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**REPRESENTATION THEORY OF $GL(n)$ OVER A p -ADIC
DIVISION ALGEBRA AND UNITARITY IN THE
JACQUET-LANGLANDS CORRESPONDENCE**

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Let F be a local nonarchimedean field of characteristic 0, and let A be an F -central division algebra of dimension d_A over F . In this paper, we first develop some parts of the representation theory of $GL(m, A)$, assuming the conjecture that unitary parabolic induction is irreducible for $GL(m, A)$'s. Among others, we obtain the formula for characters of irreducible unitary representations of $GL(m, A)$ in terms of standard characters. Then we study the Jacquet–Langlands correspondence on the level of Grothendieck groups of $GL(pd_A, F)$ and $GL(p, A)$. Using this character formula, we get explicit formulas for the Jacquet–Langlands correspondence of irreducible unitary representations of $GL(n, F)$ (assuming the conjecture to hold). As a consequence, we get that the Jacquet–Langlands correspondence sends irreducible unitary representations of $GL(n, F)$ either to zero or to irreducible unitary representations, up to a sign.

Introduction

Functoriality is a key aspect of the Langlands program [1970]. One of the first examples of functoriality which were studied in the local case was the connection between representations of various inner forms of $GL(n)$ (see [Knapp and Rogawski 1997]). The first example, studied already in [Jacquet and Langlands 1970], was the connection between irreducible representations of $GL(2)$ over a local field F and irreducible representations of the multiplicative group of the quaternion algebra over F . (The L groups of these two groups are both $GL(2, \mathbb{C}) \times \text{Gal}(\bar{F}/F)$, and the functoriality considered here corresponds to the identity mapping.)

Let F be a local nonarchimedean field of characteristic 0 and let A be an F -central division algebra of rank d_A over F . For each positive integer p , P. Deligne,

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D. Kazhdan and M.-F. Vignéras established bijections

$$\text{LJ}_{pd_A}$$

between irreducible essentially square-integrable representations of $\text{GL}(pd_A, F)$ and $\text{GL}(p, A)$ ([Rogawski 1983] takes care of the case $p = 1$). The crucial requirement which holds for these bijections, and which characterizes them uniquely, is that the characters Θ_δ and $\Theta_{\text{LJ}_{pd_A}(\delta)}$ of the representations δ and $\text{LJ}_{pd_A}(\delta)$ satisfy the character identity

$$(-1)^{pd_A} \Theta_\delta(g) = (-1)^p \Theta_{\text{LJ}_{pd_A}(\delta)}(g')$$

whenever g and g' have the same characteristic polynomials, and when this polynomial is separable. These bijections are called Jacquet–Langlands correspondences.

A. I. Badulescu observed that Jacquet–Langlands correspondences extend in a very natural way to mappings between Grothendieck groups

$$\text{LJ}_{pd_A} : \text{Groth } \text{GL}(pd_A, F) \rightarrow \text{Groth } \text{GL}(p, A),$$

such that the extensions are compatible with parabolic induction, i.e. that they commute with parabolic induction. (Essentially, such extensions are unique if we require that characters of $\text{GL}(n, F)$'s go to 0 if $d_A \nmid n$.) Moreover, these extensions satisfy the character identity above on the level of formal characters (for a precise description of the extensions, see §6.1). We shall call these mappings Jacquet–Langlands correspondences on the level of Grothendieck groups.

For the group G of rational points of a reductive group defined over a local nonarchimedean field, we denote by \tilde{G} the set of all the equivalence classes of irreducible smooth representations of G . The unitary dual \hat{G} of G consists of all the unitarizable classes in \tilde{G} .

We consider $\text{GL}(n, F)^\sim$ as a subset of $\text{Groth } \text{GL}(n, F)$ in a natural way (it forms a \mathbb{Z} -basis). An interesting question is to understand what happens with irreducible representations under the Jacquet–Langlands correspondence on the level of Grothendieck groups, and in particular, what happens with irreducible unitary representations. Already, very simple examples will show that LJ_n will carry some irreducible representations to zero. Further, it is not hard to see that an irreducible (unitarizable) representation can go to the negative of an irreducible representation.

In this paper we study what happens with irreducible unitary representations under the Jacquet–Langlands correspondence, assuming the following conjecture for general linear groups over division algebras, introduced in [Tadić 1990]:

- (U0) unitary parabolic induction is irreducible for $\text{GL}(m, A)$'s.

In other words, if π_1 and π_2 are irreducible unitary representations of $\mathrm{GL}(m_1, A)$ and $\mathrm{GL}(m_2, A)$, the parabolically induced representation $\mathrm{Ind}^{\mathrm{GL}(m_1+m_2, A)}(\pi_1 \otimes \pi_2)$ is irreducible.

Note that [Vogan 1986] implies that (U0) holds in the archimedean case. Also, J. Bernstein [1984] proved that (U0) holds if $A = F$ (unfortunately his method cannot be extended to the division algebra case).

Here we first develop some directions of the representation theory of $\mathrm{GL}(n)$ over division algebras over a local nonarchimedean field, to be able to obtain the formula for characters of irreducible unitary representations of $\mathrm{GL}(m, A)$ in terms of standard characters. (Assuming (U0) to hold, [Tadić 1990] and [Badulescu and Renard 2004] imply a classification of the unitary duals $\widehat{\mathrm{GL}(m, A)}$ of $\mathrm{GL}(m, A)$ for all $m \geq 1$.) Using this character formula, we compute explicit formulas for $\mathrm{LJ}_n(\pi)$, $\pi \in \widehat{\mathrm{GL}(n, F)}$ (Propositions 7.3, 9.5 and Section 11). As a consequence:

Corollary. *Assume that (U0) holds. Then*

$$\mathrm{LJ}_{pd_A}(\widehat{\mathrm{GL}(pd_A, F)}) \subseteq \pm \widehat{\mathrm{GL}(p, A)} \cup \{0\};$$

that is, Jacquet–Langlands correspondences send irreducible unitary representations of general linear groups over F either again to irreducible unitary representations of general linear groups over A , up to a sign, or to 0.

Remark. The following direct consequence may be of some interest. Let σ be an element of $\widehat{\mathrm{GL}(p, A)}$ (resp. of $(\mathrm{Groth} \widehat{\mathrm{GL}(p, A)}) \setminus \{0\}$). Suppose that (U0) holds and suppose that there exists $\pi \in \widehat{\mathrm{GL}(pd_A, F)}$ such that the characters of σ and π are equal up to (the same) sign on elements with same characteristic polynomials. Then σ is unitarizable (resp. $\sigma \in \pm \widehat{\mathrm{GL}(p, A)}$).

There are very strong formal similarities between the Jacquet–Langlands correspondences studied in this paper and the Kazhdan–Patterson lifting studied in [Tadić 1996].

We now give a description of the content of the paper according to sections. Section 1 goes over notation and necessary basic results for general linear groups over a local nonarchimedean field. Section 2 introduces notation and basic results for general linear groups over division algebras. In Section 3 we show that the canonical involution on irreducible representations of $\mathrm{GL}(m, A)$ (introduced by A.-M. Aubert, and by P. Schneider and U. Stuhler) preserves unitarity. We also obtain an explicit formula for the involution on irreducible unitary representations. (We assume throughout that (U0) holds.) In Section 4 we describe irreducible subquotients of ends of complementary series, obtaining in this way a character identity, which enable us to compute in Section 5 characters of irreducible unitary representations of $\mathrm{GL}(m, A)$ in terms of standard characters. In Section 6 we recall the Jacquet–Langlands correspondence on the level of Grothendieck groups. In

Section 7 we compute $\text{LJ}(\pi)$ for one of four basic types of $\pi \in \text{GL}(n, F)^\wedge$, while the unitarity of $\text{LJ}(\pi)$ is shown in **Section 8**. **Sections 9 and 10** study the same problem for the second basic type of π . In **Section 11** we compute the Jacquet–Langlands correspondence of the remaining two basic types of π , using canonical involutions.

1. Some facts from the representation theory of $\text{GL}(n, F)$

We introduce the notation and basic results that we shall need for general linear groups over a local nonarchimedean field.

1.1. We fix a local nonarchimedean field F . The modulus character of F will be denoted by $|\cdot|_F$ (it satisfies $|x|_F \int_F f(xa)da = \int_F f(a)da$ for any continuous compactly supported complex-valued function f on F , where da denotes a Haar measure of the additive group $(F, +)$ of the field).

1.2. Let G be the group of rational points of a reductive group over F . The set of equivalence classes of irreducible smooth representations of G will be denoted by

$$\tilde{G}.$$

The subset of unitarizable classes in \tilde{G} will be denoted by

$$\hat{G}.$$

A representation $\pi \in \tilde{G}$ is called essentially square-integrable if there exists a character χ of G such that $\chi\pi$ is square-integrable representation modulo the center. All the essentially square-integrable classes in \tilde{G} will be denoted by

$$\mathfrak{D}(G).$$

The Grothendieck group of the category of all representations of G of finite length will be denoted by

$$\text{Groth } G.$$

1.3. Now we introduce the Bernstein–Zelevinsky notation for the general linear group $\text{GL}(n, F)$ (for more on notation see [Zelevinsky 1980; Tadić 1986]).

For two smooth representations π_1 and π_2 of $\text{GL}(n_1, F)$ and $\text{GL}(n_2, F)$, we consider $\pi_1 \otimes \pi_2$ as a representation of $\text{GL}(n_1, F) \times \text{GL}(n_2, F)$. Identifying in a natural way $\text{GL}(n_1, F) \times \text{GL}(n_2, F)$ with the Levi factor of the parabolic subgroup

$$\left\{ \begin{bmatrix} g_1 & * \\ 0 & g_2 \end{bmatrix}; g_i \in \text{GL}(n_i, F) \text{ for } i = 1, 2 \right\},$$

we denote by

$$\pi_1 \times \pi_2$$

the smooth representation of $\mathrm{GL}(n_1 + n_2, F)$ parabolically induced by $\pi_1 \otimes \pi_2$. Then

$$(\pi_1 \times \pi_2) \times \pi_3 \cong \pi_1 \times (\pi_2 \times \pi_3).$$

1.4. The characters of F^\times will be identified with characters of $\mathrm{GL}(n, F)$ using the determinant homomorphism. The character of $\mathrm{GL}(n, F)$ corresponding to $|\cdot|_F$ will be denoted by

$$\nu.$$

For any character χ of F^\times ,

$$\chi(\pi_1 \times \pi_2) \cong (\chi\pi_1) \times (\chi\pi_2).$$

1.5. Let

$$R_{n,F} = \text{Groth } \mathrm{GL}(n, F).$$

Then $\mathrm{GL}(n, F)^\sim$ is a \mathbb{Z} -basis of $R_{n,F}$.

We can lift \times to a \mathbb{Z} -bilinear mapping $\times : R_{n_1,F} \times R_{n_2,F} \rightarrow R_{n_1+n_2,F}$ since the semisimplification of $\pi_1 \times \pi_2$ depends only on semisimplifications of π_1 and π_2 . Set

$$R_F = \bigoplus_{n \geq 0} R_{n,F}.$$

One extends \times to an operation $\times : R_F \times R_F \rightarrow R_F$ in an obvious way, and R_F becomes an associative, commutative graded ring.

Fix a character χ of F^\times . Lift the mappings $\pi \mapsto \chi\pi : R_{n,F} \rightarrow R_{n,F}$ to a \mathbb{Z} -linear map $\chi : R_F \rightarrow R_F$. In this way we get an endomorphism of the graded ring R_F .

We have a natural partial ordering on $R_{n,F}$. ($\mathrm{GL}(n, F)^\sim$ generates the cone of positive elements in $R_{n,F}$.) Then orderings on the $R_{n,F}$'s determine an ordering \leq on R_F in a natural way. An additive homomorphism $\phi : R_F \rightarrow R_F$ will be called positive if $x \in R_F, x \geq 0$ implies $\phi(x) \geq 0$.

1.6. Set

$$\mathfrak{D}_F = \bigcup_{n \geq 1} \mathfrak{D}(\mathrm{GL}(n, F)).$$

For $\delta \in \mathfrak{D}_F$ there exists a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. The representation $\nu^{-e(\delta)}\delta$ will be denoted by δ^u . In this way,

$$\delta = \nu^{e(\delta)}\delta^u,$$

where $e(\delta) \in \mathbb{R}$ and δ^u is unitarizable.

1.7. We now describe the Langlands classification for general linear groups. Let $M(\mathfrak{D}_F)$ be the set of all finite multisets in \mathfrak{D}_F and $d = (\delta_1, \delta_2, \dots, \delta_k) \in M(\mathfrak{D}_F)$. Let γ be a permutation of $\{1, 2, \dots, k\}$ such that $e(\delta_{\gamma(1)}) \geq e(\delta_{\gamma(2)}) \geq \dots \geq e(\delta_{\gamma(k)})$.

