\alpha|^2 = \langle \alpha, \alpha \rangle$, and $A_\alpha \in \mathfrak{a}$ is determined as usual by $\langle H, A_\alpha \rangle = \alpha(H)$ for $H \in \mathfrak{a}$, where $\langle , \rangle$ is the inner product on $\mathfrak{a}$, $\mathfrak{a}^*$ induced by the Killing form; see, e.g., [Helgason 1984, p. 536]. The most complete result is proved in [Kostant 2004, Theorem 2.28], namely M is actually the direct product $M_e \times F_s$, where $F_s \subset F_M$ is a product of $\mathbb{Z}_2$ factors, $F_s = \mathbb{Z}_2^l$.

Let $\mathfrak{b}$ be maximal abelian in $\mathfrak{m}$; then $\mathfrak{h} = \mathfrak{b} \oplus i\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{u}$. We define roots and weights of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Roots and weights are real-valued on $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{b}$, and define members of $\mathfrak{h}_{\mathbb{R}}^*$ by restriction. We order $\mathfrak{a}^*$ lexicographically, thereby determining a system $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{a})$ of positive restricted roots. We extend this ordering to an ordering of $\mathfrak{h}_{\mathbb{R}}^*$ by requiring that $\mathfrak{a}^*$ come before $(i\mathfrak{b})^*$, and we call $\Delta^+ = \Delta^+(\mathfrak{u}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ the resulting system of positive roots. Then a restricted root α is in Σ^+ if and only if all of the roots β such that $\beta|_{\mathfrak{a}} = \alpha$ are in Δ^+.

Let Λ be the set of dominant integral forms on $\mathfrak{h}^{\mathbb{C}}$. Since U is simply connected we have $\widehat{U} \simeq \hat{\mathfrak{u}} \simeq \Lambda$ for the unitary duals of U and $\mathfrak{u}$. For each $\lambda \in \Lambda$ let δ_λ be an irreducible representation of U (U-type) with highest weight λ, acting in H_λ. The differential of this representation is also denoted δ_λ.

Let $\Lambda_{\mathfrak{m}}$ be the set of dominant integral forms on $\mathfrak{b}^{\mathbb{C}}$, and let Λ_{M_e} be the subset of all $\eta \in \Lambda_{\mathfrak{m}}$ that are analytically integral for M_e. In other words, Λ_{M_e} is the set of highest weights of the $\mathfrak{m}$-types which exponentiate to M_e-types.

An element $\lambda \in \mathfrak{a}^*$ or $(i\mathfrak{b})^*$ is considered as an element of $\mathfrak{h}_{\mathbb{R}}^*$ by extending it to zero on $i\mathfrak{b}$ or $\mathfrak{a}$, respectively. We decompose each $\lambda \in \Lambda \subset \mathfrak{h}_{\mathbb{R}}^*$ in terms of its restrictions to $\mathfrak{a}$ and $i\mathfrak{b}$ as

$$\lambda = \mu + \eta, \quad \text{where} \quad \mu = \lambda|_{\mathfrak{a}}, \ \eta = \lambda|_{i\mathfrak{b}}.$$

Then μ is the so-called highest restricted weight of δ_λ, and η is in Λ_{M_e} (as easily seen). The meaning of η is that $\mathfrak{m}$, M_e, act irreducibly on the highest restricted weight subspace V_μ of H_λ, defined as

$$V_\mu = \{v \in H_\lambda : \delta_\lambda(H)v = \mu(H)v, \ \forall H \in \mathfrak{a}\},$$

and this irreducible representation $\sigma_\eta = \delta_\lambda(M_e)|_{V_\mu}$ has highest weight η. The group $M = M_e F_M$ also acts irreducibly on V_μ by the M-type $\sigma_\lambda = \delta_\lambda(M)|_{V_\mu}$. This M-type σ_λ extends the M_e-type σ_η and it is a scalar on F_M, since we have

$$\sigma_\lambda(\gamma_\alpha) = \delta_\lambda(\gamma_\alpha)|_{V_\mu} = \exp(2\pi i \mu(A_\alpha)/|\alpha|^2)\,\mathrm{Id}, \quad \forall \alpha \in \Sigma.$$

The map $\lambda \to \sigma_\lambda$ from $\Lambda \simeq \widehat{U}$ to $\widehat{M}$ is surjective, by [Kostant 2004, Theorem 2.33].

The classical Cartan–Helgason theorem describes the set $\widehat{U}(\tau_0)$ of (equivalence classes of) irreducible spherical representations of U, that is, the U-types that contain the trivial K-type τ_0 upon restriction to K. According to this theorem, if $\delta_\lambda|_K$ contains τ_0, then σ_λ is equivalent to the trivial M-type σ_0, i.e., the group M acts trivially on the highest weight vector v_λ of δ_λ. Conversely, if v_λ is M-fixed, then there is a K-fixed vector $v_K \in H_\lambda$, that is, $\delta_\lambda|_K$ contains the trivial K-type τ_0. The first characterization of the set $\widehat{U}(\tau_0)$ of spherical U-types is then

$$\widehat{U}(\tau_0) = \{\delta_\lambda \in \widehat{U} : \sigma_\lambda \sim \sigma_0\}.$$

It is well known that τ_0 occurs only once in each $\delta_\lambda \in \widehat{U}(\tau_0)$.

An equivalent characterization of $\widehat{U}(\tau_0)$ in terms of the highest weight λ of δ_λ is

$$\widehat{U}(\tau_0) = \{\delta_\lambda \in \widehat{U} : \lambda|_{i\mathfrak{b}} = 0 \text{ and } \lambda|_{\mathfrak{a}} \in \Lambda_{\text{sph}}\},$$

where the set Λ_{sph} of highest restricted spherical weights is given by

$$\Lambda_{\text{sph}} = \left\{\mu \in \mathfrak{a}^* : \frac{\langle \mu, \alpha \rangle}{|\alpha|^2} \in \mathbb{Z}^+ \text{ for } \alpha \in \Sigma^+\right\}.$$

Conversely any linear form λ on $\mathfrak{h}_{\mathbb{R}}$ such that $\lambda|_{i\mathfrak{b}} = 0$ and $\lambda|_{\mathfrak{a}} \in \Lambda_{\text{sph}}$ is the highest weight of some $\delta \in \widehat{U}(\tau_0)$; see [Helgason 1984, Theorem 4.1 p. 535].

Suppose we now replace the trivial K-type τ_0 by an arbitrary K-type τ, and ask for a similar description of the set $\widehat{U}(\tau)$ of the U-types δ that contain τ upon restriction to K (with multiplicity $m(\tau, \delta) > 0$).

Evidently, to know explicitly $\widehat{U}(\tau)$ and the multiplicity $m(\tau, \delta)$ for any τ and any $\delta \in \widehat{U}(\tau)$ is tantamount to knowing the branching theorem for $U \supset K$. In other words, the information contained in the branching law can be separated into two parts: given τ we first determine the set $\widehat{U}(\tau)$, and then for each $\delta \in \widehat{U}(\tau)$ we compute $m(\tau, \delta)$.

The multiplicity function $m(\tau, \delta)$ is, in general, a complicated object. (See [Kostant 2004, Theorem 2.3] for a recent result.) On the other hand, the results in [Kostant 2004] make it possible to give a general description of the set $\widehat{U}(\tau)$ independently of the multiplicity function.

First, it is easy to prove that if $\delta_\lambda|_K$ contains τ then $\tau|_M$ contains σ_λ, but the multiplicities are not the same in general, namely we have $m(\tau, \delta_\lambda) \leq m(\sigma_\lambda, \tau)$ [Camporesi 2005, Proposition 2.2].

This result says that if δ_λ is in $\widehat{U}(\tau)$ then σ_λ is in $\widehat{M}(\tau)$, the finite set of the M-types that occur in $\tau|_M$. Then $\widehat{U}(\tau)$ is clearly the disjoint union

$$\widehat{U}(\tau) = \bigcup_{\sigma \in \widehat{M}(\tau)} \widehat{U}_\sigma(\tau), \quad \text{where } \widehat{U}_\sigma(\tau) = \{\delta_\lambda \in \widehat{U}(\tau) : \sigma_\lambda \sim \sigma\}.$$

Let $\Lambda_\sigma(\tau)$ be the set of highest restricted weights of all U-types in $\widehat{U}_\sigma(\tau)$, and let η_σ be the highest weight of $\sigma|_{M_e}$. Then each $\delta_\lambda \in \widehat{U}_\sigma(\tau)$ has highest weight λ of the form $\mu + \eta_\sigma$, with $\mu \in \Lambda_\sigma(\tau)$, and we have an obvious parametrization for $\widehat{U}(\tau)$:

$$\widehat{U}(\tau) = \bigcup_{\sigma \in \widehat{M}(\tau)} \{\delta_\lambda \in \widehat{U} : \lambda \in \eta_\sigma + \Lambda_\sigma(\tau)\}.$$

The problem is then to find an explicit description of the set $\Lambda_\sigma(\tau)$, analogous to the Cartan–Helgason theorem in the case $\tau = \tau_0$.

Let $\mathscr{F}_\sigma$ be the set of all $\lambda \in \Lambda$ such that $\sigma_\lambda \sim \sigma$. In other words $\mathscr{F}_\sigma$ is the fiber over $\sigma \in \widehat{M}$ of the map $\lambda \to \sigma_\lambda$ from $\Lambda \simeq \widehat{U}$ to $\widehat{M}$. Then $\Lambda = \bigcup_{\sigma \in \widehat{M}} \mathscr{F}_\sigma$ (disjoint union); see [Kostant 2004]. Obviously $\eta_\sigma + \Lambda_\sigma(\tau)$ is a subset of $\mathscr{F}_\sigma$ for each $\sigma \in \widehat{M}(\tau)$ — in fact $\eta_\sigma + \Lambda_\sigma(\tau)$ is just $\mathscr{F}_\sigma \cap \widehat{U}(\tau)$. Moreover, if σ is fixed and τ varies over the K-types that contain σ, we have clearly

$$\mathscr{F}_\sigma = \eta_\sigma + \bigcup_{\tau \supset \sigma} \Lambda_\sigma(\tau). \tag{1–1}$$

Kostant [2004, Theorem 3.5] proves that $\mathscr{F}_\sigma$ is just a translate of Λ_{sph}, namely there exists a unique minimal element $\eta_\sigma + \mu_\sigma \in \mathscr{F}_\sigma$ (relative to the partial ordering of Λ defined by $\lambda' \geq \lambda \iff \lambda' - \lambda \in \Lambda$, or also relative to the partial ordering of Λ defined just before Theorem 3.4 of [Kostant 2004] — the two being equivalent within each fiber $\mathscr{F}_\sigma$ as a consequence of that theorem) such that (in our notation)

$$\mathscr{F}_\sigma = \eta_\sigma + \mu_\sigma + \Lambda_{\text{sph}}. \tag{1–2}$$

The element $\mu_\sigma \in \mathfrak{a}^*$ can be computed explicitly [Kostant 2004, formula (194)]. Kostant refers to (1–2) as a generalization of the Cartan–Helgason theorem.

