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We consider complete noncompact simply connected nonpositively curved surfaces that are twice continuously differentiably embedded in Euclidean three space. If such a surface has square integrable second fundamental form, then it must be a plane.

1. Introduction

The study of minimal surfaces has a long tradition and has been extended in many directions: surfaces of constant mean curvature, isometric immersion of surfaces into hyperbolic spaces or other higher dimensional spaces, and so on. Here, we try to go in another direction. Consider the condition

$$\int |B|^2 \, da < \infty, \tag{1}$$

where $|B|$ is the length of second fundamental form. Now, $|B|^2 = 4|H|^2 - 2K$, where H is the mean curvature and K is the Gauss curvature. The Gauss curvature of minimal surfaces is nonpositive. We examine what minimal surface results continue to hold if we extend the minimal condition $H \equiv 0$ to $K \leq 0$ but require (1) instead. We recall:

Theorem [Xavier 2001]. *Let $M \subset \mathbb{R}^3$ be a complete simply connected embedded minimal surface whose Gaussian curvature is bounded from below. If there is a plane whose intersection with M is transversal and connected then M is a plane or a helicoid.*

Theorem [Meeks and Rosenberg 2005]. *A properly embedded simply connected minimal surface in $\mathbb{R}^3$ is either a plane or a helicoid.*

Minimal surfaces can be divided into two classes: those of finite total curvature and those of infinite total curvature. For the case of finite total curvature we have $\int |B|^2 = -2 \int K < \infty$, so condition (1) is satisfied. In this paper we prove:

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Theorem 1. *A complete simply connected embedded C^2-surface M in $\mathbb{R}^3$ with $K \leq 0$ and (1) is a plane.*

We can apply the proof of Theorem 1 to recover the finite total curvature case of Meeks and Rosenberg's Theorem.

S. Bernstein [1915–17] proved that if M is an entire minimal graph, then M must be a plane. This theorem has been generalized; see [Fujimoto 1988; Osserman 1971; Xavier 1981]. In [Chan 2000], we showed:

Theorem 2. *Suppose M is a complete, oriented, one-ended, nonpositively curved Riemannian surface with an isolated set of parabolic points $\{p \in M : K(p) = 0\}$. Then M cannot be C^2 isometrically immersed in $\mathbb{R}^3$ with (1) and one embedded end.*

The stipulation of one embedded end cannot be weakened, because of the catenoid and Enneper's surface. The conditions in Theorem 2, apart from the isolatedness of parabolic points, are satisfied by the following example. Let (x, y, z) be the usual coordinates of $\mathbb{R}^3$. Let S be the locus of points satisfying

$$(1-z)\big(y^2 + z(1+z)\big) = (1+z)\big(x^2 - z(1-z)\big).$$

By the Implicit Function Theorem, S is a surface in $\mathbb{R}^3$. In fact, S lies in the region $|z| \leq 1$ and is not simply connected (its genus is 1). In [Chan and Treibergs 2001], we improved our theorem as follows:

Theorem 3. *Suppose M is a complete, oriented, nonpositively curved Riemannian surface C^2 isometrically immersed in $\mathbb{R}^3$ satisfying (1) and having one embedded end near infinity. Then M must lie in a slab (a region between two parallel planes).*

Since the plane stays in the slab, this generalizes Bernstein's Theorem. In that paper, we also provided explicit examples of nonpositively curved surfaces of genus n. In cylindrical coordinated of $\mathbb{R}^3$, for $n \geq 2$ an integer, the surface

$$r^n(z - \cos n\theta) = z - z^3$$

has genus $n-1$ and lies in the slab $|z| \leq 1$. Thus, simple connectivity is a critical condition for the Theorem 1.

2. Proof of Theorem 1.

Morse's theorems relate the topology of manifolds to the singularities of Morse functions on them. In this section, we apply Morse theory to noncompact complete surfaces in $\mathbb{R}^3$ by finding Morse functions that have no critical points outside a large compact set.

Let M be a complete, oriented, connected, nonpositively curved surface C^2 immersed in $\mathbb{R}^3$. White [1987] proved that if (1) is satisfied, the Gauss map extends continuously to one point near infinity and M is properly immersed near infinity.

The simple connectivity and embeddedness of the surface M in Theorem 1 imply that M has one embedded end. Let p denote the point at infinity of the surface. By White's theorem, (1) implies that the Gauss image $g(p)$ is one point and M is properly embedded. Then there exists a punctured disk $U_\epsilon(p)$ that is embedded and such that $|g(x) - g(p)| < \epsilon$ for all $x \in U_\epsilon(p)$. The outside of a big compact set of M can be considered as a graph of a function over the xy-plane P.

We divide the proofs into two cases, $K \equiv 0$ and $K \not\equiv 0$. Assume that $K \equiv 0$ on M in Theorem 1. According to [Hartman and Nirenberg 1959], M is a generalized cylinder. By (1), M is a plane. Assume instead that $K \not\equiv 0$. Then there is a point p_0 such that $K(p_0) < 0$. We may assume that the tangent plane $T_{p_0}M$ is not parallel to P and that the distance function to $T_{p_0}M$ (the height function above $T_{p_0}M$) is a Morse function. Since the normal directions of a neighborhood of p_0 form a set of full measure on the unit sphere, there is a point p_1 in a small neighborhood of p_0 such that the distance function of the tangent plane $T_{p_1}M$ is a Morse function and $T_{p_1}M$ is not parallel to P. Then p_1 is a critical point of f. Because the curvature is nonpositive at the critical points, all indices are 1.

For a Morse function on a compact surface $\overline{M}$, Morse's formula [Milnor 1963] says that the Euler characteristic is the algebraic sum of the numbers C_λ of critical points of index λ:

$$\chi(\overline{M}) = \sum_{\lambda} (-1)^\lambda C_\lambda. \tag{2}$$

In the case of a surface, λ takes the values 0, 1, and 2, so $\chi(\overline{M}) = C_0 - C_1 + C_2$.

To prove Theorem 1, we need to modify Morse's formula. Let f be the distance function from $T_{p_1}M$ on M. Choose ϵ so small that $|g(p_1) - g(p)| > \epsilon$. Then no critical points of f occur in $U_\epsilon(p)$. Because the Gauss map extends continuously to p, the point at infinity, and because M is the graph of a function in $U_\epsilon(p)$, we can cut out a disk about p in U_ϵ and glue back a spherical cap containing exactly two new critical points of f, of indices 0 and 2 respectively. Call the new surface $\widehat{M}$. After gluing, $\chi(\widehat{M}) = \chi(M) + 1$. Thus the Morse formula for noncompact surfaces yields

$$\chi(M) = \chi(\widehat{M}) - 1 = 1 - C_1 + 1 - 1 = 1 - C_1.$$

There is at least one critical point p_1 whose index is 1 for f, so $\chi(M)$ has a upper bound of zero, contradicting simple connectivity. Thus M is a plane. This completes the proof.

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