The representation

$$\lambda(d) = \delta_{\gamma(1)} \times \delta_{\gamma(2)} \times \cdots \times \delta_{\gamma(k)}$$

has a unique irreducible quotient. Its class depends only on d (not on γ as above). This unique irreducible quotient will be denoted by $L(d)$ or $L(\delta_1, \delta_2, \dots, \delta_k)$. From §1.4 it follows that for a character χ of F^\times ,

$$\chi L(\delta_1, \delta_2, \dots, \delta_k) \cong L(\chi\delta_1, \chi\delta_2, \dots, \chi\delta_k).$$

To shorten the notation, we shall often denote the semisimplification of $\lambda(d)$ simply by $\lambda(d) \in R_F$. This will produce no confusion. From the properties of the Langlands classification it is well-known that the $\lambda(d) \in R_F, d \in M(\mathfrak{D}_F)$, form a basis of R_F .

Proposition [Zelevinsky 1980]. *The ring R_F is a polynomial ring over \mathfrak{D}_F .* □

1.8. One defines addition of elements of $M(\mathfrak{D}_F)$ by

$$(\delta_1, \delta_2, \dots, \delta_k) + (\delta'_1, \delta'_2, \dots, \delta'_{k'}) = (\delta_1, \delta_2, \dots, \delta_k, \delta'_1, \delta'_2, \dots, \delta'_{k'}).$$

Proposition [Rodier 1982]. *For $d_1, d_2 \in M(\mathfrak{D}_F)$, $L(d_1 + d_2)$ is a subquotient of $L(d_1) \times L(d_2)$. The multiplicity is one.* □

1.9. Let \mathcal{C}_F be the set of all the equivalence classes of irreducible cuspidal representations of all general linear groups $GL(n, F), n \geq 1$. For $\rho \in \mathcal{C}_F$ and $k \in \mathbb{Z}, k \geq 0$, the set

$$[\rho, v^k \rho] = \{\rho, v\rho, v^2\rho, \dots, v^k\rho\}$$

is called a segment of irreducible cuspidal representations. A segment of the form $[v^{k_1}\rho, v^{k_2}\rho]$ (where $k_1, k_2 \in \mathbb{R}$ are such that $k_2 - k_1 \in \mathbb{Z}$ and $k_2 - k_1 \geq 0$) will also be denoted by

$$[k_1, k_2]^\rho.$$

The set of all such segments will be denoted by \mathcal{S}_F . The set of all finite multisets in \mathcal{S}_F will be denoted by $M(\mathcal{S}_F)$. We consider the partial ordering \leq on $M(\mathcal{S}_F)$ introduced in [Zelevinsky 1980, §7.1], defined by linking segments.

1.10. Let $\Delta = [\rho, v^k \rho] = \{\rho, v\rho, v^2\rho, \dots, v^k\rho\} \in \mathcal{S}_F$. The representation

$$\rho \times v\rho \times v^2\rho \times \cdots \times v^k\rho$$

has a unique irreducible quotient, denoted by

$$\delta([\rho, v^k \rho]),$$

and a unique irreducible subrepresentation, denoted by

$$\delta([\rho, v^k \rho])^t.$$

The representations $\delta([\rho, v^k \rho])$ are essentially square-integrable. Further, the representations $\delta([\rho, v^k \rho])^t$ are called Zelevinsky’s segment representations.

By Bernstein–Zelevinsky theory, the mapping

$$\delta : \mathcal{S}_F \rightarrow \mathcal{D}_F, \quad \Delta \mapsto \delta(\Delta),$$

is a bijection.

We can state this also in the following way. For $n \in \mathbb{N}$ and $\rho \in \mathcal{C}_F$, set

$$\delta(\rho, n) = \delta([- (n - 1)/2, (n - 1)/2]^\rho).$$

Then $(\rho, n) \mapsto \delta(\rho, n)$ is a bijection of $\mathcal{C}_F \times \mathbb{N}$ onto \mathcal{D}_F .

We lift $\Delta \mapsto \delta(\Delta)$ naturally to a bijection

$$M(\delta) : M(\mathcal{S}_F) \rightarrow M(\mathcal{D}_F), \quad (\Delta_1, \dots, \Delta_k) \mapsto (\delta(\Delta_1), \dots, \delta(\Delta_k)).$$

Using this bijection we get the Langlands classification in terms of $M(\mathcal{S}_F)$.

For $a \in M(\mathcal{S}_F)$ we set

$$L(a) = L(M(\delta)(a)) \quad \text{and} \quad \lambda(a) = \lambda(M(\delta)(a)).$$

1.11. Note that §1.10 and the proposition in §1.7 imply that R_F is a polynomial algebra over $\delta(\Delta)$, $\Delta \in \mathcal{S}_F$. Therefore the mapping

$$\delta(\Delta) \mapsto \delta(\Delta)^t, \quad \Delta \in \mathcal{S}_F,$$

extends uniquely to a ring morphism ${}^t : R_F \rightarrow R_F$. This ring morphism is involutive. A fundamental fact is that it carries irreducible representations into irreducible ones; see [Aubert 1995; Schneider and Stuhler 1997].

Obviously, for a character χ of F^\times , $(\chi \delta(\Delta))^t \cong \chi(\delta(\Delta)^t)$ for $\Delta \in \mathcal{S}_F$. Therefore, ${}^t : R_F \rightarrow R_F$ and $\chi : R_F \rightarrow R_F$ commute, since they commute on generators.

1.12. For an irreducible representation π of a general linear group, there exists a unique $(\rho_1, \dots, \rho_k) \in M(\mathcal{C}_F)$ such that

$$\pi \hookrightarrow \rho_1 \times \dots \times \rho_k.$$

The multiset (ρ_1, \dots, ρ_k) is called the cuspidal support of π and it is denoted by

$$\text{supp}(\pi).$$

It is well-known (and one easily sees it) that ${}^t : R_F \rightarrow R_F$ preserves the cuspidal support of irreducible representations.

1.13. Denote the set of all unitarizable classes in \mathcal{D}_F by \mathcal{D}_F^u (so \mathcal{D}_F^u consists of the square-integrable classes). For $\delta \in D^u$ and a positive integer n denote

$$u(\delta, n) = L(v^{(n-1)/2} \delta, v^{(n-3)/2} \delta, \dots, v^{-(n-1)/2} \delta).$$

The following theorem describes irreducible unitarizable representations.

Theorem [Tadić 1986]. *Let*

$$\mathfrak{B}_F = \{u(\delta, n), v^\alpha u(\delta, n) \times v^{-\alpha} u(\delta, n) \mid \delta \in \mathfrak{D}_F^u, n \in \mathbb{N}, 0 < \alpha < \frac{1}{2}\}.$$

- (i) *If $\sigma_1, \dots, \sigma_k \in \mathfrak{B}_F$, then $\sigma_1 \times \dots \times \sigma_k$ is an irreducible unitarizable representation of some general linear group over F .*
- (ii) *If π is an irreducible unitarizable representation of some general linear group over F , there exist $\sigma_1, \dots, \sigma_m \in \mathfrak{B}_F$, unique up to a permutation, such that*

$$\pi \cong \sigma_1 \times \dots \times \sigma_m. \quad \square$$

1.14. Let $\rho \in \mathcal{C}_F$. Fix positive integers d and n . Set

$$\begin{aligned} \langle 1, d \rangle^\rho &= [v^{-(d-1)/2} \rho, v^{(d-1)/2} \rho] \in \mathcal{S}_F, \\ \langle n, d \rangle^\rho &= (\langle 1, d \rangle^{v^{-(n-1)/2} \rho}, \langle 1, d \rangle^{v^{1-(n-1)/2} \rho}, \dots, \langle 1, d \rangle^{v^{(n-1)/2} \rho}) \in M(\mathcal{S}_F). \end{aligned}$$

We take (formally) $\langle 0, d \rangle^\rho$ to be the empty multiset (then $L(\langle 0, d \rangle^\rho)$ is the one-dimensional representation of the trivial group $\text{GL}(0, F)$, which is the identity of R_F). Similarly, we also take $\langle n, 0 \rangle^\rho$ to be the empty multiset (so again $L(\langle n, 0 \rangle^\rho)$ is the identity in R). Observe that

$$\begin{aligned} [v^{k_1} \rho, v^{k_2} \rho] &= [k_1, k_2]^\rho = \langle 1, k_2 - k_1 + 1 \rangle^{v^{(k_1+k_2)/2} \rho}, \\ \langle 1, d \rangle^{v^\alpha \rho} &= [-(d-1)/2 + \alpha, (d-1)/2 + \alpha]^\rho. \end{aligned}$$

From §1.7 follows that for a character χ of F^\times

$$\chi L(\langle n, d \rangle^\rho) \cong L(\langle n, d \rangle^{\chi \rho}).$$

Further

$$u(\delta(\rho, d), n) = L(\langle n, d \rangle^\rho).$$

1.15. There are two important distinguished bases of R_F , irreducible representations and standard modules $\lambda(d), d \in M(\mathfrak{D}_F)$. The theorem in §1.13 implies that the following theorem solves the problem of expressing irreducible unitary representations in terms of standard modules, and irreducible unitary characters in terms of standard characters. It is convenient to present it in the following form:

Theorem [Tadić 1995]. *Let $n, d \in \mathbb{Z}, n, d \geq 1$. Let W_n be the group of permutations of the set $\{1, 2, \dots, n\}$. Set $W_n^{(d)} = \{w \in W_n; w(i) + d \geq i \text{ for all } 1 \leq i \leq n\}$. Then*

$$L(\langle [v^\rho, v^d \rho], \dots, [v^n \rho, v^{d+n-1} \rho] \rangle) = \sum_{w \in W_n^{(d)}} (-1)^w \prod_{i=1}^n \delta([v^i \rho, v^{w(i)+(d-1)} \rho])$$

in R_F , where $(-1)^w$ denotes the sign of the permutation w . □

1.16. From the following theorem one can get all the irreducible subquotients of ends of complementary series. (This is crucial information for determining the topology of the unitary dual.)

Theorem [Tadić 1987]. *For positive integers n and d , and $\rho \in \mathcal{C}_F$, we have*

$$\begin{aligned} & \nu^{1/2}L(\langle n, d \rangle^\rho) \times \nu^{-1/2}L(\langle n, d \rangle^\rho) \\ &= L(\langle n+1, d \rangle^\rho) \times L(\langle n-1, d \rangle^\rho) + L(\langle n, d+1 \rangle^\rho) \times L(\langle n, d-1 \rangle^\rho). \quad \square \end{aligned}$$

1.17. The following theorem implies that the involution t carries class of irreducible unitary representations to itself. Moreover, it implies an explicit formula for the involution on irreducible unitary representations.

Theorem [Tadić 1986]. *For positive integers n and d , and $\rho \in \mathcal{C}_F$, we have*

$$(L(\langle n, d \rangle^\rho))^t = L(\langle d, n \rangle^\rho).$$

2. Representations of $GL(n)$ over a division algebra A

We now introduce necessary notation and basic results on general linear groups over division algebras. Since the situation is very similar to the case of general linear groups over a field, we only point out the differences between these two cases. (More details can be found in [Tadić 1990].)

We assume that the characteristic of F is 0.

2.1. Let A be a finite dimensional division algebra over F whose center is F . Let

$$\dim_F A = d_A^2.$$

Let $\text{Mat}(n \times n, A)$ be the algebra of all $n \times n$ matrices with entries in A . Then $GL(n, A)$ is the group of invertible matrices with the natural topology. The commutator subgroup is denoted by $SL(n, A)$. We denote by

$$\det : GL(n, A) \rightarrow GL(1, A) / SL(1, A)$$

the determinant homomorphism, as defined by J. Dieudonné (for $n = 1$ this is just the quotient map). The kernel is $SL(n, A)$.

The reduced norm of $\text{Mat}(n \times n, A)$ will be denoted by RN . We identify characters of $GL(n, A)$ with characters of F^\times using RN . Let

$$\nu = |RN|_F : GL(n, A) \rightarrow \mathbb{R}^\times.$$

2.2. We now comment on modifications that need to be made to the material in Sections 1.3 to 1.12 so it applies also to the case of general linear groups over division algebras. (More details can be found in [Tadić 1990].) Only small modifications are required for §1.3–§1.8; we deal with them first.