Now (1–1) suggests that we look for a similar description of the set $\Lambda_\sigma(\tau)$. We did so for U/K of rank one and τ arbitrary, and using the results of [Kostant 2004] we proved the following in an earlier article:

Theorem 1.1 [Camporesi 2005, Proposition 2.3 and Theorem 2.4]. *Let U/K be a compact Riemannian symmetric space of rank one with U simply connected and K connected, and let τ be any K-type. For each $\sigma \in \widehat{M}(\tau)$ there is a unique minimal element $\mu_{\sigma,\tau} \in \Lambda_\sigma(\tau)$ such that*

$$\Lambda_\sigma(\tau) = \mu_{\sigma,\tau} + \Lambda_{\text{sph}}. \tag{1–3}$$

Thus we have

$$\begin{aligned}\widehat{U}(\tau) &= \{\delta_\lambda \in \widehat{U} : \sigma_\lambda \sim \sigma \textit{ for some } \sigma \in \widehat{M}(\tau) \textit{ and } \lambda|_{\mathfrak{a}} \in \mu_{\sigma,\tau} + \Lambda_{\text{sph}}\} \\ &= \{\delta_\lambda \in \widehat{U} : \lambda|_{i\mathfrak{b}} = \eta_\sigma \textit{ for some } \sigma \in \widehat{M}(\tau) \textit{ and } \lambda|_{\mathfrak{a}} \in \mu_{\sigma,\tau} + \Lambda_{\text{sph}}\} \\ &= \bigcup_{\sigma \in \widehat{M}(\tau)} \{\delta_\lambda \in \widehat{U} : \lambda \in \eta_\sigma + \mu_{\sigma,\tau} + \Lambda_{\text{sph}}\}.\end{aligned}$$

Moreover $\mathscr{F}_\sigma \setminus (\eta_\sigma + \Lambda_\sigma(\tau))$ *is a finite set, consisting of the weights* $\lambda = \eta_\sigma + \mu$ *with* $\mu_\sigma \le \mu < \mu_{\sigma,\tau}$. *Conversely, any linear form* λ *on* $\mathfrak{h}_{\mathbb{R}}$ *such that* $\lambda|_{i\mathfrak{b}} = \eta_\sigma$ *for some* $\sigma \in \widehat{M}(\tau)$ *and* $\lambda|_{\mathfrak{a}} \in \mu_{\sigma,\tau} + \Lambda_{\text{sph}}$ *is the highest weight of a* U*-type* $\delta \in \widehat{U}(\tau)$. *Finally,*

$$\mu_\sigma = \min_{\tau \supset \sigma} \mu_{\sigma,\tau}. \tag{1–4}$$

At the time we did not give an explicit formula for $\mu_{\sigma,\tau}$. With such a formula the theorem above yields a generalization of the Cartan–Helgason theorem (for U/K of rank one) which holds for any K-type τ and is more refined than (1–2).

Here we obtain an explicit formula for the minimal element $\mu_{\sigma,\tau}$ in the case of $U/K = S^n$, $P^n(\mathbb{C})$, $P^n(\mathbb{H})$ and for τ arbitrary. Our method is based on a case-by-case direct evaluation of $\mu_{\sigma,\tau}$ by putting together the known branching theorems for $U \supset K$ and $K \supset M$.

For $U/K = S^n$, $P^n(\mathbb{C})$ we only need the so-called interlacing conditions on the highest weights, which are necessary and sufficient for $\tau \in \widehat{K}$ to occur in $\delta \in \widehat{U}$.

In the quaternionic case the branching theorems for $U \supset K$ and $K \supset M$ are more complicated. The first was given in [Lepowsky 1971]. The double interlacing conditions on the highest weights of $\tau \in \widehat{K}$ and $\delta \in \widehat{U}$ stated in this theorem are still necessary but no longer sufficient for δ to contain τ. To find the minimal element $\mu_{\sigma,\tau}$ we shall also need the multiplicity formula of Lepowsky.

Finally, a remark about the higher rank case. For U/K of higher rank the set $\Lambda_\sigma(\tau)$ has, in general, more than one minimal element. There can be at most a finite number of such minimal elements, $\mu_{\sigma,\tau}^{(j)}$, $j = 1, \dots, k_{\sigma,\tau}$. We then have

$$\Lambda_\sigma(\tau) = \bigcup_j \left(\mu_{\sigma,\tau}^{(j)} + \Lambda_{\text{sph}}\right),$$

where the union is not necessarily disjoint. It is an interesting open problem to find a general formula for these Λ_{sph}-generators of $\Lambda_\sigma(\tau)$.

2. The case of spheres

Let $U/K = S^d$ ($d \ge 2$), with $U = \text{Spin}(d+1)$, $K = \text{Spin}(d)$. The linear realization of the spin groups is of course more complicated than that of the orthogonal groups $\text{SO}(d)$. However it is enough to work at the Lie algebra level, where we can use the well known isomorphism $\mathfrak{spin}(d) \simeq \mathfrak{so}(d)$.

We can treat the even and odd cases in a unified way, up to some point, including the definition of $\mathfrak{a}$ and $\mathfrak{m}$, as follows. We start with the noncompact form $\mathfrak{g}$ and take $\mathfrak{k}$ embedded from below. Thus let $\mathfrak{g} = \mathfrak{so}(1,d) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{so}(d) \end{pmatrix} \simeq \mathfrak{so}(d) \quad \text{and} \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & X^t \\ X & \mathbf{0}_d \end{pmatrix},\ X \in \mathbb{R}^d \right\}.$$

The compact form of $\mathfrak{g}$ is

$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p} = \left\{ \begin{pmatrix} 0 & iX^t \\ iX & Y \end{pmatrix},\ X \in \mathbb{R}^d,\ Y \in \mathfrak{so}(d) \right\}.$$

The Lie algebra $\mathfrak{u}$ is Lie isomorphic to $\mathfrak{u}' = \mathfrak{so}(d+1)$ realized as

$$\mathfrak{u}' = \mathfrak{so}(d+1) = \left\{ \begin{pmatrix} 0 & -X^t \\ X & Y \end{pmatrix},\ X \in \mathbb{R}^d,\ Y \in \mathfrak{so}(d) \right\}.$$

Indeed $\mathfrak{u}$ and $\mathfrak{u}'$ are conjugate in $\mathrm{SU}(d+1)$, i.e., there is an element $g \in \mathrm{SU}(d+1)$ such that $g\mathfrak{u}'g^{-1} = \mathfrak{u}$. For example, let g be the element

$$g = \begin{pmatrix} a^{-d} & 0 \\ 0 & a\mathbf{1}_d \end{pmatrix},$$

where a is any complex number such that $a^{d+1} = i$. It is easily checked that

$$g \begin{pmatrix} 0 & -X^t \\ X & Y \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & iX^t \\ iX & Y \end{pmatrix}, \quad \forall X \in \mathbb{R}^d,\ Y \in \mathfrak{so}(d). \tag{2–1}$$

We fix the maximal abelian subspace of $\mathfrak{p}$ given by

$$\mathfrak{a} = \mathbb{R}e_1 = \mathbb{R} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & & & & \\ 0 & & & & \\ \vdots & & & \mathbf{0}_d & \\ 0 & & & & \end{pmatrix}.$$

Then $\mathfrak{m}$ is given by

$$\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = \begin{pmatrix} \mathbf{0}_2 & 0 \\ 0 & \mathfrak{so}(d-1) \end{pmatrix} \simeq \mathfrak{so}(d-1).$$

We take the standard Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{u}'$ given by

$$\mathfrak{h}' = \left\{ \begin{pmatrix} 0 & & & & & & \\ & 0 & ih_1 & & & & \\ & -ih_1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & & 0 & ih_n \\ & & & & & -ih_n & 0 \end{pmatrix},\ h_j \in i\mathbb{R} \right\}$$

for $d = 2n$, and by

$$\mathfrak{h}' = \left\{ \begin{pmatrix} 0 & ih_1 & & & \\ -ih_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & ih_n \\ & & & -ih_n & 0 \end{pmatrix}, \; h_j \in i\mathbb{R} \right\}$$

for $d = 2n - 1$. In both cases $\mathfrak{h}' = \mathfrak{b}_1 \oplus \mathfrak{b}$, where $\mathfrak{b}$ is the Cartan subalgebra of $\mathfrak{m}$ given for any $n \geq 2$ by

$$\mathfrak{b} = \left\{ \begin{pmatrix} \mathbf{0} & & & & & \\ & 0 & ih_2 & & & \\ & -ih_2 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & ih_n \\ & & & & -ih_n & 0 \end{pmatrix}, \; h_j \in i\mathbb{R} \right\}$$

(here $\mathbf{0} = \mathbf{0}_3$ for $d = 2n$ and $\mathbf{0} = \mathbf{0}_2$ for $d = 2n - 1$) and $\mathfrak{b}_1$ is the orthogonal complement of $\mathfrak{b}$ in $\mathfrak{h}'$ with respect to the Killing form; it consists of the elements $\begin{pmatrix} B & \\ & \mathbf{0}_{2n-2} \end{pmatrix}$, where B is of the form

$$\begin{pmatrix} 0 & & \\ & 0 & ih_1 \\ & -ih_1 & 0 \end{pmatrix} \text{ for } d = 2n \quad \text{or} \quad \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} \text{ for } d = 2n - 1,$$

and $h_1 \in i\mathbb{R}$. For $d = 2$ we have $\mathfrak{h}' = \mathfrak{b}_1$, $\mathfrak{m} = \mathfrak{b} = \mathbf{0}_3$, and the group $M \simeq \mathrm{Spin}(1) \simeq \mathbb{Z}_2$ is not connected.

For $d = 2n$ we have $\operatorname{rank} \mathfrak{u}' = \operatorname{rank} \mathfrak{k}$ and $\mathfrak{h}' \subset \mathfrak{k} \subset \mathfrak{u}$, so $\mathfrak{h}'$ is also a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{u}$.

For $d = 2n-1$ we have $\operatorname{rank} \mathfrak{u}' > \operatorname{rank} \mathfrak{k} = \operatorname{rank} \mathfrak{m}$, so $\mathfrak{b}$ is also a Cartan subalgebra of $\mathfrak{k}$, while $\mathfrak{h}'$ is no longer contained in $\mathfrak{u}$.

In both cases we take as Cartan subalgebra of $\mathfrak{u}$

$$\mathfrak{h} = i\mathfrak{a} \oplus \mathfrak{b}.$$

For $d = 2n - 1$ the element g given above conjugates $\mathfrak{h}'$ with $\mathfrak{h}$, and the map $\mathrm{Ad}(g) = g(\,\cdot\,)g^{-1}$ is actually the identity on $\mathfrak{b}$, and exchanges $\mathfrak{b}_1$ with $i\mathfrak{a}$ bijectively; see (2–1).

For $d = 2n$ the element g cannot of course conjugate $\mathfrak{h}'$ with $\mathfrak{h}$ since it fixes $\mathfrak{k}$, so it fixes $\mathfrak{h}' \subset \mathfrak{k}$. However $\mathfrak{h}'$ and $\mathfrak{h}$ are two Cartan subalgebras of the compact Lie algebra $\mathfrak{u}$, thus there exists $u_0 \in U$ such that $\mathrm{Ad}(u_0)\mathfrak{h}' = \mathfrak{h}$. The transformation $\mathrm{Ad}(u_0)$ is essentially a Cayley transform. Moreover we can always choose u_0 so that $\mathrm{Ad}(u_0)$ acts as the identity on $\mathfrak{b}$ and sends $\mathfrak{b}_1$ bijectively onto $i\mathfrak{a}$. To unify the notation, we shall denote this u_0 by g.

In both cases we then have an isomorphism $\mathrm{Ad}(g)$ (of $\mathfrak{u}$ into itself for $d = 2n$, of $\mathfrak{u}'$ into $\mathfrak{u}$ for $d = 2n-1$) such that

$$\mathrm{Ad}(g)\mathfrak{h}' = \mathfrak{h}, \quad \mathrm{Ad}(g)|_{\mathfrak{b}} = \mathrm{Id}, \quad \mathrm{Ad}(g)\mathfrak{b}_1 = i\mathfrak{a}.$$

Moreover if B_1 is the basis of $\mathfrak{b}_1$ given by the element $\begin{pmatrix} B & \\ & \mathbf{0}_{2n-2} \end{pmatrix}$, where

$$B = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix} \text{ for } d = 2n \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ for } d = 2n-1,$$

we can always arrange that

$$\mathrm{Ad}(g)B_1 = ie_1.$$

The point is now as follows. Let the choice of Cartan subalgebras be $\mathfrak{h}'$ for $\mathfrak{u}'$, $\mathfrak{b}$ for $\mathfrak{m}$, and $\mathfrak{h}_k = \mathfrak{h}'$ ($d = 2n$) or $\mathfrak{h}_k = \mathfrak{b}$ ($d = 2n-1$) for $\mathfrak{k}$. Then the branching rules for $\mathfrak{u}' \supset \mathfrak{k}$ and for $\mathfrak{k} \supset \mathfrak{m}$ are classical and well known (see below).

