Pacific
Journal of
Mathematics

A SURFACE OF GENERAL TYPE WITH $p_g = q = 2$ AND $K_X^2 = 5$

JUNGKAI A. CHEN AND CHRISTOPHER D. HACON

Volume 223 No. 2 February 2006

PACIFIC JOURNAL OF MATHEMATICS
Vol. 223, No. 2, 2006

A SURFACE OF GENERAL TYPE WITH $p_g = q = 2$ AND $K_X^2 = 5$

JUNGKAI A. CHEN AND CHRISTOPHER D. HACON

We give an example of a minimal complex surface of general type with $p_g = q = 2$ and $K_X^2 = 5$.

1. Introduction

Recently, there has been considerable interest in understanding the geometry of irregular complex projective surfaces with $\chi(X, \omega_X) = 1 + p_g(X) - q(X) = 1$, and in particular of surfaces with $p_g = q = 2$. Let X be a smooth minimal complex surface of general type. If $\chi(X, \omega_X) = 1$, then one has the bound $1 \le K_X^2 \le 9$. If, in addition, the surface is irregular, that is, $q(X) = h^0(X, \Omega_X^1) > 0$, then $K_X^2 \ge 2p_g(X)$ and so $p_g(X) \le 4$. In [Debarre 1982], it is shown that the case $p_g = q = 4$ corresponds to the product of two curves of genus 2. In [Hacon and Pardini 2002] and [Pirola 2002], surfaces with $p_g = q = 3$ are completely classified. When $K_X^2 = 2p_g(X) = 6$ they are symmetric products of curves of genus 3 and when $K_X^2 = 8$ they admit an irrational pencil. The case $p_g = q = 2$ seems far more delicate. At any rate Catanese suggests that, analogously to the $p_g = q = 3$ case, a surface of general type with $p_g = q = 2$ and with no fibration over an elliptic curve is a degree 2 covering of a principally polarized abelian surface (A, Θ) branched along a divisor in the linear series $|2\Theta|$.

Zucconi [2003] has classified surfaces of general type with $p_g = q = 2$ which admit an irrational pencil. Manetti [2003] showed that a minimal surface of general type with K_X ample and $K_X^2 = 4$, is a degree 2 covering of a principally polarized abelian surface (A, Θ) branched along a divisor $D \in |2\Theta|$. Ciliberto and Mendes Lopes [2002] conjecture that this should be the case for any minimal surface of general type with $p_g = q = 2$ and $K_X^2 = 4$.

Here we give a counterexample to Catanese's conjecture above. The example we construct is birational to a triple cover of an abelian surface. Its canonical divisor K_X is ample, $p_g = q = 2$ and $K_X^2 = 5$. The construction is motivated in Section 3, where we obtain restrictions on the structure of the sheaf $\mathrm{alb}_{X,*}(\omega_X)$.

MSC2000: 14C20, 14K12.

Keywords: surfaces of general type, Fourier–Mukai transform.

Chen was partially supported by NSC 92-2115-M-002-029, Taiwan. He is a member of the Mathematics Division, National Center for Theoretical Sciences at Taipei.

2. Construction and verification

We will need some results from the theory of Mukai transforms. Let $\hat{A}$ be the dual abelian variety of A and $\mathcal{P}$ be the normalized Poincaré line bundle on $A \times \hat{A}$. Following [Mukai 1981], define the functor $\hat{\mathcal{S}}$ of $\mathcal{O}_A$-modules into the category of $\mathcal{O}_{\hat{A}}$-modules by

$$\hat{\mathcal{S}}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M).$$

The derived functor $R\hat{\mathcal{S}}$ of $\hat{\mathcal{S}}$ then induces an equivalence of categories between the two derived categories $D(A)$ and $D(\hat{A})$. More precisely, from [Mukai 1981] we know that *there are isomorphisms of functors*

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-g] \quad \text{and} \quad R\hat{\mathcal{S}} \circ R\mathcal{S} \cong (-1_{\hat{A}})^*[-g],$$

where $[-g]$ denotes "shift the complex g places to the right". The Weak Index Theorem (WIT) holds for a coherent sheaf $\mathcal{F}$ on A if there exists an integer $i(\mathcal{F})$ such that for all $j \neq i(\mathcal{F})$, one has $R^j\hat{\mathcal{S}}(\mathcal{F}) = 0$. The coherent sheaf $R^{i(\mathcal{F})}\hat{\mathcal{S}}(\mathcal{F})$ is denoted simply by $\hat{\mathcal{F}}$.

