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GORAN MUIĆ

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REDUCIBILITY OF STANDARD REPRESENTATIONS

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In this paper we determine the reducibility of standard representations for classical p-adic groups in terms of the classification of discrete series due to Mœglin and Tadić.

Introduction

We describe the reducibility of standard representations for classical p-adic groups in terms of the classification of discrete series due to Mœglin [2002] and Tadić [2002] and tempered representations done by Goldberg [1994]. Their results, and consequently ours, are complete only assuming that discrete series have generic supercuspidal support, thanks to the work of Shahidi [1990] on rank-one supercuspidal reducibilities. The present paper completes our work on reducibility of generalized principal series [Muić 2004; 2005].

Many authors have studied this subject, including Ban, Jantzen, Goldberg, and those just cited. The main application of our results is to the determination of the unitary duals of classical p-adic groups, and especially to the determination of ends of complementary series. The reducibility problem for standard modules for generic inducing data was solved in [Muić 1998; 2001], with the proof of the Casselman–Shahidi conjecture; there we used only the L-function theory of Shahidi [1990]. A nice application was given in [Lapid et al. 2004] with the classification of the generic unitary dual via a reduction (using the results of [Muić 2001]) to the determination of complementary series.

To describe the results of the present paper, we introduce some notation. Let G_n be a symplectic or (full) orthogonal group having split rank n. Let F be a nonarchimedean field of characteristic different from 2. Let $\delta \in \operatorname{Irr}\operatorname{GL}(m_\delta, F)$ (this defines m_δ) be an essentially square-integrable representation. Then we can write $\delta \simeq |\det|^{e(\delta)} \otimes \delta^u$, where $e(\delta)$ is real and δ^u is unitary. A standard representation is an induced representation of the form $\delta'_1 \times \cdots \times \delta'_m \rtimes \tau$, where $\delta'_i \in \operatorname{Irr}\operatorname{GL}(m_{\delta_i}, F)$, $i = 1, \ldots, m$, are essentially square-integrable representations such that $e(\delta'_1) \geq \cdots \geq e(\delta'_m) > 0$ and τ is tempered. The importance of standard

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representations stems from their occurrence in the Langlands classification of irreducible admissible representations. In our case, $\delta_1' \times \cdots \times \delta_m' \rtimes \tau$ has a unique proper maximal subrepresentation, whose corresponding quotient will be denoted by $\mathrm{Lang}(\delta_1' \times \cdots \times \delta_m' \rtimes \tau)$. As first realized by Langlands in the case of real reductive groups, then extended by Borel and Wallach [1980] to p-adic (Zariski-) connected reductive groups, and then by Ban and Jantzen [2001] to nonconnected reductive groups, the quotient $\mathrm{Lang}(\delta_1' \times \cdots \times \delta_m' \rtimes \tau)$ is isomorphic to the image of a particular integral intertwining-operator,

$$\delta_1' \times \cdots \times \delta_m' \rtimes \tau \to \tilde{\delta}_1' \times \cdots \times \tilde{\delta}_m' \rtimes \tau,$$

called the long-intertwining operator. (If π is an representation, $\tilde{\pi}$ stands for its contragredient representation.)

In this paper we determine the reducibility of $\delta_1' \times \cdots \times \delta_k' \rtimes \tau$. First, there is a standard reduction to the maximal parabolic case outlined by Speh and Vogan [1980] and used extensively by many authors, including Jantzen, Tadić, Shahidi, and the author. It can be stated as follows in our case (see for example [Jantzen 1996] for the proof):

Theorem. $\delta_1' \times \cdots \times \delta_k' \rtimes \tau$ *reduces if and only if one of the following holds*:

(i) $\delta_i' \times \delta_j'$ *reduces for some pair* $i \neq j$.

(ii) $\delta_i' \times \tilde{\delta}_j'$ *reduces for some pair* $i \neq j$.

(iii) $\delta_i' \rtimes \tau$ *reduces for some* i.

The work of Bernstein and Zelevinsky attaches to each δ_i' a segment and (i) and (ii) can be rephrased in terms of segments. In this paper we determine the reducibility in (iii). Thus, let $\delta \in \mathrm{Irr}\,\mathrm{GL}(m_\delta, F)$ be an essentially square-integrable representation. According to [Zelevinsky 1980], δ is attached to a segment. We may (and will) write this segment as $[\nu^{-l_1}\rho,\ \nu^{l_2}\rho]$, with $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 \in \mathbb{Z}_{\geq 0}$, and $\rho \in \mathrm{Irr}\,\mathrm{GL}(m_\rho, F)$ unitary. In this paper we describe the reducibility of the following induced representation (see [Tadić 1998] for notation):

$$\delta \rtimes \tau.$$

Since reducibility for unitary δ is an integral part of the classification of discrete series and tempered representations (see Theorem 1.1 as well as [Goldberg 1994; Mœglin 2002; Mœglin and Tadić 2002]), we consider only nonunitary generalized principal series. Thus, we may assume $l_2 - l_1 > 0$, since $\delta \rtimes \tau$ and $\tilde{\delta} \rtimes \tau$ have the same composition factors.

By the work of Harish-Chandra (see [Waldspurger 2003]) there exist discrete series $\delta_1, \ldots, \delta_k, \sigma$ such that $\tau \hookrightarrow \delta_1 \times \cdots \times \delta_k \rtimes \sigma$. In this introduction, we say that τ is *basic* if all δ_i, $i = 1, \ldots, k$, are pairwise nonequivalent and $\delta_i \rtimes \sigma$ reduces, for

$i = 1, \dots, k$. (See [Mœglin and Tadić 2002] or Theorem 4.1 for the reformulation in terms of admissible triple attached to σ.) Any other tempered representation is of the form $\tau \simeq \delta_1'' \times \cdots \times \delta_{k''}'' \rtimes \tau''$, where τ'' is basic and $\delta_1'', \dots, \delta_{k''}''$ are in discrete series. In this case, we again directly apply the idea of [Speh and Vogan 1980] to see that $\delta \rtimes \tau$ reduces if and only if one of the following holds:

$$\delta \times \delta_i'' \text{ reduces for some } i;$$

$$\delta \times \tilde{\delta}_i'' \text{ reduces for some } i;$$

$$\delta \rtimes \tau''.$$

This is done in Lemma 2.1. Therefore we have reduced the study of reducibility to the case when τ is basic; this case will occupy the main portion of this paper.

We mention that in the case when G_n is a symplectic group, being basic means being an elliptic tempered representation; see [Herb 1993].

Definition 0.1. We say that τ is *elementary* if one of the following holds:

(1) $\tau \simeq \sigma$.

(2) $\tau \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$.

(3) $\tau \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$.

(4) $\tau \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$.

As it can be seen, the notion of a elementary tempered representation depends on δ; we use it only in this introduction, to explain the results of the paper. Of course any elementary representation is also basic.

Sections 2 and 3 reduce the determination of the reducibility of $\delta \rtimes \tau$ for τ basic to the case $\delta \rtimes \tau$ for τ elementary. We then have (see Lemma 3.1):

Theorem. *Assume τ is basic. There exist a unique sequence of discrete series $\delta_1'', \dots, \delta_{k''}''$ and an elementary representation τ'' such that*

(a) $\tau \hookrightarrow \delta_1'' \times \cdots \times \delta_{k''}'' \rtimes \tau''$,

(b) $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]),\ \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \notin \{\delta_1'', \dots, \delta_{k''}''\}$.

Moreover, $\delta \rtimes \tau$ reduces if and only if one of the following holds:

$$\delta \times \delta_i'' \text{ reduces for some } i;$$

$$\delta \times \tau'' \text{ reduces.}$$

This proof is based on certain results of multiplicity-one type in Jacquet modules, coming from the deep fact that if $\delta \rtimes \sigma$ reduces, then apart from its Langlands quotient it has one more representation appearing in its composition series with multiplicity one. This follows from a careful analysis of the reducibility of $\delta \rtimes \sigma$, carried out in [Muić 2005] and summarized in Section 4; see Theorem 4.3.

Thus, we have reduced to the case that τ is elementary. The case $\tau \simeq \sigma$ is treated in [Muić 2005] and recapitulated here in Theorem 4.3. The remaining cases in Definition 0.1 are treated in Sections 5, 6, and 7, respectively. To describe the results let Jord_ρ be the set of all Jordan blocks (a, ρ) from $\mathrm{Jord} = \mathrm{Jord}(\sigma)$ (see Section 4 for definition).

We describe the elementary cases. If $\delta \rtimes \sigma$ is irreducible, then $\delta \rtimes \tau''$ is irreducible for any elementary representation τ'' (see Lemma 2.4 and the discussion following its proof). Otherwise, $\delta \rtimes \sigma$ *reduces* (see Theorem 4.1) and we are in one of the three remaining cases of Definition 0.1:

Case (2): Write $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \simeq \tau_1 \oplus \tau_2$. (Theorem 4.1 implies that $2l_1+1 \notin \mathrm{Jord}_\rho$.) We have several cases (Lemmas 5.1, 5.3, 5.4, and 5.5):

- Assume $\mathrm{Jord}_\rho \cap\,]2l_1+1, 2l_2+1] = \varnothing$. Then $\delta \rtimes \tau_1$ and $\delta \rtimes \tau_2$ both reduce.
- Assume $\mathrm{Jord}_\rho \cap\,]2l_1+1, 2l_2+1] = \{2l_2+1\}$. Then exactly one of $\delta \rtimes \tau_1$ and $\delta \rtimes \tau_2$ reduces.
- Assume $\mathrm{Jord}_\rho \cap\,]2l_1+1, 2l_2+1[\, \neq \varnothing$. Take the smallest element in that intersection, say $2l_{\min}+1$. We can distinguish τ_1 from τ_2 using the previous case:
$$\delta([\nu^{-l_1}\rho, \nu^{l_{\min}}\rho]) \rtimes \tau_1 \text{ is irreducible;}$$
$$\delta([\nu^{-l_1}\rho, \nu^{l_{\min}}\rho]) \rtimes \tau_2 \text{ reduces.}$$
Then $\delta \rtimes \tau_2$ reduces and $\delta \rtimes \tau_1$ reduces if and only if $\delta([\nu^{l_{\min}+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ reduces. (The last reducibility is described completely in Theorem 4.1. Recall that we have assumed that $\delta \rtimes \sigma$ reduces.)

Case (3): Write $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma \simeq \tau_1 \oplus \tau_2$. (Theorem 4.1 implies that $2l_2+1$ does not lie in Jord_ρ.) We have several cases, treated in Lemmas 6.2, 6.4, 6.5, 6.6:

- Assume that $l_1 \geq 0$ and $\mathrm{Jord}_\rho \cap [2l_1+1, 2l_2+1[\, \neq \varnothing$. Then $\delta \rtimes \tau_1$ and $\delta \rtimes \tau_2$ both reduce.
- Assume that $l_1 \geq 0$ and $\mathrm{Jord}_\rho \cap [2l_1+1, 2l_2+1[\, = \{2l_1+1\}$, or that $l_1 < 0$ and $\mathrm{Jord}_\rho \cap [-2l_1-1, 2l_2+1[\, = \{-2l_1-1\}$, or that $l_1 = -\frac{1}{2}$ and $\mathrm{Jord}_\rho \cap [-2l_1-1, 2l_2+1[\, = \varnothing$. Then exactly one of $\delta \rtimes \tau_1$ and $\delta \rtimes \tau_2$ reduces.
- Assume that $l_1 \geq 0$ and $\mathrm{Jord}_\rho \cap [2l_1+1, 2l_2+1[\, \neq \varnothing$, or that $l_1 < 0$ and $\mathrm{Jord}_\rho \cap [-2l_1-1, 2l_2+1[\, \neq \varnothing$. Write $2l_{\max}+1$ for the largest element of that intersection. (It depends on the sign of l_1.) We can distinguish τ_1 from τ_2 using the previous case:
$$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1 \text{ is irreducible;}$$
$$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2 \text{ reduces.}$$

Then $\delta \rtimes \tau_2$ reduces and $\delta \rtimes \tau_1$ reduces if and only if $\delta([\nu^{-l_1}\rho, \nu^{l_{\max}}\rho]) \rtimes \sigma$ reduces. (The last reducibility is described completely in Theorem 4.1. Recall that $\delta \rtimes \sigma$ reduces.)

Case (4): Write $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \simeq \tau_1 \oplus \tau_2$ and $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \tau_i \simeq \tau_{i1} \oplus \tau_{i2}$. (Theorem 4.1 implies that $2l_1{+}1, 2l_2{+}1 \notin \mathrm{Jord}_\rho$.) We again have several cases, listed in Lemma 7.1:

- Assume $\mathrm{Jord}_\rho \cap \,]2l_1{+}1, 2l_2{+}1[\, = \varnothing$. Then exactly one of the representations $\delta \rtimes \tau_{ij}$ (i fixed, $j = 1, 2$) is irreducible.
- Assume $\mathrm{Jord}_\rho \cap \,]2l_1{+}1, 2l_2{+}1[\, \neq \varnothing$. Let $2l_{\max}{+}1$ be the largest element of that intersection. The representations τ_{ij} (i fixed, $j = 1, 2$) can be distinguished as follows:

$$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_{i1} \text{ is irreducible;}$$
$$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_{i2} \text{ is reducible.}$$

  Then $\delta \rtimes \tau_{i1}$ is reducible if and only if $\delta([\nu^{-l_1}\rho, \nu^{l_{\max}}\rho]) \rtimes \tau_i$ reduces. Moreover, $\delta \rtimes \tau_{i2}$ reduces.

1. Preliminaries

Let F be a nonarchimedean field of characteristic different from 2. We will look at towers of orthogonal or symplectic groups $G_n = G(V_n)$ that are groups of isometries of F-spaces $(V_n, (\ ,\ ))$, $n \geq 0$, where the form $(\ ,\ )$ is nondegenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise. We fix a set of standard parabolic subgroups in the usual way. See [Mœglin et al. 1987] for details.