Our aim is to find the branching rule for $\mathfrak{u} \supset \mathfrak{k}$ using for $\mathfrak{u}$ the Cartan subalgebra $\mathfrak{h} = i\mathfrak{a} \oplus \mathfrak{b}$. This branching rule will involve the branching rule for $\mathfrak{k} \supset \mathfrak{m}$ plus a condition characterizing the highest restricted weights μ. More precisely, we shall find that a U-type δ with highest weight $\lambda = \mu + \eta$ contains a K-type τ with highest weight ν if and only if τ contains the M-type σ with highest weight η, and moreover the highest restricted weight μ is of the form $\mu_{\sigma,\tau} + \mu_0$, with μ_0 a highest spherical weight and $\mu_{\sigma,\tau}$ a suitable element of $\mathfrak{a}^*$ (to be determined below).

Let ε_j ($j = 1, \ldots, n$) be the linear form on $\mathfrak{h}'^{\mathbb{C}}$ which equals h_j when acting on the elements of $\mathfrak{h}'$ given above. We denote the restriction of ε_j to $\mathfrak{b}^{\mathbb{C}}$ still by ε_j. Then we have the following systems of positive roots:

$$\begin{aligned}
\Delta^+_{\mathfrak{u}'} &= \Delta^+(\mathfrak{u}'^{\mathbb{C}}, \mathfrak{h}'^{\mathbb{C}}) = \Delta^+_{\mathfrak{so}(d+1)}, \\
\Delta^+_{\mathfrak{k}} &= \Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}_{\mathfrak{k}}) = \Delta^+_{\mathfrak{so}(d)}, \\
\Delta^+_{\mathfrak{m}} &= \Delta^+(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}}) = \Delta^+_{\mathfrak{so}(d-1)},
\end{aligned}$$

where

$$\begin{aligned}
\Delta^+_{\mathfrak{so}(2n+1)} &= \{\varepsilon_i \pm \varepsilon_j,\ 1 \le i < j \le n\} \cup \{\varepsilon_k,\ 1 \le k \le n\}, \\
\Delta^+_{\mathfrak{so}(2n)} &= \{\varepsilon_i \pm \varepsilon_j,\ 1 \le i < j \le n\},
\end{aligned}$$

and similar expressions hold for $\Delta^+_{\mathfrak{so}(2n-1)}$ and $\Delta^+_{\mathfrak{so}(2n-2)}$ with the indices running from 2 to n. For $d = 2n$, $\Delta^+_{\mathfrak{u}'}$ is also the set of roots of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}'^{\mathbb{C}}$.

The standard parametrization of $\widehat{\mathfrak{u}'}$, $\hat{\mathfrak{k}}$ and $\widehat{\mathfrak{m}}$ is as follows. The dominant integral forms for $\mathfrak{u}'$ are the linear functionals

$$\lambda' = \sum_{j=1}^{n} a_j \varepsilon_j, \text{ with } 2a_j \in \mathbb{Z},\ a_i - a_j \in \mathbb{Z},\ \forall i, j, \text{ and } \begin{cases} a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 & \text{for } d = 2n, \\ a_1 \geq a_2 \geq \cdots \geq |a_n| \geq 0 & \text{for } d = 2n-1. \end{cases}$$

For $d = 2n$ these are also the dominant integral forms for $\mathfrak{u}$ with respect to $\mathfrak{h}'$.

The dominant integral forms for $\mathfrak{k}$ are the linear functionals

$$\nu = \begin{cases} \sum_1^n b_j \varepsilon_j & \text{for } d = 2n, \\ \sum_2^n b_j \varepsilon_j & \text{for } d = 2n-1, \end{cases} \text{ with } 2b_j \in \mathbb{Z},\ b_i - b_j \in \mathbb{Z},\ \forall i, j, \text{ and } \begin{cases} b_1 \geq b_2 \geq \cdots \geq |b_n| \geq 0 & \text{for } d = 2n, \\ b_2 \geq b_3 \geq \cdots \geq b_n \geq 0 & \text{for } d = 2n-1. \end{cases}$$

The dominant integral forms for $\mathfrak{m}$ are the linear functionals (for all $n \geq 2$)

$$\eta = \sum_{j=2}^{n} c_j \varepsilon_j, \text{ with } 2c_j \in \mathbb{Z},\ c_i - c_j \in \mathbb{Z},\ \forall i, j, \text{ and } \begin{cases} c_2 \geq c_3 \geq \cdots \geq c_n \geq 0 & \text{for } d = 2n, \\ c_2 \geq c_3 \geq \cdots \geq |c_n| \geq 0 & \text{for } d = 2n-1. \end{cases}$$

For $d = 2$ we have $\widehat{M} = \{\sigma_0, \sigma_1\}$, where σ_0 and σ_1 are the trivial and nontrivial representations of $M \simeq \mathbb{Z}_2$.

The branching theorem for $\mathfrak{u}' \supset \mathfrak{k}$ says that (with obvious notations)

$$\lambda' \supset \nu \iff \begin{cases} a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq b_{n-1} \geq a_n \geq |b_n| & \text{for } d = 2n, \\ a_1 \geq b_2 \geq a_2 \geq b_3 \geq \cdots \geq a_{n-1} \geq b_n \geq |a_n| & \text{for } d = 2n-1, \end{cases}$$

and $a_j - b_j \in \mathbb{Z}$, $\forall j$. Moreover the multiplicity is always one.

The branching theorem for $\mathfrak{k} \supset \mathfrak{m}$ says that ($\forall n \geq 2$)

$$\nu \supset \eta \iff \begin{cases} b_1 \geq c_2 \geq b_2 \geq c_3 \geq \cdots \geq b_{n-1} \geq c_n \geq |b_n| & \text{for } d = 2n, \\ b_2 \geq c_2 \geq b_3 \geq c_3 \geq \cdots \geq c_{n-1} \geq b_n \geq |c_n| & \text{for } d = 2n-1, \end{cases}$$

and $b_j - c_j \in \mathbb{Z}$, $\forall j$. The multiplicity is again always one. For $d = 2$ the representation of $K = \mathrm{Spin}(2)$ with weight $\nu = b_1 \varepsilon_1$ (where $2b_1 \in \mathbb{Z}$) contains σ_0 (resp. σ_1) if and only if $b_1 \in \mathbb{Z}$ (resp. $b_1 \in \mathbb{Z} + \frac{1}{2}$).

Now the map $\mathrm{Ad}(g) : \mathfrak{h}' \to \mathfrak{h}$ induces a map $\lambda' \to g \cdot \lambda'$ from the linear forms λ' on $\mathfrak{h}'^{\mathbb{C}}$ to those on $\mathfrak{h}^{\mathbb{C}}$ given by

$$(g \cdot \lambda')(H) = \lambda'(\mathrm{Ad}(g^{-1})H), \quad \forall H \in \mathfrak{h}^{\mathbb{C}}. \tag{2–2}$$

Since $\mathrm{Ad}(g)$ is the identity on $\mathfrak{b}$ and since $\mathrm{Ad}(g^{-1})e_1 = -iB_1$, we get in both cases

$$g \cdot \varepsilon_j = \varepsilon_j, \ \ \forall\, 2 \le j \le n, \qquad g \cdot \varepsilon_1 = \alpha,$$

where $\alpha \in \mathfrak{a}^*$ is the (unique) positive restricted root defined by $\alpha(e_1) = 1$, and as linear forms on $\mathfrak{h}^{\mathbb{C}}$, $\alpha|_{\mathfrak{b}} \equiv 0$, $\varepsilon_j|_{\mathfrak{a}} \equiv 0$. Let us order $\mathfrak{h}^*_{\mathbb{R}} = (i\mathfrak{h})^*$ by requiring that $\mathfrak{a}^*$ comes before $(i\mathfrak{b})^*$. Then the system of positive roots of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ is given by

$$\Delta^+ = \Delta^+(\mathfrak{u}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) = g \cdot \Delta^+_{\mathfrak{u}'} = \Delta^+_{\mathfrak{m}} \cup \Delta^+_{\alpha},$$

where

$$\Delta^+_{\alpha} = \{\beta \in \Delta^+ : \beta|_{\mathfrak{a}} = \alpha\} = \begin{cases} \{\alpha \pm \varepsilon_j,\ 2 \le j \le n\} \cup \{\alpha\} & \text{for } d = 2n, \\ \{\alpha \pm \varepsilon_j,\ 2 \le j \le n\} & \text{for } d = 2n-1. \end{cases}$$

The dominant weights of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ are obtained by applying g to the dominant weights of $\mathfrak{u}'^{\mathbb{C}}$ with respect to $\mathfrak{h}'^{\mathbb{C}}$. Note that each $\lambda' \in \widehat{\mathfrak{u}'}$ can be decomposed as

$$\lambda' = \sum_1^n a_j \varepsilon_j = a_1 \varepsilon_1 + \eta,$$

where $\eta = \sum_2^n a_j \varepsilon_j$ is in $\widehat{\mathfrak{m}}$ ($\forall n \ge 2$), as immediately seen. Then each $\lambda \in \hat{\mathfrak{u}}$ has the form

$$\lambda = g \cdot \lambda' = \mu + \eta,$$

where $\mu = a_1 \alpha$ is the highest restricted weight and $\eta = \sum_2^n a_j \varepsilon_j \in \widehat{\mathfrak{m}}$, with $a_1 \ge a_2$ and $a_1 - a_2 \in \mathbb{Z}$, i.e., $a_1 = a_2 + k$, $k \in \mathbb{Z}^+$. For $d = 2$ we have $\lambda = \mu = a_1 \alpha$, where a_1 is in $\mathbb{Z}^+$ (resp. $\mathbb{Z}^+ + \frac{1}{2}$) if and only if $\sigma_\lambda \sim \sigma_0$ (resp. σ_1). It follows that $\hat{\mathfrak{u}}$ is the disjoint union

$$\hat{\mathfrak{u}} \simeq \Lambda = \bigcup_{\sigma \in \widehat{M}} \mathcal{F}_\sigma,$$

where for σ fixed in $\widehat{M}$, with highest weight $\eta = \sum_2^n a_j \varepsilon_j$, we have for any $d > 2$ ($M \simeq \mathrm{Spin}(d-1)$ being connected in this case)

$$\begin{aligned} \mathcal{F}_\sigma &= \{\lambda \in \Lambda : \delta_\lambda(M)|_{V_\mu} \sim \sigma\} \\ &= \{\lambda \in \Lambda : \lambda|_{i\mathfrak{b}} = \eta\} \\ &= \{\lambda = \mu + \eta : \mu = (a_2 + k)\alpha,\ k \in \mathbb{Z}^+\} \\ &= \eta + \mu_\sigma + \Lambda_{\mathrm{sph}}, \end{aligned}$$

where

$$\mu_\sigma = a_2 \alpha, \qquad \Lambda_{\mathrm{sph}} = \{k\alpha,\ k \in \mathbb{Z}^+\}$$

(a_2 the first component of η). For $d = 2$ we still have $\Lambda = \mathcal{F}_{\sigma_0} \cup \mathcal{F}_{\sigma_1}$, with $\eta = 0$ and $\mu_{\sigma_0} = 0$, $\mu_{\sigma_1} = \frac{1}{2}\alpha$. This is just Kostant's result (1–2); see [Kostant 2004, Theorem 3.5].

