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Let A and B be separable C^*-algebras with actions of a locally compact second countable group by automorphisms. We construct a C^*-algebra, $\mathscr{A}_\pi$, such that the equivariant KK-groups, $KK^*_G(A, B)$, of Kasparov is isomorphic to the K-theory groups of $\mathscr{A}_\pi$.

1. Introduction

Duality results for KK-theory started with the work of W. Paschke [1981], who obtained a description of the BDF-extension group $\mathrm{Ext}^{-1}(A)$ (for which see [Brown et al. 1977]) as the K_0-group of the C^*-algebra

$$\pi(A)' \cap Q = \{q \in Q : q\pi(a) = \pi(a)q,\ a \in A\},$$

where Q is the Calkin algebra and $\pi : A \to Q$ is a certain "large" $*$-homomorphism. This showed that K-homology can be described by K-theory, which is sometimes thought of as the dual of K-homology. The work of Paschke has been generalized by others; see [Valette 1983; Skandalis 1988; Higson 1995; Thomsen 2001]. The general duality result obtained in this last reference formed the basis for the progress on the calculation of the KK-theory and the E-theory of amalgamated free products obtained in [Thomsen 2003]. Thus sufficiently general results of this kind can be very useful, and need not be justified solely by their great theoretical appeal. It is the purpose of the present paper to obtain duality results for the equivariant KK-theory of Kasparov [1988]. Specifically, we shall show that the notion of an absorbing $*$-homomorphism, which is the key to all the above-mentioned duality results, makes perfect sense in the equivariant setting and that there always exist (sufficiently nice) absorbing $*$-homomorphisms, also in this case. As a result we are able to associate to any pair of separable G-algebras, A and B, a C^*-algebra $\mathscr{A}^G_\pi$ such that

$$K_0\left(\mathscr{A}^G_\pi\right) = KK^1_G(A, B)$$

and

$$K_1\left(\mathscr{A}^G_\pi\right) = KK^0_G(A, B).$$

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As the notation should suggest, $\mathscr{A}_\pi^G$ is the fixed point algebra of a C^*-algebra $\mathscr{A}_\pi$ with an action of G by automorphisms. The π is here a $*$-homomorphism which is absorbing in an appropriate way, and it can be chosen such that the canonical forgetful maps $KK_G^i(A, B) \to KK^i(A, B)$, $i = 0, 1$, become the maps $K_i\left(\mathscr{A}_\pi^G\right) \to K_i(\mathscr{A}_\pi)$, $i = 0, 1$, induced by the embedding $\mathscr{A}_\pi^G \subseteq \mathscr{A}_\pi$.

2. *G*-algebras and Hilbert *G*-modules

This section introduces the basic definitions, and sets up notation and terminology by describing some fundamental results on Hilbert modules over G-algebras that are crucial for the following. They are all more or less known, and we omit the proofs. The main result, Theorem 2.8, which is due to R. Meyer, is the cornerstone for the results of the paper; it gives us access to genuinely equivariant stabilization results for Hilbert bimodules over G-algebras, provided the algebra acting from the left has been suitably stabilized.

When B is a C^*-algebra and E, F are Hilbert B-modules (see [Kasparov 1980a; 1980b; 1988; Lance 1995; Jensen and Thomsen 1991]), we let $\mathbb{L}_B(E, F)$ denote the Banach space of adjointable maps from E to F, and by $\mathbb{K}_B(E, F)$ the ideal in $\mathbb{L}_B(E, F)$ consisting of the "compact" operators, i.e. $\mathbb{K}_B(E, F)$ is the closed subspace generated by $\{\theta_{x,y} : x \in F, y \in E\}$, where $\theta_{x,y}(z) = x\langle y, z\rangle$. When $E = F$, $\mathbb{L}_B(E, F)$ is a C^*-algebra which we denote by $\mathbb{L}_B(E)$. Similarly, the ideal $\mathbb{K}_B(E, F)$ is denoted by $\mathbb{K}_B(E)$ in this case. Moreover, when E is the Hilbert B-module B itself, we will write $M(B)$ for the multiplier algebra $M(B) = \mathbb{L}_B(B)$ and B for $\mathbb{K}_B(B)$.

Let G be a locally compact second countable group.

Definition 2.1. A *G-algebra* is a pair (A, α) where A is a σ-unital C^*-algebra and $\alpha : G \to \operatorname{Aut} A$ is a homomorphism such that $G \ni g \mapsto \alpha_g(a)$ is norm-continuous for all $a \in A$.

In the following we shall often drop the explicit reference to α and denote the G-algebra (A, α) simply by A. We write then $g \cdot a$ for $\alpha_g(a)$. By a C^*-algebra we mean in the following a G-algebra for which the G-action is trivial. Given two G-algebras, A and B, the minimal tensor product $A \otimes B$ will be considered, unless explicitly stated otherwise, as a G-algebra with what is usually referred to as "the diagonal action": $g \cdot (a \otimes b) = g \cdot a \otimes g \cdot b$.

Definition 2.2. Let (B, β) be a G-algebra. A *Hilbert B,G-module* is a pair (E, v), where E be a Hilbert B-module and v is a representation of G as operators on E such that the map $G \times E \ni (g, e) \mapsto v_g(e)$ is continuous, and

$$\langle v_g(e), v_g(f)\rangle = \beta_g(\langle e, f\rangle), \tag{2–1}$$

for $e, f \in E$, and $g \in G$.

Although the operators v_t are not adjointable in general, they do give rise to a representation of G as automorphisms of $\mathbb{L}_B(E)$ since $v_g m v_{g^{-1}}$ is adjointable when m is, and $(v_g m v_{g^{-1}})^* = v_g m^* v_{g^{-1}}$. Although $g \mapsto v_g m v_{g^{-1}}$ is not always norm-continuous, it is when $m \in \mathbb{K}_B(E)$. More generally, there is also a natural action of G on $\mathbb{L}_B(E, F)$ given by $t \cdot L = w_t L v_{t^{-1}}$. Again this action is only norm-continuous on $\mathbb{K}_B(E, F)$, in general.

Given a G-algebra B and a Hilbert B,G-module (E, v) we make $L^2(G, E)$ into a Hilbert B,G-module $(L^2(G, E), v \otimes \lambda)$, where

$$(v \otimes \lambda)_t f(s) = v_t f(t^{-1}s). \tag{2–2}$$

Let E be a Hilbert B,G-module. In the following we will denote by E^∞ the Hilbert B,G-module which is the direct sum of a sequence of copies of E, i.e. E^∞ consists of the sequences $(e_1, e_2, e_3, \dots)$ of elements in E for which $\sum_{i=1}^{\infty} \langle e_i, e_i \rangle$ converges in norm in B, and the G-action is the obvious one:

$$t \cdot (e_1, e_2, e_3, \dots) = (t \cdot e_1, t \cdot e_2, t \cdot e_3, \dots).$$

We say that a Hilbert B,G-module E is *countably generated* when it is countably generated as a Hilbert B-module, i.e. when there is a countable set $M \subseteq E$ such that the span of MB is dense in E. Since B is required to be σ-unital it is countably generated as a Hilbert B,G-module.

In the following we let $\mathbb{K}$ denote the C^*-algebra of compact operators on the Hilbert space l_2. A G-algebra B is *stable* when $B \otimes \mathbb{K} \simeq B$ as G-algebras.

We denote by $\mathcal{K}_G$ the C^*-algebra of compact operators on $L^2(G)$, which we consider as a G-algebra; we have $\mathcal{K}_G = (\mathcal{K}_G, \operatorname{Ad}\lambda)$, where λ is the left-regular representation of G.

Theorem 2.3 [Kasparov 1980b; Mingo and Phillips 1984]. *Let B be a G-algebra and E a countably generated Hilbert B,G-module. Then*

$$L^2(G, E) \oplus L^2(G, B^\infty) \simeq L^2(G, B^\infty)$$

as Hilbert B,G-modules.

Corollary 2.4. *Let B be a G-algebra and E a countably generated Hilbert B,G-module. Assume B is stable. Then $L^2(G, E) \oplus L^2(G, B) \simeq L^2(G, B)$ as Hilbert B,G-modules.*

We denote by $\mathbb{K}_G$ the G-algebra $(\mathbb{K} \otimes \mathcal{K}_G, \operatorname{id}_{\mathbb{K}} \otimes \operatorname{Ad}\lambda)$.

Definition 2.5. A G-algebra A will be called *G-stable* when $A \otimes \mathbb{K}_G$ is isomorphic to A as G-algebras.

Note that $A \otimes \mathbb{K}_G$ is always G-stable.

Lemma 2.6. *Let A be a G-algebra. The following are equivalent*:

1. *A is G-stable.*
2. *A is stable and $A \simeq A \otimes \mathcal{K}_G$ as G-algebras.*
3. *A is stable and $A \simeq L^2(G, A)$ as Hilbert A, G-modules.*

Lemma 2.7. *Let (B, β) be a G-stable G-algebra, and let u be a unitary representation of G on l_2. Then $(B \otimes \mathbb{K}, \beta \otimes \operatorname{Ad} u)$ is $*$-isomorphic to (B, β) as G-algebras.*

Theorem 2.8 [Meyer 2000]. *Let A and B be G-algebras. Assume that A is G-stable. Let E be a countably generated Hilbert B,G-module and $\varphi : A \to \mathbb{L}_B(E)$ an equivariant $*$-homomorphism such that $\overline{\varphi(A)E} = E$. It follows that $E \oplus L^2(G, B^\infty) \simeq L^2(G, B^\infty)$ as Hilbert B,G-modules.*

Corollary 2.9. *Let A and B be G-stable G-algebras. Let E be a countably generated Hilbert B,G-module and $\varphi : A \to \mathbb{L}_B(E)$ an equivariant $*$-homomorphism such that $\overline{\varphi(A)E} = E$. It follows that $E \oplus B \simeq B$ as Hilbert B,G-modules.*

3. Stabilizing equivariant KK-theory: The even case

In this section we take the main steps towards a simplification in the definition of equivariant KK-theory for G-algebras that are also G-stable. We concentrate on the even case, i.e. on KK_G^0, since this is actually the most difficult case. The odd case is easier and will be dealt with in the next section. What we do corresponds in the nonequivariant case to the substitution of general Hilbert C^*-modules by a single canonical one; see for example [Blackadar 1986, Proposition 17.4.1]. To some extend, all we do is to show how R. Meyer's construction [2000, Lemma 3.3] can be made to work modulo operator homotopy and addition by degenerate elements, rather than homotopy.

Throughout this section A and B are G-algebras, A separable, B stable. A *graded Hilbert B,G-module* is a graded Hilbert B-module E which is also a Hilbert B,G-module with the same "inner product" such that the G-action of G on E commutes with the grading. An *even Kasparov triple* (E, φ, F) for A and B consists of a graded Hilbert B,G-module E, an equivariant $*$-homomorphism $\varphi : A \to \mathbb{L}_B(E)$ mapping into the degree 0 elements of $\mathbb{L}_B(E)$ and a degree 1 element $F \in \mathbb{L}_B(E)$ such that

$$(F^* - F)\varphi(a),\ (F^2 - 1)\varphi(a),\ F\varphi(a) - \varphi(a)F,\ (g \cdot F - F)\varphi(a) \in \mathbb{K}_B(E)$$

for all $a \in A$ and all $g \in G$. The even Kasparov triple (E, φ, F) is *degenerate* when

$$(F^* - F)\varphi(a) = (F^2 - 1)\varphi(a) = F\varphi(a) - \varphi(a)F = (g \cdot F - F)\varphi(a) = 0$$

for all g, a. Two even Kasparov triples (E_0, φ_0, F_0) and (E_1, φ_1, F_1) are *operator homotopic* when there is a family (E, φ, G_t), $t \in [0, 1]$, of even Kasparov triples for A and B such that $t \mapsto G_t$ is norm-continuous, (E, φ, G_0) is isomorphic to (E_0, φ_0, F_0) and (E, φ, G_1) is isomorphic to (E_1, φ_1, F_1).

By definition [Kasparov 1988] $KK^0_G(A, B)$ consists of the homotopy classes of even Kasparov triples. It was pointed out in [Baaj and Skandalis 1989] that $KK^0_G(A, B)$, in line with more general equivariant KK-theory groups, can also be defined as the equivalence classes of even Kasparov triples for A and B, when the equivalence is operator homotopy after addition by degenerate elements, rather than homotopy as in [Kasparov 1988]. As in the nonequivariant case the equality between the two definitions follows from the fact that the Kasparov product can be defined modulo the apparently strongest of the two equivalence relations. To obtain the description of $KK^0_G(A, B)$ as the K_1-group of a C^*-algebra we need to work entirely with the latter notion of equivalence for even Kasparov triples.

The Hilbert B,G-module $B \oplus B$ graded by the map $(x, y) \mapsto (x, -y)$ will be denoted by B^e. An even Kasparov triple (E, φ, F) for A and B will be called *elementary* when $E = B^e$ and *essential* when $\overline{\varphi(A)E} = E$. Note that the direct sum of two elementary and/or essential even Kasparov triples are isomorphic to an elementary and/or essential even Kasparov triple.

Definition 3.1. An even Kasparov triple (E, φ, F) will be called *homogeneous* when it has the form $E = E_0 \oplus E_0$, graded by $(x, y) \mapsto (x, -y)$, for some Hilbert B,G-module E_0, and $\varphi = (\psi, \psi)$, where $\psi : A \to \mathbb{L}_B(E_0)$ is an equivariant $*$-homomorphism.

