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Let Γ be a finite discrete group acting smoothly on a compact manifold X, and let D be a first-order elliptic self-adjoint Γ-equivariant differential operator acting on sections of some Γ-equivariant Hermitian vector bundle over X. We use these data to define Toeplitz operators with symbols in the transformation group C^*-algebra $C(X) \rtimes \Gamma$. If the symbol of such a Toeplitz operator is invertible, then the operator is Fredholm. In the case where X is a spin manifold and D is the Dirac operator, we give a geometric-topological formula for the index.

Let X be a smooth compact manifold without boundary, let V be a Hermitian vector bundle over X, and suppose D is a first-order elliptic self-adjoint differential operator acting on sections of V. Let P be the positive spectral projection of D. Then P is an order zero pseudodifferential operator, and it follows from standard facts about pseudodifferential operators on compact manifolds that given a smooth function f on X, the pointwise multiplication operator M_f acting on square-integrable sections $L^2(V)$ of V commutes with P modulo the ideal of compact operators. From this fact it is easy to show that if f is invertible, then the Toeplitz operator $PM_f : \operatorname{Ran} P \to \operatorname{Ran} P$ is a Fredholm operator. Furthermore, the index of PM_f can be computed using the Atiyah–Singer Index Theorem; see [Baum and Douglas 1982].

Now suppose that a discrete group Γ acts smoothly on both X and V in a compatible way, and suppose that D commutes with the action of Γ on sections of V. Then P also commutes with this action. In addition, there is a natural action ρ of the transformation group C^*-algebra $C(X) \rtimes \Gamma$ on $L^2(V)$, and P commutes with the elements of $C(X) \rtimes \Gamma$ modulo the compacts. Therefore, whenever $F \in C(X) \rtimes \Gamma$ is invertible, $T_F := P\rho(F)$ is a Fredholm operator, and it is natural to ask what the index of this operator is.

Let X be an odd-dimensional oriented spin manifold and let Γ be a finite group acting on X by isometries that preserve the orientation and spin structure of X. In [Park 2002], the case of free actions was considered; in this paper we consider

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general actions. We use the Lefschetz theorem in [Fang 2005] to prove a theorem that computes the Fredholm index of T_F in terms of the geometry and topology of X and a "Chern character" form explicitly constructed from F.

We begin by more precisely defining the objects under discussion. Let X be a smooth compact manifold, and let Γ be a discrete group acting smoothly on X from the right. Let V be a Γ-equivariant complex vector bundle over X, and equip V with a Γ-invariant Hermitian structure. Then Γ acts on the left on both the smooth sections $C^\infty(V)$ and the square-integrable sections $L^2(V)$ of V:

$$(\gamma \cdot s)(x) = s(x\gamma)\gamma^{-1}.$$

If $C(X)$ acts on $L^2(V)$ by pointwise multiplication, we have a covariant representation of $(C(X), \Gamma)$ on $L^2(V)$, and hence a representation ρ of $C(X) \rtimes \Gamma$ on $L^2(V)$.

Let D be a first-order, Γ-equivariant, elliptic self-adjoint differential operator acting on sections of V, and let $P = \chi_{[0,\infty)}(D)$ denote the positive spectral projection of D; this operator is also Γ-equivariant. For each F in $C(X) \rtimes \Gamma$, define the Toeplitz operator $T_F \in \mathcal{B}(L^2(V))$ to be $T_F = P\rho(F)P + I - P$. More generally, for each positive integer n, let $P_n \in \mathcal{B}((L^2(V))^n)$ be the matrix with P as each diagonal entry and all other entries zero, and let ρ_n denote the obvious representation of $\mathrm{M}(n, C(X) \rtimes \Gamma)$ on $(L^2(V))^n$ determined by ρ. Then for each F in $\mathrm{M}(n, C(X) \rtimes \Gamma)$, let $T_F = P_n\rho_n(F)P_n + I - P_n$. We note that while Toeplitz operators are typically defined as operators on the range of P_n, we have opted to extend our Toeplitz operators to all of $(L^2(V))^n$ in a way that does not affect their index theory.

Proposition 1. *If F is in* $\mathrm{GL}(n, C(X) \rtimes \Gamma)$, *then T_F is Fredholm.*

Proof. It suffices to show that P commutes with elements of $C(X) \rtimes \Gamma$ modulo the compacts, for then it follows easily that $T_F T_{F^{-1}} = I = T_{F^{-1}} T_F \bmod \mathcal{K}\big((L^2(V))^n\big)$. The operator P is Γ-equivariant and therefore commutes with $\rho(\gamma)$ for each γ in Γ. On the other hand, for all f in $C(X)$, the commutator $[P, \rho(f)]$ is in $\mathcal{K}(L^2(V))$, by [Baum and Douglas 1982, Lemma 2.10]. These elements are dense in $C(X) \rtimes \Gamma$, so the desired conclusion follows. □

Our goal is to find a geometric-topological formula for the index of T_F that can be computed directly from F. To this end, we will show that the Fredholm index of T_F is equal to the Γ-*invariant index* of a certain Γ-equivariant operator.

