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We give examples of bihermitian compact surfaces (M, g) whose Ricci tensor ρ satisfies $\nabla_X \rho(X, X) = \frac{1}{3} X\tau g(X, X)$. We construct one-parameter families of such metrics on all the Hirzebruch surfaces Σ_k, for $k \geq 0$.

Introduction

Let (M, g) be a riemannian manifold with Ricci tensor ρ satisfying

$$\nabla_X \rho(X, X) = \frac{2}{n+2} X\tau g(X, X), \tag{0–1}$$

where τ is the scalar curvature of (M, g) and $n = \dim M$. A. Gray [1978] called such manifolds $\mathcal{A}C^{\perp}$ *manifolds*. Many interesting manifolds are of this type, including (compact) Einstein–Weyl manifolds [Jelonek 1999], weakly self-dual Kähler surfaces [Jelonek 2002a; Apostolov et al. 2003] and D'Atri spaces.

In [Jelonek 2002a] we showed that every Kähler surface has a harmonic anti-self-dual part W^- of the Weyl tensor W (i.e. such that $\delta W^- = 0$) if and only if it is an $\mathcal{A}C^{\perp}$-manifold. Later [2002b] we gave an example of a Kähler $\mathcal{A}C^{\perp}$-metric on a Hirzebruch surface Σ_1; this example was independently found by Apostolov, Calderbank and Gauduchon [Apostolov et al. 2003], who additionally proved that this is the only compact Kähler $\mathcal{A}C^{\perp}$-surface with nonconstant scalar curvature. We show here that for hermitian surfaces the situation is different: there are many examples of hermitian $\mathcal{A}C^{\perp}$-surfaces with nonconstant scalar curvature.

In [Jelonek 2002a] we also showed that any simply connected 4-dimensional $\mathcal{A}C^{\perp}$-manifold (M, g) whose Ricci tensor has exactly two eigenvalues of multiplicity 2 admits two mutually opposite hermitian structures commuting with the Ricci tensor. Surfaces admitting two oppositely oriented complex structures will be called *bihermitian surfaces*. (The reader should be warned that this term has been used differently in [Apostolov et al. 2003], where it means a surface admitting two positively oriented hermitian structures).

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Proposition 0.1 [Jelonek 2002b]. *Let (M, g) be a compact 4-manifold with even first Betti number admitting two opposite to each other hermitian structures J, $\bar{J}$ which commute with the Ricci tensor ρ of (M, g). Then M is a ruled surface or is locally a product of two riemannian surfaces.* □

Here we construct examples of $\mathcal{A}C^{\perp}$-metrics with nonconstant scalar curvature and bihermitian Ricci tensor. We shall call such surfaces with the appropriate riemannian structure the Gray surfaces. We shall construct our examples on Hirzebruch surfaces Σ_k which are ruled surfaces of genus 0.

Using the methods of Bérard-Bergery [1982] (see also [Sentenac 1981; Page 1978]) we reduce the problem to a certain ODE of the second order. We show that this equation has a positive solution satisfying the appropriate boundary conditions and we shall prove in this way the existence of bihermitian $\mathcal{A}C^{\perp}$-metrics. In this way we also give new examples of compact 4-dimensional $\mathcal{A}C^{\perp}$-manifolds; compare [Besse 1987, p. 433].

1. Hermitian 4-manifolds

Let (M, g, J) be an almost hermitian manifold. We say that (M, g, J) is a hermitian manifold if its almost hermitian structure J is integrable. In the sequel we shall consider 4-dimensional hermitian manifolds (M, g, J), which we shall also call hermitian surfaces. Such manifolds are always oriented and we choose an orientation in such a way that the Kähler form $\Omega(X, Y) = g(JX, Y)$ is self-dual (that is, $\Omega \in \wedge^{+}M$). The vector bundle of self-dual forms admits a decomposition

$$\wedge^{+}M = \mathbb{R}\Omega \oplus LM,$$

where $LM = \{\Phi \in \wedge M : \Phi(JX, JY) = -\Phi(X, Y)$ is the bundle of real J-skew invariant 2-forms. LM is a complex line bundle over M with the complex structure $\mathcal{J}$ defined by $(\mathcal{J}\Phi)(X, Y) = -\Phi(JX, Y)$. For a 4-dimensional hermitian manifold the covariant derivative of the Kähler form Ω is locally expressed by

$$\nabla\Omega = a \otimes \Phi + \mathcal{J}a \otimes \mathcal{J}\Phi, \tag{1–1}$$

where $\mathcal{J}a(X) = -a(JX)$.