- 1.3: We define by the same formula as in §1.3 the multiplication \times of smooth representations of general linear groups over A . Again as in §1.3, we have $(\pi_1 \times \pi_2) \times \pi_3 \cong \pi_1 \times (\pi_2 \times \pi_3)$.
- 1.4: In §2.1 we identified characters of $\mathrm{GL}(n, A)$ with characters of F^\times using the reduced norm RN . The character identified with $|\cdot|_F$ was again denoted by ν . Again, for a character χ of F^\times , we have $\chi(\pi_1 \times \pi_2) \cong (\chi\pi_1) \times (\chi\pi_2)$.
- 1.5: Define

$$R_{n,A} = \mathrm{Groth} \mathrm{GL}(n, A)$$

(recall that $R_F = \bigoplus_{n \geq 0} R_{n,F}$). One defines on R_A the structure of an (associative, commutative) ring in the same way as in §1.5 was done for the field case. Also, characters of F^\times lift to automorphisms of R_A (as in §1.5).

- 1.6: All essentially square-integrable classes in $\bigcup_{n \geq 1} \mathrm{GL}(n, A)^\sim$ are denoted by \mathcal{D}_A . One defines $e(\delta)$ and δ^u for $\delta \in \mathcal{D}_A$ as in §1.6.
- 1.7: For $d \in M(\mathcal{D}_A)$ we define $\lambda(d)$ and $L(d)$ in the same way as in §1.7. The Langlands classification for general linear groups over division algebras have the same expression as in the field case. (The parameters are in $M(\mathcal{D}_A)$.)
Here R_A is a polynomial algebra over \mathcal{D}_A .
- 1.8: The proposition holds in the same form for general linear groups over division algebras; see [Tadić 1990, Proposition 2.3].

2.3. By [Deligne et al. 1984] there exists a bijection

$$\mathrm{JL}_p : \mathcal{D}(\mathrm{GL}(p, A)) \rightarrow \mathcal{D}(\mathrm{GL}(pd_A, F)), \quad \delta' \leftrightarrow \delta,$$

such that characters $\Theta_{\delta'}$ and Θ_δ satisfy

$$(-1)^p \Theta_{\delta'}(g') = (-1)^{pd_A} \Theta_\delta(g)$$

whenever g' and g have same characteristic polynomials, and when this polynomial is separable.

This bijection is uniquely determined by the character requirement above and is called the Jacquet–Langlands correspondence between irreducible essentially square-integrable representations of $\mathrm{GL}(p, A)$ and $\mathrm{GL}(pd_A, F)$.

This bijection commutes with twisting with characters (see [Badulescu 2002]).

Take $\delta' \in \mathcal{D}(\mathrm{GL}(p, A))$ cuspidal and let δ' correspond to $\delta \in \mathcal{D}(\mathrm{GL}(pd_A, F))$ by the correspondence above. We know that

$$\delta = \delta(\rho, q)$$

for some positive integer q which divides pd_A and for some irreducible cuspidal representation ρ of $\mathrm{GL}((pd_A)/q, F)$. Further, it is known that $q \mid d_A$ and that p is relatively prime to q (pd_A is the lowest common multiple of d_A and $(pd_A)/q$).

We define

$$s_{\delta'} = q,$$

and a character

$$v_{\delta'} = v^{s_{\delta'}}$$

of F^\times . As we noted in §2.1, we use RN to identify the characters of F^\times with characters of $\mathrm{GL}(n, A)$. Therefore, we can view $v_{\delta'}$ as a character of any $\mathrm{GL}(k, A)$. (Note that in this definition, δ' is cuspidal; soon we shall give a definition also in the case that δ' is essentially square-integrable.)

2.4. We continue with the modifications required so the material in §1.9 to §1.12 can apply to general linear groups over division algebras.

1.9: We denote by \mathcal{C}_A the set of equivalence classes of irreducible cuspidal representation of all $\mathrm{GL}(n, A)$, $n \geq 1$. For $\rho' \in \mathcal{C}_A$ and $k \in \mathbb{Z}$, $k \geq 0$, the set

$$[\rho', v_{\rho'}^k \rho'] = \{\rho', v_{\rho'} \rho', v_{\rho'}^2 \rho', \dots, v_{\rho'}^k \rho'\}$$

is called a segment of irreducible cuspidal representations of general linear groups over division algebras. In this case, we also denote the segment $[v_{\rho'}^{k_1} \rho', v_{\rho'}^{k_2} \rho']$ (for $k_1, k_2 \in \mathbb{R}$, $k_2 - k_1 \in \mathbb{Z}$ and $k_2 - k_1 \geq 0$) by

$$[k_1, k_2]^{\rho'}.$$

The set of all such segments will be denoted by \mathcal{S}_A . The set of all finite multisets in \mathcal{S}_A will be denoted by $M(\mathcal{S}_A)$ and we consider the partial ordering \leq on $M(\mathcal{S}_A)$ introduced in [Tadić 1990, Section 4] (defined again by linking segments).

1.10: For $\Delta' = [\rho', v_{\rho'}^k \rho'] = \{\rho', v_{\rho'} \rho', v_{\rho'}^2 \rho', \dots, v_{\rho'}^k \rho'\} \in \mathcal{S}_A$, the representation $\rho' \times v_{\rho'} \rho' \times v_{\rho'}^2 \rho' \times \dots \times v_{\rho'}^k \rho'$ has a unique irreducible quotient, denoted by $\delta([\rho', v_{\rho'}^k \rho'])$, and it has a unique irreducible subrepresentation, denoted by $\delta([\rho', v_{\rho'}^k \rho'])^t$.

The mapping

$$\delta : \mathcal{S}_A \rightarrow \mathcal{D}_A, \quad \Delta' \mapsto \delta(\Delta'),$$

is a bijection. If we let

$$\delta(\rho', n) = \delta([- (n - 1) / 2, (n - 1) / 2]^{\rho'}),$$

then we can restate this as follows: $(\rho', n) \mapsto \delta(\rho', n)$ is a bijection of $\mathcal{C}_A \times \mathbb{N}$ onto \mathcal{D}_A .

We define $v_{\delta(\rho', n)}$ to be $v_{\rho'}$, i.e.

$$v_{\delta(\rho', n)} = v_{\rho'}.$$

We lift $\Delta' \mapsto \delta(\Delta')$ naturally to a bijection $M(\delta) : M(\mathcal{S}_A) \rightarrow M(\mathcal{D}_A)$. This gives the Langlands classification for general linear groups over a division algebra A in terms of $M(\mathcal{S}_A)$. For $a \in M(\mathcal{S}_A)$ let $L(a) = L(M(\delta)(a))$ and $\lambda(a) = \lambda(M(\delta)(a))$ as before.

- 1.11: Since R_A is a polynomial algebra over $\delta(\Delta')$, $\Delta' \in \mathcal{S}_A$, the mapping $\delta(\Delta') \mapsto \delta(\Delta')^t$, $\Delta \in \mathcal{S}_A$, extends uniquely to the ring morphism $^t : R_A \rightarrow R_A$, which carries irreducible representations into irreducible ones; see [Aubert 1995; Schneider and Stuhler 1997]. This homomorphism of rings is an involution. Again, for a character χ of F^\times , $^t : R_A \rightarrow R_A$ and $\chi : R_A \rightarrow R_A$ commute.
- 1.12: One defines the cuspidal support of an irreducible representation in the same way as before (it is an element of $M(\mathcal{C}_A)$). The involution t preserves the cuspidal support.

2.5. Let \mathcal{D}_A^u be the set of all the unitarizable classes in \mathcal{D}_A . Let

$$u(\delta', n) = L(v_{\delta'}^{(n-1)/2} \delta', v_{\delta'}^{(n-3)/2} \delta', \dots, v_{\delta'}^{-(n-1)/2} \delta').$$

for $\delta' \in \mathcal{D}_A^u$ and a positive integer n .

We first recall a conjecture from [Tadić 1990]:

- (U0) If π_1 and π_2 are irreducible unitarizable representations of general linear groups over A , then $\pi_1 \times \pi_2$ is irreducible.

Now [Tadić 1990, Section 6] and [Badulescu and Renard 2004] imply:

Proposition. Assume that (U0) holds. Let

$$\mathcal{B}_A = \{u(\delta', n), v_{\delta'}^\alpha u(\delta', n) \times v_{\delta'}^{-\alpha} u(\delta', n) \mid \delta' \in \mathcal{D}_A^u, n \in \mathbb{N}, 0 < \alpha < \frac{1}{2}\}.$$

Then

- (i) If $\sigma_1, \dots, \sigma_k \in \mathcal{B}_A$, then $\sigma_1 \times \dots \times \sigma_k$ is an irreducible unitarizable representation of some general linear group over A .
- (ii) If π is an irreducible unitarizable representation of some general linear group over A , then there exist $\sigma_1, \dots, \sigma_m \in \mathcal{B}_A$, unique up to a permutation, such that

$$\pi \cong \sigma_1 \times \dots \times \sigma_m. \quad \square$$

2.6. For $\rho' \in \mathcal{C}_A$ and positive integers d, n , set

$$\langle 1, d \rangle^{\rho'} = [v_{\rho'}^{-(d-1)/2} \rho', v_{\rho'}^{(d-1)/2} \rho'] \in \mathcal{S}_A,$$

$$\langle n, d \rangle^{\rho'} = (\langle 1, d \rangle^{\rho'} v_{\rho'}^{-(n-1)/2}, \langle 1, d \rangle^{\rho'} v_{\rho'}^{1-(n-1)/2}, \dots, \langle 1, d \rangle^{\rho'} v_{\rho'}^{(n-1)/2}) \in M(\mathcal{S}_A).$$

Further, $\langle 0, d \rangle^{\rho'}$ and $\langle n, 0 \rangle^{\rho'}$ are empty multisets (so $L(\langle 0, d \rangle^{\rho'})$ and $L(\langle n, 0 \rangle^{\rho'})$ are both the identity in R_A). As before, we have

$$\begin{aligned} [v_{\rho'}^{k_1} \rho', v_{\rho'}^{k_2} \rho'] &= [k_1, k_2]^{\rho'} = \langle 1, k_2 - k_1 + 1 \rangle^{\nu_{\rho'}^{(k_1+k_2)/2} \rho'}, \\ \langle 1, d \rangle^{\nu_{\rho'}^{\alpha} \rho'} &= [-(d-1)/2 + \alpha, (d-1)/2 + \alpha]^{\rho'}. \end{aligned}$$

For a character χ of F^\times we have $\chi L(\langle n, d \rangle^{\rho'}) \cong L(\langle n, d \rangle^{\chi \rho'})$. Also

$$u(\delta(\rho', d), n) = L(\langle n, d \rangle^{\rho'}).$$

3. Involution and unitarity on unitary duals of $GL(n, A)$

3.1. We call an irreducible representation π of a general linear group over A rigid if

$$e(\rho')/s_{\rho'} \in \frac{1}{2}\mathbb{Z}$$

for all ρ' in the cuspidal support of π .

Lemma. Assume that (U0) holds. For $d, n \in \mathbb{N}$, $\rho' \in \mathcal{C}_A$ we have

$$L(\langle n, d \rangle^{\rho'})^t = L(\langle d, n \rangle^{\rho'}).$$

Proof. Since t commutes with character automorphisms $\chi : R_A \rightarrow R_A$, it is enough to prove the equality for unitary $\rho' \in \mathcal{C}_A$.

The proof goes in several steps.

First we prove that $L(\langle n, d \rangle^{\rho'})^t$ is unitarizable. We prove it by induction with respect to n . For $n = 1$ we know $L(\langle 1, d \rangle^{\rho'})^t = L(\langle d, 1 \rangle^{\rho'})$, which is unitarizable by the proposition in §2.5.

Let $n \geq 1$ and suppose that we have proved the unitarity of the representations $L(\langle n, d \rangle^{\rho'})^t$ for that n . The division algebra version of the proposition from §1.8 (see §2.2), together with (U0), implies that

$$L(\langle n+1, d \rangle^{\rho'}) \times L(\langle n-1, d \rangle^{\rho'}) \leq \nu_{\rho'}^{-1/2} L(\langle n, d \rangle^{\rho'}) \times \nu_{\rho'}^{1/2} L(\langle n, d \rangle^{\rho'}).$$

Applying the involution $^t : R_A \rightarrow R_A$ to this relation, we get

$$L(\langle n+1, d \rangle^{\rho'})^t \times L(\langle n-1, d \rangle^{\rho'})^t \leq \nu_{\rho'}^{-1/2} L(\langle n, d \rangle^{\rho'})^t \times \nu_{\rho'}^{1/2} L(\langle n, d \rangle^{\rho'})^t.$$

First observe that $L(\langle n, d \rangle^{\rho'})^t$ is rigid (since t preserves the cuspidal support). This fact, the unitarity of $L(\langle n, d \rangle^{\rho'})^t$ and The proposition in §2.5 imply that $L(\langle n, d \rangle^{\rho'})^t$ is a product of elements of the form $L(\langle n', d' \rangle^{\rho''})$, where $\rho'' \in \mathcal{C}_A$ are unitarizable. The same proposition implies that all representations

$$\nu_{\rho''}^{-\alpha} L(\langle n', d' \rangle^{\rho''}) \times \nu_{\rho''}^{\alpha} L(\langle n', d' \rangle^{\rho''}), \quad 0 < \alpha < \frac{1}{2},$$

are unitarizable. This implies that the representations

$$v_{\rho'}^{-\alpha} L(\langle n, d \rangle^{\rho'})^t \times v_{\rho'}^{\alpha} L(\langle n, d \rangle^{\rho'})^t, \quad 0 < \alpha < \frac{1}{2},$$

are also unitarizable. Recall that all the irreducible subquotients at the end of complementary series are unitarizable ([Miličić 1973], see also [Tadić 1986] and [Tadić 1988]). This implies that all the irreducible subquotients of $v_{\rho'}^{-1/2} L(\langle n, d \rangle^{\rho'})^t \times v_{\rho'}^{1/2} L(\langle n, d \rangle^{\rho'})^t$ are unitarizable. In particular,

$$L(\langle n+1, d \rangle^{\rho'})^t \times L(\langle n-1, d \rangle^{\rho'})^t$$

is unitarizable (and irreducible, since t carries irreducible representations to the irreducible ones).