Comparing the two branching rules for $\mathfrak{u}' \supset \mathfrak{k}$ and for $\mathfrak{k} \supset \mathfrak{m}$, and using the above parametrization of $\hat{\mathfrak{u}}$, we obtain the following branching rule for $\mathfrak{u} \supset \mathfrak{k}$ ($\forall d > 2$):

$$\lambda = \mu + \eta \supset \nu \iff \nu \supset \eta \text{ and } \mu = a_1\alpha,\ a_1 = \begin{cases} b_1 + k, & d = 2n, \\ b_2 + k, & d = 2n-1, \end{cases} \quad k \in \mathbb{Z}^+,$$

where again b_1 (for $d = 2n$) or b_2 (for $d = 2n - 1$) is the first component of $\nu = \sum b_j \varepsilon_j$. For $d = 2$ we get $\lambda = a_1\alpha \supset \nu = b_1\varepsilon_1$ if and only if $a_1 = |b_1| + k$, $k \in \mathbb{Z}^+$.

If $\delta \in \widehat{U}$ has highest weight $\lambda = \mu + \eta$, if $\tau \in \widehat{K}$ has highest weight ν, and if $\sigma \in \widehat{M}$ has highest weight η, then we get the following rule for branching from U to K in terms of branching from K to M:

$$\delta|_K \supset \tau \iff \tau|_M \supset \sigma \text{ and } \mu \in \mu_{\sigma,\tau} + \Lambda_{\mathrm{sph}}, \tag{2–3}$$

where

$$\mu_{\sigma,\tau} = \begin{cases} b_1\alpha, & d = 2n, \\ b_2\alpha, & d = 2n-1. \end{cases}$$

(For $d = 2$, $\mu_{\sigma,\tau} = |b_1|\alpha$.) This agrees with the general rank-one result (1–3). Note that in this case $\mu_{\sigma,\tau}$ is the same for all σ in $\widehat{M}(\tau)$ and depends on τ only. Finally, since $b_1 \geq a_2$ (for $d = 2n$) and $b_2 \geq a_2$ (for $d = 2n - 1$), we see that for σ fixed and τ varying over the K-types that contain σ we have, in agreement with (1–4),

$$\min_{\tau \supset \sigma} \mu_{\sigma,\tau} = a_2\alpha = \mu_\sigma.$$

3. The case of complex projective spaces

Let $U/K = P^n(\mathbb{C})$ ($n \geq 2$), with $U = \mathrm{SU}(n+1)$ and $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ embedded as

$$K = \left\{ \begin{pmatrix} B & 0 \\ 0 & b \end{pmatrix},\ B \in \mathrm{U}(n),\ b \in \mathrm{U}(1),\ b \det B = 1 \right\}.$$

The group K is isomorphic to $\mathrm{U}(n)$.

At the Lie algebra level, consider the noncompact form $\mathfrak{g} = \mathfrak{su}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ Y \in \mathfrak{u}(n),\ y \in \mathfrak{u}(1) = i\mathbb{R},\ y + \operatorname{tr} Y = 0 \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} \mathbf{0}_n & Z \\ \bar{Z}^t & 0 \end{pmatrix},\ Z \in \mathbb{C}^n \right\}.$$

Then the compact form $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ coincides, in this case, with the Lie algebra $\mathfrak{su}(n+1)$ of $(n+1) \times (n+1)$ antihermitian traceless matrices:

$$\mathfrak{u} = \left\{ \begin{pmatrix} Y & Z \\ -\bar{Z}^t & y \end{pmatrix},\ Z \in \mathbb{C}^n,\ Y \in \mathfrak{u}(n),\ y \in \mathfrak{u}(1),\ y + \operatorname{tr} Y = 0 \right\}.$$

We fix the maximal abelian subspace of $\mathfrak{p}$ given by

$$\mathfrak{a} = \mathbb{R}e_1 = \mathbb{R}\begin{pmatrix} & & & & 1 \\ & & & & 0 \\ & & \mathbf{0}_n & & \vdots \\ & & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The group A is then

$$A = \exp \mathfrak{a} = \left\{ \begin{pmatrix} \operatorname{ch} t & 0 & \operatorname{sh} t \\ 0 & \mathbf{1}_{n-1} & 0 \\ \operatorname{sh} t & 0 & \operatorname{ch} t \end{pmatrix},\ t \in \mathbb{R} \right\},$$

and the centralizer of A in K is

$$M = \left\{ \begin{pmatrix} b & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & b \end{pmatrix},\ B \in \mathrm{U}(n-1),\ b \in \mathrm{U}(1),\ \ b^2 \det\ B = 1 \right\},$$

with Lie algebra

$$\mathfrak{m} = \left\{ \begin{pmatrix} y & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & y \end{pmatrix},\ Y \in \mathfrak{u}(n-1),\ y \in \mathfrak{u}(1),\ 2y + \operatorname{tr} Y = 0 \right\}.$$

The group M is connected and isomorphic to a double cover of $\mathrm{U}(n-1)$.

As in the case of S^{2n} we have rank $\mathfrak{u} =$ rank $\mathfrak{k}$. Let $\mathfrak{h}_{\mathfrak{k}}$ be the Cartan subalgebra of $\mathfrak{u}$ which is contained in $\mathfrak{k}$ and consists of the diagonal matrices. Let $\mathfrak{b} \subset \mathfrak{h}_{\mathfrak{k}}$ be the Cartan subalgebra of $\mathfrak{m}$ consisting of the diagonal elements. Then $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{b}_1 \oplus \mathfrak{b}$, where $\mathfrak{b}_1$ consists of the matrices of the form $\operatorname{diag}(h, 0, \ldots, 0, -h)$ with $h \in i\mathbb{R}$.

The classical branching rule for $U \supset K$ with respect to the Cartan subalgebra $\mathfrak{h}_{\mathfrak{k}}$ is well known (see below). We will find the branching rule for $U \supset K$ using for $\mathfrak{u}$ the Cartan subalgebra

$$\mathfrak{h} = i\mathfrak{a} \oplus \mathfrak{b} = \left\{ \begin{pmatrix} h_0 & & 0 & & ix \\ & h_2 & & & \\ 0 & & \ddots & & 0 \\ & & & h_n & \\ ix & & 0 & & h_0 \end{pmatrix},\ x \in \mathbb{R},\ h_j \in i\mathbb{R},\ 2h_0 + h_2 + \cdots + h_n = 0 \right\}.$$

Again this branching rule will involve the branching rule for $K \supset M$, which is known, plus a condition on the highest restricted weights. In order to relate the roots and weights of U in the two different Cartan subalgebras, we need an element that conjugates $\mathfrak{h}_{\mathfrak{k}}$ with $\mathfrak{h}$. It is easy to check that the element

$$g = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & \mathbf{1}_{n-1} & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \in U \tag{3–1}$$

satisfies

$$\mathrm{Ad}(g)\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h}, \quad \mathrm{Ad}(g)|_{\mathfrak{b}} = \mathrm{Id}, \quad \mathrm{Ad}(g)\mathfrak{b}_1 = i\mathfrak{a}. \tag{3–2}$$

Moreover if B_1 is the basis of $\mathfrak{b}_1$ given by $B_1 = \mathrm{diag}(i, 0, \dots, 0, -i)$, we verify that $\mathrm{Ad}(g)B_1 = ie_1$.

Let ε_j be the linear functional on $\mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}}$ defined by $\varepsilon_j(\mathrm{diag}(h_1, \dots, h_{n+1})) = h_j$, for $1 \le j \le n+1$. Then $\varepsilon_1 + \cdots + \varepsilon_{n+1} = 0$, and each linear form $\lambda' \in (i\mathfrak{h}_{\mathfrak{k}})^*$ can be written in a unique way as

$$\lambda' = \sum_{j=1}^{n+1} a_j \varepsilon_j, \quad \text{with} \quad \sum_{1}^{n+1} a_j = 0. \tag{3–3}$$

The positive roots (in the standard ordering) of the pairs $(\mathfrak{u}^{\mathbb{C}}, \mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}})$, $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}})$, and $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, are the linear forms $\varepsilon_i - \varepsilon_j$, with $1 \le i < j \le n+1$ for $\Delta_{\mathfrak{u}}^+$, $1 \le i < j \le n$ for $\Delta_{\mathfrak{k}}^+$, and $2 \le i < j \le n$ for $\Delta_{\mathfrak{m}}^+$. (The restriction of ε_j to $\mathfrak{b}^{\mathbb{C}}$ is still denoted ε_j.)

We have the following parametrizations of $\widehat{U}$, $\widehat{K}$, and $\widehat{M}$:

$$\widehat{U} \simeq \left\{ \lambda' = \sum_{j=1}^{n+1} a_j \varepsilon_j : a_j \in \frac{\mathbb{Z}}{n+1},\ a_i - a_j \in \mathbb{Z},\ \forall\, i, j,\ a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \right\},$$

$$\widehat{K} \simeq \left\{ \nu = \sum_{j=1}^{n+1} b_j \varepsilon_j : b_j \in \frac{\mathbb{Z}}{n+1},\ b_i - b_j \in \mathbb{Z},\ \forall\, i, j,\ b_1 \ge b_2 \ge \cdots \ge b_n \right\},$$

$$\widehat{M} \simeq \left\{ \eta = c_0(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{j=2}^{n} c_j \varepsilon_j : c_j \in \frac{\mathbb{Z}}{n+1},\ c_i - c_j \in \mathbb{Z},\ \forall\, 2 \le i, j \le n, \right.$$
$$\left. 2c_0 \in \frac{\mathbb{Z}}{n+1},\ 2(c_0 - c_j) \in \mathbb{Z},\ \forall\, 2 \le j \le n,\ c_2 \ge c_3 \ge \cdots \ge c_n \right\}.$$

In all cases it is understood that the sum of the components of the weights is zero; compare (3–3). For M we have $2c_0 + c_2 + \cdots + c_n = 0$.

Given these parametrizations, we have the following simple branching rules. For $U \supset K$ we have (with obvious notations)

$$\lambda' \supset \nu \iff \begin{cases} a_j - b_j \in \mathbb{Z},\ \forall\, 1 \le j \le n+1, \\ a_1 \ge b_1 \ge a_2 \ge b_2 \ge \cdots \ge a_n \ge b_n \ge a_{n+1}. \end{cases}$$

For $K \supset M$ we have

$$\nu \supset \eta \iff \begin{cases} b_j - c_j \in \mathbb{Z}, \ \forall\, 2 \le j \le n, \\ b_1 \ge c_2 \ge b_2 \ge c_3 \ge \cdots \ge c_n \ge b_n. \end{cases}$$

In both cases the multiplicity is one. (For the first see [Ikeda and Taniguchi 1978, Proposition 5.1], for example. For the second see [Baldoni Silva 1979, Theorem 4.4] and note that the additional condition required there is automatically satisfied in our parametrization, in view of (3–3).)