An opposite (almost) hermitian structure on a hermitian 4-manifold (M, g, J) is an (almost) hermitian structure $\overline{J}$ whose Kähler form (with respect to g) is anti-self-dual.

On a riemannian manifold a distribution $\mathcal{D} \subset TM$ is called *umbilical* [Jelonek 2000] if $\nabla_X X_{|\mathcal{D}^{\perp}} = g(X, X)\xi$ for every $X \in \Gamma(\mathcal{D})$, where $X_{|\mathcal{D}^{\perp}}$ is the $\mathcal{D}^{\perp}$ component of X with respect to the orthogonal decomposition $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$. The vector field ξ is called the mean curvature normal of $\mathcal{D}$. An involutive distribution $\mathcal{D}$ is tangent to a foliation, which is called totally geodesic if its every leaf is a totally geodesic submanifold of (M, g), i.e., $\nabla_X X \in \mathcal{D}$ if X is a section of a vector bundle $\mathcal{D} \subset TM$. In the sequel we shall not distinguish between $\mathcal{D}$ and a tangent foliation and we shall also say that $\mathcal{D}$ is totally geodesic in such a case.

On any hermitian non-Kähler 4-manifold (M, g, J) there are two natural distributions $\mathcal{D} = \{X \in TM : \nabla_X J = 0\}$ and $\mathcal{D}^\perp$ defined in the open set $U = \{x : |\nabla J_x| \neq 0\}$. We shall call $\mathcal{D}$ the *nullity distribution* of (M, g, J). From (1–1) it is clear that $\mathcal{D}$ is J-invariant and that $\dim \mathcal{D} = 2$ in $U = \{x \in M : \nabla J_x \neq 0\}$. By $\mathcal{D}^\perp$ we shall denote the orthogonal complement of $\mathcal{D}$ in U. On U we can define the opposite almost hermitian structure $\bar{J}$ by formulas $\bar{J}X = JX$ if $X \in \mathcal{D}^\perp$ and $\bar{J}X = -JX$ if $X \in \mathcal{D}$, which we shall call natural opposite almost hermitian structure. It is not difficult to check that for the famous Einstein hermitian manifold $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ with the Page metric [1978] (see also [Bérard-Bergery 1982; Sentenac 1981; Koda 1993; LeBrun 1997]) the opposite structure $\bar{J}$ is hermitian and extends to the global opposite hermitian structure.

By an $\mathcal{A}C^\perp$*-manifold* [Gray 1978] we mean a riemannian manifold (M, g) satisfying

$$\mathfrak{C}_{XYZ} \nabla_X \rho(Y, Z) = \frac{2}{\dim M + 2} \mathfrak{C}_{XYZ} X\tau g(Y, Z),$$

where ρ is the Ricci tensor of (M, g) and $\mathfrak{C}$ means the cyclic sum. A riemannian manifold (M, g) is an $\mathcal{A}C^\perp$ manifold if and only if the Ricci endomorphism Ric of (M, g) is of the form $\mathrm{Ric} = S + \frac{2}{n+2}\tau\, \mathrm{Id}$, where S is a Killing tensor, τ is the scalar curvature and $n = \dim M$. Recall that a $(1, 1)$-tensor S on a riemannian manifold (M, g) is called a Killing tensor if $g(\nabla S(X, X), X) = 0$ for all $X \in TM$. It is not difficult to prove the following lemma:

Lemma. *Let $S \in End(TM)$ be a $(1, 1)$-tensor on a riemannian 4-manifold (M, g). Assume that S has exactly two everywhere different eigenvalues λ, μ of the same multiplicity 2, i.e., $\dim D_\lambda = \dim D_\mu = 2$, where D_λ, D_μ are eigendistributions of S corresponding to λ, μ respectively. Then S is a Killing tensor if and only if both distributions D_λ and D_μ are umbilical with mean curvature normal equal respectively*

$$\xi_\lambda = \frac{\nabla \mu}{2(\lambda - \mu)}, \qquad \xi_\mu = \frac{\nabla \lambda}{2(\mu - \lambda)}.$$