Now consider (A, M), a simple polarized abelian surface of type $(1, 2)$. Assume M is symmetric, i.e., $(-1)^*M \cong M$. The linear series $|M|$ has 4 isolated base points $\{o, p, q, r\}$. We may assume that o is the identity of the abelian surface and that p, q, r are 2-torsion elements with $r = p + q$ (see [Barth 1987], for instance). Each divisor $D \in |M|$ is either a nonsingular curve of genus 3 or a singular curve with a simple node distinct from the base points. $M^\vee$ satisfies the WIT of index 2. Let

$$\mathcal{F} = \widehat{M^\vee} := R^2\hat{\mathcal{S}}(M^\vee)$$

be the Fourier–Mukai transform of $M^\vee$. The vector bundle $\mathcal{F}$ has rank 2. Let $\mathcal{E} = \mathcal{F}^\vee$. One can check that

$$\dim \operatorname{Hom}\bigl(S^3\mathcal{E}, \textstyle\bigwedge^2\mathcal{E}\bigr) = h^0\bigl(\hat{A}, (S^3\mathcal{E})^\vee \otimes \textstyle\bigwedge^2\mathcal{E}\bigr) = 2.$$

By Miranda's triple covering construction [1985], there is a 2-dimensional family of triple coverings $\hat{f} : \hat{X} \to \hat{A}$ with

$$\hat{f}_*\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{A}} \oplus \mathcal{E}.$$

The idea is this: to construct a triple covering $\hat{f} : \hat{X} \to \hat{A}$ over $\hat{A}$ with *Tschirnhausen module* $\mathcal{E}$ [Miranda 1985], we first construct a triple covering $f : X \to A$ with Tschirnhausen module $\phi_{M^\vee}^*\mathcal{E}$. In Claim 1 we identify those coverings of this type that descend to a triple covering $f : \hat{X} \to \hat{A}$. In Claim 2 we verify that for a general such covering the singularities of X are rational. It follows that the singularities of $\hat{X}$ are also rational. Finally, we compute the invariants of $\hat{X}$ via the invariants of X.

Let $\phi_{M^\vee} : A \to \hat{A}$ be the isogeny defined by $M^\vee$. We have the commutative diagram

$$\begin{tikzcd} X = A \times_{\hat{A}} \hat{X} \arrow[r, "\phi"] \arrow[d, "f"'] & \hat{X} \arrow[d, "\hat{f}"] \\ A \arrow[r, "\phi_{M^\vee}"] & \hat{A} \end{tikzcd}$$

where $\phi : X \to \hat{X}$ is a 4 : 1 étale covering and $f : X \to A$ is a triple covering determined by a section of

$$\phi_{M^\vee}^* \operatorname{Hom}\big(S^3\mathscr{E}, \textstyle\bigwedge^2\mathscr{E}\big) \subset \operatorname{Hom}\big(S^3\phi_{M^\vee}^*\mathscr{E}, \textstyle\bigwedge^2\phi_{M^\vee}^*\mathscr{E}\big).$$

By [Mukai 1981], $\phi_{M^\vee}^*\mathscr{E} \cong M^\vee \oplus M^\vee$. Thus

$$\operatorname{Hom}\big(S^3\phi_{M^\vee}^*\mathscr{E}, \textstyle\bigwedge^2\phi_{M^\vee}^*\mathscr{E}\big) \cong H^0(A, M)^{\oplus 4}.$$