We will use freely the main results and standard notation of [Zelevinsky 1980] on the representation theory of general linear groups. In particular, we write ν for the character obtained by the composition of the determinant character and the absolute value of F (normalized as usual).

If $\rho \in \mathrm{Irr}\,\mathrm{GL}(M_\rho, F)$ is a supercuspidal representation and $k \in \mathbb{Z}_{\geq 0}$, we define a segment $[\rho, \nu^k\rho]$ as the set $\{\rho, \nu\rho, \dots, \nu^k\rho\}$. This segment has associated to it a unique essentially square-integrable representation $\delta([\rho, \nu^k\rho])$ given as a unique irreducible subrepresentation of $\nu^k\rho \times \cdots \times \nu\rho \times \rho$.

Next we discuss briefly tempered representations in $\mathrm{Irr}' = \cup_{n\geq 1} \mathrm{Irr}\, G_n$. Our main reference is [Goldberg 1994] in the connected case and [Mœglin and Tadić 2002] in the general case (full orthogonal groups). Of course, everything is based on deep results of Harish-Chandra (see [Waldspurger 2003]) on the theory of R-groups in the connected case. Mackey theory can be used to extend this results to nonconnected case [Mœglin and Tadić 2002; Lapid et al. 2004].

Theorem 1.1. (i) *Let $\delta_1, \dots, \delta_k, \sigma$ be a sequence of discrete series. Then the induced representation $\delta_1 \times \cdots \times \delta_k \rtimes \sigma$ is a direct sum of pairwise nonequivalent tempered representations. It has a length 2^l where l is the number of nonequivalent δ_i such that $\delta_i \rtimes \sigma$ reduces* (see Theorem 4.1).

(ii) *Let $\tau \in \mathrm{Irr}'$ be a tempered representation. Then there exist discrete series $\delta_1, \dots, \delta_k, \sigma$ such that $\tau \hookrightarrow \delta_1 \times \cdots \times \delta_k \rtimes \sigma$. If $\delta_1', \dots, \delta_k', \sigma'$ is also a sequence of discrete series such that $\tau \hookrightarrow \delta_1' \times \cdots \times \delta_k' \rtimes \sigma$, then $\sigma \simeq \sigma'$ and sequence $\delta_1', \dots, \delta_k'$ is obtained form sequence $\delta_1, \dots, \delta_k$ permuting terms and taking replacing terms with its contragredient. The multiset $\{\delta_1, \dots, \delta_k, \tilde{\delta}_1, \dots, \tilde{\delta}_k, \sigma\}$ is called the tempered support of τ.*

Corollary 1.2. *Let $\delta \in \mathrm{Irr}\,\mathrm{GL}(m_\delta, F)$ be a discrete series and let $\tau \in \mathrm{Irr}'$ be a tempered representation.*

(i) *If δ appears in the tempered support of τ or $\delta \rtimes \sigma$ is irreducible, then $\delta \rtimes \tau$ is irreducible.*

(ii) *If δ does not appear in the tempered support of τ and $\delta \rtimes \sigma$ is reducible, then $\delta \rtimes \tau$ is a direct sum of two nonequivalent tempered representations.*

Proof. Part (i) follows directly form Theorem 1.1(i). Part (ii) also follows from Theorem 1.1(i); see [Lapid et al. 2004, Corollary 2] for details. □

Lemma 1.3. *Assume that $\tau \in \mathrm{Irr}'$ is a tempered representation, that $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, and that $\rho \in \mathrm{Irr}$ is an irreducible unitary supercuspidal representation. If there exists $\tau' \in \mathrm{Irr}'$ such that $\tau \hookrightarrow \nu^{m_1}\rho \times \cdots \times \nu^{-m_1}\rho \rtimes \tau'$, then τ' is tempered and $\tau \hookrightarrow \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \rtimes \tau'$. In particular, $\delta([\nu^{-m_1}\rho, \nu^{m_1}\rho])$ appears in the tempered support of τ.*

Proof. We first prove that $\tau \hookrightarrow \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \rtimes \tau'$. If not, take the smallest $k \geq 1$ for which there exists a sequence $m_1 > a_1 > \cdots > a_k > -m_1$ such that

$$\tau \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{a_2+1}\rho, \nu^{a_1}]) \times \cdots \times \delta([\nu^{-m_1}\rho, \nu^{a_k}\rho]) \rtimes \tau'. \tag{1–1}$$

The minimality of k implies that we can permute essentially square-integrable representations in (1–1) as we want, and the inclusion is still preserved. In particular,

$$\tau \hookrightarrow \delta([\nu^{-m_1}\rho, \nu^{a_k}\rho]) \times \delta([\nu^{a_1+1}\rho, \nu^{b}m_1]) \times \delta([\nu^{a_2+1}\rho, \nu^{a_1}]) \\ \times \cdots \times \delta([\nu^{a_k+1}\rho, \nu^{a_{k-1}}\rho]) \rtimes \tau'.$$

Since $a_k < m_1$, this violates the temperedness criterion for τ, proving that $\tau \hookrightarrow \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \rtimes \tau'$. Now, we show that τ' is tempered. If not, by the Langlands classification, we can find a unitary supercuspidal representation ρ', real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha - \beta < 0$ and $\alpha + \beta \in \mathbb{Z}_{\geq 0}$, and a representation $\tau'' \in \mathrm{Irr}'$, such that

$$\tau' \hookrightarrow \delta([\nu^{-\beta}\rho', \nu^{\alpha}\rho']) \rtimes \tau''.$$

Hence

$$\tau \hookrightarrow \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-\beta}\rho', \nu^{\alpha}\rho']) \rtimes \tau''.$$

The segments $[\nu^{-m_1}\rho, \nu^{m_1}\rho]$ and $[\nu^{-\beta}\rho', \nu^{\alpha}\rho']$ must be linked; otherwise, by [Zelevinsky 1980], the second intertwining operator in

$$\tau \hookrightarrow \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-\beta}\rho', \nu^{\alpha}\rho']) \rtimes \tau'' \to \delta([\nu^{-\beta}\rho', \nu^{\alpha}\rho']) \times \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \rtimes \tau'' \tag{1–2}$$

is an isomorphism, and this violates the temperedness criterion for τ. Now, since the segments $[\nu^{-m_1}\rho, \nu^{m_1}\rho]$ and $[\nu^{-\beta}\rho', \nu^{\alpha}\rho']$ are linked, we must have $\rho' \simeq \rho$ and $\alpha - m_1 \in \mathbb{Z}$. We also must have $m_1 > \alpha$ and $-\beta < -m_1 \leq \alpha+1$, and again using (1–2), we see that τ must satisfy

$$\tau \hookrightarrow \delta([\nu^{-m_1}\rho, \nu^{\alpha}\rho]) \times \delta([\nu^{-\beta}\rho, \nu^{m_1}\rho]) \rtimes \tau''.$$

Again, this violates the temperedness criterion for τ. □

Our main tool in the subsequent analysis is Tadić's theory of Jacquet modules. We conclude this section by recalling his basic result.

Let $R(G_n)$ be the Grothendieck group of admissible representations of finite length. Define

$$R(G) = \bigoplus_{n \geq 0} R(G_n), \qquad R(\mathrm{GL}) = \bigoplus_{n \geq 0} R(\mathrm{GL}(n, F)).$$

We write $\geq$ or $\leq$ for the natural order on $R(G)$. Explicitly, we have $\pi_1 \leq \pi_2$ for $\pi_1, \pi_2 \in R(G)$ if and only if $\pi_2 - \pi_1$ is a linear combination of irreducible representations with nonnegative coefficients.

Let $\sigma \in \mathrm{Irr}\, G_n$. For each standard proper maximal parabolic subgroup (see [Mœglin et al. 1987]) P_j with Levi factor $\mathrm{GL}(j, F) \times G_{n-j}$, where $1 \leq j \leq n$, we can identify $R_{P_j}(\sigma)$ with its semisimplification in $R(\mathrm{GL}(j, F)) \otimes R(G_{n-j})$. Thus, we can consider

$$\mu^*(\sigma) = 1 \otimes \sigma \ + \sum_{j=1}^{n} R_{P_j}(\sigma) \in R(\mathrm{GL}) \otimes R(G).$$

The basic result of Tadić is the following (see [Mœglin and Tadić 2002] and references there):

Theorem 1.4. *Let $\sigma \in \mathrm{Irr}\, G_n$. Consider the decomposition into irreducible constituents*

$$\mu^*(\sigma) = \sum_{\delta', \sigma_1} \delta' \otimes \sigma_1 \in R(\mathrm{GL}) \otimes R(G)$$

(*with repetitions possible*). *Assume that* $l_1, l_2 \in \mathbb{R}$, *that* $l_1 + l_2 + 1 \in \mathbb{Z}_{>0}$, *and that* $\rho \in \operatorname{Irr}\operatorname{GL}(m_\rho, F)$ *is a supercuspidal representation. Then*

$$\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big) = \sum_{\delta',\sigma_1} \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} \delta([\nu^{i-l_2}\tilde{\rho}, \nu^{l_1}\tilde{\rho}]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta' \otimes \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1.$$

(*We omit* $\delta[\nu^\alpha\rho, \nu^\beta\rho])$ *if* $\alpha > \beta$.)

2. Basic reductions

In this section we perform basic reductions in the determination of reducibility.

Let $\tau \in \operatorname{Irr}'$ be a tempered irreducible representation. We assume that

$$\tau \hookrightarrow \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma, \tag{2–1}$$

where $\delta_1, \ldots, \delta_k, \sigma$ are discrete series. By [Zelevinsky 1980], δ_i is attached to a segment. We may (and will) write this segment as

$$[\nu^{-m_i}\rho_i, \nu^{m_i}\rho_i],$$

with $2m_i \in \mathbb{Z}_{\geq 0}$ and $\rho_i \in \operatorname{Irr}\operatorname{GL}(m_{\rho_i}, F)$ unitary and supercuspidal.

Let $\delta \in \operatorname{Irr}\operatorname{GL}(m_\delta, F)$ be a nonunitary essentially square-integrable representation. We study the reducibility and the composition series of $\delta \rtimes \tau$. Again, δ is attached to a segment, which we write as

$$[\nu^{-l_1}\rho, \nu^{l_2}\rho], \tag{2–2}$$

with $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 \in \mathbb{Z}_{\geq 0}$, and $\rho \in \operatorname{Irr}\operatorname{GL}(m_\rho, F)$ unitary and supercuspidal.

Next, since $\delta \rtimes \tau = \tilde{\delta} \rtimes \tau$ in $R(G)$, we also assume

$$l_2 - l_1 > 0 \tag{2–3}$$

In this way is $\delta \rtimes \tau$ becomes a standard representation, whose Langlands quotient we denote by $\operatorname{Lang}(\delta \rtimes \tau)$.

The first reduction in the study of the reducibility of $\delta \rtimes \tau$ follows from the factorization of the long-intertwining operator $\delta \rtimes \tau \to \tilde{\delta} \rtimes \tau$.

Lemma 2.1. *Assume that there exists* i_0, $1 \leq i_0 \leq k$, *such that* $\delta_{i_0} \rtimes \sigma$ *is irreducible, or that there exist* j_0, $1 \leq j_0 \leq k$, *and* $i_0 \neq j_0$ *such that* $\delta_{i_0} \cong \delta_{j_0}$. *Then there exists a tempered irreducible representation* $\tau' \hookrightarrow \times_{i=1, i\neq i_0}^{k} \delta_i \rtimes \sigma$ *such that*

$$\tau \simeq \delta_{i_0} \rtimes \tau'.$$

Moreover $\delta \rtimes \tau$ *reduces if and only if at least one of the following holds*:

(i) $\delta \times \delta_{i_0}$ *reduces* (see [Zelevinsky 1980]).

(ii) $\delta \times \tilde{\delta}_{i_0}$ *reduces.*

(iii) $\delta \rtimes \tau'$ *reduces.*

Proof. The existence of τ' with the required properties follows from Corollary 1.2. Next, all induced representations $\delta \rtimes \tau$, $\delta \rtimes \tau'$, $\delta \times \delta_{i_0}$, $\delta_{i_0} \times \tilde{\delta}$ are standard representations (that is, they admit Langlands quotients). Therefore, we have a factorization of the long-intertwining operator into long-intertwining operators:

$$\delta \rtimes \tau \simeq \delta \times \delta_{i_0} \rtimes \tau' \to \delta_{i_0} \times \delta \rtimes \tau' \to \delta_{i_0} \times \tilde{\delta} \rtimes \tau' \to \tilde{\delta} \times \delta_{i_0} \rtimes \tau' \simeq \tilde{\delta} \rtimes \tau.$$

The lemma follows by the standard argument. □

Lemma 2.1 enables us to assume that

$$\text{(2–4)} \qquad \delta_i \not\simeq \delta_j \text{ if } i \neq j \quad \text{and} \quad \delta_i \rtimes \sigma \text{ reduces,} \quad \text{for } i, j = 1, \dots, k.$$

We use (2–4) in the remainder of the paper (sometimes without mentioning it explicitly). In particular, since $\delta_i \rtimes \sigma$ reduces we must have

$$\delta_i \simeq \tilde{\delta}_i$$

and $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$ is a multiplicity-free representation of length 2^k, by Theorem 1.1.