2. Gray surfaces

Let (M, g_0) be a compact riemannian surface of constant curvature $K \in \mathbb{R}$ and let $p : P \to M$ be a principal circle bundle over M with a connection form θ such that $d\theta = cp^*\omega$, where ω is the volume form of (M, g) and $c \in \mathbb{R}$. The manifold P with the metric $g_P = \theta \otimes \theta + p^* g_0$ is a 3-dimensional $\mathcal{A}$-manifold. Let $\theta^\sharp$ be a vector field dual to θ with respect to g_P. Consider a local orthonormal frame $\{X, Y\}$ on

(M, g_0) and let X^h, Y^h be horizontal lifts of X, Y with respect to $p : P \to M$ (so that $\theta(X^h) = \theta(Y^h) = 0$ and $p(X^h) = X,\ p(Y^h) = Y$). Set $H = \partial/\partial t$.

Now consider the manifold $Q = \mathbb{R} \times P$ with the metric

$$g_{f,h} = dt \otimes dt + f(t)^2 \theta \otimes \theta + h(t)^2 p^* g_0, \tag{2–1}$$

where $f, h \in C^\infty((a, b)) \cap C([a, b])$ and $f > 0$ on (a, b) and $h > 0$ on $[a, b]$. We define two almost hermitian structures $J, \bar{J}$ on Q as follows:

$$JH = \frac{1}{f}\theta^\sharp,\ JX^h = Y^h,\ \ \bar{J}H = -\frac{1}{f}\theta^\sharp,\ \bar{J}X^h = Y^h.$$

Proposition 2.1 [Jelonek 2002b]. *Let $\mathfrak{D}$ be a distribution spanned by the fields $\{\theta^\sharp, H\}$. Then $\mathfrak{D}$ is a totally geodesic foliation with respect to the metric $g_{f,h}$. Both structures $J, \bar{J}$ are hermitian and $\mathfrak{D}$ is contained in the nullity of J and $\bar{J}$. The distribution $\mathfrak{D}^\perp$ is umbilical with the mean curvature normal $\xi = -\nabla \ln h$.* □

Proposition 2.2. *Let (M, g) be a 4-dimensional riemannian manifold whose Ricci tensor ρ has two eigenvalues $\lambda_0(x), \mu_0(x)$ of the same multiplicity 2 at every point x of M. Assume that the eigendistribution $\mathfrak{D}_\lambda = \mathfrak{D}$ corresponding to λ_0 is a totally geodesic foliation and the eigendistribution $\mathfrak{D}_\mu = \mathfrak{D}^\perp$ corresponding to μ_0 is umbilical. Then (M, g) is an $\mathcal{A}C^\perp$-manifold if and only if $\lambda_0 - 2\mu_0$ is constant and $\nabla\tau \in \Gamma(\mathfrak{D})$. The distributions $\mathfrak{D}, \mathfrak{D}^\perp$ determine two hermitian structures $J, \bar{J}$ which are opposite to each other and commute with ρ. Both structures $J, \bar{J}$ are hermitian and $\mathfrak{D}$ is contained in the nullity of J and $\bar{J}$.*

Proof. Let S_0 be the Ricci endomorphism of (M, g), so that $\rho(X, Y) = g(S_0 X, Y)$. Let the tensor S be defined by the formula

$$S_0 = S + \frac{\tau}{3}\,\mathrm{id}\,.$$

Then $\operatorname{tr} S = -\frac{1}{3}\tau$. Let λ_0, μ_0 be eigenfunctions of S_0 and assume that