Definition 2. Let V be a Γ-equivariant complex vector bundle over X, and let R be a Γ-equivariant elliptic pseudodifferential operator acting on sections of V. The Γ-*invariant index* of R is denoted by $\mathrm{Ind}_{\Gamma-inv}(R)$, and is the dimension of the Γ-invariant subspace of $\ker R$ minus the dimension of the Γ-invariant subspace of $\ker R^*$.

Proposition 3. *Let R be a Γ-equivariant elliptic operator. Then*

$$\mathrm{Ind}_{\Gamma-inv}(R) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{Ind}_\gamma(R),$$

where $\mathrm{Ind}_\gamma(R)$ *is the trace of the Γ-equivariant index of R evaluated at γ.*

Proof. Let σ be a representation of Γ on a finite-dimensional complex vector space W, and decompose σ as

$$\sigma = n_0 \mathbf{1} + \sum_{i=1}^{k} n_i \sigma_i,$$

where the σ_i are irreducible and distinct, and $\mathbf{1}$ denotes the trivial representation. Let W^Γ be the subspace of W that is fixed by Γ. The fact that each of the σ_i fixes only 0 implies that $n_0 = \dim W^\Gamma$. Then, by [Serre 1977, Exercise 2.5],

$$\dim W^\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(\gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{Tr}(\sigma(\gamma)),$$

whence the result follows. □

For each natural number n, let $\mathrm{Map}(\Gamma, \mathrm{M}(n, C(X)))$ denote the C^*-algebra of all functions from the group Γ to $\mathrm{M}(n, C(X))$. The algebra $\mathrm{M}(n, C(X))$ acts on $\mathrm{Map}(\Gamma, \mathrm{M}(n, C(X)))$ via pointwise multiplication; let $\mathscr{L}\big(\mathrm{Map}(\Gamma, \mathrm{M}(n, C(X)))\big)$ be the algebra of $\mathrm{M}(n, C(X))$-linear maps on $\mathrm{Map}(\Gamma, \mathrm{M}(n, C(X)))$. We define a homomorphism

$$\mu : \mathrm{M}(n, C(X)) \rtimes \Gamma \longrightarrow \mathscr{L}\big(\mathrm{Map}(\Gamma, \mathrm{M}(n, C(X)))\big)$$

as follows: for all ψ in $\mathrm{Map}(\Gamma, \mathrm{M}(n, C(X)))$, all functions f in $\mathrm{M}(n, C(X))$, and all elements α and γ in Γ, set $(\mu(f)\psi)(\alpha) = \mu(\alpha \cdot f)\psi(\alpha)$ and $(\mu(\gamma))(\alpha) = \psi(\gamma^{-1}\alpha)$, and then extend μ to all of $\mathrm{M}(n, C(X)) \rtimes \Gamma$ by stipulating that μ be an algebra homomorphism. If $\mathrm{M}(n, C(X))$ acts on $(L^2(V))^n$ by pointwise multiplication and if λ denotes the left regular representation of $\mathrm{M}(n, C(X))$ on the Hilbert space $\mathrm{Map}\big(\Gamma, (L^2(V))^n\big)$, then $\lambda(F)$ is matrix multiplication by $\mu(F)$ for every F in $\mathrm{M}(n, C(X) \rtimes \Gamma$.

The group Γ acts on $\mathrm{Map}\big(\Gamma, (L^2(V))^n\big)$ by the formula

$$(\gamma \cdot \psi)(\alpha) = \gamma \cdot (\psi(\gamma^{-1}\alpha)),$$

and the subspace $\mathrm{Map}\big(\Gamma, (L^2(V))^n\big)^\Gamma$ of elements fixed by Γ contains precisely the ψ for which $\gamma \cdot (\psi(e)) = \psi(\gamma)$ for all γ in Γ; here e denotes the identity element of Γ. Thus the elements of $\mathrm{Map}\big(\Gamma, (L^2(V))^n\big)^\Gamma$ are determined by their value at e. Conversely, specifying a value at e uniquely determines an element of $\mathrm{Map}\big(\Gamma, (L^2(V))^n\big)^\Gamma$.

Define $U : (L^2(V))^n \to \operatorname{Map}\bigl(\Gamma, (L^2(V))^n\bigr)^\Gamma$ as

$$(U(s_1, s_2, \dots, s_n))(\alpha) = \frac{1}{\sqrt{|\Gamma|}}(\alpha \cdot s_1, \alpha \cdot s_2, \dots, \alpha \cdot s_n).$$

Because the inner product on $L^2(V)$ has been chosen to be Γ-invariant, U is a unitary operator, and $U^*\psi = \sqrt{|\Gamma|}\psi(e)$ for every ψ.