Given a homogeneous even Kasparov triple $(E, \varphi, F) = (E_0 \oplus E_0, \varphi, F)$ as above, there is a canonical way to form a homogeneous and degenerate even Kasparov triple; namely, $(E, \varphi, \hat{1})$, where $\hat{1} \in \mathbb{L}_B(E)$ is

$$\hat{1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

Lemma 3.2. *Let $\mathscr{E} = (E, \varphi, F)$ be an even Kasparov triple. There is then a degenerate even Kasparov triple $\mathscr{D}$ such that $\mathscr{E} \oplus \mathscr{D}$ is isomorphic to a homogeneous even Kasparov triple. When $\mathscr{E}$ is elementary we can choose $\mathscr{D}$ to be both elementary and degenerate.*

Proof. Let $\mathscr{D}_0 = (B^e, 0, \hat{1})$. It follows from Kasparov's stabilization theorem [1980b] that $\mathscr{E} \oplus \mathscr{D}_0$ is isomorphic to an even Kasparov triple of the form

$$(E_0 \oplus E_0, (\psi_+, \psi_-), F_0)\,,$$

where $E_0 \oplus E_0 = B \oplus B$ is graded by $(x, y) \mapsto (x, -y)$, and each of the two B-summands carry an action by G which gives it the structure of a Hilbert B,G-module (not necessarily the canonical such structure). By adding to $\mathscr{D}_0$ the degenerate even Kasparov triple $(B \oplus B, 0, 0)$ with the two G-actions interchanged we may assume that the G-actions on the two B-summands agree (but not that they are the canonical one). The infinite direct sum $\bigoplus_1^\infty E_0$ is a Hilbert B,G-module, and the triples

$$\mathscr{E}_+ = \left(\left(\bigoplus_1^\infty E_0\right) \oplus \left(\bigoplus_1^\infty E_0\right), \left(\bigoplus_1^\infty \psi_+\right) \oplus \left(\bigoplus_1^\infty \psi_+\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$$

and

$$\mathscr{E}_- = \left(\left(\bigoplus_1^\infty E_0\right) \oplus \left(\bigoplus_1^\infty E_0\right), \left(\bigoplus_1^\infty \psi_-\right) \oplus \left(\bigoplus_1^\infty \psi_-\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$$

are both degenerate even Kasparov triples. The direct sum $\mathscr{E} \oplus \mathscr{D}_0 \oplus \mathscr{E}_+ \oplus \mathscr{E}_-$ is isomorphic to an even Kasparov triple of the form

$$\left(\left(\bigoplus_{-\infty}^\infty E_0\right) \oplus \left(\bigoplus_{-\infty}^\infty E_0\right), \Phi_+ \oplus \Phi_-, F'\right), \tag{3–1}$$

where

$$\Phi_+ = (\ldots\ldots, \psi_+, \psi_+, \psi_+, \psi_-, \psi_-, \ldots\ldots)$$
$$\Phi_- = (\ldots\ldots, \psi_+, \psi_+, \psi_-, \psi_-, \psi_-, \ldots\ldots).$$

Let $S \in \mathbb{L}_B\left(\bigoplus_{-\infty}^\infty E_0\right)$ be the two-sided shift; specifically, when $e = (e_i)_{i\in\mathbb{Z}} \in \bigoplus_{-\infty}^\infty E_0$, $S(e)$ is given by $S(e)_i = e_{i+1}$. Then

$$T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & S \end{smallmatrix}\right) \in \mathbb{L}_B\left(\left(\bigoplus_{-\infty}^\infty E_0\right) \oplus \left(\bigoplus_{-\infty}^\infty E_0\right)\right)$$

is a G-invariant unitary of degree 0 such that $\operatorname{Ad} T \circ (\Phi_+ \oplus \Phi_-) = \Phi_+ \oplus \Phi_+$. Thus T is an isomorphism between the Kasparov triple (3–1) and

$$\left(\left(\bigoplus_{-\infty}^\infty E_0\right) \oplus \left(\bigoplus_{-\infty}^\infty E_0\right), \Phi_+ \oplus \Phi_+, TF'T^*\right),$$

which is a homogeneous even Kasparov triple. It follows that $\mathscr{D} = \mathscr{D}_0 \oplus \mathscr{E}_+ \oplus \mathscr{E}_-$ has the required property. When $\mathscr{E}$ is an elementary even Kasparov triple we can take $\mathscr{D}_0 = 0$, and then $\mathscr{D}$ will be (isomorphic to) an elementary even Kasparov triple since B is stable. □

Let E be a Hilbert B,G-module. Let $\varphi : A \to \mathbb{L}_B(E)$ be an equivariant $*$-homomorphism. Since A is a Hilbert A-module in itself, we can form the internal tensor product $A \otimes_\varphi E$ which is a graded Hilbert B-module E'; see, for instance, [Jensen and Thomsen 1991, 2.1.4]. Since φ is equivariant, E' is actually a graded Hilbert B,G-module in a canonical way; compare the proof of Theorem 2.8. Following [Connes and Skandalis 1984], we introduce the notion of *connections* in this setting. For every $a \in A$ we can define an adjointable map $T_a \in \mathbb{L}_B(E, E')$ such that $T_a(y) = a \otimes_\varphi y$. The adjoint T_a^* is determined by the condition that $T_a^*(b \otimes_\varphi e) = \varphi(a^*b)e$, and we set

$$\widetilde{T}_a = \begin{pmatrix} 0 & T_a^* \\ T_a & 0 \end{pmatrix} \in \mathbb{L}_B\left(E \oplus E'\right).$$

Let $F \in \mathbb{L}_B(E)$. An *F-connection* is an element $F' \in \mathbb{L}_B\left(E'\right)$ such that

$$[\widetilde{T}_a, F \oplus F'] \in \mathbb{K}_B\left(E \oplus E'\right) \quad \text{for all } a \in A.$$

This condition is equivalent to

$$T_a F - F' T_a \in \mathbb{K}_B(E, E'), \tag{3–2}$$

and

$$F T_a^* - T_a^* F' \in \mathbb{K}_B(E', E) \tag{3–3}$$

for all $a \in A$. In particular, there *is* an F-connection, by [Connes and Skandalis 1984]. When E is graded and F has degree 1 there is an F-connection of degree 1.

Lemma 3.3 [Meyer 2000, Lemma 3.1]. *In the setting above, assume that A is G-stable, that $\overline{\varphi(A)E} = E$, and that*

$$[\varphi(a), F], (F^* - F)\varphi(a), (g \cdot F - F)\,\varphi(a) \in \mathbb{K}_B(E)$$

for all $a \in A$ and all $g \in G$. There is then a G-invariant F-connection. □

Lemma 3.4. *In the setting above, assume that $\mathscr{E} = (E, \varphi, F)$ is an even Kasparov triple. Define $\varphi' : A \to \mathbb{L}_B\left(E'\right)$ by $\varphi'(a)(a_1 \otimes_\varphi e) = aa_1 \otimes_\varphi e$, and let $F' \in \mathbb{L}_B\left(E'\right)$ be an F-connection of degree 1. It follows that (E', φ', F') is an even Kasparov triple.*

Proof. This is stated as part of [Meyer 2000, Lemma 3.3], but is really one of the fundamental steps in the construction of the Kasparov product of $[E, \varphi, F] \in KK_G^0(A, B)$ with $[\mathrm{id}_A] \in KK_G^0(A, A)$. The details of the argument can be found in [Jensen and Thomsen 1991, Lemma 2.2.6], for example. □

Note that the even Kasparov triple (E', φ', F') of Lemma 3.4 is essential, and that E' is isomorphic, as a graded Hilbert B,G-module, to $E_{\mathrm{ess}} = \overline{\varphi(A)E}$ under the map given by $a \otimes_\varphi e \mapsto \varphi(a)e$. Under this isomorphism F' turns into a degree 1 operator which we denote by F_{ess}. The defining relations for F', (3–2) and (3–3), turn into the conditions

$$\varphi(a)F - F_{\mathrm{ess}}\varphi(a) \in \mathbb{K}_B(E, E_{\mathrm{ess}}) \tag{3–4}$$

and

$$F\varphi(a) - \varphi(a)F_{\mathrm{ess}} \in \mathbb{K}_B(E_{\mathrm{ess}}, E). \tag{3–5}$$

In particular we see that F_{ess} is determined up to "a compact perturbation", in the sense that $\varphi(a)\left(F_{\mathrm{ess}} - F''\right) \in \mathbb{K}_B(E_{\mathrm{ess}})$ for all $a \in A$, when F'' is another operator in $\mathbb{L}_B(E_{\mathrm{ess}})$ which satisfies (3–5).

Set $\varphi_{\mathrm{ess}}(a) = \varphi(a)|_{E_{\mathrm{ess}}}$, and note that (E', φ', F') is then isomorphic, as an even Kasparov triple, to the essential even Kasparov triple $\mathscr{E}_{\mathrm{ess}} = (E_{\mathrm{ess}}, \varphi_{\mathrm{ess}}, F_{\mathrm{ess}})$. Note also that $\mathscr{E}_{\mathrm{ess}}$ is both essential and homogeneous when $\mathscr{E}$ is homogeneous. It was shown by Meyer [2000, Lemma 3.3] that $\mathscr{E}$ and $\mathscr{E}_{\mathrm{ess}}$ are homotopic and hence define the same element of $KK_G(A, B)$. We can therefore conclude that the even Kasparov triples $\mathscr{E}$ and $\mathscr{E}_{\mathrm{ess}}$ are operator homotopic after addition by degenerate even Kasparov triples. Since we need to know what the involved degenerate triples look like, we have to obtain a more explicit proof of this fact. For this we need the following lemma.

Lemma 3.5. *Let A and B be a σ-unital G-algebras and $\varphi : A \to B$ an equivariant surjection. Consider a separable closed self-adjoint subspace $\mathscr{F} \subseteq M(B)$ and a finite subgroup $G_0 \subseteq G$. Then canonical extension $\overline{\varphi} : M(A) \to M(B)$ maps $\{m \in M(A) : mf - fm \in \ker\varphi,\ f \in \mathscr{F},\ g\cdot m - m \in \ker\varphi, g \in G,\ g_0\cdot m = m, g_0 \in G_0\}$ onto $M(B)^G \cap \overline{\varphi}(\mathscr{F})'$.*

Proof. This is an equivariant version of a result from [Olsen and Pedersen 1989]. The proof presented in [Jensen and Thomsen 1991, Theorem 1.1.26] can be easily adopted to the equivariant case by use of an asymptotically G-invariant approximate unit. We leave the details to the reader. □

Lemma 3.6. *Let $\mathscr{E} = (E, \varphi, F)$ be a homogeneous even Kasparov triple. Set $\mathfrak{X}_1 = (E_{\mathrm{ess}}, 0, F_{\mathrm{ess}}) \oplus (E, \varphi, \hat{1}) \oplus (E_{\mathrm{ess}}, 0, \hat{1}) \oplus (E, 0, \hat{1}) \oplus (E_{\mathrm{ess}}, \varphi_{\mathrm{ess}}, \hat{1})$ and $\mathfrak{X}_2 = (E, 0, F) \oplus (E, \varphi, \hat{1}) \oplus (E_{\mathrm{ess}}, 0, \hat{1}) \oplus (E, \varphi, \hat{1}) \oplus (E_{\mathrm{ess}}, 0, \hat{1})$. Then $\mathscr{E} \oplus \mathfrak{X}_1$ is operator homotopic to $\mathscr{E}_{\mathrm{ess}} \oplus \mathfrak{X}_2$.*

Proof. Consider the even Kasparov triple $(\widetilde{E}, \psi, H)$ for $M_2(A)$ and B, where $\widetilde{E} = E \oplus E_{\mathrm{ess}}$, $H = F \oplus F_{\mathrm{ess}}$, and

$$\psi : M_2(A) \to \mathbb{L}_B(E \oplus E_{\mathrm{ess}}) = \begin{pmatrix} \mathbb{L}_B(E) & \mathbb{L}_B(E_{\mathrm{ess}}, E) \\ \mathbb{L}_B(E, E_{\mathrm{ess}}) & \mathbb{L}_B(E_{\mathrm{ess}}) \end{pmatrix}$$

is given by

$$\psi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \varphi(a_{11}) & \varphi(a_{12}) \\ \varphi(a_{21}) & \varphi(a_{22}) \end{pmatrix}.$$

For $t \in [0, 1]$, set

$$R_t = \begin{pmatrix} -t & \sqrt{1-t^2} \\ \sqrt{1-t^2} & t \end{pmatrix},$$

which we consider as a unitary multiplier of $M_2(A)$ in the obvious way. Set $\psi_t = \psi \circ \mathrm{Ad}\, R_t$ and $\iota(a) = \begin{pmatrix} a & \\ & 0 \end{pmatrix}$. Then $(\widetilde{E}, \psi_t \circ \iota, H)$, $t \in [0, 1]$, is a path of even Kasparov triples connecting $(E, \varphi, F) \oplus (E_{\mathrm{ess}}, 0, F_{\mathrm{ess}})$ and $(E, 0, F) \oplus (E_{\mathrm{ess}}, \varphi, F_{\mathrm{ess}})$. Since

$$\begin{aligned}(\widetilde{E}, \psi_1 \circ \iota, H) &\oplus (\widetilde{E}, \psi \circ \iota, \hat{1}) \oplus (\widetilde{E}, \psi_0 \circ \iota, \hat{1}) \\ &= (E, \varphi, F) \oplus (E_{\mathrm{ess}}, 0, F_{\mathrm{ess}}) \oplus (E, \varphi, \hat{1}) \oplus (E_{\mathrm{ess}}, 0, \hat{1}) \oplus (E, 0, \hat{1}) \oplus (E_{\mathrm{ess}}, \varphi_{\mathrm{ess}}, \hat{1})\end{aligned}$$

and

$$(\widetilde{E}, \psi_0 \circ \iota, H) \oplus (\widetilde{E}, \psi \circ \iota, \hat{1}) \oplus (\widetilde{E}, \psi_1 \circ \iota, \hat{1})$$
$$= (E, 0, F) \oplus (E_{\text{ess}}, \varphi_{\text{ess}}, F_{\text{ess}}) \oplus (E, \varphi, \hat{1}) \oplus (E_{\text{ess}}, 0, \hat{1}) \oplus (E, \varphi, \hat{1}) \oplus (E_{\text{ess}}, 0, \hat{1}),$$

it suffices to show that $(\widetilde{E}, \psi_1, H) \oplus (\widetilde{E}, \psi, \hat{1}) \oplus (\widetilde{E}, \psi_0, \hat{1})$ is operator homotopic to $(\widetilde{E}, \psi_0, H) \oplus (\widetilde{E}, \psi, \hat{1}) \oplus (\widetilde{E}, \psi_1, \hat{1})$. Let $p : \mathbb{L}_B(\widetilde{E}) \to \mathbb{L}_B(\widetilde{E})/\mathbb{K}_B(\widetilde{E})$ be the quotient map. Let X be the C^*-subalgebra of $\mathbb{L}_B(\widetilde{E})$ generated by

$$\mathbb{K}_B(\widetilde{E}) \cup \psi(M_2(A)) \bigcup_{n \in \mathbb{N}} H^n \psi(M_2(A)).$$