$$\lambda_0 - 2\mu_0 = 3C$$

is constant. S also has two eigenfunctions, which we denote by λ, μ. It is easy to see that $\lambda_0 = \frac{1}{3}\tau + C$ and $\mu_0 = \frac{1}{6}\tau - C$. Then $\mu = -\frac{1}{6}\tau - C$ and $\lambda = C$ is constant. Since the distribution $\mathfrak{D}^\perp$ is umbilical we have $\nabla_X X_{|\mathfrak{D}} = g(X, X)\xi$ for any $X \in \Gamma(\mathfrak{D}^\perp)$ where ξ is the mean curvature normal of $\mathfrak{D}^\perp$. Since the distribution $\mathfrak{D}$ is totally geodesic we also have $\nabla_X X_{|\mathfrak{D}^\perp} = 0$ for any $X \in \Gamma(\mathfrak{D})$. Let $\{E_1, E_2, E_3, E_4\}$ be a local orthonormal basis of TM such that $\mathfrak{D}^\perp = \operatorname{span}\{E_1, E_2\}$ and $\mathfrak{D} = \operatorname{span}\{E_3, E_4\}$. Then $\nabla_{E_1} E_{1|\mathfrak{D}} = \nabla_{E_2} E_{2|\mathfrak{D}} = \xi$. Consequently, noting that $\nabla\mu_{|\mathfrak{D}^\perp}$ vanishes if and only if $\nabla\tau_{|\mathfrak{D}^\perp}$ does, we get

$$\operatorname{tr}_g \nabla S = 2\nabla S(E_1, E_1) = -2(S - \mu\,\mathrm{Id})(\nabla_{E_1} E_1) + \nabla\mu_{|\mathfrak{D}^\perp} = -2(\lambda - \mu)\xi,$$

if we assume that $\nabla\tau_{|\mathscr{D}^\perp} = 0$. On the other hand, $\mathrm{tr}_g \nabla S_0 = \frac{1}{2}\nabla\tau$ and $\mathrm{tr}_g \nabla S = \mathrm{tr}_g \nabla S_0 - \frac{1}{3}\nabla\tau$. Consequently

$$\mathrm{tr}_g \nabla S = \frac{\nabla\tau}{6} = -\nabla\mu.$$

Thus

$$\xi = -\frac{1}{2(\mu-\lambda)}\nabla\mu.$$

From the lemma it follows that (M, g) is an $\mathscr{A}C^\perp$-manifold if $\lambda_0 - 2\mu_0$ is constant and $\nabla\tau \in \Gamma(\mathscr{D})$. These conditions are also necessary, since $\nabla\lambda = 0$ if (M, g) is $\mathscr{A}C^\perp$-manifold and D_λ is totally geodesic. Analogously

$$\xi = -\frac{1}{2(\mu-\lambda)}\nabla\mu \quad \text{and} \quad \nabla\mu = -\frac{1}{6}\nabla\tau \in \Gamma(\mathscr{D}),$$

where ξ is the mean curvature normal of an umbilical distribution D_μ, if (M, g) is an $\mathscr{A}C^\perp$-manifold. □

We now construct examples of compact $\mathscr{A}C^\perp$-surfaces (M, g, J) with nonconstant scalar curvature on a ruled surface $\Sigma_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. Let $a, b \in \mathbb{R}$ be any two real numbers such that $a < b$. Consider a metric $g_{f,h}$ on a product $(a, b) \times S^1 \times S^2$ given by the formula

$$g_{f,h} = dt^2 + g_t, \tag{2–2}$$

where $g_t = f^2(t)\theta^2 + h(t)^2 can$ is the metric on $S^1 \times S^2$ parameterized by t, can denotes the canonical metric on S^2 of constant curvature 1 and $f, h \in C^\infty(a, b)$ are positive functions defined on (a, b).

Proposition 2.3 [Bérard-Bergery 1982; Sentenac 1981]. *The metric $g_{f,h}$ defined on $(a, b) \times S^1 \times S^2$ extends to a smooth metric on the surface $\Sigma_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ if the following conditions are satisfied*:

(a) $f(a) = f(b) = 0$, $f'(a) = 1$, $f'(b) = -1$, *and* $f^{(2k)}(a) = f^{(2k)}(b) = 0$ *for* $k \in \mathbb{N}$.