We now proceed as in the case of spheres. If g is the element (3–1), we define a map $\lambda' \to g \cdot \lambda'$ from the linear forms λ' on $\mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}}$ to those on $\mathfrak{h}^{\mathbb{C}}$ given by (2–2). By (3–2) we find

$$g \cdot \varepsilon_j = \varepsilon_j,\ \forall\, 2 \le j \le n, \quad g \cdot (\varepsilon_1 + \varepsilon_{n+1}) = \varepsilon_1 + \varepsilon_{n+1}, \quad g \cdot (\varepsilon_1 - \varepsilon_{n+1}) = 2\alpha, \tag{3–4}$$

where now $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\alpha, \pm 2\alpha\}$, the shorter root α being defined again by $\alpha(e_1) = 1$, and as linear forms on $\mathfrak{h}^{\mathbb{C}}$, $\alpha|_{\mathfrak{b}} \equiv 0$, $(\varepsilon_1 + \varepsilon_{n+1})|_{\mathfrak{a}} \equiv 0$, $\varepsilon_j|_{\mathfrak{a}} \equiv 0$. With the usual ordering, we get the following system of positive roots of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$:

$$\Delta^+ = \Delta^+(\mathfrak{u}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) = g \cdot \Delta^+(\mathfrak{u}^{\mathbb{C}}, \mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}}) = \Delta_{\mathfrak{m}}^+ \cup \{2\alpha\} \cup \Delta_\alpha^+,$$

where

$$\Delta_\alpha^+ = \{\alpha - \varepsilon_j + \tfrac{1}{2}(\varepsilon_1 + \varepsilon_{n+1}),\ \alpha + \varepsilon_j - \tfrac{1}{2}(\varepsilon_1 + \varepsilon_{n+1}),\ \ 2 \le j \le n\}.$$

The element g then relates the dominant weights of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}}$ to the dominant weights of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Note that any $\lambda' \in \widehat{U}$ can be written as

$$\lambda' = \textstyle\sum_1^{n+1} a_j \varepsilon_j = \tfrac{1}{2}(a_1 - a_{n+1})(\varepsilon_1 - \varepsilon_{n+1}) + \eta, \tag{3–5}$$

$$\eta = \tfrac{1}{2}(a_1 + a_{n+1})(\varepsilon_1 + \varepsilon_{n+1}) + \textstyle\sum_2^n a_j \varepsilon_j. \tag{3–6}$$

It is easy to check that η is in $\Lambda_{M_e} \simeq \widehat{M}$. Applying g we find that any highest weight λ of U with respect to $\mathfrak{h}^{\mathbb{C}}$ can be written as

$$\lambda = g \cdot \lambda' = \mu + \eta, \tag{3–7}$$

where $\mu = (a_1 - a_{n+1})\alpha$ is the highest restricted weight and $\eta \in \widehat{M}$ as above. To fully parametrize the weights as $\lambda = \mu + \eta$, we need a condition relating the quantity $a_1 - a_{n+1}$ to the components a_j of η $(2 \le j \le n)$.

From the first parametrization of $\widehat{U}$ we have $a_1 \ge a_2$ and $a_1 - a_2 \in \mathbb{Z}$, whence $a_1 = a_2 + k'$, $k' \in \mathbb{Z}^+$. On the other hand we also have $a_n \ge a_{n+1}$ and $a_{n+1} = -\sum_1^n a_j$, whence $k' \ge -a_2 - a_n - \sum_2^n a_j$. Putting together the two conditions we see that k' must satisfy

$$k' \ge \max\Bigl(0,\ -a_2 - a_n - \sum_2^n a_j\Bigr).$$

With this condition we get

$$\begin{aligned} a_1 - a_{n+1} &= a_1 + (a_1 + a_2 + \cdots + a_n) \\ &= 2a_1 + a_2 + \cdots + a_n \quad (\text{using } a_1 = a_2 + k') \\ &= 2a_2 + 2k' + a_2 + \cdots + a_n \quad (\text{with } k' \text{ as above}) \\ &= a_2 - a_n + \left| a_2 + a_n + \sum_2^n a_j \right| + 2k, \ \ k \in \mathbb{Z}^+, \end{aligned}$$

as immediately checked. This gives a condition on $a_1 - a_{n+1}$ as a function of $a_2, \ldots, a_n$ in order for λ' to be in $\widehat{U}$. Thus we get Kostant's result that

$$\widehat{U} \simeq \Lambda = \bigcup_{\sigma \in \widehat{M}} \mathcal{F}_\sigma,$$

where for σ fixed in $\widehat{M}$, with highest weight $\eta = a_0(\varepsilon_1 + \varepsilon_{n+1}) + \sum_2^n a_j \varepsilon_j$, we have

$$\begin{aligned} \mathcal{F}_\sigma &= \{\lambda \in \Lambda : \lambda|_{i\mathfrak{b}} = \eta\} \\ &= \{\lambda = \mu + \eta : \ \ \mu = \mu_\sigma + 2k\alpha, \ k \in \mathbb{Z}^+\} \\ &= \eta + \mu_\sigma + \Lambda_{\mathrm{sph}}, \end{aligned}$$

where $\mu_\sigma = \left(a_2 - a_n + \left|a_2 + a_n + \sum_2^n a_j\right|\right)\alpha = (a_2 - a_n + |a_2 + a_n - 2a_0|)\alpha$ and

$$\Lambda_{\mathrm{sph}} = \{2k\alpha, \ k \in \mathbb{Z}^+\}.$$

Next, comparing the branching rules for $U \supset K$ and $K \supset M$, we see that if $\lambda = \mu + \eta \in \widehat{U}$ contains $\nu \in \widehat{K}$, then ν must contain $\eta \in \widehat{M}$. We need now a condition relating μ with ν and η.

By going over the same steps as in the computation of the element μ_σ, we find that the highest restricted weights of the U-types in $\mathcal{F}_\sigma$ that contain the K-type τ with highest weight $\nu = \sum_1^{n+1} b_j \varepsilon_j$ must have the form $\mu = \mu_{\sigma,\tau} + 2k\alpha$, $k \in \mathbb{Z}^+$, where

$$\mu_{\sigma,\tau} = (b_1 - b_n + |b_1 + b_n + a_2 + \cdots + a_n|)\, \alpha.$$

This agrees with (1–3), and we again get the rule (2–3). In this case $\mu_{\sigma,\tau}$ depends explicitly on both σ and τ.

It is easy to see that $\mu_{\sigma,\tau} \geq \mu_\sigma$, with equality holding only for $a_2 = b_1$ and $a_n = b_n$, which are, respectively, the highest possible value of a_2 and the lowest of a_n (regarding τ as fixed and σ as varying over $\widehat{M}(\tau)$).

If we instead fix $\sigma \in \widehat{M}$ and let τ vary over the K-types that contain σ, then $b_1 = a_2$ is the lowest possible value of b_1 and $b_n = a_n$ the highest of b_n. Comparing the formulas for μ_σ and $\mu_{\sigma,\tau}$ we get then (1–4).

4. The case of quaternionic projective spaces

Let $U/K = P^n(\mathbb{H})$ ($n \geq 2$), with $U = \mathrm{Sp}(n+1)$, $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$. We adopt the notations of [Baldoni Silva 1979], which the reader should consult for background; see especially pp. 240–241 there for the definition of $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{m}$, and H.

The noncompact form is $\mathfrak{g} = \mathfrak{sp}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$. We fix $\mathfrak{a} = \mathbb{R}H$, so that $\mathfrak{m} \simeq \mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1)$. The group $M \simeq \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$ is connected.

Let $\mathfrak{h}_\mathfrak{k}$ be the Cartan subalgebra of $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ that is contained in $\mathfrak{k}$ and consists of the diagonal matrices. We fix the basis $\{X_j\}_{j=1}^{n+1}$ of $\mathfrak{h}_\mathfrak{k}^\mathbb{C}$ as in [Baldoni Silva 1979], and let $\{\varepsilon_j\}_{j=1}^{n+1}$ be the dual basis.

Let $\mathfrak{b} \subset \mathfrak{h}_\mathfrak{k}$ be the Cartan subalgebra of $\mathfrak{m}$ consisting of the diagonal matrices. Then $\mathfrak{h}_\mathfrak{k} = \mathfrak{b}_1 \oplus \mathfrak{b}$, where $\mathfrak{b}_1 = \mathbb{R}B_1$, B_1 the $2(n+1) \times 2(n+1)$ matrix given by

$$B_1 = \mathrm{diag}(i, 0, \ldots, 0, -i, -i, 0, \ldots, 0, i) = i(X_1 - X_{n+1}).$$

We denote by the same symbol ε_j the restriction of ε_j to $\mathfrak{b}^\mathbb{C}$.

Consider the other Cartan subalgebra $\mathfrak{h} = i\mathfrak{a} \oplus \mathfrak{b}$ of $\mathfrak{u}$. Let $g \in U$ be an element such that (3–2) holds with $\mathrm{Ad}(g)B_1 = iH$. Transporting g to the linear forms as usual, we find again (3–4), where $\alpha(H) = 1$ defines again the shorter restricted positive root α.

The root systems of the pairs $(\mathfrak{u}^\mathbb{C}, \mathfrak{h}_\mathfrak{k}^\mathbb{C})$, $(\mathfrak{k}^\mathbb{C}, \mathfrak{h}_\mathfrak{k}^\mathbb{C})$, and $(\mathfrak{m}^\mathbb{C}, \mathfrak{b}^\mathbb{C})$, are

$$\begin{aligned}
\Delta_\mathfrak{u} &= \Delta(\mathfrak{u}^\mathbb{C}, \mathfrak{h}_\mathfrak{k}^\mathbb{C}) = \{\pm\varepsilon_i \pm \varepsilon_j,\ 1 \leq i < j \leq n+1\} \cup \{\pm 2\varepsilon_j,\ 1 \leq j \leq n+1\},\\
\Delta_\mathfrak{k} &= \Delta(\mathfrak{k}^\mathbb{C}, \mathfrak{h}_\mathfrak{k}^\mathbb{C}) = \{\pm\varepsilon_i \pm \varepsilon_j,\ 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_j,\ 1 \leq j \leq n+1\},\\
\Delta_\mathfrak{m} &= \Delta(\mathfrak{m}^\mathbb{C}, \mathfrak{b}^\mathbb{C}) = \{\pm(\varepsilon_1 + \varepsilon_{n+1})\} \cup \{\pm\varepsilon_i \pm \varepsilon_j,\ 2 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_j,\ 2 \leq j \leq n\}.
\end{aligned}$$

We make the following choice of positive roots for $\mathfrak{m}$:

$$\Delta_\mathfrak{m}^+ = \{\varepsilon_1 + \varepsilon_{n+1}\} \cup \{\varepsilon_i \pm \varepsilon_j,\ 2 \leq i < j \leq n\} \cup \{2\varepsilon_j,\ 2 \leq j \leq n\}.$$

In the usual ordering of $\mathfrak{h}_\mathbb{R}^* = (\mathfrak{a} \oplus i\mathfrak{b})^*$ in which $\mathfrak{a}^*$ comes before $(i\mathfrak{b})^*$, we have the following system of positive roots of the pair $(\mathfrak{u}^\mathbb{C}, \mathfrak{h}^\mathbb{C})$:

$$\Delta^+ = \Delta^+(\mathfrak{u}^\mathbb{C}, \mathfrak{h}^\mathbb{C}) = \Delta_\mathfrak{m}^+ \cup \Delta_{2\alpha}^+ \cup \Delta_\alpha^+,$$

where

$$\begin{aligned}
\Delta_{2\alpha}^+ &= \{\beta \in \Delta^+ : \beta|_\mathfrak{a} = 2\alpha\} = \{2\alpha,\ 2\alpha + \varepsilon_1 + \varepsilon_{n+1},\ 2\alpha - (\varepsilon_1 + \varepsilon_{n+1})\},\\
\Delta_\alpha^+ &= \{\beta \in \Delta^+ : \beta|_\mathfrak{a} = \alpha\} = \{\alpha + \tfrac{1}{2}(\varepsilon_1 + \varepsilon_{n+1}) \pm \varepsilon_j,\ \alpha - \tfrac{1}{2}(\varepsilon_1 + \varepsilon_{n+1}) \pm \varepsilon_j,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \leq j \leq n\}.
\end{aligned}$$

(Note that $m_{2\alpha} = |\Delta_{2\alpha}^+| = 3$, $m_\alpha = |\Delta_\alpha^+| = 4(n-1)$, and $m_{2\alpha} + m_\alpha + 1 = 4n = \dim P^n(\mathbb{H})$.)