Then $H^* X \cup X H^* \cup H X \cup X H \subseteq X$. We can therefore consider H as a multiplier of X. Note that $p(X)$ is generated by $p \circ \psi(M_2(A)) \bigcup_{n \in \mathbb{N}} p(H^n) p \circ \psi(M_2(A))$ and that $\overline{p \circ \psi(M_2(A)) p(X)} = p(X)$. In particular, it follows that $p \circ \psi$ extends to a unital $*$-homomorphism of the multiplier algebras:

$$\overline{p \circ \psi} : M(M_2(A)) \to M(p(X)).$$

We set $\overline{T}_t = \overline{p \circ \psi}(R_t) \in M(p(X))$, which is a symmetry for each $t \in [0, 1]$. Observe that each $\overline{T}_t$ is invariant under the action of $\mathbb{Z}_2 \oplus G$, coming from the grading of $\widetilde{E}$ and the representation of G, and that

$$\begin{aligned} \overline{T}_t p(H) p \circ \psi(a) &= \overline{T}_t p \circ \psi(a) p(H) = p \circ \psi(R_t a) p(H) \\ &= p(H) p \circ \psi(R_t a) = p(H) \overline{T}_t p \circ \psi(a) \end{aligned}$$

for all $t \in [0, 1]$, $a \in M_2(A)$. It follows that $\overline{T}_t$ and $p(H)$ commute in $M(p(X))$. Since $\overline{T}_t$, $t \in [0, 1]$, is a norm-continuous path of unitaries in the connected component of 1 in the unitary group of $M(p(X))^G \cap p(H)'$, it follows from Lemma 3.5 that we can find a norm-continuous path T_t, $t \in [0, 1]$, of degree 0 unitaries in $M(X)$ such that $\overline{p}(T_t) = \overline{T}_t$, while $g \cdot T_t - T_t$, $T_t H - H T_t \in \mathbb{K}_B(\widetilde{E})$ for all $t \in [0, 1]$ and all $g \in G$. Since $\mathbb{K}_B(\widetilde{E})$ is an essential ideal in X there is a unique degree 0 unitary $S_t \in M(\mathbb{K}_B(\widetilde{E})) = \mathbb{L}_B(\widetilde{E})$ such that $S_t x = T_t x$ for all $x \in X$. Since T_t depends norm-continuously on t so does S_t, and

$$S_t H - H S_t \in \mathbb{K}_B(\widetilde{E}), \tag{3–6}$$

since this is true for T_t. In addition

$$\psi(a) S_t - \psi(a R_t), \;\; S_t \psi(a) - \psi(R_t a) \in \mathbb{K}_B(\widetilde{E}) \tag{3–7}$$

and

$$g \cdot S_t - S_t \in \mathbb{K}_B(\widetilde{E}) \tag{3–8}$$

for all $t \in [0, 1]$, $g \in G$ and $a \in M_2(A)$. It follows that

$$(\widetilde{E}, \psi, S_t^* H S_t),\ t \in [0, 1], \tag{3–9}$$

is an operator homotopy. Let $\sim_{\mathrm{OH}}$ denote an operator homotopy. By (3–8) and (3–7) there is an operator homotopy

$$(\widetilde{E}, \psi, S_t^* H S_t) \oplus (\widetilde{E}, \psi_t, \hat{1}) \ \sim_{\mathrm{OH}}\ (\widetilde{E}, \psi, \hat{1}) \oplus (\widetilde{E}, \psi_t, H) \tag{3–10}$$

for all $t \in [0, 1]$, obtained by rotating $\begin{pmatrix} H & \\ & \hat{1} \end{pmatrix}$ to $\begin{pmatrix} \hat{1} & \\ & H \end{pmatrix}$. Then

$$\begin{aligned}
&(\widetilde{E}, \psi_0, H) \oplus (\widetilde{E}, \psi, \hat{1}) \oplus (\widetilde{E}, \psi_1, \hat{1}) \\
&\quad \sim_{\mathrm{OH}} (\widetilde{E}, \psi, S_0^* H S_0) \oplus (\widetilde{E}, \psi_0, \hat{1}) \oplus (\widetilde{E}, \psi_1, \hat{1}) \quad \text{(by (3–10) applied with } t = 0) \\
&\quad \sim_{\mathrm{OH}} (\widetilde{E}, \psi, S_1^* H S_1) \oplus (\widetilde{E}, \psi_0, \hat{1}) \oplus (\widetilde{E}, \psi_1, \hat{1}) \quad \text{(by (3–9))} \\
&\quad \sim_{\mathrm{OH}} (\widetilde{E}, \psi_0, \hat{1}) \oplus (\widetilde{E}, \psi, \hat{1}) \oplus (\widetilde{E}, \psi_1, H) \quad \text{(by (3–10) applied with } t = 1).
\end{aligned}$$

□

Lemma 3.7. *Let A and B be G-algebras. Assume that B is G-stable and separable. There is an equivariant $*$-homomorphism $\varphi : A \to M(B)$ such that $\overline{\varphi(A)B} = B$.*

Proof. Since B is stable, $B \simeq B \otimes \mathbb{K}$ as G-algebras. Let (π, u) be a covariant nondegenerate unitary representation of A on l_2, i.e. $\pi : A \to \mathbb{B}(l_2) = M(\mathbb{K})$ is a $*$-homomorphism, u is a continuous unitary representation of G on l_2, $\overline{\pi(A)l_2} = l_2$ and $u_g \pi(a) u_g^* = \pi(g \cdot a)$ for all g, a. Such a pair (π, u) exists; see [Pedersen 1979], for example. We can then define an equivariant $*$-homomorphism $\pi_0 : (A, \alpha) \to (M(B \otimes \mathbb{K}), \overline{\beta} \otimes \mathrm{Ad}\, u)$, where $\overline{\beta}$ is the canonical extension of the given action of G on B, such that $\pi_0(a)(b \otimes k) = b \otimes \pi(a)k$. Note that $\overline{\pi_0(A)(B \otimes \mathbb{K})} = B \otimes \mathbb{K}$. By Lemma 2.7 there is an equivariant $*$-isomorphism $\theta : (B \otimes \mathbb{K}, \beta \otimes \mathrm{Ad}\, u) \to (B, \beta)$. Set $\varphi = \overline{\theta} \circ \pi_0$, where $\overline{\theta} : M(B \otimes \mathbb{K}) \to M(B)$ is the canonical extension of θ. □

Lemma 3.8. *Let (E, φ, F_t), $t \in [0, 1]$, be an operator homotopy of even Kasparov triples. It follows that there is an operator homotopy $(E_{\mathrm{ess}}, \varphi_{\mathrm{ess}}, H_t)$, $t \in [0, 1]$, such that $H_0 = (F_0)_{\mathrm{ess}}$ and $H_1 = (F_1)_{\mathrm{ess}}$.*

Proof. This follows from the construction of an F_t-connection (see [Connes and Skandalis 1984] or [Jensen and Thomsen 1991, Proposition 2.2.5]), and the fact that F_{ess} is unique up to compact perturbation. □

Theorem 3.9. *Let A and B be G-algebras, A separable. Assume that A and B are G-stable.*

(a) *Every element of $KK_G^0(A, B)$ is represented by an even Kasparov triple for A and B which is both elementary and essential.*

(b) *Two elementary and essential even Kasparov triples, $\mathscr{E}_1$ and $\mathscr{E}_2$, for A and B, define the same element of $KK^0_G(A, B)$ if and only if there are degenerate even Kasparov triples, $\mathscr{D}_1$ and $\mathscr{D}_2$, for A and B which are both elementary and essential, such that $\mathscr{E}_1 \oplus \mathscr{D}_1$ is operator homotopic to $\mathscr{E}_2 \oplus \mathscr{D}_2$.*

Proof. (a) By Lemma 3.2 and Lemma 3.6 every element of $KK_G(A, B)$ is represented by an even Kasparov triple $\mathscr{E} = (E, \varphi, F)$ which is both homogeneous and essential. By Lemma 3.7 there is a $*$-homomorphism $\pi : A \to M(B)$ such that $\overline{\pi(A)B} = B$. Then

$$\mathscr{Z} = \left(B^e, \pi \oplus \pi, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right) \tag{3–11}$$

is a degenerate even Kasparov triple for A and B. It follows from Corollary 2.9 that $E \oplus B^e$ is isomorphic to B^e as graded Hilbert B,G-modules. Thus $\mathscr{E} \oplus \mathscr{Z}$ is isomorphic to an elementary and essential even Kasparov triple for A and B.

(b) Let $\mathscr{E}_i = 1, 2$, be elementary and essential even Kasparov triples for A and B representing the same element in $KK^0_G(A, B)$. It follows that there are degenerate even Kasparov triples $\mathscr{D}_i, i = 1, 2$, such that $\mathscr{E}_1 \oplus \mathscr{D}_1$ is operator homotopic to $\mathscr{E}_2 \oplus \mathscr{D}_2$. It follows then from Lemma 3.8 that $(\mathscr{E}_1 \oplus \mathscr{D}_1)_{\text{ess}}$ is operator homotopic to $(\mathscr{E}_2 \oplus \mathscr{D}_2)_{\text{ess}}$. Since $\mathscr{E}_i$ is essential, $(\mathscr{E}_i \oplus \mathscr{D}_i)_{\text{ess}}$ is isomorphic to $\mathscr{E}_i \oplus (\mathscr{D}_i)_{\text{ess}}$, $i = 1, 2$. It follows from Corollary 2.9 that $(\mathscr{D}_i)_{\text{ess}} \oplus \mathscr{Z}$ is isomorphic to a degenerate even Kasparov triple which is both elementary and essential. This completes the proof. □

Theorem 3.10. *Let A and B be G-algebras, A separable. Assume that A and B are G-stable.*

(a) *Every element of $KK^0_G(A, B)$ is represented by an elementary even Kasparov triple.*

(b) *Two elementary even Kasparov triples, $\mathscr{X}$ and $\mathscr{Y}$, for A and B, define the same element of $KK^0_G(A, B)$ if and only if there are degenerate and elementary even Kasparov triples, $\mathscr{D}_1$ and $\mathscr{D}_2$, for A and B, such that $\mathscr{X} \oplus \mathscr{D}_1$ is operator homotopic to $\mathscr{Y} \oplus \mathscr{D}_2$.*

Proof. Part (a) follows from Theorem 3.9(a).

(b) It follows from Theorem 3.9(b) that it suffices to consider an elementary even Kasparov triple $\mathscr{E}$ and show that there are degenerate and elementary even Kasparov triples $\mathscr{D}_i, i = 1, 2$, such that $\mathscr{E} \oplus \mathscr{D}_1$ is operator homotopic to $\mathscr{X} \oplus \mathscr{D}_2$, where $\mathscr{X}$ is an elementary and essential even Kasparov triple. After an application of Lemma 3.2 we may assume that $\mathscr{E}$ is both elementary and homogeneous. We can then consider the degenerate even Kasparov triples $\mathscr{Z}_i, i = 1, 2$, constructed from $\mathscr{E}$ as in Lemma 3.6, so that $\mathscr{E} \oplus \mathscr{Z}_1$ is operator homotopic to $\mathscr{E}_{\text{ess}} \oplus \mathscr{Z}_2$ by that lemma. Note that because $\mathscr{E}$ is elementary, $\mathscr{X}_2$ is (up to isomorphism) the sum of two degenerate even Kasparov triples — one which thanks to Corollary 2.9 can be stabilized in an equivariant way, namely, $\big(E_{\text{ess}}, 0, \hat{1}\big) \oplus \big(E_{\text{ess}}, 0, \hat{1}\big)$, and another which is (isomorphic to) an elementary degenerate even Kasparov triple, namely, $(E, 0, F) \oplus (E, \varphi, \hat{1}) \oplus (E, \varphi, \hat{1})$. It follows therefore from Corollary 2.9 that $\mathscr{X}_2 \oplus \mathscr{X}$ is isomorphic to an elementary degenerate even Kasparov triple, where $\mathscr{X}$ is the even Kasparov triple (3–11). Similarly, $\mathscr{X}_1 \oplus \mathscr{X}$ is also isomorphic to an elementary degenerate even Kasparov triple. Since $\mathscr{E} \oplus \mathscr{X}_2 \oplus \mathscr{X} \oplus \mathscr{X}$ is operator homotopic to $\mathscr{E}_{\text{ess}} \oplus \mathscr{X}_1 \oplus \mathscr{X} \oplus \mathscr{X}$, and since $\mathscr{E}_{\text{ess}} \oplus \mathscr{X}$ is isomorphic to an essential and elementary even Kasparov triple by Lemma 3.6, we can take $\mathscr{D}_1$ to be an isomorphic copy of $\mathscr{X}_1 \oplus \mathscr{X} \oplus \mathscr{X}$, $\mathscr{K}$ to be an isomorphic copy of $\mathscr{E}_{\text{ess}} \oplus \mathscr{X}$, and $\mathscr{D}_2$ to be an isomorphic copy of $\mathscr{X}_2 \oplus \mathscr{X}$. □

4. Stabilizing equivariant *KK*-theory: The odd case

This section is the odd analog of the previous section. The development is basically identical with the even case, but there are a few more shortcuts that we can exploit. The first comes from the description of KK_G^1 given in [Thomsen 2000].

Throughout this section A and B are G-algebras, A separable and B stable. An *odd Kasparov triple* (E, φ, P) for A and B consists of a Hilbert B,G-module E, an equivariant $*$-homomorphism $\varphi : A \to \mathbb{L}_B(E)$ and an element $P \in \mathbb{L}_B(E)$ such that

$$(P^* - P)\varphi(a), (P^2 - P)\varphi(a), P\varphi(a) - \varphi(a)P, (g \cdot P - P)\varphi(a) \in \mathbb{K}_B(E)$$

for all $a \in A$ and all $g \in G$. The odd Kasparov triple (E, φ, P) is *degenerate* when

$$(P^* - P)\varphi(a) = (P^2 - P)\varphi(a) = P\varphi(a) - \varphi(a)P = (g \cdot P - P)\varphi(a) = 0$$

for all g, a. Two odd Kasparov triples (E_0, φ_0, P_0) and (E_1, φ_1, P_1) are *operator homotopic* when there is a family (E, φ, G_t), $t \in [0, 1]$, of odd Kasparov triples for A and B such that $t \mapsto G_t$ is norm-continuous, (E, φ, G_0) is isomorphic to (E_0, φ_0, P_0) and (E, φ, G_1) is isomorphic to (E_1, φ_1, P_1).

The odd equivariant KK-group $KK_G^1(A, B)$ consists of the homotopy classes of odd Kasparov triples. As in the even case, $KK_G^1(A, B)$ can also be defined as the equivalence classes of odd Kasparov triples for A and B, when the equivalence is operator homotopy after addition by degenerate elements. This description of $KK_G^1(A, B)$ differs from Kasparov's original definition [1988], but it is not difficult to see that they are equivalent. In fact, as shown in [Thomsen 2000], it suffices to consider Hilbert B,G-modules E, which as a Hilbert B-module is B itself, but with a "twisted" G-action.

We say that an odd Kasparov triple (E, φ, F) is *essential* when $\overline{\varphi(A)E} = E$ and *elementary* when $E = B$.