(b) $h(a) \neq 0 \neq h(b)$, $h'(a) = h'(b) = 0$, *and* $h^{(2k+1)}(a) = h^{(2k+1)}(b) = 0$ *for* $k \in \mathbb{N}$. □

Theorem 2.4. *On the surface $\Sigma_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ there exists a one-parameter family $\{g_\alpha : \alpha > 1\}$ of bihermitian $\mathscr{A}C^\perp$-metrics. The Ricci tensor $\rho = \rho_\alpha$ of (Σ_0, g_α) is bihermitian and has two eigenvalues, which are everywhere different.*

Proof. Consider the metric (2–2) on $(a, b) \times S^1 \times S^2$. We shall find the conditions on f, h to obtain the warped product metric $(\mathbb{CP}^1, g_f) \times_h (\mathbb{CP}^1, 4\,\mathrm{can})$, where $g_f = dt^2 + f^2(t)\theta^2$ is the metric on the first copy of $\mathbb{CP}^1$ and can is the standard Fubini–Studi metric on the second copy of $\mathbb{CP}^1$. Then the Ricci tensor of (U, g_h)

has two eigenvalues λ, μ corresponding to the eigendistributions $D_\lambda = \mathcal{D}$, $D_\mu = \mathcal{D}^\perp$ which are given by the following formulas [Jelonek 2000; Madsen et al. 1997]:

$$\lambda_1 = -2\frac{h''}{h} - \frac{f''}{f}, \quad \lambda_2 = -\frac{f''}{f} - 2\frac{f'h'}{fh}, \quad \mu = -\frac{h''}{h} - \left(\frac{h'}{h}\right)^2 - \frac{f'h'}{fh} + \frac{1}{h^2}.$$

Since $\lambda_1 = \lambda_2$ we obtain

$$f = Dh'.$$

Note that D_λ is totally geodesic and D_μ is totally umbilical (since $g_{f,h}$ is the warped product metric).

To obtain an $\mathcal{A}C^\perp$-metric, λ, μ have to satisfy

$$\lambda - 2\mu = 3C$$

for some constant $C \in \mathbb{R}$. Thus we obtain an equation

$$\frac{h'''}{h'} - 2\frac{h''}{h} - 2\left(\frac{h'}{h}\right)^2 + \frac{2}{h^2} + 3C = 0;$$

introducing $h' = \sqrt{P(h)}$, this becomes

$$h^2 P''(h) - 2P'(h)h - 4P(h) + 4 + 6Ch^2 = 0.$$

Consequently

$$P(h) = \frac{A}{h} + Bh^4 + Ch^2 + 1,$$

where $A, B \in \mathbb{R}$ are arbitrary.

Now let $D = 1$ and consider the equations

$$P(x) = 0, \; P(y) = 0, \quad P'(x) = 2, \; P'(y) = -2.$$

We are looking for unknown real numbers A, B, C and (x, y), where $0 < x < y$. We have

$$\frac{A}{x} + Bx^4 = -Cx^2 - 1, \qquad \frac{A}{y} + By^4 = -Cy^2 - 1,$$
$$-\frac{A}{x^2} + 4Bx^3 = 2 - 2Cx, \qquad -\frac{A}{y^2} + 4By^3 = -2 - 2Cy.$$

Then

$$A = \frac{xy\big(x^4 - y^4 + Cx^2y^2(x^2 - y^2)\big)}{y^5 - x^5},$$

$$B = \frac{x - y + C(x^3 - y^3)}{y^5 - x^5},$$

$$C = \frac{y^2 + x^2 + 2(x - y)}{y^3 - x^3},$$

and

$$x = \frac{(\alpha-1)(\alpha^2+3\alpha+1)}{\alpha(2\alpha^2+\alpha+2)}, \quad y = \alpha x = \frac{(\alpha-1)(\alpha^2+3\alpha+1)}{(2\alpha^2+\alpha+2)} \quad \text{for some } \alpha > 1.$$

Note that $x, y > 0$ and $C > 0$, $A, B < 0$.