For the positive roots $\Delta_{\mathfrak{k}}^+$ and $\Delta_{\mathfrak{u}}^+$ of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathfrak{k}}^{\mathbb{C}}$ we make the choice

$$\Delta_{\mathfrak{k}}^+ = \{\varepsilon_i \pm \varepsilon_j,\ 1 \le i < j \le n\} \cup \{2\varepsilon_j,\ 1 \le j \le n\} \cup \{-2\varepsilon_{n+1}\},$$
$$\Delta_{\mathfrak{u}}^+ = \Delta_{\mathfrak{k}}^+ \cup \Delta_{\mathfrak{p}}^+, \quad \text{where} \quad \Delta_{\mathfrak{p}}^+ = \{\varepsilon_1 \pm \varepsilon_{n+1}\} \cup \{-\varepsilon_{n+1} \pm \varepsilon_j,\ 2 \le j \le n\}.$$

It is easily checked using (3–4) that with this choice one has

$$g \cdot \Delta_{\mathfrak{u}}^+ = \Delta^+.$$

The notion of dominance is then preserved by g, and g relates the dominant weights of $\mathfrak{u}^{\mathbb{C}}$ in the two different Cartan subalgebras.

We have the following parametrizations of $\widehat{U}$, $\widehat{K}$, and $\widehat{M}$:

$$\widehat{U} \simeq \{\lambda' = \sum_{j=1}^{n+1} a_j \varepsilon_j : a_j \in \mathbb{Z},\ \forall\, j,\ \ a_1 \ge -a_{n+1} \ge a_2 \ge \cdots \ge a_n \ge 0\},$$

$$\widehat{K} \simeq \{\nu = \sum_{j=1}^{n+1} b_j \varepsilon_j : b_j \in \mathbb{Z},\ \forall\, j,\ \ b_1 \ge b_2 \ge \cdots \ge b_n \ge 0,\ \ b_{n+1} \le 0\},$$

$$\widehat{M} \simeq \{\eta = c_0(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{j=2}^{n} c_j \varepsilon_j : c_j \in \mathbb{Z},\ \forall\, 2 \le j \le n,\ \ 2c_0 \in \mathbb{Z},$$
$$c_2 \ge c_3 \ge \cdots \ge c_n \ge 0,\ \ c_0 \ge 0\}.$$

By proceeding as in the complex case, we decompose any $\lambda' \in \widehat{U}$ as in (3–5), with η given by (3–6). Then $\eta \in \widehat{M}$, as easily seen. Applying g and using (3–4), we find that any highest weight λ of U with respect to $\mathfrak{h}^{\mathbb{C}}$ can be written as in (3–7), where again $\mu = (a_1 - a_{n+1})\alpha$ is the highest restricted weight and $\eta \in \widehat{M}$.

Let σ be a fixed M-type with highest weight

$$\eta_\sigma = a_0(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{j=2}^{n} a_j \varepsilon_j.$$

Let $\lambda = g \cdot \lambda' = \eta_\sigma + \mu$ be in $\mathscr{F}_\sigma$, then $\lambda' = \sum_1^{n+1} a_j \varepsilon_j$ with $a_1 + a_{n+1} = 2a_0$ (fixed with σ). To find the minimal element of the restricted weights $\mu = (a_1 - a_{n+1})\alpha$ (for $\lambda \in \mathscr{F}_\sigma$) write

$$a_1 - a_{n+1} = a_1 + a_{n+1} - 2a_{n+1}, \tag{4–1}$$

and observe that since $a_1 + a_{n+1}$ is fixed and $-a_{n+1} \ge a_2$, the minimum of $a_1 - a_{n+1}$ is attained when $-a_{n+1} = a_2$. Thus we get

$$\min \mathscr{F}_\sigma = \eta_\sigma + \mu_\sigma, \quad \text{where} \quad \mu_\sigma = (a_1 + a_{n+1} + 2a_2)\alpha = 2(a_0 + a_2)\alpha. \tag{4–2}$$

(Note that μ_σ is not necessarily in Λ_{sph} since a_0 can be half-odd-integer.)

The decomposition (4–1) can then be written as

$$a_1 - a_{n+1} = (a_1 + a_{n+1} + 2a_2) + 2(-a_{n+1} - a_2),$$

and since $k = -a_{n+1} - a_2 \in \mathbb{Z}^+$, we get

$$\mu = \mu_\sigma + 2k\alpha \in \mu_\sigma + \Lambda_{\text{sph}},$$

which is Kostant's result (1–2).

To find the minimal element $\mu_{\sigma,\tau}$ of $\Lambda_\sigma(\tau)$ we need the branching theorems for $U \supset K$ and $K \supset M$. The first is given in [Lepowsky 1971, Theorem 2], the second in [Baldoni Silva 1979, Theorem 5.5]. By adapting these theorems to our case (in particular to our choice of ordering) we obtain the following statements.

Theorem 4.1 (Lepowsky branching theorem for $\mathrm{Sp}(n+1) \supset \mathrm{Sp}(n) \times \mathrm{Sp}(1)$). *Let* $\lambda' = \sum_1^{n+1} a_j \varepsilon_j \in \widehat{U}$ *and* $\nu = \sum_1^{n+1} b_j \varepsilon_j \in \widehat{K}$. *Define*

$$\begin{aligned} A_1 &= a_1 - \max(-a_{n+1}, b_1), \\ A_2 &= \min(-a_{n+1}, b_1) - \max(a_2, b_2), \\ A_3 &= \min(a_2, b_2) - \max(a_3, b_3), \\ &\vdots \\ A_n &= \min(a_{n-1}, b_{n-1}) - \max(a_n, b_n), \\ A_{n+1} &= \min(a_n, b_n). \end{aligned}$$

Then the multiplicity $m(\nu, \lambda')$ *vanishes unless*

$$-b_{n+1} + \sum_1^{n+1} A_j \in 2\mathbb{Z} \tag{4–3}$$

(*or*, *equivalently*, $-a_{n+1} - b_{n+1} + \sum_1^n (a_j + b_j) \in 2\mathbb{Z}$) *and*

$$A_1 \geq 0,\ A_2 \geq 0,\ \dots,\ A_n \geq 0 \tag{4–4}$$

($A_{n+1} \geq 0$ *automatically*). *Under these conditions we have*

(4–5)
$$m(\nu, \lambda') = \sum_{L \subset \{1,2,\dots,n+1\}} (-1)^{|L|} \binom{n - 1 - |L| + \frac{1}{2}\big(b_{n+1} + \sum_1^{n+1} A_j\big) - \sum_{j \in L} A_j}{n-1},$$

where the binomial coefficient $\binom{x}{y}$ *is defined to be zero if* $x < y$.

Keeping in mind the conditions of dominance on λ' and ν, it is easy to see that (4–4) is equivalent to the following double interlacing conditions on the highest weights:

$$
(4\text{–}6)\qquad
\begin{aligned}
a_1 \geq b_1 \geq a_2 \geq b_3 \geq a_4 \geq \cdots \geq &\begin{cases} b_{n-1} \geq a_n \text{ if } n \text{ is even,} \\ a_{n-1} \geq b_n \text{ if } n \text{ is odd,} \end{cases} \\
-a_{n+1} \geq b_2 \geq a_3 \geq b_4 \geq a_5 \geq \cdots \geq &\begin{cases} a_{n-1} \geq b_n \text{ if } n \text{ is even,} \\ b_{n-1} \geq a_n \text{ if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

What makes the quaternionic case more complicated is that these conditions are only necessary but not sufficient, in general, for λ' to contain ν. This is due to the alternating sum formula (4–5), which involves a great deal of cancellation and may give zero even if λ' satisfies (4–6).

Theorem 4.2 (Baldoni Silva branching theorem for $\mathrm{Sp}(n)\times\mathrm{Sp}(1)\supset\mathrm{Sp}(n-1)\times\mathrm{Sp}(1)$). *Let* $\nu = \sum_1^{n+1} b_j\varepsilon_j \in \widehat{K}$ *and* $\eta = a_0(\varepsilon_1+\varepsilon_{n+1}) + \sum_2^n a_j\varepsilon_j \in \widehat{M}$. *Define*

$$
\begin{aligned}
A'_1 &= b_1 - \max(a_2, b_2), \\
A'_2 &= \min(a_2, b_2) - \max(a_3, b_3), \\
A'_3 &= \min(a_3, b_3) - \max(a_4, b_4), \\
&\ \vdots \\
A'_{n-1} &= \min(a_{n-1}, b_{n-1}) - \max(a_n, b_n), \\
A'_n &= \min(a_n, b_n).
\end{aligned}
$$

Then the multiplicity $m(\eta,\nu)$ *vanishes unless*

$$
(4\text{–}7)\qquad A'_1 \geq 0,\ A'_2 \geq 0,\ \ldots,\ A'_{n-1} \geq 0
$$

($A'_n \geq 0$ *automatically*) *and*

$$
(4\text{–}8)\qquad 2a_0 = -b_{n+1} + c_1 - 2l \quad \textit{for some} \quad l = 0, 1, \ldots, \min(-b_{n+1}, c_1),
$$

where c_1 *satisfies* $c_1 \in \mathbb{Z}^+$ *and*

$$
(4\text{–}9)\qquad c_1 + \sum_1^n A'_j \in 2\mathbb{Z}
$$

(*or, equivalently,* $c_1 + b_1 + \sum_2^n (a_j + b_j) \in 2\mathbb{Z}$). *Under these conditions we have*

$$
(4\text{–}10)\qquad m(\eta,\nu) = \sum_{c_1} \sum_{L\subset\{1,2,\ldots,n\}} (-1)^{|L|} \binom{n-2-|L|+\frac{1}{2}\left(-c_1+\sum_1^n A'_j\right) - \sum_{j\in L} A'_j}{n-2},
$$

where the outer sum is over all values of c_1 *satisfying* (4–8).

Again (4–7) is equivalent to the following double interlacing conditions:

$$
\begin{aligned}
&b_1 \geq a_2 \geq b_3 \geq a_4 \geq \cdots \geq \begin{cases} b_{n-1} \geq a_n \text{ if } n \text{ is even,} \\ a_{n-1} \geq b_n \text{ if } n \text{ is odd,} \end{cases} \\
&b_2 \geq a_3 \geq b_4 \geq a_5 \geq \cdots \geq \begin{cases} a_{n-1} \geq b_n \text{ if } n \text{ is even,} \\ b_{n-1} \geq a_n \text{ if } n \text{ is odd.} \end{cases}
\end{aligned} \tag{4–11}
$$

To better understand the condition (4–8), note first that it is equivalent to

$$2a_0 = -b_{n+1} + c_1, -b_{n+1} + c_1 - 2, \ldots, |-b_{n+1} - c_1|,$$

that is,

$$|-b_{n+1} - c_1| \leq 2a_0 \leq -b_{n+1} + c_1, \tag{4–12}$$

with $2a_0$ changing by steps of 2 and having the same parity as $-b_{n+1} + c_1$. By (4–12) we get similar inequalities involving $-b_{n+1}$ and c_1, namely

$$|2a_0 - c_1| \leq -b_{n+1} \leq 2a_0 + c_1, \tag{4–13}$$

$$|2a_0 + b_{n+1}| \leq \quad c_1 \quad \leq 2a_0 - b_{n+1}, \tag{4–14}$$

with $-b_{n+1}$ and c_1 changing by steps of 2 and having the same parity as $2a_0 + c_1$ and $2a_0 - b_{n+1}$, respectively.

The value of c_1 may also be required to satisfy the additional condition

$$-c_1 + \sum_1^n A'_j \geq 0,$$

for otherwise the sum over L in (4–10) gives zero. Thus the integer c_1 must satisfy

$$0 \leq c_1 \leq \sum_1^n A'_j = b_1 - \max(a_2, b_2) + \sum_2^n A'_j \equiv k_0, \tag{4–15}$$

and must have the same parity as the integer k_0, by (4–9).