Lemma 2.2. *Assume that* (2–4) *holds and that* $\delta \times \delta_i$ *reduces for some* i, $1 \leq i \leq k$. *Then* $\delta \rtimes \tau$ *reduces.*

Proof. Without loss of generality, we may assume $i = 1$. Since $\delta \times \delta_1$ reduces, by [Zelevinsky 1980], $\rho \simeq \rho_1$ and the segments $[\nu^{-m_1}\rho,\ \nu^{m_1}\rho]$, $[\nu^{-l_1}\rho,\ \nu^{l_2}\rho]$ are linked. Hence, using (2–2) and (2–3), we obtain $l_2 > m_1 \geq -l_1 - 1$ and $-m_1 < -l_1$. Again, by [Zelevinsky 1980], the long-intertwining operator $\delta \times \delta_1 \to \delta_1 \times \delta$ has kernel isomorphic to

$$\delta\left([\nu^{-m_1}\rho, \nu^{l_2}\rho]\right) \times \delta\left([\nu^{-l_1}\rho, \nu^{m_1}\rho]\right) \simeq \delta\left([\nu^{-l_1}\rho, \nu^{m_1}\rho]\right) \times \delta\left([\nu^{-m_1}\rho, \nu^{l_2}\rho]\right).$$

Thus, for any tempered irreducible representation τ', the induced representation $\delta\left([\nu^{-m_1}\rho, \nu^{l_2}\rho]\right) \times \delta\left([\nu^{-l_1}\rho, \nu^{m_1}\rho]\right) \rtimes \tau'$ has a Langlands quotient.

Theorem 1.1 implies that $\delta_2 \times \cdots \times \delta_k \rtimes \sigma$ is a direct sum of 2^{k-1} pairwise nonequivalent tempered representations. Moreover there exists a unique one among them, say τ', such that

$$\tau \hookrightarrow \delta_1 \rtimes \tau'.$$

Hence, by Frobenius reciprocity,

$$\mu^*(\tau) \geq \delta_1 \otimes \tau'.$$

Therefore, Theorem 1.4 shows that

$$\mu^*(\delta \rtimes \tau) \geq \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta_1 \otimes \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \tau'. \tag{2–5}$$

Now, the multiplicity of

$$\delta\left([\nu^{-l_2}\rho, \nu^{m_1}\rho]\right) \times \delta\left([\nu^{-m_1}\rho, \nu^{l_1}\rho]\right) \otimes \tau' \tag{2–6}$$

in $\mu^*(\delta \rtimes \tau)$ is at least 1. (Take $i = j = 0$ in (2–5).) Hence, to complete the proof we just need the next lemma.

Lemma 2.3. (i) *The Langlands quotient*

$$\mathrm{Lang}\big(\delta\left([\nu^{-m_1}\rho, \nu^{l_2}\rho]\right) \times \delta\left([\nu^{-l_1}\rho, \nu^{m_1}\rho]\right) \rtimes \tau'\big)$$

appears in $\delta \times \delta_1 \rtimes \tau'$ *with multiplicity exactly two.*

(ii) *The multiplicity of* (2–6) *in* $\mu^*(\delta \times \delta_1 \rtimes \tau')$ *is exactly two.*

Proof. The proof is similar to that of [Muić 2005, Theorem 3.1], but there are some differences since we are considering the tempered case. We expand

$$\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \rtimes \tau'\big)$$

using Theorem 1.4. Thus, we take indices $0 \leq j \leq i \leq l_1+l_2+1$, $0 \leq j' \leq i' \leq 2m_1+1$, and irreducible constituents $\delta' \otimes \tau_1$ of $\mu^*(\tau')$, and we obtain

$$\delta([\nu^{-l_2}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-m_1}\rho, \nu^{l_1}\rho]) \leq \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{i'-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{m_1+1-j'}\rho, \nu^{m_1}\rho]) \times \delta' \tag{2–7}$$

and

$$\tau' \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{m_1+1-i'}\rho, \nu^{m_1-j'}\rho]) \rtimes \tau_1.$$

Equation (2–7) shows that δ' is nondegenerate. In particular, it is fully induced from the tensor product of essentially square-integrable representations; see [Zelevinsky 1980]. Now, if $i > 0$, (2–7) shows that one of those essentially square-integrable representations must be attached to a segment of the form $[\nu^{-l_2}\rho, \nu^{k}\rho]$, for some $k < l_2$. Since $\mu^*(\tau') \geq \delta' \otimes \tau_1$, we obtain $\tau' \hookrightarrow \nu^k\rho \times \cdots \times \nu^{-l_2}\rho \rtimes \tau_1'$, for some irreducible representation τ_1'. This contradicts the temperedness criterion for σ'. Thus, $i = 0$, and since $0 \leq j \leq i$, we obtain $j = 0$. Then (2–7) becomes

$$\delta([\nu^{-l_2}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-m_1}\rho, \nu^{l_1}\rho]) \leq \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{i'-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{m_1+1-j'}\rho, \nu^{m_1}\rho]) \times \delta'. \tag{2–8}$$

This implies that the only possible terms in a supercuspidal support of δ' are $\nu^{-m_1}\rho, \ldots, \nu^{m_1}\rho$, without repetition.

If $i' = 2m_1+1$ and $j' = 0$, we obtain $\delta' \simeq \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho])$ since δ' is nondegenerate. Since $\mu^*(\tau') \geq \delta' \otimes \tau_1$, we obtain $\tau' \hookrightarrow \nu^{m_1}\rho \times \cdots \times \nu^{-m_1}\rho \rtimes \tau_1'$, for some irreducible representation τ_1'. Applying Lemma 1.3 we arrive at a contradiction.

Next, if $j' > 0$, then $i' = 2m_1+1$ since $\nu^{m_1}\rho$ appears exactly once in the supercuspidal support of the representation on the left-hand side of (2–8). If j' were less than $2m_1+1$, then $\nu^{-m_1}\rho, \ldots, \nu^{m_1-j'}$ (without repetition) would form the supercuspidal support of δ'. Since $j' > 0$, as before, this would violate the temperedness criterion for τ'. Thus $j' = 2m_1+1$, and our term (2–6) arises exactly once. The case $i' < 2m_1+1$ is analogous, and yields the term (2–6) exactly once. This proves (ii).

Now, we prove (i). Write L for the Langlands quotient defined in (i). We show that it appears at least twice in $\delta \times \delta_1 \rtimes \tau'$; then (ii) implies that it appears exactly twice. We have

$$\begin{aligned}(2\text{–}9)\quad &\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \rtimes \tau' \to \\ &\quad \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau' \to \\ &\quad\quad \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \tau' \to \\ &\quad\quad\quad \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \rtimes \tau'\end{aligned}$$

Now, applying [Zelevinsky 1980], the third intertwining operator above has kernel isomorphic to

$$\delta([\nu^{-l_2}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-m_1}\rho, \nu^{l_1}\rho]) \rtimes \tau',$$

and this representation has L as a unique irreducible subrepresentation (Langlands subrepresentation). Also, applying [Zelevinsky 1980], the first intertwining operator in (2–9) has kernel isomorphic to

$$\delta([\nu^{-m_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{m_1}\rho]) \rtimes \tau'.$$

Obviously L is its quotient.

We prove that the copies of L differ. First, by Corollary 1.2(ii), we decompose $\delta_1 \rtimes \tau' \simeq \bigoplus_{i=1}^2 \tau_i$ (multiplicity-free). Inducing in stages, (2–9) can be considered as a factorization of the long-intertwining operator

$$\bigoplus_{i=1}^2 \delta \rtimes \tau_i \to \bigoplus_{i=1}^2 \tilde{\delta} \rtimes \tau_i.$$

Thus, its image is isomorphic to

$$\bigoplus_{i=1}^2 \operatorname{Lang}(\delta \rtimes \tau_i).$$

Hence, it is also the image of the composition of all three intertwining operators in (2–9).

We show that the image of the composition of the first and second intertwining operators in (2–9) intersects the kernel of the third. If not, we see that

$$\delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \tau' \tag{2–10}$$

has at least three different irreducible subrepresentations. Thus, by Frobenius reciprocity,

$$\delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \otimes \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \otimes \tau' \tag{2–11}$$

appears at least three times in an appropriate Jacquet module of (2–10). We show that this is not the case, combining Theorem 1.4 and the transitivity of Jacquet modules. First, we express

$$\mu^*\big(\delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \tau'\big)$$

using Theorem 1.4, and arguing as before we can easily see that only the following term, appearing with multiplicity two,

$$\delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \otimes \tau'$$

can have (2–10) in an appropriate Jacquet module. Finally, the Jacquet modules of

$$\delta([\nu^{-m_1}\rho, \nu^{m_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho])$$

can be computed easily and explicitly [Zelevinsky 1980], showing that (2–11) appears with multiplicity one therein. □

The next lemma elucidates further reductions.

Lemma 2.4. *Assume that (2–4) holds, that $\delta \times \delta_i$ is irreducible for all $i = 1, \dots, k$, and that $\delta \rtimes \sigma$ is irreducible. Then $\delta \rtimes \tau$ is irreducible.*

Proof. This follows from the factorization of the long-intertwining operator $\delta \rtimes \tau \to \tilde{\delta} \rtimes \tau$ given by

$$\begin{aligned} \delta \rtimes \tau &\hookrightarrow \delta \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma \\ &\simeq \delta_1 \times \delta \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma \simeq \cdots \simeq \delta_1 \times \delta_2 \times \cdots \times \delta_k \times \delta \rtimes \sigma \\ &\simeq \delta_1 \times \delta_2 \times \cdots \times \delta_k \times \tilde{\delta} \rtimes \sigma \simeq \cdots \simeq \tilde{\delta} \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma. \end{aligned} \tag{2–12}$$

Hence the long-intertwining operator has no kernel, and $\delta \rtimes \tau \simeq \mathrm{Lang}(\delta \rtimes \tau)$. □

We conclude this section by explaining how we will study reducibility in the remainder of the paper.

Assuming the truth of (2–4), Lemma 2.4 shows that a necessary condition for $\delta \rtimes \tau$ to be reducible is that $\delta \times \delta_i$ be reducible for some i, $1 \le i \le k$, or that $\delta \rtimes \sigma$

reduces. The reducibility of $\delta \rtimes \sigma$ is described in [Muić 2005]. We recall this result in Section 4. Because of Lemma 2.2, in our further investigation of reducibility we may assume that

$$\delta \times \delta_i \quad \text{is irreducible for } i = 1, \ldots, k, \tag{2–13}$$

that (2–4) holds, and that

$$\delta \rtimes \sigma \quad \text{reduces.} \tag{2–14}$$

It is implicit in [Muić 2005] that when $\delta \rtimes \sigma$ reduces, it contains, apart from its Langlands quotient (appearing with multiplicity one), one more subquotient, appearing with multiplicity one (see Section 4). Then Lemma 3.1 reduces the investigation of reducibility to the determination of three basic cases:

$k = 1$ and $\delta_1 = \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho])$ (only for $l_1 \geq 0$),
$k = 1$ and $\delta_1 = \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho])$,
$k = 2$ and $\delta_1 = \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho])$ $\delta_2 = \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho])$ (only for $l_1 \geq 0$).

Here (2–4), (2–13), (2–14) hold; see (2–1).

3. Main reduction

In this section we prove the key lemma just described.

Lemma 3.1. *Assume the following*:

(a) (2–4) *holds*.

(b) $\tau' \in \mathrm{Irr}'$ *is a tempered representation and* $\delta'_1, \ldots, \delta'_l$ *are discrete series such that* $\tau \hookrightarrow \delta'_1 \times \cdots \times \delta'_l \rtimes \tau'$. *We write* $\delta'_i = \delta([\nu^{-m'_i}\rho'_i, \nu^{m'_i}\rho'_i])$, *with* $m'_i \in \mathbb{Z}_{\geq 0}$ *and* ρ'_i *unitary and supercuspidal*.

(c) $(2l_1+1, \rho), (2l_2+1, \rho) \notin \{(2m'_i+1, \rho'_i);\ i = 1, \ldots, l\}$.

(d) $\delta \rtimes \tau'$ *contains an irreducible subquotient* π *other than* $\mathrm{Lang}(\delta \rtimes \tau)$ *appearing with multiplicity one in its composition series*.

(e) $[\nu^{-m'_i}\rho'_i, \nu^{m'_i}\rho'_i]$ *and* $[\nu^{-l_1}\rho, \nu^{l_2}\rho]$ *are not linked*, *for all* $i = 1, \ldots, l$.

Then $\delta \rtimes \tau$ *reduces*.

Proof. Upon several applications of Corollary 1.2(ii), equation (2–4) shows that $\delta'_1 \times \cdots \times \delta'_l \rtimes \tau'$ decomposes into a direct sum of pairwise inequivalent tempered representations:

$$\delta'_1 \times \cdots \times \delta'_l \rtimes \tau' \simeq \bigoplus_{i=1}^{2^l} \tau_i. \tag{3–1}$$

Thus we have, in the appropriate Grothendieck group,

$$(3\text{–}2)\qquad \sum_{i=1}^{2^l} \delta \rtimes \tau_i = \delta \times \delta'_1 \times \cdots \times \delta'_l \rtimes \tau' \geq \begin{cases} \delta \times \delta'_1 \times \cdots \times \delta'_l \rtimes \pi \\ \delta \times \delta'_1 \times \cdots \times \delta'_l \rtimes \mathrm{Lang}(\delta \times \tau). \end{cases}$$

It will follow from Lemma 3.2 below that $\delta \rtimes \tau_i$ for all $i = 1, \ldots, 2^l$, has a common irreducible subquotient with both induced representations on the right-hand side of (3–2), and hence that $\delta \rtimes \tau_i$ reduces. But this implies the conclusion of Lemma 3.1. □

Lemma 3.2. *The multiplicity of* $\delta'_1 \times \cdots \times \delta'_l \otimes \pi$ *in* $\mu^*(\delta \times \delta'_1 \times \cdots \times \delta'_l \rtimes \tau')$ *and in* $\mu^*(\delta'_1 \times \cdots \times \delta'_l \rtimes \pi)$ *is equal to* 2^l (see (3–1)). *Moreover, it is one in* $\mu^*(\delta \rtimes \tau_i)$, *for all* $i = 1, \ldots, 2^l$. *The same holds upon replacing* π *by* $\mathrm{Lang}(\delta \rtimes \tau')$.

To prove this, we first show:

Claim 3.3. *The multiplicity of* $\delta'_1 \times \cdots \times \delta'_l \otimes \pi$ *in* $\mu^*(\delta \times \delta'_1 \times \cdots \times \delta'_l \rtimes \tau')$ *is exactly* 2^l.