Define $\widetilde{\mathrm{D}}$ on $\operatorname{Map}\bigl(\Gamma, (L^2(V))^n\bigr)$ by the formula $(\widetilde{\mathrm{D}}\,\psi)(\alpha) = D(\psi(\alpha))$. Let $\widetilde{\mathrm{P}}$ be the positive spectral projection of $\widetilde{\mathrm{D}}$, and for each F in $\mathrm{M}(n, C(X) \rtimes \Gamma)$, define

$$\widetilde{\mathrm{T}}_F = \widetilde{\mathrm{P}}\,\lambda(F)\,\widetilde{\mathrm{P}} + I - \widetilde{\mathrm{P}}.$$

We can express the Fredholm index of Toeplitz operators T_F in terms of the $\widetilde{\mathrm{T}}_F$:

Proposition 4. *For every F in* $\mathrm{GL}(n, C(X) \rtimes \Gamma)$,

$$\operatorname{Ind} T_F = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \operatorname{Ind}_\gamma \widetilde{\mathrm{T}}_F\,.$$

Proof. For each F, the operator $\widetilde{\mathrm{T}}_F$ is Γ-equivariant with symbol $\mu(F)$. Furthermore, when $\widetilde{\mathrm{T}}_F$ is restricted to the Hilbert space $\operatorname{Map}\bigl(\Gamma, (L^2(V))^n\bigr)^\Gamma$, we have $\widetilde{\mathrm{T}}_F = U T_F U^*$. Thus the result follows immediately from Propositions 1 and 3. □

For the remainder of the paper, we will assume that X is a $(2m+1)$-dimensional spin manifold with spinor bundle S and Dirac operator D, and that Γ acts on X by orientation-preserving isometries that preserve the spin structure. In this case, we can combine Proposition 4 with the index theorem in [Fang 2005] to get a geometric-topological index formula for the index of T_F.

Theorem 5. *For each γ in Γ, let $X_1^\gamma, X_2^\gamma, \dots, X_{K_\gamma}^\gamma$ be the connected components of the fixed point set of γ, and for each $1 \le k \le K_\gamma$, let NX_k^γ denote the normal bundle of X_k^γ in X. Then for all F in* $\mathrm{GL}(n, C^\infty(X) \rtimes \Gamma)$,

$$\operatorname{Ind} T_F = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{k=1}^{K_\gamma}\Bigl(\frac{-i}{2\pi}\Bigr)^{(1+\dim X_k^\gamma)/2}\int_{X_k^\gamma} \widehat{A}(X_k^\gamma)\,\mathrm{ch}(\mu(F))\Lambda^{-1},$$

where

$$\mathrm{ch}(\mu(F)) = \sum_{k=0}^{\infty}(-1)^k\frac{k!}{(2k+1)!}\operatorname{Tr}\bigl((\mu(F))^{-1}\,d\mu(F))^{2k+1}\bigr)$$

and

$$\Lambda = \operatorname{Pf}\Bigl(2\sin\Bigl(\frac{i}{2}\bigl(R(NX_k^\gamma) + \ln J(X_k^\gamma)\bigr)\Bigr)\Bigr);$$

here Pf *is the Pfaffian, $R(NX_k^\gamma)$ is the curvature of NX_k^γ, and $J(X_k^\gamma)$ is the Jacobian matrix of the action of γ on NX_k^γ.*

For this theorem to be useful to us, we need to be able to express $\operatorname{ch}(\mu(F))$ in terms of F.

The action of F on X induces an action on the algebra $\Omega^*(X)$ of smooth differential forms, and we can extend our map μ to an algebra homomorphism

$$\mu : \mathrm{M}(n, \Omega^*(X)) \rtimes \Gamma \longrightarrow \mathscr{L}\big(\mathrm{Map}(\Gamma, \mathrm{M}(n, \Omega^*(X)))\big).$$

Take $\sum_{\gamma\in\Gamma} \omega_\gamma \gamma$ in $\mathrm{M}(n, \Omega^*(X)) \rtimes \Gamma$. Then for all ψ in $\mathrm{Map}\big(\Gamma, \mathrm{M}(n, \Omega^*(X))\big)$,

$$\left(\mu\left(\sum_{\gamma\in\Gamma} \omega_\gamma \gamma\right)\psi\right)(\alpha) = \sum_{\gamma\in\Gamma} (\alpha \cdot \omega_\gamma)\psi(\alpha\gamma).$$