Consider the differential equation

$$\frac{d^2h}{dt^2} = \tfrac{1}{2}P'(h), \quad h'(0) = 0, \quad h(0) = x = \frac{(\alpha-1)(\alpha^2+3\alpha+1)}{\alpha(2\alpha^2+\alpha+2)}, \tag{2–3}$$

where $P = P_\alpha$ and $h = h_\alpha$ depend on the parameter $\alpha > 1$. This equation is equivalent to

$$\frac{dh}{dt} = \sqrt{P(h)}, \quad h(0) = x = \frac{(\alpha-1)(\alpha^2+3\alpha+1)}{\alpha(2\alpha^2+\alpha+2)},$$

if $t \in D = \{t \geq 0 : h'(t) \geq 0\}$. One can also check that

$$P''(h) = \frac{2}{h^2}(P(h) - 1 + 5Bh^2).$$

Consequently $P''(h_0) < 0$ if $P(h_0) = 0$. It follows that $P = P_\alpha$, where $(\alpha > 1)$, has exactly two positive roots $\{x, y\}$ and $P(t) > 0$ if $t \in (x, y)$. Note that equation (2–3) admits a smooth periodic solution h defined on the whole of $\mathbb{R}$ and such that $\operatorname{img} h = [x, y]$. Let b be the smallest positive number such that

$$h(b) = y = \frac{(\alpha-1)(\alpha^2+3\alpha+1)}{(2\alpha^2+\alpha+2)}.$$

Let us take $a = 0$. Then it is easy to check that $h''(a) = 1$ and $h''(b) = -1$ since $P'(x) = 2$ and $P'(y) = -2$. Note also that $h'(a) = h'(b) = 0$ and consequently $h^{(2k+1)}(a) = h^{(2k+1)}(b) = 0$. Thus the metric g_h extends to a smooth warped product metric on the whole of the surface Σ_0. Now it is easy to check that $\lambda = -10Bh^2 - 3C$ and $\mu = -5Bh^2 - 3C$. The tensor $\rho - \frac{1}{3}\tau g$ is a Killing tensor with eigenvalues C and $5Bh^2 + C$, corresponding to $\mathfrak{D}$ and $\mathfrak{D}^\perp$ respectively. It follows that we obtained a one-parameter family of bihermitian $\mathcal{A}C^\perp$-metrics $\{g_\alpha : \alpha > 1\}$ on Σ_0. □

Remark. In that way we have constructed examples of compact $\mathcal{A}C^\perp$ bihermitian surfaces whose Ricci tensor is bihermitian. Our examples are of cohomogeneity 1 under the action of the group $G = S^1 \times \mathrm{SO}(3)$ of isometries with principal orbit $S^1 \times \mathbb{CP}^1$ and two special orbits $\mathbb{CP}^1$; see [Madsen et al. 1997]. They do not have a harmonic Weyl tensor, hence they are proper $\mathcal{A}C^\perp$ manifolds, meaning that their Ricci tensor is not a Codazzi tensor; compare [Besse 1987].

Next we shall give the examples with the symmetry group $U(2)$. As in [Madsen et al. 1997], $L(k,1)$ (where $k\in\mathbb{N}$) will denote the a lens space. By Σ_k we denote the $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ associated with the principal bundle

$$p : P(k) = L(k,1) \to \mathbb{CP}^1$$

(it is the space of cohomogeneity 1 under an action of $U(2)$ with principal orbit $L(k,1)$ and two special orbits $\mathbb{CP}^1$). The diffeomorphism type of Σ_k depends only on the parity of k: if k is even , then Σ_k is diffeomorphic to $S^2\times S^2$, for k odd, Σ_k is diffeomorphic to $\mathbb{CP}^2\#\bar{\mathbb{CP}}^2$. Let us consider the metric (2–1) where $P=P(k)$, $g_0=g_{FS}$ is the Fubini–Study metric on $\mathbb{CP}^1$ and $d\theta=2kp^*\omega_{FS}$ where ω_{FS} is the Kähler form of $(\mathbb{CP}^1, g_{FS})$.

Theorem 2.5. *On the surfaces Σ_k, $k\geq 1$ there exists a one-parameter family $\{g_x : x\in(0,\epsilon_k)\}$ of bihermitian $\mathcal{A}C^\perp$-metrics. The Ricci tensor $\rho=\rho_x$ of (Σ_k, g_x) is bihermitian and has two eigenvalues, which are everywhere different.*