Let $\mathcal{E} = (E, \varphi, P)$ be an odd Kasparov triple. Set $E' = A \otimes_\varphi E$, and define $\varphi' : A \to \mathbb{L}_B\left(E'\right)$ by $\varphi'(a)(a_1 \otimes_\varphi e) = aa_1 \otimes_\varphi e$, and let $P' \in \mathbb{L}_B\left(E'\right)$ be a P-connection. It follows that (E', φ', P') is an odd Kasparov triple. So see this, one can for example apply Lemma 3.4 to the even Kasparov triple

$$\big(E \oplus E,\ (\varphi, \varphi),\ (2P - 1, 1 - 2P)\big),$$

where $E \oplus E$ is graded by $(x, y) \mapsto (y, x)$, or take a look at [Jensen and Thomsen 1991, Lemma 2.2.6]. Set $E_{\text{ess}} = \overline{\varphi(A)E}$, $\varphi_{\text{ess}}(a) = \varphi(a)|_{E_{\text{ess}}}$, and let $P_{\text{ess}} \in \mathbb{L}_B\,(E_{\text{ess}})$ be the image of P' under the isomorphism $A \otimes_\varphi E \simeq E_{\text{ess}}$ given by $a \otimes_\varphi e \mapsto \varphi(a)e$. Then $(E_{\text{ess}}, \varphi_{\text{ess}}, P_{\text{ess}})$ is an essential odd Kasparov triple. The following lemma is proved in the same way as Lemma 3.6.

Lemma 4.1. *Let $\mathcal{E} = (E, \varphi, P)$ be an odd Kasparov triple. Set*

$$\mathfrak{X}_1 = (E_{\text{ess}}, 0, P_{\text{ess}}) \oplus (E, \varphi, 0) \oplus (E_{\text{ess}}, 0, 0) \oplus (E, 0, 0) \oplus (E_{\text{ess}}, \varphi_{\text{ess}}, 0),$$
$$\mathfrak{X}_2 = (E, 0, P) \oplus (E, \varphi, 0) \oplus (E_{\text{ess}}, 0, 0) \oplus (E, \varphi, 0) \oplus (E_{\text{ess}}, 0, 0).$$

Then $\mathcal{E} \oplus \mathfrak{X}_1$ is operator homotopic to $\mathcal{E}_{\text{ess}} \oplus \mathfrak{X}_2$.

Similarly, the proofs of Theorem 3.9 and Theorem 3.10 carry over to the odd case with only the obvious changes. E.g. the even Kasparov triple $\mathfrak{X}$ which plays a prominent role in both proofs can be substituted by the odd Kasparov triple $(B, \pi, 0)$, where $\pi : A \to M(B)$ is an equivariant $*$-homomorphism such that $B = \overline{\pi(A)B}$; see Lemma 3.7.

Theorem 4.2. *Let A and B be G-stable G-algebras, with A separable.*

(a) *Every element of $KK_G^1(A, B)$ is represented by an odd Kasparov triple for A and B which is both elementary and essential.*

(b) *Two elementary and essential odd Kasparov triples, $\mathcal{E}_1$ and $\mathcal{E}_2$, for A and B, define the same element of $KK_G^1(A, B)$ if and only if there are degenerate odd Kasparov triples, $\mathfrak{D}_1$ and $\mathfrak{D}_2$, for A and B which are both elementary and essential, such that $\mathcal{E}_1 \oplus \mathfrak{D}_1$ is operator homotopic to $\mathcal{E}_2 \oplus \mathfrak{D}_2$.*

Theorem 4.3. *Let A and B be G-stable G-algebras, with A separable.*

(a) *Every element of $KK_G^1(A, B)$ is represented by an elementary odd Kasparov triple.*

(b) *Two elementary odd Kasparov triples, $\mathcal{X}$ and $\mathcal{Y}$, for A and B, define the same element of $KK_G^1(A, B)$ if and only if there are degenerate and elementary odd Kasparov triples, $\mathfrak{D}_1$ and $\mathfrak{D}_2$, for A and B, such that $\mathcal{X} \oplus \mathfrak{D}_1$ is operator homotopic to $\mathcal{Y} \oplus \mathfrak{D}_2$.*

5. Equivariantly absorbing homomorphisms

In this section we define and establish the existence of absorbing $*$-homomorphisms in the equivariant case. The main tools are Kasparov's methods [1980b; 1988], the ideas from [Thomsen 2001], and Corollary 2.9 above.

When X is a normed vector space with an action of G by linear transformations we will in the following say that a net $\{x_\alpha\} \subseteq X$ is *asymptotically G-invariant* when $\lim_\alpha g \cdot x_\alpha - x_\alpha = 0$, uniformly on compact subsets of G. Note that it follows from [Kasparov 1988, page 152, lemma] that a G-algebra always contains an asymptotically G-invariant approximate unit with other convenient properties. We state only the part that we shall need.

Lemma 5.1 [Kasparov 1988]. *Let A be a G-algebra, and $D \subseteq M(A)$ a separable C^*-subalgebra. Then A contains an approximate unit $\{u_n\}$ which is asymptotically G-invariant and asymptotically commutes with D, i.e.* $\lim_{n\to\infty} u_n d - d u_n = 0$ *for all $d \in D$.*

The multiplier algebra $M(B)$ of a G-algebra is not always a G-algebra because the action of G on $M(B)$ is only continuous for the strict topology in general. The set of elements on which the G-action is continuous is a C^*-subalgebra of $M(B)$, and it *is* a G-algebra. We denote this G-algebra by $M(B)_G$. The image of $M(B)_G$ in $Q(B)$ will be denoted by $Q(B)_G$. In contrast the fixed point algebra in $M(B)$ and $Q(B)$ will be denoted by $M(B)^G$ and $Q(B)^G$, respectively.

Let E be a Hilbert B-module. A $*$-homomorphism $\pi : M(A) \to \mathbb{L}_B(E)$ will be called *strictly continuous*, when π is strictly continuous on norm-bounded sets. A $*$-homomorphism $\pi : A \to \mathbb{L}_B(E)$ will be called *quasi-unital* when there is a projection $p \in \mathbb{L}_B(E)$ such that $pE = \overline{\pi(A)E}$. It is known that $\pi : A \to \mathbb{L}_B(E)$ is quasi-unital if and only if π admits a strictly continuous extension $\overline{\pi} : M(A) \to \mathbb{L}_B(E)$; see [Lance 1995].

Theorem 5.2. *Let A and B be G-algebras, A separable and B G-stable. Let $s : A \to M(B)_G$ be a completely positive contraction. There is then a strictly continuous equivariant $*$-homomorphism $\pi : M(A) \to M(B)$ and a sequence of contractions $\{W_n\}$ in $M(B)$ such that* $\lim_{n\to\infty} W_n^* \pi(a) W_n = s(a)$ *for all $a \in A$.*

Proof. Let α and β be the given actions of G on A and B, respectively. We begin by constructing two Hilbert B,G-modules, E_0 and E. Give the algebraic tensor product $A \otimes C_c(G, B)$ a right B-module structure such that $(a \otimes f)b = a \otimes fb$ where $fb(g) = f(g)\beta_g(b)$ and make it into a semi-inner-product B-module in the sense of [Lance 1995], such that

$$\langle a \otimes \psi, a_1 \otimes \psi_1 \rangle = \int_G g^{-1} \cdot \big(\psi(g)^* s(g \cdot (a^* a_1))\psi_1(g)\big)\, dg.$$

Let E_0 denote the resulting Hilbert B-module, obtained by completion. Let r be the right-regular representation of G, namely $(r_k\psi)(g) = \Delta(k)^{1/2}\psi(gk)$. Define a representation T of G on E_0 such that $T_k = \alpha_k \otimes r_k$ on $A \otimes C_c(G, B)$. Then E_0 is a Hilbert B,G-module which is countably generated since A is separable, G second countable and B σ-unital. Give next $C_c(G, A) \otimes C_c(G, B)$ the right B-module structure such that $(j \otimes f)b = j \otimes fb$, and turn it into a semi-inner-product B-module such that

$$\langle j \otimes f, j_1 \otimes f_1\rangle = \int_G \int_G g^{-1} \cdot \big(f(g)^* s(g \cdot (j(g^{-1}h)^* j_1(g^{-1}h))) f_1(g)\big)\, dg\, dh.$$

By completion we obtain a Hilbert B-module E. Set $(\alpha \otimes \lambda)_k j(g) = \alpha_k(j(k^{-1}g))$, and define a representation S of G as linear operators on E such that

$$S_k(j \otimes f) = (\alpha \otimes \lambda)_k\,(j) \otimes r_k(f).$$

Define a linear map $\Phi : C_c(G, A) \otimes C_c(G, B) \to C_c(G, E_0)$ such that $\Phi(j \otimes f)(h) = j(h) \otimes f$. Then

$$\int_G \big\langle \Phi(j \otimes f)(h), \Phi(j_1 \otimes f_1)(h)\big\rangle\, dh = \langle j \otimes f, j_1 \otimes f_1\rangle;$$

hence Φ extends to an isomorphism $\Phi : E \to L^2(G, E_0)$ of Hilbert B,G-modules. Since $\Phi \circ S_k = (\lambda_k \otimes T_k) \circ \Phi$, we see that Φ itself is an isomorphism of Hilbert B,G-modules. Define $\pi' : M(A) \to \mathbb{L}_B(E)$ such that

$$\pi'(a)(j \otimes f) = \pi_0(a) j \otimes f,$$

where $(\pi_0(a)j)(g) = aj(g)$. Then π' is a strictly continuous equivariant $*$-homomorphism. Let $\kappa_n,\ n \in \mathbb{N}$, be a sequence of nonnegative functions in $C_c(G)$ such that $\operatorname{supp}\kappa_{n+1} \subseteq \operatorname{supp}\kappa_n$ and $\int_G \kappa_n(g)^2\, dg = 1$ for all n, and

$$\bigcap_n \operatorname{supp}\kappa_n = \{e\}. \tag{5–1}$$

Let $\{u_n\}$ be an approximate unit in A. For $n, m \in \mathbb{N}$, set $W_{n,m}b = \kappa^a \otimes \kappa_n^b \in C_c(G, A) \otimes C_c(G, B)$, where $\kappa^a(g) = \kappa_1(g)u_n$, and $\kappa_m^b(g) = \kappa_m(g)\beta_g(b)$. We claim that $W_{n,m}$ is adjointable, as a map $W_{n,m} : B \to E$. To see this, define first $T : C_c(G, A) \otimes C_c(G, B) \to B$ such that

$$T(j \otimes f) = \int_G \int_G \kappa_1(g^{-1}h)\kappa_m(g) g^{-1} \cdot \big(s(g \cdot (u_n j(g^{-1}h))) f(g)\big)\, dg\, dh.$$

It is then straightforward to check that

$$\langle W_{n,m}b, j \otimes f\rangle = \langle b, T(j \otimes f)\rangle. \tag{5–2}$$

Let $\sum_i j_i \otimes f_i$ be a finite sum of simple tensors in $C_c(G, A) \otimes C_c(G, B)$. By using the Schwarz inequality for completely positive maps, as in [Lance 1995], for instance, we find that

$$\begin{aligned}
&\Big\| T\Big(\sum_i j_i \otimes f_i\Big)\Big\| \\
&\quad = \Big\| \int_G \int_G \kappa_1(g^{-1}h)\kappa_m(g) \sum_i g^{-1} \cdot \big(s(g \cdot (u_n j_i(g^{-1}h))) f_i(g)\big)\, dg\, dh \Big\| \\
&\quad \le \Big\| \int_G \int_G g^{-1} \cdot \Big(\sum_i f_i(g)^* s(g \cdot (j_i(g^{-1}h)^* u_n)) \\
&\qquad\qquad \times \sum_l s(g \cdot (u_n j_l(g^{-1}h))) f_l(g) \Big)\, dg\, dh \Big\|^{1/2} \\
&\qquad\qquad\qquad \times \Big(\int_G \int_G \kappa_1(g^{-1}h)^2 \kappa_m(g)^2\, dg\, dh \Big)^{1/2} \\
&\quad \le \Big\| \int_G \int_G g^{-1} \cdot \Big(\sum_{i,l} f_i(g)^* s(g \cdot (j_i(g^{-1}h)^* u_n^2 j_l(g^{-1}h))) f_l(g) \Big)\, dg\, dh \Big\|^{1/2} \\
&\quad \le \Big\| \int_G \int_G g^{-1} \cdot \Big(\sum_{i,l} f_i(g)^* s(g \cdot (j_i(g^{-1}h)^* j_l(g^{-1}h))) f_l(g) \Big)\, dg\, dh \Big\|^{1/2} \\
&\quad = \Big\| \sum_i j_i \otimes f_i \Big\|.
\end{aligned}$$

It follows that T extends to a linear contraction $T : E \to B$ which then, by (5–2), is $W^*_{n,m}$. Note that

$$W^*_{n,m} \pi'(a) W_{n,m} b = \int_G \int_G \kappa_1(g^{-1}h)^2 \kappa_m(g)^2 g^{-1} \cdot (s(g \cdot (u_n a)))\, b\, dg\, dh, \tag{5–3}$$

so that

$$\begin{aligned}
&\| W^*_{n,m} \pi(a) W_{n,m} b - s(a) b \| \\
&\qquad = \Big\| \int_G \int_G \kappa_1(g^{-1}h)^2 \kappa_m(g)^2 \big(g^{-1}(s(g \cdot (u_n a))) - s(a)\big) b\, dg\, dh \Big\| \\
&\qquad \le \|b\| \sup\{ \|g^{-1} \cdot s(g \cdot (u_n a)) - s(a)\| : g \in \operatorname{supp} \kappa_m \} \\
&\qquad \le \|b\| \|s(u_n a) - s(a)\| + \|b\| \sup\{ \|g^{-1} \cdot s(g \cdot (u_n a)) - s(u_n a)\| : g \in \operatorname{supp} \kappa_m \}
\end{aligned}$$

for all $b \in B$, $a \in A$. Moreover $g \mapsto g^{-1} \cdot s(g \cdot a)$ is norm-continuous because s takes values in $M(B)_G$. Thus it follows from (5–1) and the preceding bound that

$$\lim_{n \to \infty} \big\| W^*_{n,k_n} \pi(a) W_{n,k_n} - s(a) \big\| = 0$$

for all $a \in A$, when $k_1 < k_2 < k_3 < \cdots$ is an appropriately chosen sequence in $\mathbb{N}$. Since $E \oplus B \simeq L^2(G, E_0) \oplus L^2(G, B) \simeq L^2(G, B) \simeq B$ by Corollary 2.4 and Lemma 2.6, there is an equivariant adjointable isometry $S : E \to B$ of Hilbert B,G-modules. Set $\pi(a) = S\pi'(a)S^*$ and $W_n = SW_{n,k_n}$. □

Proposition 5.3. *Let A and B be G-stable G-algebras. Assume that A is separable. Let $\varphi : A \to M(B)$ be an equivariant completely positive contraction. It follows that there is a strictly continuous equivariant $*$-homomorphism $\pi : M(A) \to M(B)$ and an asymptotically G-invariant sequence $\{S_n\}$ of contractions in $M(B)$ such that*

$$\lim_{n\to\infty} S_n^*\pi(a)S_n = \varphi(a)$$

for all $a \in A$.