For an irreducible representation π denote by π^+ the Hermitian contragredient of π . Then $\pi \mapsto \pi^+$ lifts to an automorphism of R_A . Observe that t carries the class of irreducible Hermitian representations to itself, since the automorphisms t and $\pi \mapsto \pi^+$ of R_A commute (one directly checks this on generators). Therefore, $L(\langle n+1, d \rangle^{\rho'})^t \otimes L(\langle n-1, d \rangle^{\rho'})^t$ is Hermitian. Now (d) in section 3 of [Tadić 1993] implies that $L(\langle n+1, d \rangle^{\rho'})^t \otimes L(\langle n-1, d \rangle^{\rho'})^t$ is (irreducible) unitarizable, which implies that $L(\langle n+1, d \rangle^{\rho'})^t$ is unitarizable. Therefore, we have proved the inductive step.

So, we have proved that representations $L(\langle n, d \rangle^{\rho'})^t$ for $\rho' \in \mathcal{C}_A$ unitarizable, are unitarizable in general.

We will now get an explicit formula for $L(\langle n, d \rangle^{\rho'})^t$.

First note that $L(\langle n, d \rangle^{\rho'})$ is not induced from a proper parabolic subgroup by an irreducible unitarizable representation (see Proposition 2.5). Therefore, $L(\langle n, d \rangle^{\rho'})^t$ is also not induced in that way. The proposition in §2.5 implies that $(L(\langle n, d \rangle^{\rho'}))^t = L(\langle n', d' \rangle^{\rho''})$ for some n', d' and ρ'' . Since t preserves the cuspidal support, one gets directly $\rho' \cong \rho''$ and $\{n, d\} = \{n', d'\}$. This implies the lemma if $n = d$.

It remains to consider the case $n \neq d$. Actually, in this case it is enough to prove

$$L(\langle n, d \rangle^{\rho'})^t \neq L(\langle n, d \rangle^{\rho'}).$$

Since t is an involution, our previous discussion implies that it is enough to prove this relation in the case

$$d < n.$$

We do this by induction on d . For $d = 1$,

$$L(\langle n, 1 \rangle^{\rho'})^t = L(\langle 1, n \rangle^{\rho'}),$$

which is different from $L(\langle n, 1 \rangle^{\rho'})$ since $n > 1$.

Suppose $d \geq 1$ and that we have proved the claim for $d' \leq d$. Let $d + 1 < n$. We have already observed that $L(\langle n + 1, d \rangle^{\rho'}) \times L(\langle n - 1, d \rangle^{\rho'}) \leq v_\rho^{-1/2} L(\langle n, d \rangle^{\rho'}) \times v_\rho^{1/2} L(\langle n, d \rangle^{\rho'})$. Applying t to this relation and using the inductive assumption, we get

$$L(\langle n + 1, d \rangle^{\rho'})^t \times L(\langle d, n - 1 \rangle^{\rho'}) \leq v_\rho^{-1/2} L(\langle d, n \rangle^{\rho'}) \times v_\rho^{1/2} L(\langle d, n \rangle^{\rho'}).$$

Suppose

$$L(\langle n + 1, d \rangle^{\rho'})^t = L(\langle n + 1, d \rangle^{\rho'}).$$

Then

$$L(\langle n + 1, d \rangle^{\rho'}) \times L(\langle d, n - 1 \rangle^{\rho'}) \leq v_\rho^{-1/2} L(\langle d, n \rangle^{\rho'}) \times v_\rho^{1/2} L(\langle d, n \rangle^{\rho'}).$$

Then by the definition of the ordering \leq on $M(\mathcal{P}_A)$, we can not have on the left hand side more segments than on the right hand side (since ordering is generated by linking segments). This implies $n + 1 + d \leq 2d$, i.e., $n + 1 \leq d$ which implies $n < d$. This contradicts $d + 1 < n$. Thus $L(\langle n + 1, d \rangle^{\rho'})^t \neq L(\langle n + 1, d \rangle^{\rho'})$, what we needed to prove. \square

3.2. Corollary. *Assume that (U0) holds. Then t carries the class of irreducible unitary representations to itself.*

4. On ends of complementary series of $GL(n, A)$; character identities

4.1. The following proposition describes irreducible subquotients in the ends of complementary series. Besides the fact that this is crucial information for determining the topology of the unitary dual, this result (essentially character identity) will be crucial for us in obtaining formulas for (characters of) irreducible unitary representations in terms of (characters of) standard modules.

Proposition. *Assume that (U0) holds. Then for $n, d \in \mathbb{N}$, $\rho' \in \mathcal{C}_A$ we have in R_A*

$$v_{\rho'}^{-1/2} L(\langle n, d \rangle^{\rho'}) \times v_{\rho'}^{1/2} L(\langle n, d \rangle^{\rho'}) = L(\langle n - 1, d \rangle^{\rho'}) \times L(\langle n + 1, d \rangle^{\rho'}) + L(\langle n, d - 1 \rangle^{\rho'}) \times L(\langle n, d + 1 \rangle^{\rho'}).$$

Proof. It is enough to prove the equality for ρ' unitarizable. Further, by [Tadić 1990, Proposition 4.3], it is enough to prove the proposition for $n \geq 2$. Applying involution t , we conclude that it is enough to consider only the case $d \geq 2$.

Let

$$\pi = v_{\rho'}^{-1/2} L(\langle n, d \rangle^{\rho'}) \times v_{\rho'}^{1/2} L(\langle n, d \rangle^{\rho'}).$$

We know that $L(\langle n - 1, d \rangle^{\rho'}) \times L(\langle n + 1, d \rangle^{\rho'})$ is a subquotient of multiplicity one in π (see §2.2).

Applying the same argument to $\pi' = v_{\rho'}^{-1/2} L(\langle d, n \rangle^{\rho'}) \times v_{\rho'}^{1/2} L(\langle d, n \rangle^{\rho'})$, we get that $L(\langle d-1, n \rangle^{\rho'}) \times L(\langle d+1, n \rangle^{\rho'})$ is a subquotient of π' of multiplicity one.

Applying t to π' , we get from the lemma in §3.1 that $L(\langle n, d-1 \rangle^{\rho'}) \times L(\langle n, d+1 \rangle^{\rho'})$ is a subquotient of $\pi = (\pi')^t$ of multiplicity one. Therefore, to complete the proof, it is enough to prove that there are no additional irreducible subquotients besides these two.

Let σ be an irreducible subquotient of π different from the two subquotients above. Since π is an end of complementary series, σ must be unitarizable. Since π is rigid, σ must be rigid. This easily implies that

$$\sigma = L(\langle n_1, d_1 \rangle^{\rho'}) \times \dots \times L(\langle n_k, d_k \rangle^{\rho'})$$

for some n_i 's and d_i 's. After renumbering, we can (and shall) assume that

$$n_1 + d_1 \geq n_2 + d_2 \geq \dots \geq n_k + d_k.$$

Look at the cuspidal representation $v_{\rho'}^{-(n+d)/2+1-1/2} \rho' = v_{\rho'}^{-(n+d)/2+1/2} \rho'$. This is the first representation (from the negative left hand side) in the cuspidal support of π . Then the cuspidal support tells

$$n_1 + d_1 = n + d + 1.$$

(We must have $n_1 + d_1 > n_2 + d_2$, since the multiplicity of $v_{\rho'}^{-(n+d)/2+1/2} \rho'$ in the cuspidal support of π is one.)

The rules for linking segments imply

$$d \leq d_1.$$

(Since $v_{\rho'}^{-(n+d)/2+1/2} \rho'$ is the left end of only one segment in π , and there are no segments which are more to the left, the segment starting with $v_{\rho'}^{-(n+d)/2+1/2} \rho'$ must be at least of length d .) Applying t , invoking the lemma in §3.1, and repeating the preceding argument, we get

$$n \leq n_1.$$

The three relations above imply

$$(n_1, d_1) = (n+1, d) \quad \text{or} \quad (n_1, d_1) = (n, d+1),$$

that is,

$$L(\langle n_1, d_1 \rangle^{\rho'}) = L(\langle n+1, d \rangle^{\rho'}) \quad \text{or} \quad L(\langle n_1, d_1 \rangle^{\rho'}) = L(\langle n, d+1 \rangle^{\rho'}).$$

Now the first remaining representation in the cuspidal support is

$$v_{\rho'}^{-(n+d)/2+1/2+1} \rho = v_{\rho'}^{-(n+d)/2+3/2} \rho.$$

(As above, looking at the cuspidal support of π , we can conclude that $n_2 + d_2 > n_3 + d_3$ if $k \geq 3$.) This implies

$$n_2 + d_2 = n + d - 1.$$

Now looking at the rules for linking segments, one gets directly

$$d - 1 \leq d_2.$$

(Note that $v_{\rho'}^{-(n+d)/2+3/2} \rho'$ must be the beginning of a segment in $\langle n_2, d_2 \rangle^{\rho'}$, and the shortest segment that can have this beginning is of length $d - 1$, which one gets by intersecting the most left segment in π with the segment in π starting at $v_{\rho'}^{-(n+d)/2+1/2} \rho'$.)

Repeating the preceding argument in the case of π^t and using the lemma in §3.1 we get

$$n - 1 \leq n_2.$$

The three relations above imply

$$(n_2, d_2) = (n - 1, d) \quad \text{or} \quad (n_2, d_2) = (n, d - 1),$$

that is,

$$L(\langle n_2, d_2 \rangle^{\rho'}) = L(\langle n - 1, d \rangle^{\rho'}) \quad \text{or} \quad L(\langle n_2, d_2 \rangle^{\rho'}) = L(\langle n, d - 1 \rangle^{\rho'}).$$

We now have four possibilities for the first two factors of σ . We shall analyze two possibilities. Let

$$\sigma' = \begin{cases} L(\langle n_3, d_3 \rangle^{\rho'}) \times \cdots \times L(\langle n_k, d_k \rangle^{\rho'}) & \text{if } k \geq 3, \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that σ is isomorphic to

$$L(\langle n + 1, d \rangle^{\rho'}) \times L(\langle n, d - 1 \rangle^{\rho'}) \times \sigma' \quad \text{or} \quad L(\langle n, d + 1 \rangle^{\rho'}) \times L(\langle n - 1, d \rangle^{\rho'}) \times \sigma'.$$

The first representation cannot be a subquotient of π since it corresponds to at least $2n + 1$ segments, while π is defined by $2n$ segments. For the second representation, observe that

$$(L(\langle n, d + 1 \rangle^{\rho'}) \times L(\langle n - 1, d \rangle^{\rho'}) \times \sigma')^t = L(\langle d + 1, n \rangle^{\rho'}) \times L(\langle d, n - 1 \rangle^{\rho'}) \times (\sigma')^t$$

is a subquotient of $\pi^t = v_{\rho'}^{-1/2} L(\langle d, n \rangle^{\rho'}) \times v_{\rho'}^{1/2} L(\langle d, n \rangle^{\rho'})$. This is impossible, for the same reason as in the first case.