By (4–14) and (4–15) we see that the allowed values of c_1 must satisfy

$$|2a_0 + b_{n+1}| \leq c_1 \leq \min(2a_0 - b_{n+1}, k_0). \tag{4–16}$$

For example, suppose $\nu \in \widehat{K}$ is fixed and we want to compute the M-types of ν. By (4–11) we determine the possible values of $(a_2, a_3, \ldots, a_n)$ (a finite number of $(n-1)$-tuples). For each such $(n-1)$-tuple we find the allowed values of $2a_0$ using (4–12) with c_1 subject to (4–9) and (4–15). A given value of $2a_0$ will be obtained for different values of c_1, namely those satisfying (4–16). The sum over c_1 in the multiplicity formula (4–10) will then be over these values. On the other hand if $\eta \in \widehat{M}$ is fixed, we use instead (4–13) to find the allowed values of $-b_{n+1}$, again with c_1 subject to (4–9) and (4–15).

For later use note the following. From (4–16) we get the inequality

$$|2a_0 + b_{n+1}| \le k_0. \tag{4–17}$$

If we let

$$2k = 2a_0 + b_{n+1} - k_0 + 2\sum_2^n A'_j, \tag{4–18}$$

then (4–17) implies that k is an integer and that

$$-(b_1 - \max(a_2, b_2)) \le k \le \sum_2^n A'_j. \tag{4–19}$$

We are now ready to prove the following result, which gives the minimal element of $\Lambda_\sigma(\tau)$ explicitly in most (though not all) of the cases.

Theorem 4.3. *Let τ be a fixed K-type with highest weight $\nu = \sum_1^{n+1} b_j\varepsilon_j$, and let σ be a fixed M-type with highest weight $\eta_\sigma = a_0(\varepsilon_1 + \varepsilon_{n+1}) + \sum_2^n a_j\varepsilon_j$ such that $\sigma \subset \tau|_M$. For each λ in $\mathscr{F}_\sigma$ write $\lambda = g \cdot \lambda'$ with $\lambda' = \sum_1^{n+1} a_j\varepsilon_j$, so that $a_1 + a_{n+1} = 2a_0$ is fixed with σ, and the highest restricted weight of λ is $\mu = (a_1 - a_{n+1})\alpha$. Define the elements*

$$\begin{aligned} r_{\sigma,\tau} &= b_1 - b_{n+1} + \max(a_2, b_2) - \sum_3^{n+1} A_j, & \lambda_0 &= \eta_\sigma + r_{\sigma,\tau}\alpha,\\ s_{\sigma,\tau} &= 2a_0 + 2\max(a_2, b_2), & \lambda_1 &= \eta_\sigma + s_{\sigma,\tau}\alpha,\\ t_{\sigma,\tau} &= 2b_1 - 2a_0, & \lambda_2 &= \eta_\sigma + t_{\sigma,\tau}\alpha. \end{aligned}$$

Then

$$a_1 - a_{n+1} \ge \max(r_{\sigma,\tau},\ s_{\sigma,\tau},\ t_{\sigma,\tau}), \quad \forall\lambda \in \mathscr{F}_\sigma \cap \widehat{U}(\tau). \tag{4–20}$$

If $\max(r_{\sigma,\tau}, s_{\sigma,\tau}, t_{\sigma,\tau}) = r_{\sigma,\tau}$ *then the minimal element of $\Lambda_\sigma(\tau)$ is*

$$\mu_{\sigma,\tau} = r_{\sigma,\tau}\alpha, \quad \textit{with} \quad m(\tau, \delta_{\lambda_0}) = 1. \tag{4–21}$$

If $\max(r_{\sigma,\tau}, s_{\sigma,\tau}, t_{\sigma,\tau}) = s_{\sigma,\tau}$ *then $\mu_{\sigma,\tau} = (s_{\sigma,\tau} + 2p)\alpha$, where p is the first integer with $0 \le p \le b_1 - \max(a_2, b_2)$ such that the element $\lambda = \eta_\sigma + (s_{\sigma,\tau} + 2p)\alpha$ satisfies*

$$m(\tau, \delta_\lambda) = \sum_{L\subset\{1,2,\dots,n+1\}} (-1)^{|L|}\binom{n-1-|L|+\frac{1}{2}(s_{\sigma,\tau}-r_{\sigma,\tau})+p-\sum_{j\in L}A_j}{n-1} \ne 0.$$

(*In most cases $p = 0$, but there are some special cases where $p > 0$; see below.*) *A similar conclusion holds if* $\max(r_{\sigma,\tau}, s_{\sigma,\tau}, t_{\sigma,\tau}) = t_{\sigma,\tau}$, *with $s_{\sigma,\tau}$ replaced by $t_{\sigma,\tau}$ and* $0 \le p \le 2a_0$.

Proof. We first observe that for all $\lambda \in \mathscr{F}_\sigma$, $A_j = A'_{j-1}$, $\forall j = 3, \dots, n+1$ (in the notations of Theorems 4.1 and 4.2). Therefore the quantities A_j, $j \ge 3$, are the same for all λ in $\mathscr{F}_\sigma$ since they depend on σ and τ only (not on the highest restricted weight of λ). The interlacing conditions $A_j \geq 0$ ($j \geq 3$), as well as the condition $b_1 \geq \max(a_2, b_2)$, then follow immediately from the branching law for $K \supset M$ and the fact that $\sigma \subset \tau|_M$. The other conditions $A_1 \geq 0$ and $A_2 \geq 0$ for $\lambda \in \mathcal{F}_\sigma \cap \widehat{U}(\tau)$ give (compare (4–6)):

$$a_1 \geq b_1, \quad -a_{n+1} \geq \max(a_2, b_2),$$

whence

$$a_1 - a_{n+1} = 2a_0 - 2a_{n+1} \geq 2a_0 + 2\max(a_2, b_2) = s_{\sigma,\tau},$$

and

$$a_1 - a_{n+1} = 2a_1 - 2a_0 \geq 2b_1 - 2a_0 = t_{\sigma,\tau}.$$

Thus

$$a_1 - a_{n+1} \geq \max(s_{\sigma,\tau}, t_{\sigma,\tau}), \quad \forall \lambda \in \mathcal{F}_\sigma \cap \widehat{U}(\tau). \tag{4–22}$$

Secondly, from the Lepowsky multiplicity formula (4–5), we see that $m(\nu, \lambda')$ vanishes unless

$$b_{n+1} + \sum_1^{n+1} A_j \geq 0. \tag{4–23}$$

This condition may be regarded as an additional interlacing condition necessary for $m(\nu, \lambda') > 0$. Unlike (4–6), (4–23) involves the parameter b_{n+1}, which is the highest weight of the representation $\tau|_{\mathbf{1}\times \mathrm{Sp}(1)}$. Using (4–3) we rewrite (4–23) as

$$b_{n+1} + \sum_1^{n+1} A_j \in 2\mathbb{Z}^+, \quad \forall \lambda \in \mathcal{F}_\sigma \cap \widehat{U}(\tau). \tag{4–24}$$

To gain more information from (4–23)–(4–24), we divide the elements of $\mathcal{F}_\sigma$ into two classes, namely we say $\lambda \in \mathcal{F}_\sigma$ is in class 1 if $-a_{n+1} > b_1$, in class 2 if $-a_{n+1} \leq b_1$. These two classes are separated by the element λ_3 with $-a_{n+1} = b_1$, i.e., $\lambda_3 = \eta_\sigma + 2(a_0 + b_1)\alpha$. Class 1 is certainly nonempty and actually infinite. (If we had $-a_{n+1} \leq b_1$ for all $\lambda \in \mathcal{F}_\sigma$, then $\mathcal{F}_\sigma$ would be bounded by λ_3, whereas we know that $\mathcal{F}_\sigma = \eta_\sigma + \mu_\sigma + \Lambda_{\mathrm{sph}}$ by Kostant's result.)

For λ in class 1 or for $\lambda = \lambda_3$ we have $A_1 = a_1 + a_{n+1} = 2a_0$ and $A_2 = b_1 - \max(a_2, b_2)$, and the double interlacing conditions (4–4) are automatically satisfied. Since A_1 and A_2 (like A_j, $j \geq 3$) depend on σ and τ only, all λ in class 1 have the same A_j as λ_3, $\forall j$. The same holds for the quantity

$$\begin{aligned} b_{n+1} + \sum_1^{n+1} A_j &= b_{n+1} + 2a_0 + b_1 - \max(a_2, b_2) + \sum_3^{n+1} A_j \\ &= 2(a_0 + b_1) - r_{\sigma,\tau} = s_{\sigma,\tau} - r_{\sigma,\tau} + 2(b_1 - \max(a_2, b_2)). \end{aligned} \tag{4–25}$$

It then follows from (4–5) that all δ_λ with λ in class 1 must contain τ with the same multiplicity as δ_{λ_3}. This multiplicity cannot be zero, for otherwise $\mathscr{F}_\sigma \cap \widehat{U}(\tau)$ would be finite (class 2 being finite, see below), whereas we know that $\mathscr{F}_\sigma \cap \widehat{U}(\tau) = \eta_\sigma + \mu_{\sigma,\tau} + \Lambda_{\text{sph}}$ by Theorem 1.1. In conclusion, we have

$$m(\tau, \delta_\lambda) = m(\tau, \delta_{\lambda_3}) > 0, \qquad \forall\lambda \text{ in class } 1,$$

and the minimal element $\eta_\sigma + \mu_{\sigma,\tau}$ must be $\leq \lambda_3$. Moreover the quantity $b_{n+1} + \sum_1^{n+1} A_j$ in (4–25) must be ≥ 0 and actually in $2\mathbb{Z}^+$ by (4–24). This can easily be checked independently using the branching rule for $K \supset M$. In fact the right hand side of (4–25) equals $2a_0 + b_{n+1} + k_0$ (where k_0 is defined in (4–15)), and our claim follows easily from (4–17).

Formula (4–25) then implies

$$a_1 - a_{n+1} > (a_1 - a_{n+1})_{\lambda=\lambda_3} = 2a_0 + 2b_1 \geq r_{\sigma,\tau}, \qquad \forall\lambda \text{ in class } 1. \tag{4–26}$$

Class 2 consists of those $\lambda \in \mathscr{F}_\sigma$ such that $\mu_\sigma \leq \mu \leq 2(a_0 + b_1)\alpha$, i.e.,

$$2(a_0 + a_2) \leq a_1 - a_{n+1} \leq 2(a_0 + b_1).$$

For λ in class 2 we have $A_1 = a_1 - b_1$ and $A_2 = -a_{n+1} - \max(a_2, b_2)$, so that

$$\begin{aligned} b_{n+1} + \sum_1^{n+1} A_j &= b_{n+1} + a_1 - a_{n+1} - b_1 - \max(a_2, b_2) + \sum_3^{n+1} A_j \\ &= (a_1 - a_{n+1}) - r_{\sigma,\tau}. \end{aligned} \tag{4–27}$$

Still for λ in class 2, if $m(\tau, \delta_\lambda)$ is positive, (4–23) and (4–27) yield

$$a_1 - a_{n+1} \geq r_{\sigma,\tau}.$$

Now this and (4–26) imply

$$a_1 - a_{n+1} \geq r_{\sigma,\tau}, \qquad \forall\lambda \in \mathscr{F}_\sigma \cap \widehat{U}(\tau),$$

which, together with (4–22), proves (4–20).

Note that the quantity $s_{\sigma,\tau} - r_{\sigma,\tau}$ must be in $2\mathbb{Z}$ since the right hand side of (4–25) is in $2\mathbb{Z}^+$. In fact $s_{\sigma,\tau} - r_{\sigma,\tau}$ is just the right hand side of (4–18), so that $s_{\sigma,\tau} - r_{\sigma,\tau} = 2k \in 2\mathbb{Z}$, with k satisfying (4–19). Thus $s_{\sigma,\tau}$ can be greater, equal or less than $r_{\sigma,\tau}$, in general.