Proof. Make $A \otimes B$ into a pre-Hilbert B-module such that $(a \otimes b)b_1 = a \otimes bb_1$ and such $\langle a \otimes b, a_1 \otimes b_1\rangle = b^*\varphi(a^*a_1)b_1$. Let E denote the resulting Hilbert B-module. E is countably generated since A is separable and B σ-unital. Because φ is equivariant we can make E into a Hilbert B,G-module by introducing the representation T of G on E given by $T_k(a \otimes b) = k \cdot a \otimes k \cdot b$. Let $\{u_n\}$ be an asymptotically G-invariant approximate unit in A, as in Lemma 5.1, and define $V_n : B \to E$ by $V_nb = u_n \otimes b$. Then V_n is an adjointable contraction such that

$$V_n^*(a \otimes b) = \varphi(u_na)b.$$

Since $T_kV_nk^{-1} \cdot b = k \cdot u_n \otimes b$, we see that $\{V_n\}$ is asymptotically G-invariant because $\{u_n\}$ is. Define $\pi_0 : M(A) \to \mathbb{L}_B(E)$ such that $\pi_0(m)(a \otimes b) = ma \otimes b$, and note that π_0 is a strictly continuous equivariant $*$-homomorphism, and that $\overline{\pi_0(A)E} = E$. Furthermore, $\lim_{n\to\infty} V_n^*\pi_0(a)V_n = \lim_{n\to\infty} \varphi(u_nau_n) = \varphi(a)$ for all $a \in A$. Since A and B are G-stable it follows from Corollary 2.9 that $E \oplus B \simeq B$ as Hilbert B,G-modules. It follows that there is an adjointable equivariant isometry $W : E \to B$. Set $S_n = WV_n$ and $\pi(m) = W\pi_0(m)W^*$. □

Let A and B be G-algebras, B stable. Then $M(B)$ contains a pair of G-invariant isometries V_1, V_2 with the property that $V_1V_1^* + V_2V_2^* = 1$, since this is clearly the case of $B \otimes \mathbb{K}$. Using these isometries we can add maps from A to $M(B)$ in the usual way: $(\varphi \oplus \psi)(a) = V_1\varphi(a)V_1^* + V_2\psi(a)V_2^*$. The map $\varphi \oplus \psi$ is equivariant iff φ and ψ both are. Note that this addition is independent of the choice of isometries V_1 and V_2 up to conjugation by a G-invariant unitary. Since B is stable there is a sequence $\{S_i\}_{i=1}^\infty$ of G-invariant isometries in $M(B)$ such that $\sum_{i=1}^\infty S_iS_i^* = 1$ in the strict topology. An equivariant $*$-homomorphism $\pi : A \to M(B)$ is *saturated* when there is a G-invariant unitary U in $M(B)$ such that

$$U\pi(a)U^* = \sum_{i=1}^{\infty} S_{2i}\pi(a)S_{2i}^* \tag{5–4}$$

for all $a \in A$. Thus a saturated $*$-homomorphism $A \to M(B)$ is one which is unitarily equivalent to the sum of infinitely many copies of itself plus the zero homomorphism. From the technical point of view, one of the main features of a saturated $*$-homomorphism, π, that we shall need is that there is a sequence $W_i, i \in \mathbb{N}$, of G-invariant isometries in $M(B)$ such that $W_i^* W_j = 0$ when $i \neq j$, $W_i^* \pi(a) W_j = \delta(i, j)\pi(a)$ for all a, i, j, and such that $\lim_{i\to\infty} W_i^* b = 0$ for all $b \in B$. Indeed, in the notation of (5–4), $W_i = U^* S_{2i}$ will be such a sequence.

Any equivariant $*$-homomorphism can be saturated: When $\varphi : A \to M(B)$ is an equivariant $*$-homomorphism, $\varphi'(\,\cdot\,) = \sum_{i=1}^{\infty} S_{2i}\varphi(\,\cdot\,)S_{2i}^*$ is saturated, and we call it the *saturation* of φ.

Theorem 5.4. *Let A and B be G-algebras, A separable, B stable. Let $\pi : A \to M(B)$ be a saturated equivariant $*$-homomorphism. Consider the following five conditions on π:*

(1) *For any completely positive equivariant contraction $\varphi : A \to M(B)$ there is an asymptotically G-invariant sequence of contractions $\{W_n\} \subseteq M(B)$ such that $\lim_{n\to\infty} \|\varphi(a) - W_n^* \pi(a) W_n\| = 0$ for all $a \in A$.*

(2) *For any completely positive equivariant contraction $\varphi : A \to M(B)$ there is an asymptotically G-invariant sequence $\{V_n\}$ of isometries in $M(B)$ such that*

   (a) $V_n^* \pi(a) V_n - \varphi(a) \in B,\ n \in \mathbb{N}\ ,\ a \in A\ ,$
   (b) $\lim_{n\to\infty} \|V_n^* \pi(a) V_n - \varphi(a)\| = 0,\ a \in A,$
   (c) $g \cdot V_n - V_n \in B$ *for all* $n \in \mathbb{N}$ *and all* $g \in G$.

(3) *For any completely positive equivariant contraction $\varphi : A \to M(B)$ there is an asymptotically G-invariant and norm-continuous path, $V_t, t \in [1, \infty)$, of isometries in $M(B)$ such that*

   (a) $V_t^* \pi(a) V_t - \varphi(a) \in B,\ t \in [1, \infty[\ ,\ a \in A\ ,$
   (b) $\lim_{t\to\infty} \|V_t^* \pi(a) V_t - \varphi(a)\| = 0,\ a \in A.$
   (c) $g \cdot V_t - V_t \in B$ *for all* $t \in [1, \infty[$ *and all* $g \in G$.

(4) *For any equivariant $*$-homomorphism $\varphi : A \to M(B)$ there is an asymptotically G-invariant and norm-continuous path, $U_t, t \in [1, \infty)$, of unitaries in $M(B)$ such that*

   (a) $U_t(\pi(a) \oplus \varphi(a))U_t^* - \pi(a) \in B,\ t \in [1, \infty),\ a \in A,$
   (b) $\lim_{t\to\infty} \|U_t(\pi(a) \oplus \varphi(a))U_t^* - \pi(a)\| = 0,\ a \in A\ ,$
   (c) $g \cdot U_t - U_t \in B$ *for all* $t \in [1, \infty)$ *and all* $g \in G$.

(5) *For any equivariant $*$-homomorphism $\varphi : A \to M(B)$ there is an asymptotically G-invariant sequence $\{U_n\}$ of unitaries $U_n \in M(B)$ such that*

$$\lim_{n\to\infty} \|U_n(\pi(a) \oplus \varphi(a))U_n^* - \pi(a)\| = 0,\ a \in A.$$

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). *When, in addition, A and B are G-stable,* (5) $\Rightarrow$ (1) *so that* (1)-(5) *are equivalent in this case.*

Proof. (1) $\Rightarrow$ (2): Let $\varphi : A \to M(B)$ be a completely positive equivariant contraction. By (1) there is an asymptotically G-invariant sequence $\{W_n\}$ of contractions in $M(B)$ such that $\lim_{n\to\infty} W_n^*\pi(a)W_n = \varphi(a),\ a \in A$. Set

$$X_n = \begin{pmatrix} W_n & 0 \\ \sqrt{1 - W_n^* W_n} & 0 \end{pmatrix}$$

which is a partial isometry in $M_2(M(B))$ such that $X_n^* X_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for all n and

$$\lim_{n\to\infty} X_n^* \begin{pmatrix} \pi(a) & \\ & 0 \end{pmatrix} X_n = \begin{pmatrix} \varphi(a) & \\ & 0 \end{pmatrix},\ a \in A.$$

Furthermore, $\{X_n\}$ is asymptotically G-invariant since $\{W_n\}$ is. Since B is stable there is a G-invariant element $V \in M_2(M(B))$ such that $VV^* = 1$ and $V^*V = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then $R_n = X_n V^*$ is an isometry and

$$\lim_{n\to\infty} R_n^* \begin{pmatrix} \pi(a) & \\ & 0 \end{pmatrix} R_n = V \begin{pmatrix} \varphi(a) & \\ & 0 \end{pmatrix} V^*,\ a \in A.$$

Set

$$U = \begin{pmatrix} V^* & 1 - V^*V \\ 0 & V \end{pmatrix},$$

which is a G-invariant unitary in $M_4(M(B))$ with the property that

$$U \begin{pmatrix} V \begin{pmatrix} \varphi(a) & \\ & 0 \end{pmatrix} V^* & \\ & 0 \end{pmatrix} U^* = \begin{pmatrix} \varphi(a) & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Set $S_n = \begin{pmatrix} R_n & \\ & R_n \end{pmatrix} U^*$ which is an isometry in $M_4(M(B))$ for all n such that

$$\lim_{n\to\infty} S_n^* \begin{pmatrix} \pi(a) & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} S_n = \begin{pmatrix} \varphi(a) & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

for all $a \in A$. $\{S_n\}$ is asymptotically G-invariant since $\{W_n\}$ is. Since $B^3 \simeq B$ as Hilbert B,G-modules we can use an isomorphism $M_4(B) \simeq M_2(B)$ of G-algebras to get an asymptotically G-invariant sequence $\{S_n'\}$ of isometries in $M(M_2(B))$ such that

$$\lim_{n\to\infty} S_n'^* \begin{pmatrix} \pi(a) & \\ & 0 \end{pmatrix} S_n' = \begin{pmatrix} \varphi(a) & \\ & 0 \end{pmatrix}$$

for all $a \in A$. Since B is stable there are G-invariant isometries $V_1, V_2 \in M(B)$ such that $V_1V_1^* + V_2V_2^* = 1$. Let $\Theta : M_2(M(B)) \to M(B)$ be the G-equivariant $*$-isomorphism given by

$$\Theta \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \sum_{i,j} V_i a_{ij} V_j^*. \tag{5–5}$$

Set $T'_n = \Theta(S'_n)$. Then $\Theta \circ \begin{pmatrix} \varphi & \\ & 0 \end{pmatrix} = V_1\varphi(\cdot)V_1^*$ and $\{T'_n\}$ is an asymptotically G-invariant sequence of isometries such that $\lim_{n\to\infty} T_n'^* V_1\pi(a)V_1^*T'_n = V_1\varphi(a)V_1^*$ for $a \in A$. Since π is saturated there is a G-invariant unitary U such that $U\pi(a)U^* = V_1\pi(a)V_1^*$. Then $T_n = U^*T'_nV_1$, $n \in \mathbb{N}$, is an asymptotically G-invariant sequence of isometries in $M(B)$ such that $\lim_{n\to\infty} T_n^*\pi(a)T_n = \varphi(a)$ for all $a \in A$. Note that since π is saturated we can arrange that $\lim_{n\to\infty} \|T_n^*b\| = 0$ for all $b \in B$, and that $T_i^*T_j = 0$, $T_i^*\pi(A)T_j = \{0\}$, for $i \neq j$.

Fix a compact subset X with dense span in A, an $\epsilon > 0$ and a compact subset $K \subseteq G$. Let $K = K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be a sequence of compact sets such that $G = \bigcup_n K_n$. Let $\{e_i\}_{i=1}^\infty$ be an approximate unit for B which is asymptotically G-invariant and asymptotically commutes with $\varphi(A)$; see Lemma 5.1. Let $n_1 < n_2 < n_3 < \cdots$ be a sequence in $\mathbb{N}$ and set

$$f_1 = e_{n_1}^{1/2},\ \ f_k = (e_{n_k} - e_{n_{k-1}})^{1/2},\ \ k \geq 2.$$

Let b be a strictly positive element in B. Assume that $\{n_i\}$ increases so fast that

$$\|g\cdot f_i - f_i\| \leq 2^{-i}\epsilon \tag{5–6}$$

for all $g \in K_i$ and all $i \in \mathbb{N}$,

$$\|f_kb\| \leq 2^{-k},\ \ k \geq 2, \tag{5–7}$$

and

$$\|f_k\varphi(a) - \varphi(a)f_k\| \leq 2^{-k}\epsilon \tag{5–8}$$

for all $a \in X$ and $k \in \mathbb{N}$. Now let $\{W_k\}$ be a subsequence of $\{T_k\}$ chosen such that

$$\|g\cdot W_i - W_i\| \leq 2^{-i}\epsilon,\ \ g \in K_i, \tag{5–9}$$

$$\|W_i^*b\| \leq 2^{-i}\epsilon, \tag{5–10}$$

and

$$\left\|W_i^*\pi(a)W_i - \varphi(a)\right\| \leq 2^{-i}\epsilon \tag{5–11}$$

for all $a \in X$ and all $i \in \mathbb{N}$. It follows then from (5–10) and (5–7) that $\sum_{k=1}^\infty W_kf_k$ converges in the strict topology to an isometry W in $M(B)$. Since

$$\|g\cdot(W_kf_k) - W_kf_k\| \leq 2^{-k+1}\epsilon,\ \ k \geq n,\ \ g \in K_n,$$

by (5–6) and (5–9) we see that $g\cdot W - W \in B$ for all $g \in G$ and that $\|g\cdot W - W\| \leq 2\epsilon$, $g \in K$. Note that $W^*\pi(a)W = \sum_{i=1}^\infty f_iW_i^*\pi(a)W_if_i$ for all a. Furthermore, (5–8) ensures that $\varphi(a) - \sum_{i=1}^\infty f_i\varphi(a)f_i \in B$ and that $\|\varphi(a) - \sum_{i=1}^\infty f_i\varphi(a)f_i\| \leq \epsilon$ for all $a \in X$. It follows then from (5–11) that $W^*\pi(a)W - \varphi(a) \in B$ and

$\|W^*\pi(a)W - \varphi(a)\| \leq \epsilon$ for all $a \in X$. Since X, K and $\epsilon > 0$ were arbitrary, (2) follows.

(2) $\Rightarrow$ (3): Since π is saturated there is a sequence $W_i, i \in \mathbb{N}$, of G-invariant isometries in $M(B)$ such that $W_i^* W_j = 0$ when $i \neq j$ and $W_i^* \pi(a) W_j = \delta(i, j)\pi(a)$ for all a, i, j. So by substituting $W_n V_n$ for V_n, we may assume that $V_i^* V_j = 0$ and $V_i^* \pi(a) V_j = 0$ for all $a \in A$ when $i \neq j$. Set $V_t = \sqrt{t-i}\, V_{i+1} + \sqrt{i+1-t}\, V_i$, for $t \in [i, i+1]$. Then $V_t, t \in [1, \infty)$, is an asymptotically G-invariant and norm-continuous path of isometries such that (3a)–(3c) hold.