Therefore, the only two remaining possibilities for σ are

$$L(\langle n - 1, d \rangle^{\rho'}) \times L(\langle n + 1, d \rangle^{\rho'}) \times \sigma' \quad \text{and} \quad L(\langle n, d - 1 \rangle^{\rho'}) \times L(\langle n, d + 1 \rangle^{\rho'}) \times \sigma'.$$

But since both

$$L(\langle n-1, d \rangle^{\rho'}) \times L(\langle n+1, d \rangle^{\rho'}) \quad \text{and} \quad L(\langle n, d-1 \rangle^{\rho'}) \times L(\langle n, d+1 \rangle^{\rho'})$$

have in their cuspidal supports $2nd$ representations (counted with multiplicities), which is exactly the number of representations in the cuspidal support of π (counted with multiplicities), we conclude that $\sigma' = 1$, as needed. \square

5. On characters of irreducible unitary representations of $\mathrm{GL}(n, A)$

5.1. The set of nonnegative integers is denoted by \mathbb{Z}_+ . Fix $\rho \in \mathcal{C}_F$ and $\rho' \in \mathcal{C}_A$. Let $R_F(\rho)$ be the subalgebra of R_F generated by

$$\{\delta([v^{k_1}\rho, v^{k_2}\rho]) \mid k_1, k_2 \in \frac{1}{2}\mathbb{Z}, k_2 - k_1 \in \mathbb{Z}_+\}$$

and $R_A(\rho')$ the subalgebra of R_A generated by

$$\{\delta([v_{\rho'}^{k_1}\rho', v_{\rho'}^{k_2}\rho']) \mid k_1, k_2 \in \frac{1}{2}\mathbb{Z}, k_2 - k_1 \in \mathbb{Z}_+\}.$$

Clearly, both algebras are polynomial over these sets of generators. Define an algebra isomorphism $\Psi_{\rho, \rho'} : R_F(\rho) \rightarrow R_A(\rho')$ by

$$\Psi_{\rho, \rho'} : \delta([v^{k_1}\rho, v^{k_2}\rho]) \mapsto \delta([v_{\rho'}^{k_1}\rho', v_{\rho'}^{k_2}\rho'])$$

for all $k_1, k_2 \in \frac{1}{2}\mathbb{Z}, k_2 - k_1 \in \mathbb{Z}_+$.

Lemma. *If we assume (U0), then*

$$\Psi_{\rho, \rho'} : (L(\langle n, d \rangle^{v^k \rho}) = L(\langle n, d \rangle^{v_{\rho'}^k \rho'})$$

for all $n, d \in \mathbb{N}$ and $k \in \frac{1}{2}\mathbb{Z}$.

Proof. We work by induction on n . For $n = 1$ (and all d) the lemma holds by definition of $\Psi_{\rho, \rho'}$ (see §1.14 and §2.6). Fix $n \geq 1$ and assume that the formula of the lemma holds for all $n' \leq n$. Applying $\Psi_{\rho, \rho'}$ to the formula of the theorem in §1.15 (with $v^k \rho$ instead of ρ) and using the inductive assumption we get

$$\begin{aligned} & v_{\rho'}^{1/2} L(\langle n, d \rangle^{v_{\rho'}^k \rho'}) \times v_{\rho'}^{-1/2} L(\langle n, d \rangle^{v_{\rho'}^k \rho'}) \\ &= \Psi_{\rho, \rho'}(L(\langle n+1, d \rangle^{v^k \rho})) \times L(\langle n-1, d \rangle^{v_{\rho'}^k \rho'}) + L(\langle n, d+1 \rangle^{v_{\rho'}^k \rho'}) \times L(\langle n, d-1 \rangle^{v_{\rho'}^k \rho'}). \end{aligned}$$

Subtracting this from the formula of the proposition in §4.1 (with $v_{\rho'}^k \rho'$ instead of ρ' in the formula) and using the fact that R_A is an integral domain, one gets $\Psi_{\rho, \rho'}(L(\langle n+1, d \rangle^{v^k \rho})) = L(\langle n+1, d \rangle^{v_{\rho'}^k \rho'})$, as needed. \square

5.2. From the preceding lemma and the theorem in §1.15, we immediately get:

Proposition. Assume that (U0) holds. Let $\rho' \in \mathcal{C}_A$ and $n, d \in \mathbb{Z}, n, d \geq 1$. Let W_n be the group of permutations of the set $\{1, 2, \dots, n\}$. Set $W_n^{(d)} = \{w \in W_n; w(i) + d \geq i \text{ for all } 1 \leq i \leq n\}$. Then we have in R_A

$$L(([\nu_{\rho'} \rho', \nu_{\rho'}^d \rho'], [\nu_{\rho'}^2 \rho', \nu_{\rho'}^{d+1} \rho'], \dots, [\nu_{\rho'}^n \rho', \nu_{\rho'}^{d+n-1} \rho'])) \\ = \sum_{w \in W_n^{(d)}} (-1)^w \prod_{i=1}^n \delta([\nu_{\rho'}^i \rho', \nu_{\rho'}^{w(i)+d-1} \rho']). \quad \square$$

6. Jacquet–Langlands correspondence

6.1. A. I. Badulescu [2002] studied very natural extensions of Jacquet–Langlands correspondences. We recall here some of his considerations (in a slightly different notation).

In §2.3 we recalled the Jacquet–Langlands correspondences

$$JL_p : \mathcal{D}(\mathrm{GL}(p, A)) \rightarrow \mathcal{D}(\mathrm{GL}(pd_A, F)),$$

which are uniquely determined by the requirement that the characters $\Theta_{\delta'}$ and $\Theta_{JL_p(\delta')}$ satisfy

$$(-1)^p \Theta_{\delta'}(g') = (-1)^{pd_A} \Theta_{JL_p(\delta')}(g)$$

whenever g' and g have the same characteristic polynomials, and when this polynomial is separable.

The correspondences above are bijections, so instead of the correspondences JL_p , we could consider their inverses JL_p^{-1} .

The mappings $JL_p, p \geq 1$, define in a natural way an injective mapping

$$JL : \mathcal{D}_A \rightarrow \mathcal{D}_F.$$

Since the algebras R_A and R_F are polynomial over \mathcal{D}_A and \mathcal{D}_F respectively, JL_p can be uniquely extended to a ring homomorphism of R_A into R_F , which will be again denoted by

$$JL : R_A \rightarrow R_F.$$

Clearly, the extension is also injective.

The homomorphism JL carries $\mathrm{Groth} \mathrm{GL}(p, A)$ to $\mathrm{Groth} \mathrm{GL}(pd_A, F)$, and we denote this restriction again by

$$JL_p : \mathrm{Groth} \mathrm{GL}(p, A) \rightarrow \mathrm{Groth} \mathrm{GL}(pd_A, F).$$

Then this extended JL_p again satisfies the relation

$$(-1)^p \Theta_{\pi'}(\pi') = (-1)^{pd_A} \Theta_{JL_p(\pi')}(g)$$

for any $\pi \in \mathrm{GL}(p, A)^\sim$.

Let

$$\mathcal{D}_F^{(d_A)} = \bigcup_{p \geq 1} \mathcal{D}(\mathrm{GL}(pd_A, F)).$$

Then JL defines a bijection of \mathcal{D}_A onto $\mathcal{D}_F^{(d_A)}$. Denote the inverse mapping by

$$\mathrm{LJ} : \mathcal{D}_F^{(d_A)} \rightarrow \mathcal{D}_A.$$

There exists a unique ring homomorphism $R_F \rightarrow R_A$ which extends LJ and which sends all the elements from $\mathcal{D}_F \setminus \mathcal{D}_F^{(d_A)}$ to $0 \in R_A$. This extension will be denoted again by

$$\mathrm{LJ} : R_F \rightarrow R_A.$$

If $d_A \mid m$, we denote by LJ_m the restriction

$$\mathrm{LJ}_m : \mathrm{Groth} \mathrm{GL}(m, F) \rightarrow \mathrm{Groth} \mathrm{GL}(m/d_A, A).$$

Otherwise, we take (formally) $\mathrm{LJ}_m = 0$ (as a mapping from $\mathrm{Groth} \mathrm{GL}(m, F)$ into R_A).

Let $I_{F,A}$ be the ideal in R_F generated by $\mathcal{D}_F \setminus \mathcal{D}_F^{(d_A)}$ (clearly, this ideal is graded). This is just the kernel of LJ. Therefore,

$$R_A \cong R_F / I_{F,A}.$$

Further, suppose that $\varphi \in \mathrm{Groth} \mathrm{GL}(m, F)$ is in $I_{F,A}$ and $d_A \mid m$. Then for regular semisimple $g \in \mathrm{GL}(m, F)$ we have $\Theta_\varphi(g) = 0$, where Θ_φ denotes the formal character of φ . Therefore,

$$(-1)^m \Theta_\varphi(g) = (-1)^{m/d_A} \Theta_{\mathrm{LJ}(\varphi)}(g')$$

whenever g and g' have the same characteristic polynomials, and when this polynomial is separable. Clearly,

$$\mathrm{LJ} \circ \mathrm{JL} = \mathrm{id}_{R_A}.$$

The correspondence $\mathrm{JL} : R_A \rightarrow R_F$, which we considered first, does not behave well with respect to irreducibility. Namely, one sees easily (as in the comments after [Badulescu 2002, Theorem 3.1]) that JL does not in general carry irreducible representations to irreducible ones (up to a sign). A similar situation happens with unitarity. In general even irreducible unitary representations are carried neither to irreducible unitary representations (up to a sign), nor to linear combinations of irreducible unitary representations.

Assuming (U0) holds, we shall see in the rest of the paper that the correspondence $\mathrm{LJ} : R_F \rightarrow R_A$ behave well with respect to irreducible unitary representations, i.e., that it carries such representations either again to the irreducible unitary representations (up to a sign) or to 0.

6.2. Let $\rho' \in \mathcal{C}_A$. Suppose that $\delta([\rho, v^{s_{\rho'}-1}\rho]) \in \mathcal{D}_F$ corresponds to ρ' under the Jacquet–Langlands correspondence (here $\rho \in \mathcal{C}_F$). Then $\delta([\rho, v^{s_{\rho'}k-1}\rho])$ corresponds to $\delta([\rho', v_{\rho'}^{k-1}\rho'])$.

6.3. Fix an irreducible cuspidal representation ρ of $\mathrm{GL}(m, F)$. Let $s'm$ be the smallest common multiple of m and d_A . The fact $s'm \mid d_A m$ implies

$$s' \mid d_A.$$

Note that $\delta([\rho, v^{s'-1}\rho])$ is an irreducible essentially square-integrable representation of $\mathrm{GL}(s'm, F)$. Therefore, it lifts under the Jacquet–Langlands correspondence to an irreducible essentially square-integrable representation ρ' in R_A . A short discussion implies that ρ' is cuspidal, and then $s_{\rho'} = s'$.

Now ρ' is a representation of $\mathrm{GL}(p, A)$, where $p = ms_{\rho'}/d_A$. Since $s'm$ is the smallest common multiple of m and d_A , this implies $(p, s_{\rho'}) = 1$ (if k were the greatest common divisor, then kd_A and km would divide $s'm$). Further, the smallest common multiple of d_A and $pd_A/s_{\rho'} = m$ is $s_{\rho'}m = pd_A$.

6.4. Assuming (U0) we shall compute $\mathrm{LJ}(\pi)$ for irreducible unitary representations of general linear groups over the field. Since LJ is a ring homomorphism, for this it will be enough to compute

$$\mathrm{LJ}(L(\langle r, d \rangle^\rho) \in R_A, \quad r, d \in \mathbb{N}.$$

Suppose that we are in the situation of §6.3, and suppose that $s_{\rho'} = 1$ (which means that ρ corresponds to ρ' under the Jacquet–Langlands correspondence). Then the theorem in §1.15 and the proposition in §5.2 directly imply

$$\mathrm{LJ}(L(\langle r, d \rangle^\rho)) = L(\langle r, d \rangle^{\rho'}).$$

It remains therefore to consider the case

$$s_{\rho'} \geq 2.$$

We shall assume this in the rest of the paper.

If $r = 1$, then we know $\mathrm{LJ}(L(\langle 1, d \rangle^\rho))$ by §6.2, so we can assume also $r \geq 2$.

To simplify the notation, we shall often denote below $s_{\rho'}$ by n :

$$s_{\rho'} = n.$$

7. Calculation of the Jacquet–Langlands correspondence in the unitary case

I: the case $r \leq d$ and $s_{\rho'} \mid d$

7.1. In this section, we assume $r \leq d$ and $s_{\rho'} \mid d$ (i.e., $n \mid d$).

We shall use below the following technical lemma, identical to [Tadić 1996, Lemma 3.1], whose statement we repeat here for convenience.

Denote by W'_r the group of permutations of $\{0, 1, \dots, r - 1\}$. The sign of a permutation w will be denoted by $(-1)^w$.

Lemma. Write $r = an + b$, with $a, b \in \mathbb{Z}$ such that $0 \leq b \leq n - 1$.

(i) Let

$$W'_r(n) = \{w \in W'_r; n \mid (w(i) - i) \text{ for all } 0 \leq i \leq r - 1\}.$$

For $0 \leq \ell \leq \min(n, r) - 1$, set

$$W'_r(n; \ell) = \{w \in W'_r(n); w(i) = i \text{ if } n \nmid (i - \ell)\}.$$

Then $W'_r(n)$ is a subgroup of W'_r , $W'_r(n; \ell)$ are subgroups of $W'_r(n)$ and $W'_r(n)$ is a direct product of $W'_r(n; \ell)$, for $\ell = 0, 1, 2, \dots, \min(n, r) - 1$.