For any element B of $\mathscr{L}\big(\mathrm{Map}(\Gamma, \mathrm{M}(n, \Omega^*(X)))\big)$, its trace is computed by the formula

$$\mathrm{Tr}(B) = \sum_{\alpha\in\Gamma} (B\delta_\alpha)(\alpha),$$

where δ_α is the constant function 1 when evaluated at $\alpha \in \Gamma$, and is otherwise zero. Thus

$$\begin{aligned}
\mathrm{Tr}\left(\mu\left(\sum_{\gamma\in\Gamma} \omega_\gamma \gamma\right)\right) &= \sum_{\alpha,\gamma\in\Gamma} \big(\mu(\omega_\gamma \gamma)\delta_\alpha\big)(\alpha) \\
&= \sum_{\alpha,\gamma\in\Gamma} (\alpha \cdot \omega_\gamma)\delta_\alpha(\alpha\gamma) \\
&= \sum_{\alpha\in\Gamma} \alpha \cdot \omega_e.
\end{aligned}$$

Definition 6. Let $\nu : \mathrm{M}(n, \Omega^*(X)) \rtimes \Gamma \to \Omega^*(X)$ be given by the formula

$$\nu\left(\sum_{\gamma\in\Gamma} \omega_\gamma \gamma\right) = \sum_{\alpha\in\Gamma} \alpha \cdot \omega_e,$$

and for all F in $\mathrm{GL}(n, C^\infty(X) \rtimes \Gamma)$, define

$$\widehat{\operatorname{ch}}(F) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \nu((F^{-1} dF)^{2k+1}),$$

where the exterior derivative d is extended to $\mathrm{M}(n, \Omega^*(X)) \rtimes \Gamma$ by applying d entrywise in $\mathrm{M}(n, C^\infty(X))$ and setting

$$d\left(\sum_{\gamma\in\Gamma} \omega_\gamma \gamma\right) = \sum_{\gamma\in\Gamma} (d\omega_\gamma)\gamma.$$

Combining Theorem 5 and Definition 6, we have:

Theorem 7. *For all F in* $\mathrm{GL}(n, C^\infty(X)\rtimes\Gamma)$,

$$\operatorname{Ind} T_F = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{k=1}^{K_\gamma}\Big(\frac{-i}{2\pi}\Big)^{(1+\dim X_k^\gamma)/2}\int_{X_k^\gamma}\widehat{A}(X_k^\gamma)\,\widehat{\mathrm{ch}}(F)\Lambda^{-1}.$$

This formula looks rather daunting, but in many cases it simplifies considerably.

Example 8. Let $\mathrm{SO}(2m+2)$ act on $\mathbb{R}^{2m+2}$ in the usual way, let Γ be a finite subgroup of $\mathrm{SO}(2m+2)$, and let S^{2m+1} be the unit sphere in $\mathbb{R}^{2m+2}$. The action of Γ on $\mathbb{R}^{2m+2}$ restricts to an action on S^{2m+1}, and because spheres have unique spin structures [Lawson and Michelsohn 1989], the action of Γ on S^{2m+1} trivially preserves the spin structure. Now, each γ in Γ fixes a subspace of $\mathbb{R}^{2m+2}$, and so the fixed point set X^γ of γ acting on S^{2m+1} is an equatorial sphere, and in particular, the fixed point set is connected. Furthermore, all spheres have stably trivial tangent bundles, so $\widehat{A}(X^\gamma)=1$ for all γ in Γ. Finally, a straightforward computation shows that the normal bundle of an equatorial sphere has curvature zero, and so

$$\operatorname{Ind} T_F = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\Big(\frac{-i}{2\pi}\Big)^{(1+\dim X^\gamma)/2}\int_{X^\gamma}\widehat{\mathrm{ch}}(F)\big(\mathrm{Pf}\big(2\sin\big(\tfrac{1}{2}\ln J(X^\gamma)\big)\big)\big)^{-1}.$$

Given γ in Γ, there exists an orthonormal frame of NX^γ for which the action of γ on NX^γ decomposes into blocks

$$\begin{pmatrix}\cos\theta_j^\gamma & \sin\theta_j^\gamma\\ -\sin\theta_j^\gamma & \cos\theta_j^\gamma\end{pmatrix},$$

with $0<\theta_j^\gamma<2\pi$ and $j=1,2,\dots,L^\gamma=m-\frac{1}{2}(\dim X^\gamma+1)$ (see [Lawson and Michelsohn 1989]). Incorporating this into our formula we obtain

$$\operatorname{Ind} T_F = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\Big(\frac{-i}{2\pi}\Big)^{(1+\dim X^\gamma)/2}\frac{1}{\sin(\theta_1^\gamma/2)\cdots\sin(\theta_{L^\gamma}^\gamma/2)}\int_{X^\gamma}\widehat{\mathrm{ch}}(F).$$

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