(3) $\Rightarrow$ (4): In the following we will write $\psi_1 \sim \psi_2$ between equivariant $*$-homomorphisms $\psi_1, \psi_2 : A \to M(B)$, when there is an asymptotically G-invariant and norm-continuous path $W_t, t \in [1, \infty)$, of unitaries in $M(B)$ such that

$$\lim_{t \to \infty} W_t \psi_1(a) W_t^* = \psi_2(a)$$

for all $a \in A$ and such that $W_t \psi_1(a) W_t^* - \psi_2(a) \in B$ and $g \cdot W_t - W_t \in B$ for all a, t, g. Let $\varphi : A \to M(B)$ be an equivariant $*$-homomorphism. We need to show that $\pi \oplus \varphi \sim \pi$. Let $\{V_t\}$ be an asymptotically G-invariant and norm-continuous path of isometries such that (3a), (3b) and (3c) hold. By considering the asymptotically G-invariant path of unitaries in $M_2(M(B))$ given by

$$U_t = \begin{pmatrix} V_t^* & 1 - V_t^* V_t \\ 0 & V_t \end{pmatrix},$$

we see that $\pi \oplus 0 \sim \varphi \oplus 0$. Note that $\pi \sim \pi \oplus 0 \sim \pi \oplus \pi \oplus 0$ since π is saturated. It follows that

$$\pi \oplus \varphi \sim \pi \oplus 0 \oplus \varphi \sim \pi \oplus \pi \oplus 0 \sim \pi.$$

(4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (1) (when A and B are G-stable): It follows from Proposition 5.3 that there is an equivariant $*$-homomorphism $\mu : A \to M(B)$ and an asymptotically G-invariant sequence of isometries $\{W_n\} \subseteq M(B)$ such that $\lim_{n\to\infty} W_n^* \mu(a) W_n = \varphi(a)$ for all $a \in A$. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite subsets with dense union in A and let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be a sequence of compact subsets in G whose union is G. Fix $n \in \mathbb{N}$. It follows from (5) that there is a unitary $U_n \in M(B)$ such that $\left\| U_n(\pi(a) \oplus \mu(a)) U_n^* - \pi(a) \right\| \leq 1/n$ for all $a \in F_n$ and $\|g \cdot U_n - U_n\| \leq 1/n$ for all $g \in K_n$. Let V_1 and V_2 be the isometries used to define the addition and set $S_n = U_n V_2 W_n \in M(B)$. Then $\|S_n\| \leq 1$, $\|g \cdot S_n - S_n\| \leq 1/n + \|g \cdot W_n - W_n\|$ for all $g \in K_n$, and

$$\left\| \varphi(a) - S_n^* \pi(a) S_n \right\| \leq \frac{1}{n} + \left\| W_n^* \mu(a) W_n - \varphi(a) \right\|$$

for all $a \in F_n$. □

A saturated equivariant $*$-homomorphism $\pi : A \to M(B)$ will be called *equivariantly absorbing* when it satisfies condition (5) of Theorem 5.4.

Proposition 5.5. *Let A and B be G-algebras, A separable. Assume that A and B are both G-stable. If $\varphi : A \to M(B)$ and $\pi : A \to M(B)$ are both equivariantly absorbing there is a norm-continuous path u_t, $t \in [0, \infty)$, of unitaries in $M(B)$ such that*

(i) $u_t \pi(a) u_t^* - \varphi(a) \in B$ *for all* a, t,

(ii) $g \cdot u_t - u_t \in B$ *for all* g, t,

(iii) $\lim_{t\to\infty} u_t \pi(a) u_t^* - \varphi(a) = 0$ *for all* a, *and*

(iv) $\lim_{t\to\infty} g \cdot u_t - u_t = 0$, *uniformly on compact subsets of* G.

Proof. This follows from Theorem 5.2. □

Note that the condition for an equivariant $*$-homomorphism $A \to M(B)$ to be equivariantly absorbing *does not* reduce to the condition that it is absorbing, in the sense of [Thomsen 2001], when $G = 0$. The reason is that we require an equivariantly absorbing $*$-homomorphism to be saturated, and in the nonequivariant case this means that it must be unitarily equivalent to the infinite sum of copies of itself plus the zero homomorphism. While this may not be the case of all absorbing $*$-homomorphisms, the additional requirement seems not to have any significance in practice, and as demonstrated by Proposition 5.5, saturation is a very convenient property to have.

Proposition 5.6. *Let A and B be C^*-algebras, A separable, B σ-unital and stable. Let $\pi : A \to M(B)$ be a saturated $*$-homomorphism with the following property*:

(A) *For any completely positive contraction $\varphi : A \to B$ there is a sequence of contractions $\{W_n\} \subseteq M(B)$ such that $\lim_{n\to\infty} \|\varphi(a) - W_n^* \pi(a) W_n\| = 0$ for all $a \in A$.*

It follows that π is absorbing.

Proof. By combining the $G = 0$ case of Theorem 5.4 with [Thomsen 2001, Theorem 2.5] we see that it suffices to show that π has the following property:

(B) For any completely positive contraction $\psi : A \to M(B)$ there is a sequence of contractions $\{W_n\} \subseteq M(B)$ such that $\lim_{n\to\infty} \|\psi(a) - W_n^* \pi(a) W_n\| = 0$ for all $a \in A$.

To this end, let $F \subseteq A$ be a finite set, and let $\epsilon > 0$ be given. Choose a positive element $u \in A$ such that $\|uau - a\| \leq \epsilon$ for all $a \in F$, and set $F' = \{u^2\} \cup \{uau : a \in F\}$. Let X be a compact subset of A which contains F' and spans a dense subspace in A. Let b be a strictly positive element of B. Let $\{e_i\}_{i=1}^{\infty}$ be an approximate unit for B which asymptotically commutes with $\psi(A)$. Let $n_1 < n_2 < n_3 < \cdots$ be a sequence in $\mathbb{N}$ and set $f_1 = e_{n_1}^{1/2}$ and $f_k = (e_{n_k} - e_{n_{k-1}})^{1/2}$ for $k \geq 2$. We can arrange that $\{n_i\}$ increases so fast that $\| f_k b\| \leq 2^{-k}$, $k \geq 2$, and $\| f_k \psi(a) - \psi(a) f_k\| \leq 2^{-k}\epsilon$ for all $a \in X$ and all $k \in \mathbb{N}$. It follows then that $\sum_{i=1}^{\infty} f_i \psi(a) f_i$ converges in the strict topology for all $a \in A$, and that

$$\Big\| \sum_{i=1}^{\infty} f_i \psi(a) f_i - \psi(a) \Big\| \leq \epsilon \tag{5–12}$$

for all $a \in F'$. Since π has property (A) we can find a contraction $W_i \in M(B)$ such that

$$\left\| W_i^* \pi(a) W_i - f_i^{1/2} \psi(a) f_i^{1/2} \right\| \leq 2^{-i}\epsilon \tag{5–13}$$

for all $a \in F'$. Since π is saturated we can arrange that $W_i^* W_j = 0$, $W_i^* \pi(A) W_j = \{0\}$ if $i \neq j$, and $\|W_i^* b\| \leq 2^{-i}$ for all i. Since $\| f_i^{1/2} b\| \leq \sqrt{\|b\|}\, 2^{-i/2}$ for $i \geq 2$, we see that the sum $W = \sum_{i=1}^{\infty} W_i f_i^{1/2}$ converges in the strict topology. Thanks to the properties of the W_i we find that $W^* \pi(a) W = \sum_{i=1}^{\infty} f_i^{1/2} W_i^* \pi(a) W_i f_i^{1/2}$ for all $a \in A$. It follows then from (5–13) and (5–12) that

$$\| W^* \pi(a) W - \psi(a)\| \leq 2\epsilon \tag{5–14}$$

for all $a \in F'$. Set $V = \pi(u) W$, and note that it follows from (5–14) that $\|V\| \leq \sqrt{1 + 2\epsilon}$. Furthermore, since $\|uau - a\| \leq \epsilon$ for $a \in F$, we conclude from (5–14) that $\| V^* \pi(a) V - \psi(a)\| \leq 3\epsilon$ for all $a \in F$. It follows that π has property (B), as desired. □

Theorem 5.7. *Let A and B be separable G-algebras, both G-stable. There exists a strictly continuous equivariant $*$-homomorphism $\pi : M(A) \to M(B)$ such that $\pi|_A$ is both absorbing and equivariantly absorbing.*

Proof. By [Thomsen 2001, Lemma 2.3] there is a sequence $\{s_n\}$ of completely positive contractions from A to B which is dense among all such completely positive contractions. We may assume that each s_n occurs infinitely often in the sequence. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite sets in A whose union is dense in A and let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be a sequence of compact subsets of G whose union is G. For each pair $n, l \in \mathbb{N}$, let $X_{n,l}$ denote the set of pairs (w, π), where $w \in B$, $\|w\| \leq 1$, and $\pi : M(A) \to M(B)$ is a strictly continuous equivariant $*$-homomorphism such that $\|s_n(a) - w^* \pi(a) w\| \leq 1/l$ for $a \in F_l$. Note that $X_{n,l} \neq \varnothing$ by Theorem 5.2. For each $m \in \mathbb{N}$, set

$$\delta_{n,l,m} = \inf \Big\{ \sup_{g \in K_m} \|g \cdot w - w\| : (w, \pi) \in X_{n,l} \Big\}.$$

For each triple $n, m, l \in \mathbb{N}$, choose a pair $(w_{n,l,m}, \pi_{n,l,m}) \in X_{n,l}$ such that

$$\sup_{g \in K_m} \|g \cdot w_{n,l,m} - w_{n,l,m}\| \leq \delta_{n,l,m} + \frac{1}{l}.$$

Let

$$\pi(x) = \sum_{n,l,m} S_{n,l,m} \pi_{n,l,m}(x) S^*_{n,l,m},$$

where $\{S_{n,l,m}\}$ is a family of G-invariant isometries such that $S^*_{n,i,m} S_{k,j,l} = 0$ when $(n,i,m) \neq (k,j,l)$ and $\sum_{n,i,m} S_{n,i,m} S^*_{n,i,m} = 1$ in the strict topology.

To show that the saturation of $\pi|_A$ is absorbing it suffices by Proposition 5.6 to show that $\pi|_A$ has property (A) of that proposition, and to show that the saturation of $\pi|_A$ is equivariantly absorbing it suffices to show that $\pi|_A$ has property (1) of Theorem 5.4. We will establish these two properties of $\pi|_A$ simultaneously. Since π is strictly continuous this will complete the proof. Consider therefore a completely positive contraction $\varphi : A \to M(B)$ such that either (a) $\varphi(A) \subseteq B$ or (b) φ is equivariant. Let $k \in \mathbb{N}$ and $\epsilon > 0$ be given. We will show that in both cases there is an element $W \in M(B)$ such that $\|W\| \leq \sqrt{1+\epsilon}$ and $\|\varphi(a) - W^* \pi(a) W\| \leq 3\epsilon$, $a \in F_k$. Then $\pi|_A$ will clearly satisfy condition (A). Furthermore, we will show that in case (b) W can be chosen such that $\sup_{g \in K_k} \|g \cdot W - W\| \leq 2\epsilon$. Then $\pi|_A$ will clearly satisfy condition (1) of Theorem 5.4, and we will be done.

To simplify notation, set $K = K_k$ and $F = F_k$. By using an asymptotically G-invariant approximate unit for B which asymptotically commutes with the range of φ — for which see Lemma 5.1 — we can construct a sequence $\{f_i\}$ in B such that $0 \leq f_i \leq 1$, $\|g \cdot f_i - f_i\| < 2^{-i}\epsilon$ for all i and $g \in K$, $\sum_{i=1}^{\infty} f_i^2 = 1$, $\sum_{i=1}^{\infty} f_i \varphi(a) f_i$ converge in the strict topology for all $a \in A$ and

$$\left\| \sum_{i=1}^{\infty} f_i \varphi(x) f_i - \varphi(x) \right\| \leq \epsilon, \tag{5–15}$$

for all $x \in F$. We claim that for any $i \in \mathbb{N}$, any finite set $H \subseteq A$, any $\delta > 0$ and any $N \in \mathbb{N}$, there is a pair $n, l \in \mathbb{N}$ such that $n \geq N$, and

$$\|f_i \varphi(a) f_i - w^*_{n,l,k} \pi_{n,l,k}(a) w_{n,l,k}\| \leq \delta, \tag{5–16}$$

for all $a \in H$, and

$$\|g \cdot w_{n,l,k} - w_{n,l,k}\| \leq 2^{-i}\epsilon, \tag{5–17}$$

for all $g \in K$, in case (b). To see this observe first that in case (a) Theorem 5.2 gives us a sequence $\{W_n\}$ of contractions in $M(B)$ and a strictly continuous equivariant $*$-homomorphism $\pi_0 : M(A) \to M(B)$ such that $\lim_{n\to\infty} W_n^* \pi_0(a) W_n = \varphi(a)$ for all $a \in A$. In case (b) Proposition 5.3 does the same, and more: In case (b) we can choose $\{W_n\}$ to be asymptotically G-invariant. We first choose $m \in \mathbb{N}$ so large that

$$\left\|\varphi(a) - W_m^*\pi_0(a)W_m\right\| \le \tfrac{1}{2}\delta \tag{5–18}$$

for all $a \in H$, and $\|g \cdot (W_m f_i) - W_m f_i\| \le 2^{-i}\epsilon,\ g \in K$, in case (b). Then we choose $l \in \mathbb{N}$ so large that

(i) $2/l < \delta/6$,

(ii) $H \subseteq_{\delta/6} F_l$, meaning that for all $x \in H$ there is a $y \in F_l$ such that $\|x - y\| \le \delta/6$;

in case (b) we require that l satisfy the additional condition

(iii) $\frac{1}{l} \le 2^{-i}\epsilon - \max\{\|g \cdot (W_m f_i) - W_m f_i\| : g \in K\}$.

Then choose $n \ge N$ such that

$$\|s_n(x) - f_i W_m^*\pi_0(x)W_m f_i\| \le \frac{1}{l} \tag{5–19}$$

for all $x \in F_l$. It follows from (5–19) that $\delta_{n,l,k} \le \max\{\|g \cdot (W_m f_i) - W_m f_i\| : g \in K\}$, and hence by (iii) that $\|g \cdot w_{n,l,k} - w_{n,l,k}\| \le \delta_{n,l,k} + 1/l \le 2^{-i}\epsilon$ for all $g \in K$, provided of course that we are in case (b). Thus (5–17) holds in this case. Since $\|s_n(x) - w_{n,l,k}^*\pi_{n,l,k}(x)w_{n,l,k}\| \le 1/l$ for $x \in F_l$, it follows from (5–19) that $\|f_i W_m^*\pi_0(x)W_m f_i - w_{n,l,k}^*\pi_{n,l,k}(x)w_{n,l,k}\| \le 2/l$ for all $x \in F_l$. Combining this with i) and ii) we obtain that $\|f_i W_m^*\pi_0(a)W_m f_i - w_{n,l,k}^*\pi_{n,l,k}(a)w_{n,l,k}\| \le \frac{1}{2}\delta$ for all $a \in H$. In combination with (5–18) this gives us (5–16).