Proof. Consider the metric (2–1) on $(\alpha,\beta)\times P(k)$. We shall find the conditions on f, h to obtain the metric on the whole of Σ_k. Then the Ricci tensor of (U, g_h) has two eigenvalues λ, μ corresponding to the eigendistributions $D_\lambda=\mathcal{D}$, $D_\mu=\mathcal{D}^\perp$ which are given by the following formulas (see [Besse 1987; Jelonek 2002b; 2000; Madsen et al. 1997]):

$$\begin{aligned}\lambda_1 &= -2\frac{h''}{h}-\frac{f''}{f},\\ \lambda_2 &= -\frac{f''}{f}+2\left(k^2\frac{f^2}{h^4}-\frac{f'h'}{fh}\right),\\ \mu &= -\frac{h''}{h}+\left(k^2\frac{f^2}{h^4}-\frac{f'h'}{fh}\right)-\left(\frac{h'}{h}\right)^2-3k^2\frac{f^2}{h^4}+\frac{4}{h^2}.\end{aligned}$$

Since $\lambda_1=\lambda_2$ we obtain

$$f=\pm\frac{hh'}{\sqrt{k^2+Ah^2}}.$$

We shall consider the case where $A<0$ and up to homothety of the metric we can assume that $A=-1$. Note that D_λ is totally geodesic and D_μ is umbilical. To obtain an $\mathcal{A}C^\perp$-metric λ, μ have to satisfy

$$\lambda-2\mu=C,$$

for some constant $C\in\mathbb{R}$. Thus we obtain an equation

$$\frac{f''}{f}-2\frac{h''}{h}-2\left(\frac{h'}{h}\right)^2-6k^2\frac{f^2}{h^4}+\frac{8}{h^2}+C=0, \tag{2–4}$$

with boundary conditions $h > 0$, $f(\alpha) = f''(\alpha) = h'(\alpha) = 0$, $f'(\alpha) = 1$, $f(\beta) = f''(\beta) = h'(b) = 0$, $f'(\beta) = -1$; compare [Jelonek 2002b; Madsen et al. 1997; Bérard-Bergery 1982]. Write

$$h^2 = k^2 - g^2, \quad f = g', \quad g' = \sqrt{z(g)}. \tag{2–5}$$

Then our equation reads

$$(k^2-g^2)^2 z''(g)+2g(k^2-g^2)z'(g)-4z(g)(g^2+2k^2)+16(k^2-g^2)+2C(k^2-g^2)^2=0.$$

Write $z(g) = P(g)/(k^2 - g^2)$. Then equation (2–4) reads

$$(k^2 - g^2)P''(g) + 6gP'(g) - 6P(g) + 16(k^2 - g^2) + 2C(k^2 - g^2)^2 = 0.$$

Consequently

$$P(g) = 4(k^2 + g^2) + ag + b\left(\frac{1}{5}\left(\frac{g}{k}\right)^6 - \left(\frac{g}{k}\right)^4 + 3\left(\frac{g}{k}\right)^2 + 1\right) + d(k^2 - g^2)^3$$

where $a, b \in \mathbb{R}$ and $d = C/(6k^2)$. It follows that

$$z(g) = \frac{Q(g/k)}{1 - (g/k)^2} = z_0\left(\frac{g}{k}\right),$$

where

$$Q(t) = 4(1 + t^2) + \frac{a}{k}t + \frac{b}{k^2}\left(\tfrac{1}{5}t^6 - t^4 + 3t^2 + 1\right) + dk^4(1 - t^2)^3,$$

and $z_0(t) = Q(t)/(1 - t^2)$. We shall show that our equation has a one-parameter family of solutions for every $k \in \mathbb{N}$. Let us write (for simplicity we shall write b instead of b/k^2, d instead of dk^4)

$$Q(t) = \tfrac{1}{5}(b - 5d)t^6 + (3d - b)t^4 + \bigl(4 + 3(b - d)\bigr)t^2 + 4 + b + d. \tag{2–6}$$

We shall show that for small $x \in (0, 1)$ there exist $b, d \in \mathbb{R}$ such that Q has only one positive root equal to x and $Q'(x) = -2k(1 - x^2)$. Setting $z_0(t) = Q(t)/(1 - t^2)$, this means that $z_0(x) = 0$, that x is the only positive root of z_0 and that $z_0'(x) = -2k$. The equations $Q(x) = 0$ and $Q'(x) = -2k(1 - x^2)$ are equivalent to

$$\begin{aligned} b\left(\tfrac{1}{5}x^6 - x^4 + 3x^2 + 1\right) + d(1 - x^2)^3 &= -4 - 4x^2, \\ b\left(\tfrac{6}{5}x^5 - 4x^3 + 6x\right) + d\left(-6x(1 - x^2)^2\right) &= 2kx^2 - 8x - 2k. \end{aligned}$$