(ii) Let $0 \leq \ell \leq b - 1$ (resp. $b \leq \ell \leq \min(n, r) - 1$). For $w \in W'_{a+1}$ (resp. $w \in W'_a$) define $w^* \in W'_r$ by

$$w^*(j) = \begin{cases} j & \text{if } n \nmid (j - \ell), \\ \ell + nw(i) & \text{if } j = \ell + ni. \end{cases}$$

Then $w \mapsto w^*$ is an isomorphism of W'_{a+1} (resp. W'_a) onto $W'_r(n; \ell)$. Further, $(-1)^w = (-1)^{w^*}$. □

7.2. Let

$$\Pi = L([\rho, v^{d-1}\rho], [v\rho, v^d\rho], \dots, [v^{r-1}\rho, v^{r-1+d-1}\rho]).$$

We now start the computation of

$$\begin{aligned} \text{LJ}(\Pi) &= \text{LJ}(L([\rho, v^{d-1}\rho], [v\rho, v^d\rho], \dots, [v^{r-1}\rho, v^{r-1+d-1}\rho])) \\ &= \sum_{w \in W'_r} (-1)^w \prod_{i=0}^{r-1} \text{LJ}(\delta([v^i\rho, v^{w(i)+(d-1)}\rho])) \\ &= \sum_{w \in W'_r(n)} (-1)^w \prod_{i=0}^{r-1} \text{LJ}(\delta([v^i\rho, v^{w(i)+(d-1)}\rho])), \end{aligned}$$

where we have used the fact that $n \mid d$ and $r \leq d$.

Write $r = an + b$ with $a, b \in \mathbb{Z}$ and $0 \leq b \leq n - 1$. We use the preceding lemma to modify the sum:

$$\text{LJ}(\Pi) = \sum_{w \in W'_r(n)} (-1)^w \prod_{i=0}^{r-1} \text{LJ}(\delta([v^i\rho, v^{w(i)+(d-1)}\rho]))$$

$$\begin{aligned}
&= \sum_{\substack{w_0, w_1, \dots, \\ w_{b-1} \in W'_{a+1}}} \sum_{\substack{w_b, w_{b+1}, \dots, \\ w_{\min(n,r)-1} \in W'_a}} \prod_{\ell=0}^{b-1} (-1)^{w_\ell} \prod_{i=0}^a \text{LJ}(\delta([v^{\ell+n} i \rho, v^{\ell+n} w_\ell(i)+(d-1) \rho])) \\
&\quad \times \prod_{\ell=b}^{\min(n,r)-1} (-1)^{w_\ell} \prod_{i=0}^{a-1} \text{LJ}(\delta([v^{\ell+n} i \rho, v^{\ell+n} w_\ell(i)+(d-1) \rho])) \\
&= \prod_{\ell=0}^{b-1} \sum_{w_\ell \in W'_{a+1}} (-1)^{w_\ell} \prod_{i=0}^a \text{LJ}(\delta([v^{\ell+n} i \rho, v^{\ell+n} w_\ell(i)+(d-1) \rho])) \\
&\quad \times \prod_{\ell=b}^{\min(n,r)-1} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{i=0}^{a-1} \text{LJ}(\delta([v^{\ell+n} i \rho, v^{\ell+n} w_\ell(i)+(d-1) \rho])).
\end{aligned}$$

Now assume that $\delta([\rho, v^{n-1} \rho])$ and ρ' correspond under the Jacquet–Langlands correspondence as in §6.2. Then

$$\begin{aligned}
\text{LJ}(\Pi) &= \left(\prod_{\ell=0}^{b-1} \sum_{w_\ell \in W'_{a+1}} (-1)^{w_\ell} \prod_{i=0}^a \delta([v^{\ell+n} i \rho', v^{\ell+n} i v_{\rho'}^{w_\ell(i)-i+(d/n-1)} \rho']) \right) \\
&\quad \times \prod_{\ell=b}^{\min(n,r)-1} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{i=0}^{a-1} \delta([v^{\ell+n} i \rho', v^{\ell+n} i v_{\rho'}^{w_\ell(i)-i+(d/n-1)} \rho']) \\
&= \left(\prod_{\ell=0}^{b-1} v_{\rho'}^{\ell/n} \sum_{w_\ell \in W'_{a+1}} (-1)^{w_\ell} \prod_{i=0}^a \delta([v_{\rho'}^i \rho', v_{\rho'}^{w_\ell(i)+(d/n-1)} \rho']) \right) \\
&\quad \times \prod_{\ell=b}^{\min(n,r)-1} v_{\rho'}^{\ell/n} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{i=0}^{a-1} \delta([v_{\rho'}^i \rho', v_{\rho'}^{w_\ell(i)+(d/n-1)} \rho']).
\end{aligned}$$

Note that $r \leq d$ implies $r - b \leq d$, which implies $a \leq d/n$.

If $b \geq 1$, then $r - b < d$, which implies $a < d/n$ and further $a + 1 \leq d/n$ (since $n \mid d$). Therefore

$$\begin{aligned}
\text{LJ}(\Pi) &= \prod_{\ell=0}^{b-1} v_{\rho'}^{\ell/n} L(([\rho', v_{\rho'}^{d/n-1} \rho'], [v_{\rho'} \rho', v_{\rho'}^{d/n} \rho'], \dots, [v_{\rho'}^a \rho', v_{\rho'}^{a+d/n-1} \rho'])) \\
&\quad \times \prod_{\ell=b}^{\min(n,r)-1} v_{\rho'}^{\ell/n} L(([\rho', v_{\rho'}^{d/n-1} \rho'], [v_{\rho'} \rho', v_{\rho'}^{d/n} \rho'], \dots, [v_{\rho'}^{a-1} \rho', v_{\rho'}^{a-1+d/n-1} \rho'])).
\end{aligned}$$

7.3. Now suppose that ρ is unitary and that $\delta(\rho, s_{\rho'}) = \delta(\rho, n)$ corresponds to ρ'' under the Jacquet–Langlands correspondence. Then ρ'' is unitary. Further

$$\begin{aligned} \rho'' &= \text{LJ}(\delta(\rho, n)) = \text{LJ}(v^{-(n-1)/2} \delta([\rho, v^{n-1} \rho])) = v^{-(n-1)/2} \text{LJ}(\delta([\rho, v^{n-1} \rho])) \\ &= v_{\rho'}^{-(n-1)/(2n)} \rho'. \end{aligned}$$

Note that $v_{\rho'} = v_{\rho''}$.

Now we compute (for $r \leq d$ and $n \mid d$)

$$\begin{aligned} &\text{LJ}(L(\langle r, d \rangle^{\rho})) \\ &= \text{LJ}\left(v^{-\frac{r+d}{2}+1} L([\rho, v^{d-1} \rho], [v\rho, v^d \rho], \dots, [v^{r-1} \rho, v^{r-1+d-1} \rho])\right) \\ &= v^{-\frac{r+d}{2}+1} \prod_{\ell=0}^{b-1} v_{\rho'}^{\frac{\ell}{n}} v_{\rho'}^{\frac{a+1+d/n}{2}-1} L(\langle a+1, d/n \rangle^{\rho'}) \\ &\quad \times \prod_{\ell=b}^{\min(n,r)-1} v_{\rho'}^{\frac{\ell}{n}} v_{\rho'}^{\frac{a+d/n}{2}-1} L(\langle a, d/n \rangle^{\rho'}). \end{aligned}$$

Since

$$v^{-\frac{r+d}{2}+1} = v_{\rho'}^{\frac{-r-d+2}{2n}} = v_{\rho'}^{\frac{-an-b-d+2}{2n}},$$

we have

$$\begin{aligned} &\text{LJ}(L(\langle r, d \rangle^{\rho})) \\ &= \prod_{\ell=0}^{b-1} v_{\rho'}^{\frac{-b+2+2\ell-n}{2n}} L(\langle a+1, d/n \rangle^{\rho'}) \prod_{\ell=b}^{\min(n,r)-1} v_{\rho'}^{\frac{-b+2+2\ell-2n}{2n}} L(\langle a, d/n \rangle^{\rho'}) \\ &= \prod_{\ell=0}^{b-1} v_{\rho'}^{\frac{-b+2+2\ell-n}{2n}} L(\langle a+1, d/n \rangle^{\rho'} v_{\rho'}^{\frac{n-1}{2n}} \rho'') \\ &\quad \times \prod_{\ell=b}^{\min(n,r)-1} v_{\rho'}^{\frac{-b+2+2\ell-2n}{2n}} L(\langle a, d/n \rangle^{\rho'} v_{\rho'}^{\frac{n-1}{2n}} \rho'') \\ &= \prod_{\ell=0}^{b-1} v_{\rho''}^{\frac{-b+1+2\ell}{2n}} L(\langle a+1, d/n \rangle^{\rho''}) \prod_{\ell=b}^{\min(n,r)-1} v_{\rho''}^{\frac{-b+1+2\ell-n}{2n}} L(\langle a, d/n \rangle^{\rho''}) \\ &= \prod_{\ell'=-\frac{b-1}{2}}^{\frac{b-1}{2}} v_{\rho''}^{\frac{\ell'}{n}} L(\langle a+1, d/n \rangle^{\rho''}) \prod_{\ell=b}^{\min(n,r)-1} v_{\rho''}^{\frac{(-b-n+1)/2+\ell}{n}} L(\langle a, d/n \rangle^{\rho''}). \end{aligned}$$

For an irreducible representation π of $\text{GL}(l, A)$, positive integer l and a non-negative integer k , define the string

$$\text{str}_{v_{\rho'}}(k, l, \pi)$$

as

$$(v_{\rho'}^{\frac{-(k-1)/2}{l}} \pi) \times (v_{\rho'}^{\frac{-(k-1)/2+1}{l}} \pi) \times (v_{\rho'}^{\frac{-(k-1)/2+2}{l}} \pi) \times \cdots \times (v_{\rho'}^{\frac{(k-1)/2}{l}} \pi).$$

If $k = 0$, we take the string to be the identity of R_A (i.e., the trivial representation of the trivial group $\text{GL}(0, A)$).

The next result follows directly from this:

Proposition. *Suppose that (U0) holds. Assume that $\delta(\rho, n)$ corresponds to $\rho'' \in \mathcal{C}_A$ under the Jacquet–Langlands correspondence. Let*

$$r \leq d, \quad n \mid d, \quad 1 \leq n.$$

Write

$$r = an + b, \quad a, b \in \mathbb{Z}, \quad 0 \leq b \leq n - 1.$$

Then

$$\begin{aligned} \text{LJ}(L(\langle r, d \rangle^\rho)) &= \text{str}_{v_{\rho''}}(b, n, L(\langle a + 1, d/n \rangle^{\rho''})) \\ &\quad \times \text{str}_{v_{\rho''}}(\min(n, r) - b, n, L(\langle a, d/n \rangle^{\rho''})). \quad \square \end{aligned}$$

Remark. Our definition of $\text{str}_{v_{\rho'}}(k, l, \pi)$ is more general than we need here. In this paper, whenever we use $\text{str}_{v_{\rho'}}(k, l, \pi)$, we have $l = s_{\rho'} = n$. Then $\text{str}_{v_{\rho'}}(k, n, \pi)$, which equals $\text{str}_v(k, 1, \pi)$ in this case, will be denoted simply by

$$\text{str}(k, \pi),$$

and we have

$$\text{str}(k, \pi) = (v^{-(k-1)/2} \pi) \times (v^{-(k-1)/2+1} \pi) \times (v^{-(k-1)/2+2} \pi) \times \cdots \times (v^{(k-1)/2} \pi),$$

with neither $v_{\rho'}$ nor $n = s_{\rho'}$ showing up on the right-hand side.

We shall use both notations, $\text{str}(k, \pi)$ and $\text{str}_{v_{\rho'}}(k, n, \pi)$. The later will be convenient when we study unitarity.

Now the proposition above can be rewritten:

Proposition'. *With the same assumptions as in the preceding proposition, we have*

$$\text{LJ}(L(\langle r, d \rangle^\rho)) = \text{str}(b, L(\langle a + 1, d/n \rangle^{\rho''})) \text{str}(\min(n, r) - b, L(\langle a, d/n \rangle^{\rho''})). \quad \square$$

We can rewrite this formula as

$$\begin{aligned} \text{LJ}(L(\langle r, d \rangle^\rho)) &= \text{str}(r - [r/n]n, L(\langle [r/n] + 1, d/n \rangle^{\rho''})) \\ &\quad \times \text{str}(\min(n, r) - r + [r/n]n, L(\langle [r/n], d/n \rangle^{\rho''})), \end{aligned}$$

where $[x]$ denotes the greatest integer which does not exceed x .