Proof. The discrete series δ'_i are all nonequivalent because of assumption (a). The same assumption implies that none of the δ'_i show up in the tempered support of τ'.

Next, we may assume $m'_1 \geq m'_2 \geq \cdots \geq m'_l$. We remark that (2–4) implies that if $m'_i = m'_{i+1}$, then $\rho'_i \not\simeq \rho'_{i+1}$. Now, we apply Theorem 1.4 several times computing the multiplicity of $\delta'_1 \times \cdots \times \delta'_l \otimes \pi$ in the terms that we describe now. We take irreducible constituents $\mu^*(\tau') \geq \delta' \otimes \tau'_1$, and indices $0 \leq j \leq i \leq l_1 + l_2 + 1$, $0 \leq j_a \leq i_a \leq 2m_a + 1$ $(a = 1, \ldots, l)$, so that

$$(3\text{–}3)\quad \delta'_1 \times \cdots \times \delta'_l \leq \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \prod_{a=1}^{l} \big(\delta([\nu^{i_a - m'_a}\rho'_a, \nu^{m'_a}\rho'_a]) \times \delta([\nu^{m'_a+1-j_a}\rho'_a, \nu^{m'_a}\rho'_a])\big) \times \delta'$$

and

$$\pi \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \times \prod_{a=1}^{l} \delta([\nu^{m'_a+1-i_a}\rho'_a, \nu^{m'_a-j_a}\rho'_a]) \rtimes \tau'_1.$$

We want to show that $j = 0$, that $i = l_1 + l_2 + 1$, and that, for all $a = 1, \ldots, l$, $j_a = i_a = 2m_a + 1$ or $j_a = i_a = 0$. This implies δ' is trivial and $\tau' \simeq \tau$, and in view of assumption (d) this shows the desired multiplicity.

Assume that $m'_p \geq l_2 > m'_{p+1}$ and that, if $l_1 \geq 0$, then $m'_t \geq l_1 > m'_{t+1}$. Observe that $p \leq t$. (We omit m_p if l_2 is strictly greater than all the m'_i, and similarly we omit m'_{p+1} if l_2 is strictly smaller than all the m'_i. We treat l_1 similarly.) First, we show that $j_a = i_a = 2m_a + 1$ or $j_a = i_a = 0$, for $a = 1, \ldots, p$. (We omit this if m'_p is not defined.) We do this using induction on the index a. First, let $a = 1$ and consider the term $\nu^{-m_1}\rho'_1$. It appears on the left-hand side of (3–3) exactly once.

On the right-hand side it can either be obtained in the way already described or come from δ'. (Note that $i - l_2 \geq -l_2 \geq -m_p$ and $l_2+1 - j \geq -l_1 > -l_2$; see (2–3).) If it comes from δ', then δ' ought to be nondegenerate and by [Zelevinsky 1980] it is induced from a product of essentially square-integrable representations. One of them must be attached to the segment $[\nu^{-m_1}\rho_1', \nu^s\rho_1']$, where $a \leq m_1$. Now, if $a < m_1$, this would violate the temperedness criterion for τ' (see the proof of Lemma 1.3). If $a = m_1$, then Lemma 1.3 implies that δ_1' appears twice in the tempered support of τ. This violates (2–4); see (a). This completes the proof of the base of induction. The proof of the induction step is similar. We just remark that if $a = p$ and $m_a' = l_2$, then (c) implies $\rho \not\simeq \rho_p'$. This completes the proof of the first induction. We also must have $j = 0$ since $\nu^{l_2}\rho$ does not contribute to the creation of remaining terms on the left-hand side of (3–3) since the remaining m_a' satisfy $m_a' < l_2$. Now, we show $j_a = i_a = 2m_a+1$ or $j_a = i_a = 0$, $p < a \leq t$. (We omit this if m_t' is not defined.) We do this again by induction on index a. First, let $a = p+1$ and consider the term $\nu^{-m_{p+1}}\rho_{p+1}'$. It appears on the left-hand side of (3–3) exactly once in not yet determined terms. On the right-hand side it can either be obtained in the way already described ($j_{p+1} = i_{p+1} = 2m_{p+1}+1$ or $j_{p+1} = i_{p+1} = 0$), or it may come from δ', or we may have $i - l_2 = -m_{p+1} \leq l_1$ and $\rho \simeq \rho_{p+1}'$. The second case can be treated as before. The third case is treated as follows. First, if t is not defined, l_1 is greater than all the m_i', and therefore $\nu^{l_1}\rho$ does not show up on the left-hand side of (3–3). Thus $i - l_2 = l_1+1$. This contradicts our assumption that $i - l_2 = -m_{p+1} \leq l_1$. Next we must have $i > 0$, otherwise $l_2 = m_{p+1}$, contradicting (c). Hence $-l_1 \geq m_{p+1} = l_2 - i < l_2$. This contradicts (e) if $l_1 < 0$, since the segments $[\nu^{-m'_{p+1}}\rho_{p+1}', \nu^{m'_{p+1}}\rho_{p+1}']$ and $[\nu^{-l_1}\rho, \nu^{l_2}\rho]$ are linked. If $l_1 \geq 0$, then, by the observation made above, t is defined $m'_{p+1} \geq m_t' \geq l_1$ and by construction. Since $\rho \simeq \rho_{p+1}'$, assumption (c) implies that $m'_{p+1} > l_1$. This, together with $-l_1 \geq m_{p+1} = l_2 - i < l_2$ and $\rho \simeq \rho_{p+1}'$, contradicts (e). This completes the proof of the base of the second induction. The proof of the induction step is the same as the proof of the base. This completes the proof if $l_1 < 0$ or $t+1$ is not defined. If $t+1$ is defined, we do one more easy induction on a, where $a \geq t+1$; this is left to the reader. This completes the proof of Claim 3.3. □

Now, using Theorem 1.4, it is easy to show that $\delta_1' \times \cdots \times \delta_l' \otimes \pi$ has multiplicity at least 2^l in $\mu^*(\delta_1' \times \cdots \times \delta_l' \rtimes \pi)$. Since

$$\mu^*(\delta \times \delta_1' \times \cdots \times \delta_l' \rtimes \tau') \geq \mu^*(\delta_1' \times \cdots \times \delta_l' \rtimes \pi),$$

this multiplicity is exactly 2^l. Finally, since $\tau_i \hookrightarrow \delta_1' \times \cdots \times \delta_l' \rtimes \tau'$, Frobenius reciprocity implies that

$$\mu^*(\tau_i) \geq \delta_1' \times \cdots \times \delta_l' \otimes \tau'.$$

Hence, by Theorem 1.4,

$$\mu^*(\delta \rtimes \tau_i) \geq \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \\ \times \delta'_1 \times \cdots \times \delta'_l \otimes \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \tau' \\ \geq \delta'_1 \times \cdots \times \delta'_l \otimes \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau' \geq \delta'_1 \times \cdots \times \delta'_l \otimes \pi.$$

Since this holds for all $i = 1, \ldots, 2^l$, we see that the multiplicity of $\delta'_1 \times \cdots \times \delta'_l \otimes \pi$ in $\mu^*(\delta \rtimes \tau_i)$ is exactly one. This completes the proof of Lemma 3.2.

4. Reducibility of $\delta \rtimes \sigma$

We keep the notation from Section 2. In this section we recall reducibility results for $\delta \rtimes \sigma$, starting with the basic setup for the classification of discrete series [Mœglin 2002; Mœglin and Tadić 2002]. (A brief overview can also be found in [Muić 2005, Section 1].) Let $(\mathrm{Jord}, \sigma', \varepsilon)$ be the admissible triple attached in [Mœglin 2002] to σ. It can be described as follows:

First, $\sigma' \in \mathrm{Irr}'$ is a supercuspidal representation such that there exists an irreducible representation $\pi \in \mathrm{GL}(m_\pi, F)$ such that $\sigma \hookrightarrow \pi \rtimes \sigma'$. This property determines $\sigma' \in \mathrm{Irr}'$ uniquely.

Next, Jord is defined as a set of all pairs (a, ρ) ($\rho \cong \tilde{\rho}$ is a supercuspidal representation of some $\mathrm{GL}(m_\rho, F)$, $a > 0$ is integer) such that (a) and (b) below hold:

(a) a is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L-function $L(s, \rho, r)$ is the one defined by Shahidi [1990; 1992], where $r = \bigwedge^2 \mathbb{C}^{m_\rho}$ is the exterior square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $\mathrm{GL}(m_\rho, \mathbb{C})$ if G_n is a symplectic or even-orthogonal group and $r = \mathrm{Sym}^2 \mathbb{C}^{m_\rho}$ is the symmetric-square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $\mathrm{GL}(m_\rho, \mathbb{C})$ if G_n is an odd-orthogonal group. *Any* such pair (not necessarily related to σ) is said to satisfy the *parity condition*.

(b) The induced representation $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$ is irreducible.

We write $\mathrm{Jord}_\rho = \{a;\ (a, \rho) \in \mathrm{Jord}\}$, and for $a \in \mathrm{Jord}_\rho$ we write a_- for the largest element of Jord_ρ that is strictly less than a, if one exists. (It is proved in [Mœglin 2003] that Jord is a finite set.)

Finally, ε is a function defined on a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ into $\{\pm 1\}$. The precise definition is not important here; See [Mœglin 2002], [Mœglin and Tadić 2002], or [Muić 2005, Section 1]. We just record the next two facts. We say that σ is attached to an *alternating triple* if $\varepsilon(a, \rho) \cdot \varepsilon(a_-, \rho)^{-1} = -1$ whenever a_- is defined and there is an increasing bijection $\phi_\rho : \mathrm{Jord}_\rho \to \mathrm{Jord}'_\rho(\sigma')$, where

$$\mathrm{Jord}'_\rho(\sigma') = \begin{cases} \mathrm{Jord}_\rho(\sigma') \cup \{0\} & \text{if } a \text{ is even and } \varepsilon(\min \mathrm{Jord}_\rho, \rho) = 1\ ; \\ \mathrm{Jord}_\rho(\sigma') & \text{otherwise.} \end{cases}$$

In this case σ can be described explicitly as follows: For each ρ such that $\mathrm{Jord}_\rho \neq \varnothing$, we write the elements of Jord_ρ in increasing order as $a_1^\rho < a_2^\rho < \cdots < a_{k_\rho}^\rho$. Then σ is the unique irreducible subrepresentation of

$$\times_\rho \times_{i=1}^{k_\rho} \delta([\nu^{(\phi_\rho(a_i^\rho)+1)/2}\rho, \nu^{(a_i^\rho-1)/2}\rho]) \rtimes \sigma'.$$

Suppose $(a, \rho) \in \mathrm{Jord}$ is such that a_- is defined and $\varepsilon(a, \rho) \cdot \varepsilon(a_-, \rho)^{-1} = 1$. Then there exists a unique discrete series $\sigma'' \in \mathrm{Irr}'$ such that

$$\sigma \hookrightarrow \delta([\nu^{-(a_--1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma''.$$

It is attached to the triple $(\mathrm{Jord}'', \sigma', \varepsilon)$, where

$$\mathrm{Jord}'' = \mathrm{Jord} \setminus \{(2b+1, \rho), (2b_-+1, \rho)\}$$

and ε'' is the restriction of ε to Jord''.

Removing successively some pairs (a, ρ), (a_-, ρ) with $\varepsilon(a, \rho) \cdot \varepsilon(a_-, \rho)^{-1} = 1$ we can reach an alternating triple. (Note that a_- is defined in terms of the current triple from which we remove the pair.) We say that ensuing alternating triple is *dominated* by that of σ.

Now, we recall reducibility results from [Mœglin and Tadić 2002, Section 13].

Theorem 4.1. *Assume* $l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ *and* $\rho \in \mathrm{Irr}$ *is an unitary and supercuspidal representation. Then* $\delta([\nu^{-l}\rho, \nu^{l}\rho]) \rtimes \sigma$ *is reducible if and only if* $(2l+1, \rho)$ *satisfies the parity condition and* $(2l+1, \rho) \notin \mathrm{Jord}$.

Finally, we describe nonunitary reduction [Muić 2005]. In the description of reducibility of $\delta \rtimes \sigma$ it is convenient to introduce the following definition.

Definition 4.2. We say that the pair $(2a_0^-+1, 2a_0+1) \in \mathrm{Jord}_\rho \times \mathrm{Jord}_\rho$, $a_0^- < a_0$, is a *ρ-admissible pair* if $\varepsilon(2a_0+1, \rho)\varepsilon(2a_0^-+1, \rho)^{-1} = 1$ and $]2a_0^-+1, 2a_0+1[\cap \mathrm{Jord}_\rho$ is either empty or can be divided into disjoint sets of pairs $\{2a_j^-+1, 2a_j+1\}$, $j = 1, \ldots, k$ (this defines k), such that $a_j^- < a_j$, $\varepsilon(2a_j+1, \rho)\varepsilon(2a_j^-+1, \rho)^{-1} = 1$ and there is no pair of indices $i \neq j$ such that $a_j^- < a_i^- < a_j < a_i$ or $a_i^- < a_j^- < a_i < a_j$.

Theorem 4.3. *Put*

$$\mathrm{Jord}(l_1, l_2, \rho) = \begin{cases}]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho & \text{if } l_1 \geq 0, \\ [-2l_1-1, 2l_2+1[\cap \mathrm{Jord}_\rho & \text{if } l_1 < 0. \end{cases}$$

Then $\delta \rtimes \sigma$ *reduces if and only if* $(2l_1+1, \rho)$ *satisfies the parity condition* (*hence also* $(2l_2+1, \rho)$) *and one of the following holds*:

(a) *There exists an* ρ*-admissible pair* $(2a'+1, 2a+1)$ *such that* $\{2a'+1, 2a+1\}$ *intersects* $\mathrm{Jord}(l_1, l_2, \rho)$.