For each i we choose an element $u_i \in A$, $0 \le u_i \le 1$, such that

$$\|u_i x u_i - x\| \le 2^{-i}\epsilon, \tag{5–20}$$

$x \in F$, and $\|g \cdot u_i - u_i\| \le 2^{-i}\epsilon$, $g \in K$; we can do this by Lemma 5.1. Let b be a strictly positive element of B. By using the property established above and the fact that $\pi_{n,l,k}(\,\cdot\,) = S_{n,l,k}^*\pi(\,\cdot\,)S_{n,l,k}$, we see that we can choose a sequence l_i', $i \in \mathbb{N}$, of contractions in B such that

(iv) $l_i'^*\pi(A)l_j' = \{0\},\ i \ne j$,

(v) $\|l_i'^*\pi(u_i)b\| \le 2^{-i}$,

(vi) $\|l_i'^*\pi(x)l_i' - f_i\varphi(x)f_i\| \le 2^{-i}\epsilon,\ x \in \{u_i^2\} \cup u_i F u_i$,

and in case (b),

(vii) $\|g \cdot l_i' - l_i'\| \le 2^{-i}\epsilon,\ g \in K$.

Set $l_i = \pi(u_i)l'_i$. We claim that $\sum_{i=1}^{\infty} l_i$ converges in the strict topology of $M(B)$. Since

$$\begin{aligned}
\Big\|\sum_{i=n}^{m} l_i b\Big\|^2 &\leq \|b\| \Big\|\sum_{i,j=n}^{m} l_i^* l_j b\Big\| = \|b\| \Big\|\sum_{i=n}^{m} l_i^* l_i b\Big\| \\
&= \|b\| \Big\|\sum_{i=n}^{m} l_i'^* \pi(u_i^2) l_i' b\Big\| && \text{(by (iv))}\\
&\leq \epsilon\|b\| \sum_{i=n}^{m} 2^{-i} + \|b\| \Big\|\sum_{i=n}^{m} f_i \varphi(u_i^2) f_i b\Big\| && \text{(by (vi))}\\
&\leq \epsilon\|b\| \sum_{i=n}^{m} 2^{-i} + \|b\| \sqrt{\Big\| b\Big(\sum_{i=n}^{m} f_i \varphi(u_i^2) f_i\Big)^2 b\Big\|}\\
&\leq \epsilon\|b\| \sum_{i=n}^{m} 2^{-i} + \|b\| \sqrt{\Big\| b\Big(\sum_{i=n}^{m} f_i^2\Big) b\Big\|},
\end{aligned}$$

we see that $\sum_{i=1}^{\infty} l_i b$ converges in B. Since $\|l_i^* b\| \leq 2^{-i}$ by (v), $\sum_{i=1}^{\infty} l_i^* b$ converges also in B. Since b is a strictly positive element, and since

$$\sup_m \Big\|\sum_{i=1}^{m} l_i\Big\| \leq \sqrt{1+\epsilon}$$

by (iv) and (vi), it follows that $\sum_{i=1}^{\infty} l_i$ converges in the strict topology to an element W of $M(B)$ whose norm does not exceed $\sqrt{1+\epsilon}$. For $x \in F$ we have

$$\begin{aligned}
\big\|W^*\pi(x)W - \varphi(x)\big\| &\leq \epsilon + \Big\|W^*\pi(x)W - \sum_{i=1}^{\infty} f_i\varphi(x) f_i\Big\| && \text{(by (5–15))}\\
&= \epsilon + \Big\|\sum_{i=1}^{\infty} l_i'^*\pi(u_i x u_i) l_i' - \sum_{i=1}^{\infty} f_i\varphi(x) f_i\Big\| && \text{(by (iv))}\\
&\leq 2\epsilon + \Big\|\sum_{i=1}^{\infty} l_i'^*\pi(u_i x u_i) l_i' - \sum_{i=1}^{\infty} f_i\varphi(u_i x u_i) f_i\Big\| && \text{(by (5–20))}\\
&\leq 3\epsilon && \text{(by (vi)).}
\end{aligned}$$

Finally,

$$\|g \cdot l_i - l_i\| \leq \|g \cdot l_i' - l_i'\| + \|g \cdot u_i - u_i\| \leq 2^{-i+1}\epsilon,\ \ g \in K,$$

by (vii) and the choice of u_i, in case (b). Hence $\|g \cdot W - W\| \leq 2\epsilon$ for all $g \in K$, in case (b). □

6. Duality in equivariant KK-theory

In this section we combine the results of Sections 3 and 4 with those of Section 5. In this way we obtain the duality results for equivariant KK-theory relatively painlessly.

We assume now that A and B are separable G-algebras, both G-stable.

Lemma 6.1. *Let A and B be G-algebras, B stable. Let $\pi : A \to M(B)$ be a saturated equivariant $*$-homomorphism and set*

$$E = \{m \in M(B)^G : m\pi(a) = \pi(a)m,\ a \in A\}.$$

Then $K_*(E) = \{0\}$.

Proof. The argument is well-known so we will be sketchy. Since π is saturated, E contains a sequence $\{V_i\}$ of G-invariant isometries with orthogonal ranges such that $\sum_{i=1}^{\infty} V_i V_i^* = 1$ in the strict topology. We can then define a $*$-homomorphism $\psi : E \to E$ such that $\psi(m) = \sum_{i=2}^{\infty} V_i m V_i^*$, where the sum converges in the strict topology. Then $\psi \oplus \mathrm{id}_E = \mathrm{Ad}\, U \circ \psi$ for a unitary $U \in E$, and we conclude that $\psi_* + \mathrm{id} = \psi_*$ in K-theory. It follows that $K_*(E) = \{0\}$. □

Given an equivariantly absorbing $*$-homomorphism $\pi : A \to M(B)$, set

$$\mathcal{A}_\pi = \{x \in M(B) : x\pi(a) - \pi(a)x \in B\ ,\ a \in A\},$$

and

$$\mathcal{B}_\pi = \{x \in \mathcal{A}_\pi : x\pi(A) \subseteq B\}.$$

Then $\mathcal{B}_\pi$ is a closed two-sided ideal in $\mathcal{A}_\pi$ and we set $\mathcal{D}_\pi = \mathcal{A}_\pi / \mathcal{B}_\pi$. Note that G acts by automorphisms on $\mathcal{A}_\pi$ which leave $\mathcal{B}_\pi$ globally invariant, and we get an action of G on $\mathcal{D}_\pi$. None of these actions are continuous in general.

If $\tau : A \to M(B)$ is another equivariantly absorbing $*$-homomorphism, there is a unitary $w \in M(B)$ such that $\mathrm{Ad}\, w \circ \pi(a) - \tau(a) \in B$ for all $a \in A$ and $g \cdot w - w \in B$ for all $g \in G$; see Proposition 5.5. It follows that there is an equivariant $*$-isomorphism between $\mathcal{D}_\pi$ and $\mathcal{D}_\tau$. In particular, it follows that $\mathcal{D}_\pi^G \simeq \mathcal{D}_\tau^G$.

Let u be a unitary in $M_n(\mathcal{D}_\pi^G)$. Choose $v \in M_n(\mathcal{A}_\pi)$ such that $\mathrm{id}_{M_n} \otimes q(v) = u$. Define $\pi^n : A \to \mathbb{L}_B(B^n)$ by

$$\pi^n(a)(b_1, b_2, \dots, b_n) = (\pi(a)b_1, \pi(a)b_2, \dots, \pi(a)b_n)\,.$$

Let $B^n \oplus B^n$ be graded by $(x, y) \mapsto (x, -y)$. Then

$$\left(B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right) \tag{6–1}$$

is an even Kasparov triple for A and B. It is easy to see that the class of (6–1) in $KK_G^0(A, B)$ only depends on the class of u in $K_1(\mathcal{D}_\pi^G)$. Thus the construction gives rise to a map $\Theta : K_1(\mathcal{D}_\pi^G) \to KK_G^0(A, B)$ which is easily seen to be a homomorphism.

In the following we will let 1_m and 0_m denote the unit and the zero element of $M_m(M(B))$, respectively. We will identify $M_m(M(B))$ and $M(M_m(B))$.

Theorem 6.2. *Let A and B be separable G-algebras, both G-stable. Then $\Theta : K_1(\mathscr{D}_\pi^G) \to KK_G^0(A, B)$ is an isomorphism.*

Proof. Θ *is injective*: Let $u \in M_n(\mathscr{D}_\pi^G)$ be a unitary and choose $v \in M_n(\mathscr{A}_\pi)$ such that $\mathrm{id}_{M_n} \otimes q(v) = u$. Assume that $\left[B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right] = 0$ in $KK_G^0(A, B)$. By Theorem 3.10 this means that there are elementary degenerate even Kasparov triples $\mathscr{D}_1$ and $\mathscr{D}_2$ such that $\left(B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right) \oplus \mathscr{D}_1$ is operator homotopic to $\left(B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \oplus \mathscr{D}_2$. Since $\mathscr{D}_1$ and $\mathscr{D}_2$ are degenerate we can define a new degenerate even Kasparov $A-B$-module $\mathscr{D}$ by

$$\mathscr{D} = 0 \oplus \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \cdots \quad .$$

Then $\mathscr{D}_1 \oplus \mathscr{D}$ and $\mathscr{D}_2 \oplus \mathscr{D}$ are both isomorphic to $\mathscr{D}$ and hence

$$\left(B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right) \oplus \mathscr{D}$$

is operator homotopic to $\left(B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \oplus \mathscr{D}$. Note that $\mathscr{D}$ is isomorphic to an elementary even Kasparov triple since $\mathscr{D}_1$ and $\mathscr{D}_2$ are. Up to isomorphism, it must therefore have the form

$$\mathscr{D} = \left(B^e, \begin{pmatrix} \lambda_+ & \\ & \lambda_- \end{pmatrix}, \begin{pmatrix} & a \\ b & \end{pmatrix}\right),$$

where $\lambda_\pm : A \to M(B)$ are saturated equivariant $*$-homomorphisms and $a, b \in M(B)$. By performing the same alterations to $\mathscr{D}$ as was performed to $\mathscr{E}$ in [Jensen and Thomsen 1991, pages 125–126] we may assume that $a = w$ and $b = w^*$ for some unitary $w \in M(B)$. Note that $w\lambda_-(a)w^* = \lambda_+(a)$, $(g \cdot w)\lambda_\pm(a) = w\lambda_\pm(a)$ for all $a \in A$ and all $g \in G$ since $\mathscr{D}$ is degenerate. By adding to $\mathscr{D}$ the sum

$$\left(B^e, \begin{pmatrix} \lambda_- & \\ & \lambda_+ \end{pmatrix}, \begin{pmatrix} & w^* \\ w & \end{pmatrix}\right) \oplus \left(B^e, \begin{pmatrix} \pi & \\ & \pi \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right)$$

we may assume that $\lambda_+ = \lambda_-$ and that $\lambda = \lambda_+ = \lambda_-$ is equivariantly absorbing. Now the operator homotopy between

$$\left(B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right) \oplus \left(B^e, \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}, \begin{pmatrix} & w \\ w^* & \end{pmatrix}\right)$$

and

$$\left(B^n \oplus B^n, \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \oplus \left(B^e, \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}, \begin{pmatrix} & w \\ w^* & \end{pmatrix}\right)$$

gives us a G-invariant unitary $S \in M_{n+1}(M(B))$ such that

$$S\begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix} = \begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix} S$$

for all $a \in A$, and a norm-continuous path F_t, $t \in [0, 1]$, in $M_{n+1}(M(B))$ such that $F_0 S = S\begin{pmatrix} 1_n & \\ & w \end{pmatrix}$, $F_1 = \begin{pmatrix} v & \\ & w \end{pmatrix}$, and the following inclusions are satisfied for all values of g, t, a:

$$\left(F_t F_t^* - 1_{n+1}\right)\begin{pmatrix}\pi^n(a) & \\ & \lambda(a)\end{pmatrix} \in M_{n+1}(B),$$
$$\left(F_t^* F_t - 1_{n+1}\right)\begin{pmatrix}\pi^n(a) & \\ & \lambda(a)\end{pmatrix} \in M_{n+1}(B),$$
$$(g\cdot F_t - F_t)\begin{pmatrix}\pi^n(a) & \\ & \lambda(a)\end{pmatrix} \in M_{n+1}(B),$$

and

$$F_t\begin{pmatrix}\pi^n(a) & \\ & \lambda(a)\end{pmatrix} - \begin{pmatrix}\pi^n(a) & \\ & \lambda(a)\end{pmatrix}F_t \in M_{n+1}(B).$$

As an equivariant $*$-homomorphism $A \to M_{n+1}(M(B))$, $\nu = \begin{pmatrix}\pi^n & \\ & \lambda\end{pmatrix}$ is saturated, since π and λ are. By Lemma 6.1 we can therefore find an $m \in \mathbb{N}$ and a norm-continuous path of unitaries in

$$\{x \in M_{m(n+1)}(M(B)^G) : x\nu^m(a) = \nu^m(a)x,\ a \in A\}$$

connecting $\begin{pmatrix}S & \\ & 1_{(m-1)(n+1)}\end{pmatrix}$ to $1_{m(n+1)}$. In combination with F this gives us a norm-continuous path H_t, $t \in [0, 1]$, in $M_{m(n+1)}\left(M(B)^G\right)$ such that

$$H_0 = \begin{pmatrix}1_n & & \\ & w & \\ & & 1_{(m-1)(n+1)}\end{pmatrix}, \quad H_1 = \begin{pmatrix}v & & \\ & w & \\ & & 1_{(m-1)(n+1)}\end{pmatrix},$$

and moreover the following elements lie in $M_{m(n+1)}(B)$: $\left(H_t H_t^* - 1_{m(n+1)}\right)\nu^m(a)$, $\left(H_t^* H_t - 1_{m(n+1)}\right)\nu^m(a)$, $(g\cdot H_t - H_t)\,\nu^m(a)$ and $H_t\nu^m(a) - \nu^m(a)H_t$, for all t, g, and a. Since λ and π both are equivariantly absorbing there is a unitary $w_0 \in M(B)$ such that $g\cdot w_0 - w_0 \in B$ for all $g \in G$ and $w_0\lambda(a)w_0^* - \pi(a) \in B, a \in A$; see Proposition 5.5. Set

$$W = \operatorname{diag}\underbrace{\left(1_n, w_0, 1_n, w_0, \ldots, 1_n, w_0\right)}_{m\ \text{times}} \in M_{m(n+1)}(M(B))$$

and

$$G_t = WH_tW^*.$$

Then G_t is a norm-continuous path in $M_{m(n+1)}(M(B))$ such that

$$G_0 = \begin{pmatrix}1_n & & \\ & w_0ww_0^* & \\ & & 1_{(m-1)(n+1)}\end{pmatrix}, \quad G_1 = \begin{pmatrix}v & & \\ & w_0ww_0^* & \\ & & 1_{(m-1)(n+1)}\end{pmatrix},$$

and moreover the following elements lie in $M_{m(n+1)}(B)$, again for all t, g, and a: $(G_tG_t^* - 1_{m(n+1)}\pi^{m(n+1)}(a)$, $(G_t^*G_t - 1_{m(n+1)}\pi^{m(n+1)}(a)$, $(g\cdot G_t - G_t\pi^{m(n+1)}(a)$ and $G_t\pi^{m(n+1)}(a) - \pi^{m(n+1)}(a)G_t$. Thus $\left(\mathrm{id}_{M_{m(n+1)}} \otimes q\right)(G_t)$ is a path of unitaries in $M_{m(n+1)}\left(\mathfrak{D}_\pi^G\right)$ connecting

$$\begin{pmatrix} u & & \\ & q(w_0ww_0^*) & \\ & & 1_{(m-1)(n+1)} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1_n & & \\ & q(w_0ww_0^*) & \\ & & 1_{(m-1)(n+1)} \end{pmatrix}.$$