Therefore, assuming $x \in (0, 1)$, we have

(2–7)

$$b=\frac{5}{2}\frac{(kx^4+8x^3-2kx^2+16x+k)}{x(x^4-10x^2-15)},$$

$$d=\frac{kx^7+(k+8)x^6+(8-5k)x^5-5kx^4+15kx^3+(15k-40)x^2+(5k-40)x+5k}{2x(x-1)(x+1)^2(x^4-10x^2-15)}.$$

Note that

$$\lim_{x\to 0^+}\frac{4+b+d}{x}=k>0,\quad \lim_{x\to 0^+}\frac{2d}{d-b}=1,\quad \lim_{x\to 0^+}b=-\infty,\quad \lim_{x\to 0^+}d=+\infty.$$

Consequently there exists $\epsilon>0$ (depending on k) such that $Q_x(0)>0$ and $Q_x(x)=0$ if $x\in(0,\epsilon)$ where Q_x is given by (2–6) and b, d are given by (2–7). We shall show now that if x is small then it is the only positive root of Q. We have

$$Q''(t)=2t(b-d)\biggl(\Bigl(\frac{4}{b-d}+3\Bigr)+2\Bigl(\frac{3d-b}{b-d}\Bigr)t^2+\frac{3}{5}\Bigl(\frac{b-5d}{b-d}\Bigr)t^4\biggr).$$

Consider a quadratic polynomial

$$H_x(T)=\alpha_xT^3+\beta_xT+\gamma_x,$$

where

$$\gamma_x=2\Bigl(\frac{4}{b-d}+3\Bigr),\quad \beta_x=2\Bigl(\frac{3d-b}{b-d}\Bigr),\quad \alpha_x=\frac{3}{5}\Bigl(\frac{b-5d}{b-d}\Bigr).$$

It is easy to show that

$$\lim_{x\to 0^+}\alpha_x=\tfrac{9}{5},\quad \lim_{x\to 0^+}\beta_x=-4,\quad \lim_{x\to 0^+}\gamma_x=3.$$

Consequently $H_x(T)>0$ for all $T\in\mathbb{R}$ and small x. Thus there exists $\epsilon_k>0$ such that $Q_x(t)>0$ for all $t\in(-x,x)$, $Q_x(-x)=Q_x(x)=0$, $Q'_x(-x)=2k(1-x^2)$, $Q'_x(x)=-2k(1-x^2)$ if $x\in(0,\epsilon_k)$.

Now write

$$z_x(t)=\frac{Q_x(t)}{1-t^2}.$$

If $x\in(0,\epsilon_k)$ then there exists a solution $g:(-A,A)\to(-kx,kx)$, where

$$A=\lim_{t\to kx^-}\int_0^t\frac{dg}{\sqrt{z_x(g/k)}},$$

of an equation

$$g'=\sqrt{z_x(g/k)},$$

such that $g(-A)=-kx$, $g(A)=kx$, $g'(-A)=g(A)=0$, $g''(-A)=1$, $g''(A)=-1$. If we define f,h by (2–5) then equation (2–4) and the boundary conditions are satisfied (note that $Ck^2=6d$) and the metric (2–1) on $(-A,A)\times P(k)$ determined by $x \in (0, 1)$ extends to a smooth bihermitian $\mathcal{A}C^\perp$-metric on the surface Σ_k for every $x \in (0, \epsilon_k)$. □

Remark. For other examples of manifolds of the type studied here, see [Madsen et al. 1997]; their Ricci tensor has two eigenvalues of multiplicities 1 and 3, whereas ours have Ricci tensor with two eigenvalues of the same multiplicity 2. Noncompact examples of bihermitian Kähler $\mathcal{A}C^\perp$-surfaces (also of cohomogeneity 1) were first given [Derdziński 1981] and recently the general explicit expression of such Kähler surfaces was discovered by Apostolov, Calderbank and Gauduchon [Apostolov et al. 2003].

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