Now let us suppose that $\max(r_{\sigma,\tau}, s_{\sigma,\tau}, t_{\sigma,\tau}) = r_{\sigma,\tau}$. Then the element $\lambda_0 = \eta_\sigma + r_{\sigma,\tau}\alpha$ is in $\mathscr{F}_\sigma$ and it is in class 2, by (4–26). Moreover λ_0 satisfies all of the interlacing conditions (4–6). Indeed if we solve for a_1 and $-a_{n+1}$ from the two relations $a_1 + a_{n+1} = 2a_0$, $a_1 - a_{n+1} = r_{\sigma,\tau}$, we get

$$a_1 = \tfrac{1}{2}(r_{\sigma,\tau} + 2a_0) = \tfrac{1}{2}\big(b_1 - b_{n+1} + \max(a_2, b_2) - \textstyle\sum_3^{n+1} A_j + 2a_0\big),$$
$$-a_{n+1} = \tfrac{1}{2}(r_{\sigma,\tau} - 2a_0) = \tfrac{1}{2}\big(b_1 - b_{n+1} + \max(a_2, b_2) - \textstyle\sum_3^{n+1} A_j - 2a_0\big).$$

The condition $a_1 \geq b_1$ is then equivalent to $r_{\sigma,\tau} \geq t_{\sigma,\tau}$, while the condition $-a_{n+1} \geq \max(a_2, b_2)$ is equivalent to $r_{\sigma,\tau} \geq s_{\sigma,\tau}$.

For $\lambda = \lambda_0$ we have by (4–27)

$$b_{n+1} + \sum_1^{n+1} A_j = r_{\sigma,\tau} - r_{\sigma,\tau} = 0,$$

and λ_0 satisfies (4–3), being equal to $-b_{n+1} + \sum_1^{n+1} A_j = -2b_{n+1} \in 2\mathbb{Z}^+$. By applying the multiplicity formula (4–5) to λ_0 we see that only $L = \varnothing$ contributes to the sum over L in this case, and we get

$$m(\tau, \delta_{\lambda_0}) = 1.$$

In view of (4–20), this proves (4–21).

Now let $\max(r_{\sigma,\tau}, s_{\sigma,\tau}, t_{\sigma,\tau}) = s_{\sigma,\tau}$, with $s_{\sigma,\tau} > r_{\sigma,\tau}$. Then the element $\lambda_1 = \eta_\sigma + s_{\sigma,\tau}\alpha$ (which is always in class 2) satisfies the double interlacing conditions. Indeed for $\lambda = \lambda_1$ we have $a_1 = 2a_0 + \max(a_2, b_2)$ and $-a_{n+1} = \max(a_2, b_2)$, so that

$$A_1 = 2a_0 + \max(a_2, b_2) - b_1, \qquad A_2 = 0.$$

The condition $A_1 \geq 0$ is then equivalent to $s_{\sigma,\tau} \geq t_{\sigma,\tau}$. For $\lambda = \lambda_1$ we have by (4–27)

$$b_{n+1} + \sum_1^{n+1} A_j = s_{\sigma,\tau} - r_{\sigma,\tau}, \tag{4–28}$$

which is greater than zero in this case. Actually we have $s_{\sigma,\tau} - r_{\sigma,\tau} = 2k \in 2\mathbb{Z}^+$, with $0 < k \leq \sum_3^{n+1} A_j$; compare (4–19). By (4–5) we have

$$m(\tau, \delta_{\lambda_1}) = \sum_{L \subset \{1,2,\dots,n+1\}} (-1)^{|L|} \binom{n - 1 - |L| + k - \sum_{j \in L} A_j}{n-1}, \tag{4–29}$$

with a subset L contributing to the sum if and only if

$$k \geq |L| + \sum_{j \in L} A_j. \tag{4–30}$$

For example, $L = \varnothing$ and $L = \{2\}$ always contribute to the sum. One would expect $m(\tau, \delta_{\lambda_1})$ to be always nonzero, yielding $s_{\sigma,\tau}\alpha$ as the minimal element $\mu_{\sigma,\tau}$. This is true in most of the cases but not always. For some special values of a_0, b_1 and A_j ($j \geq 3$) we actually get zero from the formula above.

Let, e.g., $k = 1$, that is, $s_{\sigma,\tau} - r_{\sigma,\tau} = 2$. By (4–30) the subsets with $|L| \geq 2$ do not contribute to the sum. Besides $L = \varnothing, \{2\}$, the subset $L = \{j\}$ contributes if and only if $A_j = 0$. The numbers $A_1, A_3, \ldots, A_{n+1}$ cannot all be zero since (4–28) would give then $b_{n+1} = 2$, while $b_{n+1} < 0$. Similarly, the numbers $A_3, \ldots, A_{n+1}$ cannot all be zero since this would conflict with (4–19) (k being 1). If q is the number of vanishing A_j, $j \neq 2$, then (4–29) gives

$$m(\tau, \delta_{\lambda_1}) = n - 1 - q.$$

This is zero if $q = n - 1$, that is, when A_1 and $n - 2$ of the $n - 1$ numbers A_j, $j \geq 3$, vanish. The nonvanishing one, A_{j_1}, will satisfy $A_{j_1} \geq 2$ since $b_{n+1} + A_{j_1} = 2$ by (4–28). We conclude that for $s_{\sigma,\tau} - r_{\sigma,\tau} = 2$ the minimal element of $\Lambda_\sigma(\tau)$ is $\mu_{\sigma,\tau} = s_{\sigma,\tau}\alpha$, except when the following condition holds:

$$\begin{cases} A_j = 0, \ \forall j \geq 3, \ j \neq j_1, \ A_{j_1} \geq 2, \\ b_1 = 2a_0 + \max(a_2, b_2). \end{cases}$$

In this case we compute

$$m(\tau, \delta_\lambda) = 1 \quad \text{for } \lambda = \eta_\sigma + (s_{\sigma,\tau} + 2)\alpha,$$

so that $\mu_{\sigma,\tau} = (s_{\sigma,\tau} + 2)\alpha$. If $k \geq 2$ we can reason in a similar way, but we get more cases in which $\mu_{\sigma,\tau} > s_{\sigma,\tau}\alpha$. In general we have then

$$s_{\sigma,\tau}\alpha \leq \mu_{\sigma,\tau} \leq 2(a_0 + b_1)\alpha,$$

i.e., $\mu_{\sigma,\tau} = (s_{\sigma,\tau} + 2p)\alpha$, where p is the first integer such that $0 \leq p \leq b_1 - \max(a_2, b_2)$ and $m(\tau, \delta_\lambda) > 0$ for $\lambda = \eta_\sigma + (s_{\sigma,\tau} + 2p)\alpha$.

Finally, let $\max(r_{\sigma,\tau}, s_{\sigma,\tau}, t_{\sigma,\tau}) = t_{\sigma,\tau}$ with $t_{\sigma,\tau} > r_{\sigma,\tau}$. Then the element $\lambda_2 = \eta_\sigma + t_{\sigma,\tau}\alpha$ is in $\mathscr{F}_\sigma$, it is in class 2, and satisfies the double interlacing conditions. Indeed for $\lambda = \lambda_2$ we get $a_1 = b_1$ and $-a_{n+1} = b_1 - 2a_0$, so that

$$A_1 = 0, \qquad A_2 = b_1 - 2a_0 - \max(a_2, b_2).$$

The condition $A_2 \geq 0$ is equivalent to $t_{\sigma,\tau} \geq s_{\sigma,\tau}$. For $\lambda = \lambda_2$ we have

$$b_{n+1} + \sum_1^{n+1} A_j = t_{\sigma,\tau} - r_{\sigma,\tau} = 2k,$$

with k a positive integer. The multiplicity $m(\tau, \delta_{\lambda_2})$ is given by the same formula (4–29), and we can repeat similar considerations as in the previous case. We get $\mu_{\sigma,\tau} = (t_{\sigma,\tau} + 2p)\alpha$, where p is the first integer such that $0 \leq p \leq 2a_0$ and $m(\tau, \delta_\lambda) > 0$ for $\lambda = \eta_\sigma + (t_{\sigma,\tau} + 2p)\alpha$. This finishes the proof of the theorem. □

Example 1. Consider the spinor K-types τ_j, with highest weights

$$\nu_j = \sum_{k=1}^{j} \varepsilon_k - (n-j)\varepsilon_{n+1} \qquad (0 \le j \le n).$$

(It is understood that $\sum_{k=a}^{b} = 0$ if $b < a$.) Let σ_j, σ'_j be the M-types with respective highest weights

$$\eta_j = \sum_{k=2}^{j+1} \varepsilon_k + \frac{n-j}{2}(\varepsilon_1 + \varepsilon_{n+1}) \qquad (0 \le j \le n-1),$$
$$\eta'_j = \sum_{k=2}^{j} \varepsilon_k + \frac{n-j-1}{2}(\varepsilon_1 + \varepsilon_{n+1}) \qquad (1 \le j \le n-1).$$

Theorem 4.2 gives the following M-decompositions of the K-types τ_j, all with multiplicity one:

$$\tau_0|_M = \sigma_0, \qquad \tau_1|_M = \sigma_1 \oplus \sigma'_1 \oplus \sigma_0,$$
$$\tau_j|_M = \sigma_j \oplus \sigma'_j \oplus \sigma_{j-1} \oplus \sigma'_{j-1} \quad (2 \le j \le n-1),$$
$$\tau_n|_M = \sigma_{n-1} \oplus \sigma'_{n-1}.$$

(See [Camporesi and Pedon 2002, Lemma 4.1]; note the misprint in the decomposition of $\tau_n|_M$, where σ_n should read σ_{n-1}.)

It is an easy matter to compute the minimal element $\mu_{\sigma,\tau}$ for each pair (σ, τ) with $\sigma \in \widehat{M}(\tau)$. The result is as follows.

For $(\sigma, \tau) = (\sigma'_1, \tau_1)$ we get

$$\mu_{\sigma,\tau} = r_{\sigma,\tau}\alpha = n\alpha > s_{\sigma,\tau}\alpha = (n-2)\alpha,$$

and $\lambda_0 = \lambda_3 > \lambda_1$. The element $\lambda_1 = \eta_\sigma + s_{\sigma,\tau}\alpha$ satisfies the double interlacing conditions (4–6) for any $n \ge 3$, but it does not contain τ (formula (4–5) gives zero since $b_{n+1} + \sum_1^{n+1} A_j = -2 < 0$). This shows that the double interlacing conditions (4–6) are not sufficient, in general, for ν to occur in λ'.

For $(\sigma, \tau) = (\sigma_{j-1}, \tau_j)$ $(2 \le j \le n)$ we get

$$\mu_{\sigma,\tau} = s_{\sigma,\tau}\alpha = (n+3-j)\alpha > r_{\sigma,\tau}\alpha = (n+1-j)\alpha,$$

and $\lambda_1 = \lambda_3 > \lambda_0$. For all remaining cases we get

$$\mu_{\sigma,\tau} = s_{\sigma,\tau}\alpha = r_{\sigma,\tau}\alpha,$$

and $\lambda_0 = \lambda_1 = \lambda_3$, except for $(\sigma, \tau) = (\sigma_0, \tau_1)$ where $\lambda_0 = \lambda_1 < \lambda_3$.

In all cases we have $\lambda_2 \le \lambda_0$ and $\lambda_2 \le \lambda_1$, except for $(\sigma, \tau) = (\sigma'_1, \tau_1)$ and $n = 2$, where $0 = \lambda_1 < \lambda_2 = \lambda_0$.

Example 2. Let τ be the K-type with highest weight $\nu = 2\varepsilon_1 + \varepsilon_2$, and σ the M-type with highest weight $\eta_\sigma = \varepsilon_2$. Theorem 4.2 implies easily that σ occurs in $\tau|_M$ with multiplicity $m(\sigma, \tau) = 1$. One computes

$$\mu_{\sigma,\tau} = t_{\sigma,\tau}\alpha = 4\alpha > s_{\sigma,\tau}\alpha = r_{\sigma,\tau}\alpha = 2\alpha,$$

and $\lambda_2 = \lambda_3 > \lambda_1 = \lambda_0$. This shows that $t_{\sigma,\tau}$ can be greater than $s_{\sigma,\tau}$ and $r_{\sigma,\tau}$.

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