(b) $\mathrm{Jord}(l_1, l_2, \rho) \neq \varnothing$ *and there is an alternating triple* $(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \varepsilon_{\mathrm{alt}})$ *dominated by* $(\mathrm{Jord}, \sigma', \varepsilon)$ *such that* $\mathrm{Jord}(l_1, l_2, \rho) \subseteq \mathrm{Jord}_{\mathrm{alt}}$ *and one of the following conditions holds*:

- $l_1 \geq 0$ *and* $2l_1+1$ *or* $2l_2+1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$.
- $l_1 < 0\,(\neq -\frac{1}{2})$ *and* $2l_2+1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$.
- $l_1 = -\frac{1}{2}$ *and* $2l_2+1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$ *and* $\varepsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = 1$.

(c) $\mathrm{Jord}(l_1, l_2, \rho) = \varnothing$, *and one of the following holds*:

- $l_1 \geq 0$ *and* $\{2l_1+1,\ 2l_2+1\} \not\subset \mathrm{Jord}_\rho$ *or* $\varepsilon(2l_2+1, \rho)\varepsilon(2l_1+1, \rho)^{-1} = 1$.
- $l_1 = -\frac{1}{2}$ *and* $2l_2+1 \in \mathrm{Jord}_\rho \implies \varepsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = 1$.

Moreover, if $\delta \rtimes \sigma$ *reduces than it has an irreducible subquotient other than* $\mathrm{Lang}(\delta \rtimes \sigma)$ *that appears in its composition series with multiplicity one.*

Proof. The reducibility part is a reformulation of the main result of [Muić 2005]. (See particularly the following results in that reference: Theorems 3.1 and 3.2, Lemmas 4.1 and 4.3, Theorem 5.1, Lemmas 6.1, 6.2 and 6.4.) The last part of the theorem is also a consequence of the results in [Muić 2005] just mentioned, except when $l_1 \geq 0$, $\mathrm{Jord}(l_1, l_2, \rho) = \varnothing$, and $2l_1+1$ or $2l_2+1 \in \mathrm{Jord}_\rho$ but not both. If σ is strongly positive this case is covered in [Muić 2004]. The general case is similar and is already contained in [Muić 2005] and [Muić 2004]. We just sketch the proof. According to [Muić 2004, Lemma 2.1] we have two cases:

If $2l_1+1 \in \mathrm{Jord}_\rho$, the possible tempered subquotient of $\delta \rtimes \sigma$ is common with $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_1$, where σ_1 is a discrete series subquotient of $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$. Going in the opposite direction, $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ reduces and besides its Langlands quotient it has a discrete series subquotient appearing with multiplicity one [Muić 2005, Lemma 6.1]. Now, one can show that $\delta \rtimes \sigma$ and $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_1$ have a common irreducible subquotient that appears with multiplicity one in their composition series. It is enough to show that $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \sigma_1$ appears with multiplicity two in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$ and with multiplicity one in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$. We expand

$$\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$$

using Theorem 1.4. Thus, we take indices $0 \leq j \leq i \leq 2l_1+1$, $0 \leq j' \leq i' \leq l_2 - l_1$, and irreducible constituents $\delta' \otimes \sigma_1'$ of $\mu^*(\sigma)$, and we obtain

$$\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \leq \delta([\nu^{i-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1-j}\rho, \nu^{l_1}\rho]) \\ \times \delta([\nu^{i'-l_2}\rho, \nu^{-l_1-1}\rho]) \times \delta([\nu^{l_2+1-j'}\rho, \nu^{l_2}\rho]) \times \delta'$$

and

$$\sigma_1 \le \delta([\nu^{l_1+1-i}\rho, \nu^{l_1-j}\rho]) \times \delta([\nu^{l_2+1-i'}\rho, \nu^{l_2-j'}\rho]) \rtimes \sigma_1'. \tag{4–1}$$

Since on the left-hand side of the first formula there are no terms $\nu^{-l_1-1}\rho$ and $\nu^{l_2}\rho$ we must have $i' = l_2 - l_1$ and $j' = 0$. Next, $\nu^{-l_1}\rho$ can not be obtained by δ'. Otherwise, since δ' is obviously nondegenerate, it is induced from the product of essentially square-integrable representations. One of them must be attached to the segment of the form $[\nu^{-l_1}\rho, \nu^a\rho]$, where $a \le l_1$. Now, using idea already applied in the proof of Lemma 1.3, we see that this violates square-integrability criterion for σ. Thus $i = 0$ (hence $j = 0$) or $j = 2l_1+1$ (hence $i = 2l_1+1$). Now, δ' is trivial and $\sigma_1' \simeq \sigma$. Now, (4–1) is

$$\sigma_1 \le \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma,$$

and as we already observed it contains σ_1 with multiplicity one. This proves the first multiplicity. To show that the multiplicity of $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \sigma_1$is one in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$, we expand $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$ using Theorem 1.4. Thus, we take indices $0 \le j \le i \le l_1 + l_2+1$, and irreducible constituents $\delta' \otimes \sigma_1'$ of $\mu^*(\sigma)$, and we obtain

$$\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \le \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta', \tag{4–2}$$

$$\sigma_1 \le \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1'. \tag{4–3}$$

As in the above multiplicity computation, we see that $j = 0$ since the left-hand of (4–2) does not contain $\nu^{l_2}\rho$. Also as in the above computation of multiplicity we conclude that $\nu^{-l_1}\rho$ cannot be obtained by δ'. Thus $i = l_2 - l_1$. Now, δ' is trivial and $\sigma_1' \simeq \sigma$. Now, formula in (4–3) is

$$\sigma_1 \le \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma,$$

and as already observed it contains σ_1 with multiplicity one. This proves the second multiplicity.

If $2l_2+1 \in \mathrm{Jord}_\rho$, then the possible tempered subquotient of $\delta \rtimes \sigma$ is common with $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1$, where σ_1 is some discrete series satisfying $\mu^*(\sigma) \ge \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1$. Now, arguing as in [Muić 2004, Lemma 4.1], we see that there exists an irreducible representation σ_1' such that

$$\sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1'. \tag{4–4}$$

First, we show that σ_1' is in a discrete series. We start as in the proof of Lemma 1.3. If σ_1' is not in a discrete series, by the Langlands classification and the theory of R-groups, we can find a unitary supercuspidal representation ρ', real numbers

α, β with $\beta-\alpha\le 0$ and $\alpha+\beta\in\mathbb{Z}_{\ge 0}$, and $\sigma_1''\in \mathrm{Irr}'$, such that

$$\sigma_1' \hookrightarrow \delta([\nu^{-\alpha}\rho', \nu^{\beta}\rho']) \rtimes \sigma_1''.$$

We look at the chain of equivariant morphisms:

$$\sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-\alpha}\rho', \nu^{\beta}\rho']) \rtimes \sigma_1'' \\ \to \delta([\nu^{-\alpha}\rho', \nu^{\beta}\rho']) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1''.$$

The representation σ must include into the kernel of the second equivariant morphism, otherwise $\beta-\alpha\le 0$ would violate the square-integrability criterion for σ. In particular, the segments $[\nu^{l_1+1}\rho, \nu^{l_2}\rho]$ and $[\nu^{-\alpha}\rho', \nu^{\beta}\rho']$ are linked. Thus, $\rho'\simeq\rho$. Further, since $\beta-\alpha\le 0$ and $\alpha+\beta\ge 0$ imply $\alpha\ge 0$, we have $l_1\ge\beta$ and

$$(4\text{–}5)\quad \sigma \hookrightarrow \delta([\nu^{l_1+1}\rho', \nu^{\beta}\rho']) \times \delta([\nu^{-\alpha}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1'' \simeq \\ \delta([\nu^{-\alpha}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho', \nu^{\beta}\rho']) \rtimes \sigma_1''.$$

If $\beta>l_1$, it follows from the first induced representation in (4–5) and [Mœglin 2002, Remark 5.1.2] that $2\beta+1\in \mathrm{Jord}(l_1, l_2, \rho)$. This is a contradiction. Otherwise, since $\alpha\le 0$, the appearance of the second induced representation in (4–5) shows that $\mathrm{Jord}_\rho\cap[2\alpha+1, 2l_2+1[\,\neq\varnothing$. This follows from [Mœglin 2002, Theorem 3.1] and again represents a contradiction since $\alpha\ge\beta=l_1$.

Now, we show that $\sigma_1'\simeq\sigma_1$ and that $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho])\otimes\sigma_1$ appears in

$$\mu^*\big(\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1\big)$$

with multiplicity one. We expand

$$\mu^*\big(\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1'\big) \ge \mu^*(\sigma) \ge \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1$$

using Theorem 1.4. Thus, we take indices $0\le j\le i\le l_2-l_1$ and an irreducible constituent $\delta'\otimes\sigma_2$ of $\mu^*(\sigma_1')$ to obtain

$$\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \le \delta([\nu^{i-l_2}\rho, \nu^{-l_1-1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta', \\ \sigma_1' \le \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_2.$$

The first of these formulas shows that $i=l_2-l_1$, since the left-hand side does not have $\nu^{-l_1-1}\rho$. Next, if δ' is nontrivial, it is nondegenerate and hence induced from the essentially square-integrable representations attached to segments that have members in $[\nu^{l_1+1}\rho, \nu^{l_2}\rho]$. Now, by [Mœglin 2002, Remark 5.1.2], as in proof of [Muić 2004, Lemma 4.1], we conclude that $\mathrm{Jord}(\sigma_1')\cap[2(l_1+1)+1, 2l_2+1]\neq\varnothing$, and this contradicts [Muić 2004, Lemma 2.1]. Thus, δ' is trivial, $l_2+1-j=l_1+1$, and $\sigma_2\cong\sigma_1$. Our assertion is now obvious.

Finally, we check that $\delta\rtimes\sigma$ and $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho])\rtimes\sigma_1$ have a common irreducible subquotient that appears in $\delta\rtimes\sigma$ with multiplicity one. It is enough to show that

$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \sigma_1$ appears with multiplicity one in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$ and in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1\big)$. We compute the first multiplicity only; the determination of the second is analogous.

First, we expand $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$ using Theorem 1.4. Thus, we take indices $0 \le j \le i \le l_1 + l_2 + 1$ and irreducible constituents $\delta' \otimes \sigma_1'$ of $\mu^*(\sigma)$, and we obtain

$$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \le \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta',$$
$$\sigma_1 \le \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1'.$$

First, $\nu^{-l_2}\rho$ cannot come from δ'. Otherwise, δ' being nondegenerate and hence induced from essentially square-integrable representations, it must have one of the segments of the form $[\nu^{-l_2}\rho, \nu^{s}\rho]$, where $s \le l_2$. This contradicts the square-integrability criterion for σ. Thus $i = 0$. Hence $j = 0$, and $\delta' \simeq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho])$. This implies that $\sigma_1' \simeq \sigma_1$, thanks to the second formula in (4–5). Finally, since $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1$ appears in $\mu^*\big(\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1\big)$ with multiplicity one, it also appears with the same multiplicity in $\mu^*(\sigma)$. □

5. The first basic case

In Sections 2 and 3 we reduced the reducibility problem to the three basic cases recorded at the end of Section 2. In this section we consider the first basic case, where $l_1 \ge 0$ and $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$ reduces. By Theorem 4.1, $(2l_1+1, \rho)$ satisfies the parity condition and is not an element of Jord. Furthermore, according to the general theory of Harish-Chandra [Waldspurger 2003], we see that

$$\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \;\simeq\; \tau_1 \oplus \tau_2,$$

where τ_1 and τ_2 are inequivalent irreducible tempered representations. We fix this notation throughout this section.

From the assumptions at the end of Section 2, the one that interests us is (2–14) (see also Theorem 4.3). Now, we have several cases:

The first case we consider is $\mathrm{Jord}_\rho \cap \,]2l_1+1, 2l_2+1] = \varnothing$. Then Jord_ρ is in fact disjoint from $[2l_1+1, 2l_2+1]$. Hence, [Muić 2004, Theorems 2.1 and 2.3] imply that in the appropriate Grothendieck group

$$\delta \rtimes \sigma = \sigma_1 + \sigma_2 + \mathrm{Lang}(\delta \rtimes \sigma),$$

where σ_1 and σ_2 are nonisomorphic discrete series representations. Moreover,

$$\mathrm{Jord}(\sigma_1) = \mathrm{Jord}(\sigma_2) = \mathrm{Jord} \cup \{(2l_1+1, \rho), (2l_2+1, \rho)\}.$$

Hence, Theorem 4.1 implies that

$$\mathcal{T}_i := \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_i \quad \text{for } i = 1, 2$$

is an irreducible tempered representation. Finally, we can relate τ_i and σ_i using [Mœglin and Tadić 2002]. That is, we can assume that

$$\sigma_i \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i \quad \text{for } i = 1, 2, \tag{5–1}$$

and that σ_i is a unique irreducible subrepresentation of $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i$.