8. Unitarity of the Jacquet–Langlands correspondence of irreducible unitary representations, I

8.1. We now show that $\text{LJ}(L(\langle r, d \rangle^\rho))$, which we have computed in the previous section, is irreducible and unitary if $\rho \in \mathcal{C}_F$ is unitary. After this, we shall show that $\text{LJ}(v^\beta(L(\langle r, d \rangle^\rho) \times v^{-\beta}(L(\langle r, d \rangle^\rho)))$ is unitary for $0 < \beta < \frac{1}{2}$.

If $b = 0$, then $\text{str}_{v_{\rho''}}(b, n, L(\langle a + 1, d/n \rangle^{\rho''}))$ is irreducible unitary since it equals 1. Suppose $b \geq 1$. Then $0 \leq (b - 1)/(2n) < n/(2n) = \frac{1}{2}$, and we conclude that $\text{str}_{v_{\rho''}}(b, n, L(\langle a + 1, d/n \rangle^{\rho''}))$ is again irreducible unitary.

If $r < n$, then $\text{str}_{v_{\rho''}}(\min(n, r) - b, n, L(\langle a, d/n \rangle^{\rho''}))$ is irreducible unitary since it equals 1. Suppose $r \geq n$. Then

$$\text{str}_{v_{\rho''}}(\min(n, r) - b, n, L(\langle a, d/n \rangle^{\rho''})) = \text{str}_{v_{\rho''}}(n - b, n, L(\langle a, d/n \rangle^{\rho''})).$$

Since now $0 \leq (n - b - 1)/(2n) \leq (n - 1)/(2n) < \frac{1}{2}$, this implies that, once more, $\text{str}_{v_{\rho''}}(\min(n, r) - b, n, L(\langle a, d/n \rangle^{\rho''}))$ is irreducible unitary. Thus $\text{LJ}(L(\langle r, d \rangle^\rho))$ is irreducible unitary.

For $0 < \beta < \frac{1}{2}$, set

$$\pi(L(\langle r, d \rangle^\rho), \beta) = v^\beta L(\langle r, d \rangle^\rho) \times v^{-\beta} L(\langle r, d \rangle^\rho).$$

We show that $\text{LJ}(\pi(L(\langle r, d \rangle^\rho), \beta))$ is irreducible unitary. Observe that

$$\begin{aligned} & \text{LJ}(\pi(L(\langle r, d \rangle^\rho), \beta)) \\ &= v^\beta \text{LJ}(L(\langle r, d \rangle^\rho)) v^{-\beta} \text{LJ}(L(\langle r, d \rangle^\rho)) \\ &= v_{\rho''}^{\beta/n} \text{str}_{v_{\rho''}}(b, n, L(\langle a + 1, d/n \rangle^{\rho''})) \times v_{\rho''}^{-\beta/n} \text{str}_{v_{\rho''}}(b, n, L(\langle a + 1, d/n \rangle^{\rho''})) \\ & \quad \times v_{\rho''}^{\beta/n} \text{str}_{v_{\rho''}}(\min(n, r) - b, n, L(\langle a, d/n \rangle^{\rho''})) \\ & \quad \times v_{\rho''}^{-\beta/n} \text{str}_{v_{\rho''}}(\min(n, r) - b, n, L(\langle a, d/n \rangle^{\rho''})). \end{aligned}$$

This implies that it is enough to show unitarity (and irreducibility, which follows from unitarity) for

$$v_{\rho''}^{\beta/n} \text{str}_{v_{\rho''}}(b, n, L(\langle a + 1, d/n \rangle^{\rho''})) \times v_{\rho''}^{-\beta/n} \text{str}_{v_{\rho''}}(b, n, L(\langle a + 1, d/n \rangle^{\rho''}))$$

if $b \geq 1$, and for

$$v_{\rho''}^{\beta/n} \text{str}_{v_{\rho''}}(n - b, n, L(\langle a, d/n \rangle^{\rho''})) \times v_{\rho''}^{-\beta/n} \text{str}_{v_{\rho''}}(n - b, n, L(\langle a, d/n \rangle^{\rho''}))$$

if $r \geq n$. For the first representation, we need to show that $0 \leq (b - 1)/(2n) + \beta/n < \frac{1}{2}$ if $b \geq 1$; for the second, that $0 \leq (n - b - 1)/(2n) + \beta/n < \frac{1}{2}$ if $r \geq n$. This obviously holds since $\beta < \frac{1}{2}$.

9. Calculation of the Jacquet–Langlands correspondence in the unitary case II: the case $r \leq d$ and $s_{\rho'}$ not dividing d

9.1. In this section we assume that $r \leq d$ and $s_{\rho'} \nmid d$ (i.e., $n \nmid d$).

If $n \nmid rd$, then one sees directly that

$$\text{LJ} \left(L \left(([\rho, v^{d-1}\rho], [v\rho, v^d\rho], \dots, [v^{r-1}\rho, v^{r-1+d-1}\rho]) \right) \right) = 0.$$

Therefore, we need only to consider the case $n \mid rd$. Soon we shall see that a stronger assumption needs to be imposed to get a nonzero result.

Write $r = an + b$, with $a, b \in \mathbb{Z}$ and $0 \leq b \leq n - 1$. Now $n \mid rd$ implies $n \mid bd$. We continue to assume

$$n \geq 2.$$

If $r = 1$, then $\text{LJ}(L([\rho, v^{d-1}\rho])) = 0$ since $n \nmid d$. Therefore, we shall assume in the sequel that

$$r \geq 2.$$

9.2. The next result is a modification of [Tadić 1996, Lemma 4.1].

Lemma. *Suppose $n \nmid d$. If the set*

$$\begin{aligned} X_r(n, d) &= \{w \in W'_r; n \mid (d + w(i) - i) \text{ for all } 0 \leq i \leq r - 1\} \\ &= \{w \in W'_r; n \mid (d + i - w^{-1}(i)) \text{ for all } 0 \leq i \leq r - 1\} \end{aligned}$$

is nonempty, then $n \mid r$.

Proof. Suppose $X_r(n, d) \neq \emptyset$. Clearly, the identity is not in $X_r(n, d)$ since $n \nmid d$.

Take some $w \in X_r(n, d)$. Note that for $0 \leq i \leq r - 2$, $n \mid (d + w(i) - i)$ and $n \mid (d + w(i+1) - i - 1)$ imply $n \mid (w(i) - w(i+1) + 1)$. Suppose $w(i) - w(i+1) + 1 = 0$ for all i as above. This implies $w(1) = w(0) + 1$, $w(2) = w(0) + 2$, \dots , $w(r - 2) = w(0) + r - 2$, which implies $w(0) = 1$ (since w cannot be identity). This implies $w(r - 1) = 0$.

Since $w \in X_r(n, d)$, we get $n \mid (d + 1)$ and $n \mid (d + w(r - 1) - (r - 1)) = (d + 1 - r)$. These two relations imply $n \mid r$.

Therefore it remains to consider the case when

$$w(i) - w(i + 1) + 1 \neq 0$$

for some $0 \leq i \leq r - 2$. If $w(i) - w(i + 1) + 1$ is negative, then $w(i) - w(i + 1) + 1 \leq -n$, which implies $w(i) + n + 1 \leq w(i + 1)$. This implies $n + 1 \leq r - 1$. If $w(i) - w(i + 1) + 1$ is positive, then $n \leq w(i) - w(i + 1) + 1$, which implies $n \leq r$. Thus we have proved (up to now) that

$$n \leq r.$$

We have written $r = an + b$ with $a, b \in \mathbb{Z}$ and $0 \leq b \leq n - 1$. We know that $a \geq 1$. Write

$$d = cn + d', \quad \text{with } c, d' \in \mathbb{Z} \text{ and } 1 \leq d' \leq n - 1 \leq r - 1,$$

which is possible since $n \nmid d$.

Since the elements of $X_r(n, d) \subseteq W'_r$ are bijections, for each $i \in \{0, 1, \dots, n-1\}$ we must have

$$\begin{aligned} \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, r - 1]) &= \text{card}(\{i - d + kn; k \in \mathbb{Z}\} \cap [0, r - 1]) \\ &= \text{card}(\{i - d' + kn; k \in \mathbb{Z}\} \cap [0, r - 1]) \\ &= \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d', d' + r - 1]). \end{aligned}$$

From this we get

$$\begin{aligned} \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d' - 1]) + \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d', r - 1]) \\ = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [d', r - 1]) + \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [r, d' + r - 1]). \end{aligned}$$

Thus for each $i \in \{0, 1, \dots, n-1\}$ we have

$$\text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d' - 1]) = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [r, d' + r - 1]).$$

The last relation can be written as

$$\text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d' - 1]) = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [an + b, an + b + d' - 1]).$$

Therefore, for each $i \in \{0, 1, \dots, n-1\}$

$$\text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [0, d' - 1]) = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [b, b + d' - 1]),$$

i.e.,

$$\text{card}(\{i\} \cap [0, d' - 1]) = \text{card}(\{i + kn; k \in \mathbb{Z}\} \cap [b, b + d' - 1]).$$

Suppose $b \geq 1$. The preceding relation for $i = d' - 1$ implies $b \leq d' - 1$, and further $b < d'$. The case $i = d'$ implies $d' < b$. This is a contradiction. The proof is now complete. \square

9.3. We also need the following result, a slight modification of [Tadić 1996, Lemma 4.2]. The proof is almost the same. Therefore, we omit it here.

Lemma. *Suppose $r = an$ ($a \in \mathbb{Z}$, $a \geq 1$) and $n \nmid d$.*

- (i) $W'_r(n)X_r(n, d)W'_r(n) = X_r(n, d)$.
- (ii) $X_r(n, d)$ normalizes $W'_r(n)$.
- (iii) For any $w \in X_r(n, d)$ we have $X_r(n, d) = wW'_r(n) = W'_r(n)w$.

(iv) For each $i \in \{0, 1, 2, \dots, r-1\}$ write $i = s(i)n + t(i)$ where $s(i), t(i) \in \mathbb{Z}$ and $0 \leq t(i) \leq n-1$. Let $d = cn + d'$, where $c, d' \in \mathbb{Z}$, $1 \leq d' \leq n-1$ ($\leq r-1$) ($d' \neq 0$ since $n \nmid d$). Define $w_{(n,d)} \in W'_r$ by

$$w_{(n,d)}(i) = \begin{cases} i + (n - d') & \text{if } t(i) \leq d' - 1, \\ i - d' & \text{if } t(i) \geq d'. \end{cases}$$

Then $w_{(n,d)} \in X_r(n, d)$ and $(-1)^{w_{(n,d)}} = (-1)^{(r/n)(n-d')d'} = (-1)^{(r/n)(n-d)d}$.

9.4. As before, let $\Pi = L([\rho, v^{d-1}\rho], [v\rho, v^d\rho], \dots, [v^{r-1}\rho, v^{r-1+d-1}\rho])$.

Recall that we assume in this section that $r \leq d$ and $s_{\rho'} = n \nmid d$. To get a nontrivial LJ(Π), we have seen that one needs to consider only the case $s_{\rho'} = n \mid rd$. The lemma in §9.2 gives further reduction to the case $s_{\rho'} = n \mid r$ (see the calculation below). This is the reason why we shall assume it in the sequel.