Θ *is surjective*: Let (E, ψ, F) be an even Kasparov triple for A and B. By Theorem 3.10 we may assume that $E = B^e$. By Lemma 3.2 we may assume that $\psi = (\varphi, \varphi)$. The constructions in [Jensen and Thomsen 1991, pages 125–126] show that $[B^e, \psi, F] \in KK_G(A, B)$ is also represented by a Kasparov triple of the form

$$\left(B^e, \begin{pmatrix} \varphi & \\ & \varphi \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right),$$

where $\varphi : A \to M(B)$ is an equivariant $*$-homomorphism and $v \in M(B)$ is a unitary. By adding on

$$\left(B^e, \begin{pmatrix} \pi & \\ & \pi \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right),$$

and using that π is equivariantly absorbing we may assume that there is a unitary $u \in M(B)$ such that $g \cdot u - u \in B$, $g \in G$, and $u\varphi(a)u^* - \pi(a) \in B$ for all $a \in A$. Let X be the graded Hilbert B,G-module which as a graded Hilbert B-module is $B \oplus B$, but with the representation of G changed to $g \cdot (b_1, b_2) = \left(u\beta_g\left(u^*b_1\right), u\beta_g\left(u^*b_2\right)\right)$. Then $\left(B^e, \begin{pmatrix} \varphi & \\ & \varphi \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right)$ is isomorphic to

$$\left(X, \begin{pmatrix} \operatorname{Ad} u \circ \varphi & \\ & \operatorname{Ad} u \circ \varphi \end{pmatrix}, \begin{pmatrix} & uvu^* \\ uv^*u^* & \end{pmatrix}\right).$$

Thanks to the properties of u a rotation argument works to show that

$$\left(X, \begin{pmatrix} \operatorname{Ad} u \circ \varphi & \\ & \operatorname{Ad} u \circ \varphi \end{pmatrix}, \begin{pmatrix} & uvu^* \\ uv^*u^* & \end{pmatrix}\right) \oplus \left(B^e, \begin{pmatrix} \pi & \\ & \pi \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right)$$

is operator homotopic to

$$\left(X, \begin{pmatrix} \operatorname{Ad} u \circ \varphi & \\ & \operatorname{Ad} u \circ \varphi \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) \oplus \left(B^e, \begin{pmatrix} \pi & \\ & \pi \end{pmatrix}, \begin{pmatrix} & uvu^* \\ uv^*u^* & \end{pmatrix}\right).$$

Thus uvu^* is a unitary in $\mathcal{A}_\pi$ such that $q(uvu^*) \in \mathfrak{D}_\pi^G$ and $\Theta\left([q\left(uvu^*\right)]\right) = [E, \psi, F]$ in $KK_G^0(A, B)$. □

Let next Q be a projection in $M_n\left(\mathfrak{D}_\pi^G\right)$, and let $P \in M_n\left(\mathcal{A}_\pi\right)$ be a lift of P. Then (B^n, π^n, Q) is an odd Kasparov for A and B and we can define a map $\Theta_0 : K_0\left(\mathfrak{D}_\pi^G\right) \to KK_G^1(A, B)$ such that $\Theta_0[Q] = [B^n, \pi^n, Q]$. By using Theorem 4.3 in place of Theorem 3.10, we can adopt the previous proof to obtain:

Theorem 6.3. *Let A and B be separable G-algebras, both G-stable. Then $\Theta_0 : K_0(\mathcal{D}_\pi^G) \to KK_G^1(A, B)$ is an isomorphism.*

Lemma 6.4. *Let A and B be separable G-algebras, both G-stable. Let $\pi : M(A) \to M(B)$ be a strictly continuous $*$-homomorphism such that $\pi|_A$ is equivariantly absorbing. It follows that $q\left(\mathcal{A}_\pi^G\right) = \mathcal{D}_\pi^G$, and that $q_* : K_*\left(\mathcal{A}_\pi^G\right) \to K_*\left(\mathcal{D}_\pi^G\right)$ is an isomorphism.*

Proof. Set $E = \pi(1)$ which is a G-invariant projection in $M(B)$ such that $EB = \overline{\pi(A)B}$. Let $x \in \mathcal{D}_\pi^G$ be self-adjoint and choose a self-adjoint $y \in \mathcal{A}_\pi$ such $q(y) = x$. By applying Lemma 3.3 to the triple (B, π, y) we see that there is a G-invariant element $z \in \mathbb{L}_B(EB)$ such that $y\pi(a) - \pi(a)z \in \mathbb{K}_B(EB, B)$ and $\pi(a)y - z\pi(a) \in \mathbb{K}_B(B, EB)$ for all $a \in A$. Define $z_0 \in M(B)$ such that $z_0 b = zEb$ for all $b \in B$. Then $z_0 \in M(B)^G$ and $y\pi(a) - \pi(a)z_0 \in B$, $\pi(a)y - z_0\pi(a) \in B$ for all $a \in A$. It follows that $z_0\pi(a) - \pi(a)z_0 \in B$ and $\pi(a)(y - z_0) \in B$ for all $a \in A$ so that $z_0 \in \mathcal{A}_\pi^G$ and $q(z_0) = q(y) = x$. Hence $q\left(\mathcal{A}_\pi^G\right) = \mathcal{D}_\pi^G$.

It suffices now to show that $K_*\left(\mathcal{B}_\pi^G\right) = 0$. To this end, consider the C^*-algebra

$$\mathcal{Y} = \left\{X \in M_2(M(B)^G) : X\begin{pmatrix}\pi(a) & \\ & 0\end{pmatrix},\ \begin{pmatrix}\pi(a) & \\ & 0\end{pmatrix}X \in M_2(B),\ a \in A\right\}.$$

Define $\kappa : \mathcal{B}_\pi^G \to \mathcal{Y}$ by $\kappa(x) = \begin{pmatrix}x & \\ & 0\end{pmatrix}$. We claim that $\kappa_* : K_*\left(\mathcal{A}_\pi^G\right) \to K_*(\mathcal{Y})$ is injective and that $\kappa_* = 0$. To prove injectivity of κ_*, note that since π absorbs 0 there is a G-invariant unitary $U \in \mathbb{L}_B(B \oplus B, B)$ such that

$$U\begin{pmatrix}\pi(a) & \\ & 0\end{pmatrix}U^* - \pi(a) \in B \tag{6–2}$$

for all $a \in B$. Then $X \mapsto UXU^*$ is an equivariant $*$-homomorphism $\lambda : \mathcal{Y} \to \mathcal{B}_\pi^G$. Define $V : B \to B$ by $Vb = U(b, 0)$, and observe that V is adjointable with adjoint $V^* : B \to B$ given by p_1U^*b, where $p_1 : B \oplus B \to B$ is the projection to the first coordinate. V is then a G-invariant isometry $V \in M(B)^G$ such that $\lambda \circ \kappa = \operatorname{Ad} U \circ \kappa = \operatorname{Ad} V$, and $V\pi(a)V^* = U\begin{pmatrix}\pi(a) & \\ & 0\end{pmatrix}U^*$. It follows from the last equality and (6–2) that $V\pi(a) - \pi(a)V \in B$ for all $a \in V$, and then that $xV \in \mathcal{B}_\pi^G$ when $x \in \mathcal{B}_\pi^G$. Therefore V is an isometry in $M\left(\mathcal{B}_\pi^G\right)$ and hence $(\operatorname{Ad} V)_* = \mathrm{id}$ in K-theory. Consequently $\lambda_* \circ \kappa_* = \mathrm{id}$ in K-theory, and κ_* must be injective. On the other hand, observe that κ is homotopic via a standard rotation argument to the $*$-homomorphism $x \mapsto \begin{pmatrix}0 & \\ & x\end{pmatrix}$, which factors through $M(B)^G$. Since the zero homomorphism is saturated, it follows from Lemma 6.1 that $K_*\left(M(B)^G\right) = 0$. Thus $\kappa_* = 0$. □

Theorem 6.5. *Let A and B be separable G-algebras, both G-stable. Let $\pi : M(A) \to M(B)$ be a strictly continuous $*$-homomorphism such that $\pi|_A$ is equivariantly absorbing. It follows that $\Theta \circ q_* : K_1\left(\mathcal{A}_\pi^G\right) \to KK_G^0(A, B)$ and $\Theta_0 \circ q_* : K_0\left(\mathcal{A}_\pi^G\right) \to KK_G^1(A, B)$ are both isomorphisms.*

Proof. Combine Theorems 6.2 and 6.3 with Lemma 6.4. □

By Theorem 5.7 we can choose the $*$-homomorphism π of Theorem 6.5 such that $\pi|_A : A \to M(B)$ is absorbing (nonequivariantly), and it follows then from [Thomsen 2001] that the K-theory of $\mathscr{A}_\pi$ gives us the nonequivariant KK-groups $KK^i(A, B)$, $i = 0, 1$. Under these identifications the canonical forgetful maps

$$KK^i_G(A, B) \to KK^i(A, B),$$

$i = 0, 1$, become the maps $K_i\left(\mathscr{A}^G_\pi\right) \to K_i(\mathscr{A}_\pi)$, $i = 0, 1$, induced by the embedding $\mathscr{A}^G_\pi \subseteq \mathscr{A}_\pi$.

References

[Baaj and Skandalis 1989] S. Baaj and G. Skandalis, "C^*-algèbres de Hopf et théorie de Kasparov équivariante", *K-Theory* **2**:6 (1989), 683–721. MR 90j:46061 Zbl 0683.46048

[Blackadar 1986] B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications **5**, Springer, New York, 1986. MR 88g:46082 Zbl 0597.46072

[Brown et al. 1977] L. G. Brown, R. G. Douglas, and P. A. Fillmore, "Extensions of C^*-algebras and K-homology", *Ann. of Math.* (2) **105**:2 (1977), 265–324. MR 56 #16399 Zbl 0376.46036

[Connes and Skandalis 1984] A. Connes and G. Skandalis, "The longitudinal index theorem for foliations", *Publ. Res. Inst. Math. Sci.* **20**:6 (1984), 1139–1183. MR 87h:58209 Zbl 0575.58030

[Higson 1995] N. Higson, "C^*-algebra extension theory and duality", *J. Funct. Anal.* **129**:2 (1995), 349–363. MR 96c:46072 Zbl 0828.46060

[Jensen and Thomsen 1991] K. K. Jensen and K. Thomsen, *Elements of KK-theory*, Birkhäuser, Boston, 1991. MR 94b:19008

[Kasparov 1980a] G. G. Kasparov, "The operator K-functor and extensions of C^*-algebras", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:3 (1980), 571–636. In Russian. MR 81m:58075 Zbl 0448.46051

[Kasparov 1980b] G. G. Kasparov, "Hilbert C^*-modules: theorems of Stinespring and Voiculescu", *J. Operator Theory* **4**:1 (1980), 133–150. MR 82b:46074 Zbl 0456.46059

[Kasparov 1988] G. G. Kasparov, "Equivariant KK-theory and the Novikov conjecture", *Invent. Math.* **91**:1 (1988), 147–201. MR 88j:58123 Zbl 0647.46053

[Lance 1995] E. C. Lance, *Hilbert C^*-modules*, London Mathematical Society Lecture Note Series **210**, Cambridge University Press, Cambridge, 1995. MR 96k:46100 Zbl 0288.46054

[Meyer 2000] R. Meyer, "Equivariant Kasparov theory and generalized homomorphisms", *K-Theory* **21**:3 (2000), 201–228. MR 2001m:19013 Zbl 0982.19004

[Mingo and Phillips 1984] J. A. Mingo and W. J. Phillips, "Equivariant triviality theorems for Hilbert C^*-modules", *Proc. Amer. Math. Soc.* **91**:2 (1984), 225–230. MR 85f:46111 Zbl 0546.46049

[Olsen and Pedersen 1989] C. L. Olsen and G. K. Pedersen, "Corona C^*-algebras and their applications to lifting problems", *Math. Scand.* **64**:1 (1989), 63–86. MR 91g:46064 Zbl 0668.46029

[Paschke 1981] W. L. Paschke, "K-theory for commutants in the Calkin algebra", *Pacific J. Math.* **95**:2 (1981), 427–434. MR 82k:46101 Zbl 0478.46056

[Pedersen 1979] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs **14**, Academic Press, London, 1979. MR 81e:46037 Zbl 0416.46043

[Skandalis 1988] G. Skandalis, "Une notion de nucléarité en K-théorie (d'après J. Cuntz)", *K-Theory* **1**:6 (1988), 549–573. MR 90b:46131 Zbl 0653.46065

[Thomsen 2000] K. Thomsen, "Equivariant KK-theory and C^*-extensions", *K-Theory* **19**:3 (2000), 219–249. MR 2001j:19004 Zbl 0945.19002

[Thomsen 2001] K. Thomsen, "On absorbing extensions", *Proc. Amer. Math. Soc.* **129**:5 (2001), 1409–1417. MR 2002c:46113 Zbl 0968.46059

[Thomsen 2003] K. Thomsen, "On the KK-theory and the E-theory of amalgamated free products of C^*-algebras", *J. Funct. Anal.* **201**:1 (2003), 30–56. MR 2004d:46086 Zbl 1034.46076

[Valette 1983] A. Valette, "A remark on the Kasparov groups $\mathrm{Ext}(A, B)$", *Pacific J. Math.* **109**:1 (1983), 247–255. MR 85f:46128 Zbl 0488.46061

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