Lemma 5.1. *Assume that* $\mathrm{Jord}_\rho \cap]2l_1+1, 2l_2+1] = \varnothing$. *In the appropriate Grothendieck group,*

$$\delta \rtimes \tau_i = \mathcal{T}_i + \mathrm{Lang}(\delta \rtimes \tau_i) \quad \textit{for } i = 1, 2.$$

In particular, $\delta \rtimes \tau_1$ *and* $\delta \rtimes \tau_2$ *reduce.*

Proof. First, (5–1) implies

$$\mathcal{T}_i \simeq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \sigma_i \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i$$
$$\to \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \tau_i.$$

The image of the second intertwining operator in this sequence is isomorphic to $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i$. To show that $\mathcal{T}_i$ is not in the kernel of that intertwining operator we show that multiplicity of $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \tau_i$ is equal to 3 in

$$\mu^*\big(\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \tau_i\big), \tag{5–2}$$

but it is at most 2 in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i\big)$ and in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_l}\rho]) \rtimes \sigma_i\big)$. We start by computing the multiplicity in (5–2). Using Theorem 1.4, we look for $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \tau_i$ in terms of the following form. We take indices $0 \le j' \le i' \le l_2 - l_1$, $0 \le j'' \le i'' \le 2l_1+1$, and irreducible constituents $\delta' \otimes \tau'$ of $\mu^*(\tau_i)$, and we obtain

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \le \delta([\nu^{i'-l_2}\rho, \nu^{-l_1-1}\rho]) \times \delta([\nu^{l_2+1-j'}\rho, \nu^{l_2}\rho])$$
$$\times \delta([\nu^{i''-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1-j''}\rho, \nu^{l_1}\rho]) \times \delta' \tag{5–3}$$

and

$$\tau_i \le \delta([\nu^{l_2+1-i'}\rho, \nu^{l_2-j'}\rho]) \times \delta([\nu^{l_1+1-i''}\rho, \nu^{l_1-j''}\rho]) \rtimes \tau'. \tag{5–4}$$

First, since the left-hand side of (5–3) does not contain $\nu^{-l_1-1}\rho$, we must have $i' = l_2 - l_1$. We claim that $j' = l_2 - l_1$. Otherwise, δ' is not trivial. Therefore, δ' is nondegenerate. Hence it is fully induced from the product of essentially square-integrable representations and one of them must have a segment that upper-ends with $\nu^a\rho$, $a \in [l_1+1, l_2]$. Since $\mu^*(\tau_i) \ge \delta' \otimes \tau'$, we have $\tau_i \hookrightarrow \nu^a\rho \rtimes \tau''$ for some irreducible representation τ''. Hence, by the Frobenius reciprocity, $\mu^*(\tau_i) \ge \nu^a\rho \otimes \tau''$. Thus

$$\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma\big) \ge \mu^*(\tau_i) \ge \nu^a\rho \otimes \tau''.$$

The first inequality here can be analyzed using Theorem 1.4. First, there are indices

$0 \le j' \le i' \le 2l_1{+}1$ and an irreducible constituent $\delta'' \otimes \sigma''$ of $\mu^*(\sigma)$ such that

$$\nu^a \rho \le \delta([\nu^{i'-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1-j'}\rho, \nu^{l_1}\rho]) \times \delta''.$$

Since $a \in [l_1{+}1, l_2]$, this implies $\delta'' \simeq \nu^a\rho$. Hence $\sigma \hookrightarrow \nu^a\rho \rtimes \sigma_1''$, for some irreducible representation σ_1''. Hence $2a{+}1 \in \mathrm{Jord}_\rho$ by [Mœglin 2002, Remark 5.1.2]. This contradicts the lemma's assumption that $\mathrm{Jord}_\rho \cap \,]2l_1{+}1, 2l_2{+}1] = \varnothing$. Thus, we have proved $i' = j' = l_2 - l_1$.

Next we look at how to obtain $\nu^{-l_1}\rho$. In view of (5–3) there are two cases. First, if $\nu^{-l_1}\rho$ is obtained from the terms $\delta([\nu^{i''-l_1}\rho, \nu^{l_1}\rho])$ and $\delta([\nu^{l_1+1-j''}\rho, \nu^{l_1}\rho])$ in (5–3), then δ' is trivial, and $\tau' \cong \tau_i$, $i'' = j'' = 0$ or $i'' = j'' = 2l_1{+}1$. Then (5–4) trivially holds, so we have obtained our term twice. The other possibility is that $\nu^{-l_1}\rho$ is obtained from δ'. Then δ' is nontrivial and clearly nondegenerate. Hence it is fully induced from the product of essentially square-integrable representations such that one of them is attached to a segment containing $\nu^{-l_1}\rho$. In view of (5–3) and the already proved fact that $i' = j' = l_2 - l_1$, we see that this segment must be of the form $[\nu^{-l_1}\rho, \nu^a\rho]$, for $a \in [-l_1, l_1]$. This violates the temperedness criterion for τ_i unless $a = l_1$. Hence $\delta' \simeq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho])$, and because of (5–3) and the already proved fact that $i' = j' = l_2 - l_1$, we must have $i'' = 2l_1{+}1$ and $j''' = 0$. To complete the proof that $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \tau_i$ appears in (5–2) with multiplicity one, we need to show the following facts. If $\mu^*(\tau_i) \ge \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \tau'$, where τ' is irreducible, then $\tau' \simeq \sigma$. Moreover, $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \sigma$ appears in $\mu^*(\tau_i)$ with multiplicity one.

First, $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma\big) \ge \mu^*(\tau_i) \ge \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \tau'$. This can be analyzed using Theorem 1.4. Thus, there are indices $0 \le j' \le i' \le 2l_1{+}1$ and an irreducible constituent $\delta'' \otimes \sigma''$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \le \delta([\nu^{i'-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1-j'}\rho, \nu^{l_1}\rho]) \times \delta'', \tag{5–5}$$

$$\tau' \le \delta([\nu^{l_1+1-i'}\rho, \nu^{l_1-j'}\rho]) \rtimes \sigma''. \tag{5–6}$$

Again we investigate how to obtain the term $\nu^{-l_1}\rho$ in (5–5). An analysis similar to the one given for (5–3) shows that this term cannot be obtained from δ'', since this would imply that δ'' is nondegenerate and one of its the segments is of the form $[\nu^{-l_1}\rho, \nu^a\rho]$ for $a \in [-l_1, l_1]$. This would then violate the square-integrability criterion for σ. Hence, in view of (5–5), $i' = j' = 0$ or $i' = j' = 2l_1{+}1$, δ'' is trivial, and $\sigma'' \simeq \sigma$. Now, (5–6) shows that $\tau' \cong \sigma$. Thus, we have proved that the multiplicity of $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \sigma$ in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma\big)$ is exactly two and by Frobenius reciprocity, exactly one in each $\mu^*(\tau_\alpha)$, for $\alpha = 1, 2$. Moreover, we have shown that if $\mu^*(\tau_i) \ge \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \tau'$, where τ' is irreducible, then $\tau' \simeq \sigma$. This completes the proof that $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \otimes \tau_i$ appears in (5–2) with multiplicity 3.

The other two claims about multiplicities can be proved similarly. We leave details to the reader.

Thus, we have proved that

$$\delta \rtimes \tau_i \geq \mathcal{T}_i + \mathrm{Lang}(\delta \rtimes \tau_i) \quad \text{for } i = 1, 2.$$

To prove that there is equality we use the factorization of the long-intertwining operator

$$\delta \rtimes \tau_1 \oplus \delta \rtimes \tau_2 \to \tilde{\delta} \rtimes \tau_1 \oplus \tilde{\delta} \rtimes \tau_2.$$

It can be factored into the long-intertwining operators (with $\delta = \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])$):

$$\begin{aligned}\delta \rtimes \tau_1 \oplus \delta \rtimes \tau_2 \simeq\ & \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \to \\ & \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \to \\ & \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma \to \\ & \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \simeq \tilde{\delta} \rtimes \tau_1 \oplus \tilde{\delta} \rtimes \tau_2.\end{aligned}$$

We denote the induced long-intertwining operators (the arrows in the preceding lines) by φ_1, φ_2, and φ_3, respectively. From [Zelevinsky 1980] we get that φ_1 and φ_3 are isomorphisms, and φ_2, by [Muić 2005, Theorem 2.1], has kernel isomorphic to $\mathcal{T}_1 \oplus \mathcal{T}_2$. This proves the lemma. □

The second case that we consider is $\mathrm{Jord}_\rho \cap\,]2l_1{+}1, 2l_2{+}1] = \{2l_2{+}1\}$. Recall that we assume the parity condition for $(2l_1{+}1, \rho) \notin \mathrm{Jord}_\rho$. The proof of Theorem 4.3 proves the next lemma.

Lemma 5.2. *Let* $\sigma \in \mathrm{Irr}'$ *be a discrete series such that* $\mathrm{Jord}_\rho \cap [2l_1{+}1, 2l_2{+}1] = \{2l_2{+}1\}$.

(i) *There exists a unique discrete series representation* $\sigma(l_2)$ (called σ_1 in the proof of Theorem 4.3) *such that* $\sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma(l_2)$. *Moreover,* $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma(l_2)$ *appears in* $\mu^*(\sigma)$ *with multiplicity one. If* $\mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma'(l_2)$, *where* $\sigma'(l_2)$ *is irreducible, then* $\sigma'(l_2) \simeq \sigma(l_2)$. *As a consequence*, $\mathrm{Jord}(\sigma(l_2)) = \mathrm{Jord} \cup \{(2l_1{+}1, \rho)\} \setminus \{(2l_2{+}1\rho)\}$ (see [Mœglin and Tadić 2002, Section 8]).

(ii) *There exists a unique common irreducible representation, say* $\sigma_{\mathrm{temp}}(l_2)$, *of* $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma(l_2)$ *and* $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$. *In the appropriate Grothendieck group*, $\delta \rtimes \sigma = \sigma_{\mathrm{temp}}(l_2) + \mathrm{Lang}(\delta \rtimes \sigma)$.

We are now ready to settle the reducibility of $\delta \rtimes \tau_i$.

Lemma 5.3. *There exists a unique* $i \in \{1, 2\}$ *such that* $\delta \rtimes \tau_i$ *is irreducible. Without loss of generality, we may take* $i = 1$. *In the appropriate Grothendieck group*,

$$\delta \rtimes \tau_2 = \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_{\mathrm{temp}}(l_2) + \mathrm{Lang}(\delta \rtimes \tau_2).$$

(The irreducibility of $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_{\text{temp}}(l_2)$ follows from [Goldberg 1994]; see also [Mœglin and Tadić 2002].)

Proof. We use the long-intertwining operator form the proof of Lemma 5.1. Again, [Zelevinsky 1980] implies that φ_1 and φ_3 are isomorphisms, and φ_2, by Lemma 5.2, has kernel isomorphic to irreducible tempered representation $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_{\text{temp}}(l_2)$. This proves the lemma. □

Finally, we consider the case $\text{Jord}_\rho \cap]2l_1+1, 2l_2+1[\neq \varnothing$. We denote by $2l+1$ the minimal element of that intersection. Then according to Lemma 5.3, we can distinguish τ_1 from τ_2 by assuming that

$$\text{(5–7)}\qquad \begin{aligned} &\delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \rtimes \tau_1 \text{ is irreducible;}\\ &\delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \rtimes \tau_2 \geq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_{\text{temp}}(l).\end{aligned}$$

Lemma 2.3 shows that $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\text{temp}}(l)$ contains

$$\text{Lang}\big(\delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \rtimes \sigma(l)\big)$$

with multiplicity one. Hence, by (5–7), in the appropriate Grothendieck group, we have

$$\text{(5–8)}\quad \begin{aligned} &\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \rtimes \tau_2\\ &\qquad \geq \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_{\text{temp}}(l)\\ &\qquad = \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\text{temp}}(l)\\ &\qquad \geq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \text{Lang}\big(\delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \rtimes \sigma(l)\big)\\ &\qquad\qquad\qquad \geq \text{Lang}\big(\delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \rtimes \mathcal{T}\big),\end{aligned}$$

where $\mathcal{T} := \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma(l)$ is irreducible and tempered. The last inequality here follows from

$$\text{(5–9)}\quad \begin{aligned} &\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \text{Lang}\big(\delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \rtimes \sigma(l)\big)\\ &\qquad \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l}\rho]) \rtimes \sigma(l)\\ &\qquad \simeq \delta([\nu^{-l_2}\rho, \nu^{l}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma(l) \simeq \delta([\nu^{-l_2}\rho, \nu^{l}\rho]) \rtimes \mathcal{T}.\end{aligned}$$

We show that $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \rtimes \tau_2$ contains

$$\text{Lang}\big(\delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \rtimes \mathcal{T}\big)$$

with multiplicity one. It is enough to show that

$$\mu^*\big(\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \rtimes \tau_2\big)$$

contains $\delta([\nu^{-l_2}\rho,\nu^{l}\rho])\otimes\mathcal{T}$ with multiplicity one. Using Theorem 1.4, we take indices $0\le j'\le i'\le l_2-l$, $0\le j''\le i''\le l_1+l+1$, and an irreducible constituent $\delta'\otimes\tau'$ of $\mu^*(\tau_2)$ to obtain

$$\delta([\nu^{-l_2}\rho,\nu^{l}\rho])\le\delta([\nu^{i'-l_2}\rho,\nu^{-l-1}\rho])\times\delta([\nu^{l_2+1-j'}\rho,\nu^{l_2}\rho])\\ \times\delta([\nu^{i''-l}\rho,\nu^{l_1}\rho])\times\delta([\nu^{l+1-j''}\rho,\nu^{l}\rho])\times\delta'$$

and

$$\mathcal{T}\le\delta([\nu^{l_2+1-i'}\rho,\nu^{l_2-j'}\rho])\times\delta([\nu^{l+1-i''}\rho,\nu^{l-j''}\rho])\rtimes\tau'. \tag{5–10}$$

The analysis of these two inequalities is similar to the one in the proof of Lemma 5.1. We just sketch the computation. First, $\nu^{-l_2}\rho$ cannot come from δ' since this would violate the temperedness criterion for τ_2. Next, $l_2+1-j''\ge l+1>-l_2$, $i''-l\ge -l>-l_2$, $l+1-j''\ge -l_1>-l_2$ imply $i'=0$. Hence $i'\ge j'\ge 0$ implies $j'=0$. Next, $\nu^{-l}\rho$ cannot come from δ' because of Lemma 1.3. Hence $i''=0$. Thus, $j''=0$. Hence $\delta'\cong\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])$. Finally, (5–10) implies that $\mathcal{T}\simeq\tau'$. Hence to complete our analysis, we show that $\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])\otimes\mathcal{T}$ occurs in $\mu^*(\tau_2)$ with multiplicity one. Since by (5–8) it occurs at least once, it is enough to show that $\mu^*(\delta([\nu^{-l_1}\rho,\nu^{l_1}\rho])\rtimes\sigma)$ contains $\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])\otimes\mathcal{T}$ with multiplicity one. We apply Theorem 1.4 and Lemma 5.2 (with l_2 replaced by l). Thus, we take indices $0\le j\le i\le l_2-l$ and an irreducible constituent $\delta'\otimes\sigma_1'$ of $\mu^*(\sigma)$, to obtain

$$\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])\le\delta([\nu^{i-l_1}\rho,\nu^{l_1}\rho])\times\delta([\nu^{l_1+1-j}\rho,\nu^{l_1}\rho])\times\delta',\\ \mathcal{T}\le\delta([\nu^{l_1+1-i}\rho,\nu^{l_1-j}\rho])\rtimes\sigma_1'.$$

The first formula implies that $i=2l_1+1$ and $j=0$. Hence $\delta\simeq\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])$, and therefore $\sigma_1'\cong\sigma(l)$. The second displayed inequality then holds by the definition of $\mathcal{T}$. Finally, the multiplicity follows from the fact that $\mu^*(\sigma)$ contains $\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])\otimes\sigma(l)$ with multiplicity one.