Now we compute in this case

$$\begin{aligned} \text{LJ}(\Pi) &= \text{LJ}(L([\rho, v^{d-1}\rho], [v\rho, v^d\rho], \dots, [v^{r-1}\rho, v^{r-1+d-1}\rho])) \\ &= \sum_{w \in W'_r} (-1)^w \prod_{i=0}^{r-1} \text{LJ}(\delta([v^i\rho, v^{w(i)+(d-1)}\rho])) \\ &= \sum_{w \in X_r(n,d)} (-1)^w \prod_{i=0}^{r-1} \text{LJ}(\delta([v^i\rho, v^{w(i)+(d-1)}\rho])) \\ &= \sum_{w \in w_{(n,d)}W'_r(n)} (-1)^w \prod_{i=0}^{r-1} \text{LJ}(\delta([v^i\rho, v^{w(i)+(d-1)}\rho])) \\ &= (-1)^{w_{(n,d)}} \sum_{w \in W'_r(n)} (-1)^w \prod_{i=0}^{r-1} \text{LJ}(\delta([v^i\rho, v^{w_{(n,d)}w(i)+(d-1)}\rho])) \\ &= (-1)^{a(n-d)d} \sum_{w \in W'_r(n)} (-1)^w \prod_{\ell=0}^{n-1} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj}\rho, v^{w_{(n,d)}w(\ell+nj)+(d-1)}\rho])). \end{aligned}$$

Thus

$$\begin{aligned} &(-1)^{a(n-d)d} \text{LJ}(\Pi) \\ &= \sum_{\substack{w'_0 \in W'_r(n;0), \dots, \\ w'_{n-1} \in W'_r(n;n-1)}} \prod_{\ell=0}^{n-1} (-1)^{w'_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj}\rho, v^{w_{(n,d)}w'_\ell(\ell+nj)+(d-1)}\rho])) \\ &= \sum_{w_0, \dots, w_{n-1} \in W'_a} \prod_{\ell=0}^{n-1} (-1)^{w_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj}\rho, v^{w_{(n,d)}(\ell+nw_\ell(j))+(d-1)}\rho])) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{w_0, \dots, w_{n-1} \in W'_a} \prod_{\ell=0}^{d'-1} (-1)^{w_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj} \rho, v^{w_{(n,d)}(\ell+nw_\ell(j))+(d-1)} \rho])) \\
 &\quad \times \prod_{\ell=d'}^{n-1} (-1)^{w_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj} \rho, v^{w_{(n,d)}(\ell+nw_\ell(j))+(d-1)} \rho])) \\
 &= \sum_{w_0, \dots, w_{n-1} \in W'_a} \prod_{\ell=0}^{d'-1} (-1)^{w_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj} \rho, v^{\ell+nw_\ell(j)+(n-d')+(d-1)} \rho])) \\
 &\quad \times \prod_{\ell=d'}^{n-1} (-1)^{w_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj} \rho, v^{\ell+nw_\ell(j)-d'+(d-1)} \rho])) \\
 &= \prod_{\ell=0}^{d'-1} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj} \rho, v^{\ell+nw_\ell(j)+(n-d')+(d-1)} \rho])) \\
 &\quad \times \prod_{\ell=d'}^{n-1} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{j=0}^{a-1} \text{LJ}(\delta([v^{\ell+nj} \rho, v^{\ell+nw_\ell(j)-d'+(d-1)} \rho])).
 \end{aligned}$$

We assume in the sequel that $\delta([v, \rho^{n-1}])$ corresponds to ρ' under the Jacquet–Langlands correspondence as in §6.2. Then the previous expression becomes

$$\begin{aligned}
 &= \prod_{\ell=0}^{d'-1} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{j=0}^{a-1} \delta([v^{\ell+nj} \rho', v^{\ell+nj} v_{\rho'}^{w_\ell(j)-j+c} \rho']) \\
 &\quad \times \prod_{\ell=d'}^{n-1} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{j=0}^{a-1} \delta([v^{\ell+nj} \rho', v^{\ell+nj} v_{\rho'}^{w_\ell(j)-j+c-1} \rho']) \\
 &= \prod_{\ell=0}^{d'-1} v_{\rho'}^{\ell/n} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{j=0}^{a-1} \delta([v_{\rho'}^j \rho', v_{\rho'}^{w_\ell(j)+c} \rho']) \\
 &\quad \times \prod_{\ell=d'}^{n-1} v_{\rho'}^{\ell/n} \sum_{w_\ell \in W'_a} (-1)^{w_\ell} \prod_{j=0}^{a-1} \delta([v_{\rho'}^j \rho', v_{\rho'}^{w_\ell(j)+c-1} \rho']) \\
 &= \prod_{\ell=0}^{d'-1} v_{\rho'}^{\ell/n} L(([\rho', v_{\rho'}^c \rho'], \dots, [v_{\rho'}^{a-1} \rho', v_{\rho'}^{a-1+c} \rho'])) \\
 &\quad \times \prod_{\ell=d'}^{n-1} v_{\rho'}^{\ell/n} L(([\rho', v_{\rho'}^{c-1} \rho'], \dots, [v_{\rho'}^{a-1} \rho', v_{\rho'}^{a-1+c-1} \rho']))
 \end{aligned}$$

since $a \leq c$, and then clearly $a < c + 1$. (Note that we assume $r \leq d$, hence $an \leq cn + d'$, hence $an < cn$, hence $a < c$; in particular, $a \leq c$.)

We have proved

$$\begin{aligned} \text{LJ}(\Pi) &= \text{LJ}(L([\rho, v^{d-1}\rho], \dots, [v^{r-1}\rho, v^{r-1+d-1}\rho])) \\ &= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} v_{\rho'}^{\ell/n} L([\rho', v_{\rho'}^c \rho'], \dots, [v_{\rho'}^{a-1} \rho', v_{\rho'}^{a-1+c} \rho']) \\ &\quad \times \prod_{\ell=d'}^{n-1} v_{\rho'}^{\ell/n} L([\rho', v_{\rho'}^{c-1} \rho'], \dots, [v_{\rho'}^{a-1} \rho', v_{\rho'}^{a-1+c-1} \rho']). \end{aligned}$$

9.5. Now suppose that ρ is unitary and that ρ'' corresponds to $\delta(\rho, s_{\rho'}) = \delta(\rho, n)$ under the Jacquet–Langlands correspondence (as at the beginning of §7.3). Then ρ'' is unitary and

$$\rho'' = \text{LJ}(\delta(\rho, n)) = \text{LJ}(v^{-(n-1)/2} \delta([\rho, v^{n-1}\rho])) = v_{\rho'}^{-(n-1)/(2n)} \rho',$$

that is,

$$\rho'' = v_{\rho'}^{-(n-1)/(2n)} \rho', \quad \rho' = v_{\rho'}^{(n-1)/(2n)} \rho'', \quad v_{\rho'} = v_{\rho''}.$$

Now (for $n \nmid d$)

$$\begin{aligned} &\text{LJ}(L(\langle r, d \rangle^\rho)) \\ &= \text{LJ}(v^{-\frac{r+d}{2}+1} L([\rho, v^{d-1}\rho], \dots, [v^{r-1}\rho, v^{r-1+d-1}\rho])) \\ &= (-1)^{a(n-d)d} v_{\rho'}^{-\frac{r+d}{2}+1} \prod_{\ell=0}^{d'-1} v_{\rho'}^{\frac{\ell}{n}} L([\rho', v_{\rho'}^c \rho'], \dots, [v_{\rho'}^{a-1} \rho', v_{\rho'}^{a-1+c} \rho']) \\ &\quad \times \prod_{\ell=d'}^{n-1} v_{\rho'}^{\frac{\ell}{n}} L([\rho', v_{\rho'}^{c-1} \rho'], \dots, [v_{\rho'}^{a-1} \rho', v_{\rho'}^{a-1+c-1} \rho']) \\ &= (-1)^{a(n-d)d} v_{\rho'}^{-\frac{a+c+d/n}{2} + \frac{1}{n}} \prod_{\ell=0}^{d'-1} v_{\rho'}^{\frac{\ell}{n}} v_{\rho'}^{\frac{a+c+1}{2}-1} L(\langle a, c+1 \rangle^{\rho'}) \\ &\quad \times \prod_{\ell=d'}^{n-1} v_{\rho'}^{\frac{\ell}{n}} v_{\rho'}^{\frac{a+c}{2}-1} L(\langle a, c \rangle^{\rho'}) \\ &= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} v_{\rho'}^{\frac{2\ell-n-d'+2}{2n}} L(\langle a, c+1 \rangle^{\rho'}) \prod_{\ell=d'}^{n-1} v_{\rho'}^{\frac{2\ell-2n-d'+2}{2n}} L(\langle a, c \rangle^{\rho'}) \\ &= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} v_{\rho'}^{\frac{2\ell-n-d'+2}{2n}} L(\langle a, c+1 \rangle^{v_{\rho'}^{\frac{n-1}{2n}} \rho''}) \prod_{\ell=d'}^{n-1} v_{\rho'}^{\frac{2\ell-2n-d'+2}{2n}} L(\langle a, c \rangle^{v_{\rho'}^{\frac{n-1}{2n}} \rho''}) \\ &= (-1)^{a(n-d)d} \prod_{\ell=0}^{d'-1} v_{\rho''}^{\frac{2\ell-d'+1}{2n}} L(\langle a, c+1 \rangle^{\rho''}) \prod_{\ell=d'}^{n-1} v_{\rho''}^{\frac{2\ell-n-d'+1}{2n}} L(\langle a, c \rangle^{\rho''}) \end{aligned}$$

$$= (-1)^{a(n-d)d} \prod_{\ell=-\frac{d'-1}{2}}^{\frac{d'-1}{2}} v_{\rho''}^{\frac{\ell}{n}} L(\langle a, c+1 \rangle^{\rho''}) \prod_{\ell=d'}^{n-1} v_{\rho''}^{\frac{(-n-d'+1)/2+\ell}{n}} L(\langle a, c \rangle^{\rho''}).$$

Therefore, we have proved:

Proposition. *Suppose that (UO) holds. Assume $n \nmid d$, $n \mid r$, and $r \leq d$. Write $d = cn + d'$, with $c, d \in \mathbb{Z}$ and $1 \leq d' \leq n - 1$. Then*

$$\begin{aligned} & \text{LJ}(L(\langle r, d \rangle^\rho)) \\ &= (-1)^{a(n-d)d} \text{str}_{v_{\rho''}}(d', n, L(\langle r/n, c+1 \rangle^{\rho''})) \text{str}_{v_{\rho''}}(n-d', n, L(\langle r/n, c \rangle^{\rho''})) \\ &= (-1)^{a(n-d)d} \text{str}(d', L(\langle r/n, c+1 \rangle^{\rho''})) \text{str}(n-d', L(\langle r/n, c \rangle^{\rho''})). \quad \square \end{aligned}$$

This expression can be also written as

$$\begin{aligned} & (-1)^{a(n-d)d} \text{str}(d - [d/n]n, L(\langle r/n, [d/n] + 1 \rangle^{\rho''})) \\ & \quad \times \text{str}(\left([d/n] + 1\right)n - d, L(\langle r/n, [d/n] \rangle^{\rho''})). \end{aligned}$$

10. Unitarity of Jacquet–Langlands correspondence of irreducible unitary representations, II

Under the assumptions of the previous section, we first show in this section that $\text{LJ}(L(\langle r, d \rangle^\rho))$ is unitary (we assume ρ to be unitary). Then we show that

$$\text{LJ}(v^\beta(L(\langle r, d \rangle^\rho)) \times v^{-\beta}(L(\langle r, d \rangle^\rho)))$$

is unitary for $0 < \beta < \frac{1}{2}$ (under the same assumptions).

For the unitarity of $\text{LJ}(L(\langle r, d \rangle^\rho))$, it is enough to show that

$$0 \leq \frac{d'-1}{2n} < \frac{1}{2} \quad \text{and} \quad 0 \leq \frac{n-d'-1}{2n} < \frac{1}{2}.$$

Both obviously hold (recall $1 \leq d' \leq n - 1$).

For the complementary series, we need to see that

$$0 \leq \frac{d'-1}{2n} + \frac{\beta}{n} < \frac{1}{2} \quad \text{and} \quad 0 \leq \frac{n-d'-1}{2n} + \frac{\beta}{n} < \frac{1}{2}$$

for $0 < \beta < \frac{1}{2}$ (see Section 9). This holds since $1 \leq d' \leq n - 1$ and $\beta < \frac{1}{2}$.

11. Jacquet–Langlands correspondence in the unitary case: the remaining cases

It remains to compute $\text{LJ}(L(\langle d, r \rangle^\rho))$ in the case $d \geq r$. Since LJ and t commute up to a sign (see [Badulescu 2002, Theorem 3.17]), we have

$$\text{LJ}(L(\langle d, r \rangle^\rho)) = \text{LJ}(L(\langle r, d \rangle^\rho)^t) = \pm \text{LJ}(L(\langle r, d \rangle^\rho))^t.$$

Therefore, we can apply our previous calculations. We have two cases.

- If $r \leq d$, $n \mid d$, $r = an + b$, $a, b \in \mathbb{Z}$, $0 \leq b \leq n - 1$, $1 \leq n$, then

$$\begin{aligned} \text{LJ}(L(\langle d, r \rangle^\rho)) &= \pm \text{LJ}(L(\langle r, d \rangle^\rho))^t \\ &= \pm \text{str}(b, L(\langle d/n, a + 1 \rangle^{\rho''})) \text{str}(\min(n, r) - b, L(\langle d/n, a \rangle^{\rho''})). \end{aligned}$$

- If $n \nmid d$, $n \mid r$, $r \leq d$, $d = cn + d'$, $c, d \in \mathbb{Z}$, $1 \leq d' \leq n - 1$, then

$$\begin{aligned} \text{LJ}(L(\langle d, r \rangle^\rho)) &= \pm \text{LJ}(L(\langle r, d \rangle^\rho))^t \\ &= \pm (-1)^{a(n-d)d} \text{str}(d', L(\langle c+1, r/n \rangle^{\rho''})) \text{str}(n-d', L(\langle c, r/n \rangle^{\rho''})). \end{aligned}$$

Here too unitarity is preserved, by the results in Sections 8 and 10, since the involution preserves unitarity. (We assume as before that (U0) holds.)

One can compute the sign in the two preceding formulas using [Badulescu 2002, Theorem 3.17]. (Our involutions differ by a sign from the ones used there.)

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