Next, by Theorem 1.4 and using $\mu^*(\tau_2)\ge\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])\otimes\mathcal{T}$ we have

$$\mu^*\big(\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\rtimes\tau_2\big)\ge\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\times\delta([\nu^{l_1+1}\rho,\nu^{l}\rho])\otimes\mathcal{T}\\ \ge\delta([\nu^{-l_2}\rho,\nu^{l}\rho])\otimes\mathcal{T}.$$

Since

$$\delta([\nu^{l+1}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{l}\rho])\rtimes\tau_2\ge\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\rtimes\tau_2,$$

we have proved the following lemma:

Lemma 5.4. $\delta\rtimes\tau_2$ *contains* $\mathrm{Lang}\big(\delta([\nu^{-l}\rho,\nu^{l_2}\rho])\rtimes\mathcal{T}\big)$ *with multiplicity one. In particular, it is reducible.*

Next we consider $\delta\rtimes\tau_1$. We decompose the long-intertwining operator $\delta\rtimes\tau_1\to\tilde{\delta}\rtimes\tau_1$ as follows:

$$\begin{aligned}(5\text{–}11)\quad \delta \rtimes \tau_1 &\hookrightarrow \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \rtimes \tau_1 \\ &\simeq \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \rtimes \tau_1 \\ &\simeq \delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1 \\ &\to \delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{-l-1}\rho]) \rtimes \tau_1.\end{aligned}$$

Lemma 5.5. *$\delta \rtimes \tau_1$ reduces if and only if $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ reduces. If $\delta \rtimes \tau_1$ reduces, it contains with multiplicity one an irreducible subquotient other than its Langlands quotient.*

Proof. By Lemma 3.1, $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ reduces if and only if $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1$ reduces. Next, (5–11) shows that if $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1$ is irreducible, so is $\delta \rtimes \tau_1$. Next, assume that $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1$ reduces. Hence $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ reduces. Let π_1 be any irreducible subquotient of $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ other than its Langlands quotient, occurring with multiplicity one (see Theorem 4.3). Applying Lemma 3.2 to $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1$ we see that the latter contains a unique irreducible subquotient, say π, containing $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \otimes \pi_1$ in its Jacquet module. It is contained with multiplicity one in the composition series of $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1$ and is the unique common irreducible subquotient of $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1$ and $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \pi_1$. The proof of Lemma 3.1 shows that

$$\pi \not\simeq \mathrm{Lang}\big(\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big).$$

Next, $\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \rtimes \pi$ is the subquotient of the kernel of the last intertwining operator in (5–11). If we show that $\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \otimes \pi$ appears with multiplicity one in $\mu^*\big(\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$ and in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$, the lemma will be proved.

We consider first $\mu^*\big(\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$. Using Theorem 1.4, we take indices $0 \le j' \le i' \le l + l_1 {+} 1$, $0 \le j'' \le i'' \le l - l_2$, and irreducible constituent $\delta' \otimes \tau'$ of $\mu^*(\tau_1)$, to obtain

$$\begin{aligned}\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \le \delta([\nu^{i'-l_1}\rho, \nu^{l}\rho]) &\times \delta([\nu^{l_1+1-j'}\rho, \nu^{l_1}\rho]) \\ &\times \delta([\nu^{i''-l_2}\rho, \nu^{-l-1}\rho]) \times \delta([\nu^{l_2+1-j''}\rho, \nu^{l_2}\rho]) \times \delta'\end{aligned}$$

and

$$(5\text{–}12)\qquad \pi \le \delta([\nu^{l_1+1-i'}\rho, \nu^{l_1-j'}\rho]) \times \delta([\nu^{l_2+1-i''}\rho, \nu^{l_2-j''}\rho]) \rtimes \tau'.$$

We just sketch the analysis of these two inequalities, which is similar to the one given in the proof of Lemma 5.1. First, we see that $i' = l + l_1{+}1$, $i'' = l_2 - l$, and $j'' = 0$. Next, since $l_1 < l$, $\nu^{-l}\rho$ cannot come from δ'. Thus $j' = l + l_1{+}1$, δ' is trivial, and $\tau' \simeq \tau_1$. Finally, π is contained in the right-hand side of the induced representation of (5–12) with multiplicity one by the construction of π. The analysis of $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$ is similar. □

6. The second basic case

In this section consider the second case: $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ reduces. By Theorem 4.1, $(2l_2+1, \rho)$ satisfies the parity condition and does not lie in Jord. From Harish-Chandra theory as before, we see that

$$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma \simeq \tau_1 \oplus \tau_2,$$

where τ_1 and τ_2 are nonisomorphic irreducible tempered representations. We fix this notation throughout this section.

The first case that we consider is $\mathrm{Jord}(l_1, l_2, \sigma) = \varnothing$ and $2l_1+1 \notin \mathrm{Jord}_\rho$ (if $l_1 \geq 0$). Since we are assuming that $\delta \rtimes \sigma$ reduces, $l_1 < 0$ implies $l_1 = -\frac{1}{2}$.

Lemma 6.1. *Under the assumptions above, in the appropriate Grothendieck group we have*

$$\delta \rtimes \sigma = \begin{cases} \sigma_1 + \sigma_2 + \mathrm{Lang}(\delta \rtimes \sigma) & \text{if } l_1 \geq 0, \\ \sigma_1 + \mathrm{Lang}(\delta \rtimes \sigma) & \text{if } l_1 = -\frac{1}{2}, \end{cases}$$

where σ_1 and σ_2 are nonisomorphic discrete series representations. Moreover,

$$\mathrm{Jord}(\sigma_1) = \begin{cases} \mathrm{Jord} \cup \{(2l_1+1, \rho), (2l_2+1, \rho)\} = \mathrm{Jord}(\sigma_2) & \text{if } l_1 \geq 0, \\ \mathrm{Jord} \cup \{(2l_2+1, \rho)\} & \text{if } l_1 = -\frac{1}{2}. \end{cases}$$

Proof. If $l_1 \geq 0$ or $l_1 = -\frac{1}{2}$ and $\mathrm{Jord}_\rho = \varnothing$, this follows from [Muić 2004, Theorems 2.1 and 2.3]. If $l_1 = -\frac{1}{2}$ and $\mathrm{Jord}_\rho = \varnothing$, the proof of [Muić 2005, Lemma 6.1] shows that all irreducible subquotients of $\delta \rtimes \sigma$ other than its Langlands quotient are in a discrete series. Hence we may apply the idea used in the proof of [Muić 2004, Theorem 2.1] to show that any of them must be actually a subrepresentation of $\delta \rtimes \sigma$. Now, it is not hard to check, using Theorem 1.4 and [Mœglin 2002, Remark 5.1.2], that $\delta \otimes \sigma$ appears in $\mu^*(\delta \rtimes \sigma)$ with multiplicity one. Since similar arguments were given in the previous section, we leave details to the reader. □

Now, Lemma 6.1 and Theorem 4.1 imply that

$$\mathcal{T}_i := \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_i \quad \text{for } i = 1, 2$$

is an irreducible tempered representation.

Lemma 6.2. *We keep the same assumptions.*

(i) *Assume $l_1 = -\frac{1}{2}$. Exactly one of the representations $\delta \rtimes \tau_1$ and $\delta \rtimes \tau_2$ reduces. Without loss of generality, we assume it is $\delta \rtimes \tau_1$. In the appropriate Grothendieck group,*

$$\delta \rtimes \tau_2 = \mathcal{T}_1 + \mathrm{Lang}(\delta \rtimes \tau_2).$$

(ii) *Assume $l_1 \geq 0$. In the appropriate Grothendieck group,*

$$\delta \rtimes \tau_i = \mathcal{T}_i + \mathrm{Lang}(\delta \rtimes \tau_i) \quad \textit{for } i = 1, 2.$$

In particular, $\delta \rtimes \tau_1$ and $\delta \rtimes \tau_2$ reduce.

Proof. The proof of (i) is similar to that of Lemma 5.3, using the factorization of the relevant long-intertwining operator (see the end the proof of Lemma 5.1). Similarly, we can conclude the proof of (ii) as in Lemma 5.3 as soon as we have proved that $\delta \rtimes \tau_i \geq \mathcal{T}_i$ for $i = 1, 2$. The latter follows from the fact that $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \otimes \sigma$ has multiplicity 6 in $\mu^*\big(\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\big)$, multiplicity 3 in $\mu^*(\delta \rtimes \tau_i)$, and multiplicity at least 2 in $\mu^*\big(\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_i\big)$. We prove the second of these estimates, leaving the simple verification of the first and third to the reader.

First, Theorem 4.3 implies that $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is irreducible. Therefore,

$$\begin{aligned}\tau_i &\hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma \\ &\hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma \\ &\simeq \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.\end{aligned}$$

Hence, Frobenius reciprocity implies

$$\mu^*(\tau_i) \geq \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \sigma + \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma.$$

Now, by Theorem 1.4, we have

$$\begin{aligned}&\mu^*(\delta \rtimes \tau_i) \\ &\geq \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \\ &\qquad\qquad \otimes \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma \\ &+ \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \\ &\qquad\qquad \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma \\ &\geq 2\,\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\otimes\sigma+\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\otimes\sigma.\end{aligned}$$

(The last inequality follows by taking $i = j = l_2 - l_1$ or $i = j = 0$ in the first sum and $i = j = 0$ in the second sum.) □

Now, consider the case $\mathrm{Jord}(l_1, l_2, \sigma) = \{-2l_1 - 1\}$ $(l_1 < 0)$ or $\mathrm{Jord}(l_1, l_2, \sigma) = \varnothing$ and $2l_1+1 \in \mathrm{Jord}_\rho$ $(l_1 \geq 0)$.

Lemma 6.3. *We keep the same assumptions.*

(i) *Assume* $l_1 \geq 0$. *In the appropriate Grothendieck group*,

$$\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma = \sigma_d + \mathrm{Lang}(\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma),$$

where σ_d *is in a discrete series. Moreover*,

$$\mathrm{Jord}(\sigma_d) = \mathrm{Jord} \cup \{(2l_2+1, \rho)\} \setminus \{(2l_1+1, \rho).$$

Further, $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_d$ *and* $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ *have a unique common irreducible subquotient say* σ_{temp}. *In the appropriate Grothendieck group*,

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma = \sigma_{\mathrm{temp}} + \mathrm{Lang}(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma).$$

(ii) *Assume* $l_1 < 0$. (Actually, our assumption $-2l_1-1 \in \mathrm{Jord}_\rho$ implies $-2l_1-1 > 0$, $l_1 < -\frac{1}{2}$.) *In the appropriate Grothendieck group*,

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma = \sigma_d + \mathrm{Lang}(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma),$$

where σ_d *is in discrete series. Moreover*,

$$\mathrm{Jord} \cup \{(2l_2+1, \rho)\} \setminus \{(-2l_1-1, \rho)\}.$$

Proof. (ii) follows from (i) replacing l_1 by $-l_1-1$. The proof of (i) follows from the proof of Theorem 4.3 (see Lemma 5.2). □

The following reducibility result is a consequence of Lemma 6.3:

Lemma 6.4. *Exactly one of* $\delta \rtimes \tau_1$ *and* $\delta \rtimes \tau_2$ *is irreducible. Without loss of generality*, *we assume it's* $\delta \rtimes \tau_1$. *In the appropriate Grothendieck group*,

$$\delta \rtimes \tau_2 = \mathrm{Lang}(\delta \rtimes \tau_2) + \begin{cases} \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\mathrm{temp}} & \text{if } l_1 \geq 0, \\ \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_d & \text{if } l_1 < 0. \end{cases}$$

Both $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\mathrm{temp}}$ *and* $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_d$ *are irreducible.*

Proof. Like Lemma 6.2(i), this follows from arguments similar to those of Lemma 5.3. □

Finally, we consider the case where $]-2l_1+1, 2l_2+1[\, \cap \mathrm{Jord}_\rho \neq \varnothing$ (with $l_1 < 0$) or $]2l_1+1, 2l_2+1[\, \cap \mathrm{Jord}_\rho \neq \varnothing$ (with $l_1 \geq 0$). Let $2l+1$ be the maximal element of the intersection. Lemma 6.3 implies that $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ contains a unique irreducible subquotient, say $\sigma_{d,l}$, in a discrete series.

We distinguish between τ_1 and τ_2:

$$\begin{aligned} &\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1 \text{ is irreducible;} \\ &\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2 \geq \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{d,l} \ \text{ (also irreducible).} \end{aligned} \tag{6–1}$$

The next lemma is an analogue of Lemma 5.5.

Lemma 6.5. *$\delta \rtimes \tau_1$ reduces if and only if $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \sigma$ reduces. If $\delta \rtimes \tau_1$ reduces, it contains with multiplicity one an irreducible other than its Langlands quotient.*

Proof. From Lemma 3.1 we know that $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \sigma$ reduces if and only if $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1$ reduces. Next, we decompose the long-intertwining operator $\delta \rtimes \tau_1 \to \tilde{\delta} \rtimes \tau_1$ as follows:

$$
\begin{aligned}
(6\text{–}2)\quad \delta \rtimes \tau_1 &\hookrightarrow \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1 \\
&\to \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \rtimes \tau_1 \\
&\simeq \delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1 \\
&\simeq \delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{-l-1}\rho]) \rtimes \tau_1.
\end{aligned}
$$

This shows that if $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1$ is irreducible, so is $\delta \rtimes \tau_1$. Assume that $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1$ reduces. Hence $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \sigma$ reduces. Let π_1 be any irreducible subquotient of $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \sigma$ other than its Langlands quotient coming with multiplicity one (see Theorem 4.3). Now Lemma 3.2, applied to $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1$, shows that it contains a unique irreducible subquotient, say π containing $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \pi_1$ in its Jacquet module. It is contained with multiplicity one in the composition series of $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1$ and is the unique common irreducible subquotient of $\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1$ and $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \pi_1$. The proof of Lemma 3.1 shows that $\pi \not\simeq \mathrm{Lang}\big(\delta([\nu^{-l_1}\rho, \nu^l\rho]) \rtimes \tau_1\big)$. Next, $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \pi$ is a subquotient of the kernel of the first intertwining operator in (6–2). If we show that $\delta([\nu^{-l_2}\rho, \nu^{-l-1}\rho]) \otimes \pi$ appears with multiplicity one in $\mu^*\big(\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$ and in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$, the lemma will be proved.

We start with $\mu^*\big(\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$. Using Theorem 1.4, we take indices $0 \le j' \le i' \le l + l_1{+}1$, $0 \le j'' \le i'' \le l_2 - l$ and an irreducible constituent $\delta' \otimes \tau'$ of $\mu^*(\tau_1)$ to obtain

$$
\begin{aligned}
\delta([\nu^{-l_2}\rho, \nu^{-l-1}\rho]) \le \delta([\nu^{i'-l_1}\rho, \nu^l\rho]) &\times \delta([\nu^{l_1+1-j'}\rho, \nu^{l_1}\rho]) \\
&\times \delta([\nu^{i''-l_2}\rho, \nu^{-l-1}\rho]) \times \delta([\nu^{l_2+1-j''}\rho, \nu^{l_2}\rho]) \times \delta'
\end{aligned}
$$

and

$$
(6\text{–}3)\qquad \pi \le \delta([\nu^{l_1+1-i'}\rho, \nu^{l_1-j'}\rho]) \times \delta([\nu^{l_2+1-i''}\rho, \nu^{l_2-j''}\rho]) \rtimes \tau'.
$$

Again, we just sketch the analysis. First we see that $i' = l + l_1{+}1$, $j' = 0$, and $j'' = 0$. Next, since $\nu^{-l_2}\rho$ cannot come from δ' (since this would violate temperedness criterion for τ), we see that $i'' = 0$. Finally, π is contained in the right-hand side of the induced representation of (6–3) with multiplicity one by construction. The analysis of $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1\big)$ is similar. □

Lemma 6.6. *$\delta \rtimes \tau_2$ contains* $\mathrm{Lang}(\delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho] \rtimes \sigma_{d,l})$ *in its composition series with multiplicity one.*

Proof. First, (6–2) implies that in appropriate Grothendieck group

$$\begin{aligned}\delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2 \\ &\geq \delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{d,l} \\ &\geq \mathrm{Lang}\big(\delta([\nu^{-l_1}\rho, \nu^{l}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{d,l}\big).\end{aligned}$$

To complete the proof of the lemma, we need only show that

$$\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \otimes \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{d,l}$$

appears with multiplicity one in $\mu^*\big(\delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2\big)$ and in $\mu^*\big(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2\big)$. We consider only the first of these; the second goes the same way.

Using Theorem 1.4, we take indices $0 \leq j' \leq i' \leq l+l_1+1$, $0 \leq j'' \leq i'' \leq l_2 - l$, and an irreducible constituent $\delta' \otimes \tau'$ of $\mu^*(\tau_2)$ to obtain

$$\begin{aligned}(6\text{–}4)\quad \delta([\nu^{-l}\rho, \nu^{l_1}\rho]) \leq \delta([\nu^{i'-l_1}\rho, \nu^{l}\rho]) \times \delta([\nu^{l_1+1-j'}\rho, \nu^{l_1}\rho]) \\ \times \delta([\nu^{i''-l_2}\rho, \nu^{-l-1}\rho]) \times \delta([\nu^{l_2+1-j''}\rho, \nu^{l_2}\rho]) \times \delta'\end{aligned}$$

and

$$(6\text{–}5)\quad \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho] \rtimes \sigma_{d,l} \leq \delta([\nu^{l_1+1-i'}\rho, \nu^{l_1-j'}\rho]) \times \delta([\nu^{l_2+1-i''}\rho, \nu^{l_2-j''}\rho]) \rtimes \tau'.$$

Since $l, l_2 > l_1$, (6–4) shows that $i' = l + l_1 + 1$, $i'' = l_2 - l$, $j'' = 0$. Next, $\nu^{-l}\rho$ cannot be arise from δ' since this would violate temperedness criterion for τ_2. Hence $j' = l + l_1 + 1$, δ' is trivial, and $\tau' \cong \tau_2$. Now, (6–5) reads

$$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{d,l} \leq \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2.$$

We need to show that the left-hand side is contained in the right-hand side with multiplicity one. We use Theorem 1.4 again, showing that $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \sigma_{d,l}$ is contained in $\mu^*\big(\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2\big)$ and in $\mu^*\big(\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{d,l}\big)$ with multiplicity two. We consider only the first case since it is the more complicated. We take indices $0 \leq j \leq i \leq l - l_2$ and an irreducible constituent $\delta' \otimes \tau'$ of $\mu^*(\tau_2)$, to obtain $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \leq \delta([\nu^{i-l_2}\rho, \nu^{-l-1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta'$ and

$$(6\text{–}6)\qquad \sigma_{d,l} \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \tau'.$$

Now, $\nu^{-l_2}\rho$ either arises from $i = 0$ or comes from δ'. If it comes from δ', as in Lemma 1.3 we conclude that $\delta \rtimes \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho])$. Hence $\tau' \cong \sigma$ and $j = 0$, producing $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \sigma_{d,l}$ once. If $\nu^{-l_2}\rho$ arises from $i = 0$, we have $j = 0$ and $\delta' \simeq \delta([\nu^{-l}\rho, \nu^{l_2}\rho])$. We show that

$$(6\text{–}7)\qquad \tau' \cong \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma,$$

and that $\delta' \otimes \tau'$ appears in $\mu^*(\tau_2)$ with multiplicity one. Then will imply that the left-hand side of (6–6) is contained in the right-hand side with multiplicity one, completing the multiplicity computation.

From Theorem 4.3 we know that $\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is irreducible. As in the proof of Lemma 6.2(ii) (replacing l by l_1), we obtain

$$\tau_1 \oplus \tau_2 \hookrightarrow \delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.$$

Hence

$$\mu^*(\tau_i) \geq \delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \otimes \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.$$

Finally, we use Theorem 1.4 to show that if

$$\mu^*(\delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma) \geq \delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \otimes \tau_1',$$

where τ_1' is irreducible, then $\tau_1' \simeq \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ and the right-hand side of 6 is contained in the left-hand side with multiplicity two. We take indices $0 \leq j \leq i \leq 2l_2{+}1$ and an irreducible constituent $\delta' \otimes \sigma_1'$ of $\mu^*(\sigma)$ to obtain

$$\delta([\nu^{-l}\rho, \nu^{l_2}\rho]) \leq \delta([\nu^{i-l_2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta'$$

and

$$\tau_1' \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1'. \qquad \text{(6–8)}$$

We check first that $\nu^{-l}\rho$ cannot come from δ'. Otherwise δ' would be nondegenerate and thus induced from essentially square-integrable representations. One of them must be attached to a segment of the form $[\nu^{-l}\rho, \nu^{s}\rho]$; moreover $s > l_1$, otherwise the square-integrability criterion for σ would be violated. Since $s \leq l_2$ as well, we see that $2s{+}1$ is the largest element of Jord_ρ that is strictly less than $2l_2{+}1$. This is a contradiction. Thus, we have two possibilities: $i = l_2 - l$ and $j = 0$ or $i = 2l_2{+}1$ and $j = l + l_2{+}1$. In both cases, δ' is trivial, $\sigma_1' \simeq \sigma$, and (6–8) in both cases implies $\tau_1' \simeq \delta([\nu^{l+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$. □

7. The third basic case

In this section we consider the final basic case, where both $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$ and $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ reduce. According to Theorem 4.1, $(2l_2{+}1, \rho)$ and $(2l_2{+}1, \rho)$ satisfy the parity condition and are not elements of Jord. From Harish-Chandra theory we see that

$$\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \;\simeq\; \tau_1 \oplus \tau_2,$$

where τ_1 and τ_2 are nonisomorphic irreducible tempered representations. We fix this notation throughout this section. Then a result of Goldberg (see [Goldberg 1994] for the connected case and [Mœglin and Tadić 2002] for the general case) and Corollary 1.2 enable us to write

$$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \tau_i \simeq \tau_{i1} \oplus \tau_{i2} \quad \text{for } i = 1, 2.$$

(Note that $\tau_{i1} \not\simeq \tau_{i2}$.) We fix this notation.

The goal of this section is to determine the reducibility of $\delta \rtimes \tau_{ij}$, $i, j = 1, 2$. The main result of this section is the next lemma.

Lemma 7.1. *We maintain the preceding assumptions.*

(i) *Assume* $\mathrm{Jord}_\rho \cap [2l_1+1, 2l_2+1] = \varnothing$. *Using the notation fixed before Lemma 5.1, exactly one of the representations* $\delta \rtimes \tau_{ij}$ (*i fixed*, $j = 1, 2$) *is irreducible. Without loss of generality we may assume that it is* $\delta \rtimes \tau_{i1}$. *Then*

$$\delta \rtimes \tau_{i2} = \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_i + \mathrm{Lang}(\delta \rtimes \tau_{i2})$$

for $i = 1, 2$. (Note that $2l_1+1, 2l_2+1 \in \mathrm{Jord}_\rho(\sigma_i)$ implies the irreducibility of $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_i$.)

(ii) *Assume* $\mathrm{Jord}_\rho \cap \,]2l_1+1, 2l_2+1[\, \neq \varnothing$. *Let* $2l_{\max}+1$ *be the largest element of that intersection. Let* σ_d *be the discrete series defined in Lemma 6.3, with* l_1 *replaced by* $l_{\max}$. *Thus* $2l_{\max}+1 \notin \mathrm{Jord}_\rho(\sigma_d)$ *and we have the decomposition* $\delta([\nu^{-l_{\max}}\rho, \nu^{l_{\max}}\rho]) \rtimes \sigma_d \simeq \tau_d^1 \oplus \tau_d^2$. *Then*:

(a) *In the appropriate Grothendieck group* (*perhaps changing the indices of* τ_d^1 *and* τ_d^2), *we have*

$$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i = \tau_d^i + \mathrm{Lang}(\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i) \quad \textit{for } i = 1, 2.$$

(b) *The representations* τ_{ij} (*i fixed*, $j = 1, 2$) *can be distinguished as follows*:

$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_{i1}$ *is irreducible*;

$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_{i2} \geq \delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_d^i$ (*irred*., *multipl. one*).

$\delta \rtimes \tau_{i1}$ *is reducible if and only if* $\delta([\nu^{-l_1}\rho, \nu^{l_{\max}}\rho]) \rtimes \tau_i$ *reduces. Next,* $\delta \rtimes \tau_{i2}$ *reduces. Finally, if* $\delta \rtimes \tau_{ij}$, $j = 1, 2$, *reduces it contains some representation, other than its Langlands quotient, in its composition series with multiplicity one*.

Proof outline. For (i) we use a decomposition of the relevant long-intertwining operator analogous to the one the proof of Lemma 5.3. (See proof of Lemma 6.2.)

For (ii)(1), we first note that Lemmas 3.1 and 6.3 imply that $\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i$ reduces, for $i = 1, 2$. Next we factor the long-intertwining operator

$$\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1 \oplus \delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2 \\ \to \delta([\nu^{-l_2}\rho, \nu^{-l_{\max}-1}\rho]) \rtimes \tau_1 \oplus \delta([\nu^{-l_2}\rho, \nu^{-l_{\max}-1}\rho]) \rtimes \tau_2$$

into the long-intertwining operators

$$\begin{aligned}
\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_1 \oplus \delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_2 \\
&\simeq \delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \\
&\to \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \\
&\to \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{-l_{\max}-1}\rho]) \rtimes \sigma \\
&\to \delta([\nu^{-l_2}\rho, \nu^{-l_{\max}-1}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma \\
&\simeq \delta([\nu^{-l_2}\rho, \nu^{-l_{\max}-1}\rho]) \rtimes \tau_1 \oplus \delta([\nu^{-l_2}\rho, \nu^{-l_{\max}-1}\rho]) \rtimes \tau_2.
\end{aligned}$$

From [Zelevinsky 1980] we know that the first and third arrows (induced long-intertwining operators) are isomorphisms. The second has kernel isomorphic to $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_d \simeq \tau_d^1 \oplus \tau_d^2$, by [Muić 2005, Theorem 2.1]. We obtain (1), since $\delta([\nu^{l_{\max}+1}\rho, \nu^{l_2}\rho]) \rtimes \tau_i$ reduces.

Next, we discuss (2). That the representations τ_{ij} can be distinguished as claimed follows from the argument used in the proof of Lemmas 5.3, 6.2 and 6.4 (again based on the factorization of the long-intertwining operator). The remaining statements of (ii) follow using ideas employed in the proof of Lemmas 6.4 and 6.5. We leave details to the reader. □

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GORAN MUIĆ
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ZAGREB
BIJENIČKA 30
10000 ZAGREB
CROATIA
gmuic@math.hr