To determine the corresponding 2-dimensional subspace, we consider the Heisenberg group action on $H^0(A, M)$. The Heisenberg group can be identified with

$$\mathscr{G}(\delta) := \{(\alpha, t, l) \mid \alpha \in k^*,\ t \in \mathbb{Z}_2,\ l \in \hat{\mathbb{Z}}_2\}$$

with group law $(\alpha, t, l)(\alpha, t', l') = (\alpha\alpha' l'(t), t+t', l+l')$. Moreover, $H^0(A, M)$ corresponds to $\operatorname{Hom}(\mathbb{Z}_2, k)$. The action of $\mathscr{G}(\delta)$ on $\operatorname{Hom}(\mathbb{Z}_2, k)$ is given by

$$(\alpha, t, l) f(x) = \alpha l(x) f(t + x).$$

Let X, Y be the sections in $H^0(A, M)$ corresponding to the characteristic functions of 0, 1 in $\operatorname{Hom}(\mathbb{Z}_2, k)$ respectively.

Claim 1. *The* 2*-dimensional subspace is determined as*

$$\phi_{M^\vee}^* \operatorname{Hom}\big(S^3\mathscr{E}, \textstyle\bigwedge^2\mathscr{E}\big) \cong \{(sX, tY, -tX, -sY) \mid s, t \in k\} \subset H^0(A, M)^{\oplus 4}.$$

Grant this for the time being. Following [Miranda 1985], we can then construct a triple covering $f : X \to A$ by using the data $a = sX$, $b = tY$, $c = -tX$, $d = -sY$. Over an affine open subset U of A, the triple covering can be described in $U \times \mathbb{A}^2$ as the covering by the 2×2 minors of

$$\begin{pmatrix} Z + a & W - 2d & c \\ b & Z - 2a & W + d \end{pmatrix}$$

where Z, W are coordinates for $\mathbb{A}^2$.

Following [Miranda 1985, §4], we have $A = s^2X^2 + stY^2$, $B = (t^2 - s^2)XY$ and $C = s^2Y^2 + stX^2$. The branch locus is defined by $D = B^2 - 4AC \in H^0(M)^{\otimes 4}$ and one can see that it corresponds to a divisor $D_1 + D_2 + D_3 + D_4$ with $D_i \in |M|$. For a general choice of s, t, the D_i are all distinct and nonsingular. It is easy to check, as in [Miranda 1985, §5], that for general choices of s, t, the only possible singularities of X lie over the 4 base points $\{p, q, r, o\}$. We remark that f is totally ramified only over these 4 base points.

Let $x \in X$ be a point lying over one of $\{p, q, r, o\}$.

Claim 2. *For general s, t, the singularity of X at x is locally isomorphic to a cone over a twisted cubic.*

Therefore, X has only rational singularities and so does $\hat{X}$. A resolution $\hat{\mu} : \hat{X}' \to \hat{X}$ can be obtained by blowing up along the singularity. The corresponding resolution $\mu : X' \to X$ is the blow up of X along the 4 points lying over $\{p, q, r, o\}$. Let $\{E_i\}_{i=1,\dots,4}$ be the exceptional divisors and $\{R_i\}_{i=1,\dots,4}$ the proper transform of the D_i. Then

$$K_{X'} = \sum_{i=1,\dots,4} R_i + \sum_{i=1,\dots,4} E_i.$$

Note that $R_i \cdot R_j = 0$, $R_i \cdot E_j = 1$ for all i, j and $E_i^2 = -3$, $E_i \cdot E_j = 0$ for $i \neq j$. Thus we have $K_{X'}^2 = 20$, and

$$\begin{aligned} p_g(X') = h^0(X', \omega_{X'}) &= h^2(X', \mathcal{O}_{X'}) = h^2(X, \mathcal{O}_X) \\ &= h^2(A, \mathcal{O}_A) + 2h^2(A, M^\vee) = 5. \end{aligned}$$

Similarly, $q(X') = 2$ and $\chi(X', \omega_{X'}) = 4$. One can also check that $K_{X'}$ is ample.

Since $X' \to \hat{X}'$ is an étale cover of degree 4, one has

$$\chi(\hat{X}', \omega_{\hat{X}'}) = 1, \quad (K_{\hat{X}'})^2 = 5,$$

and $K_{\hat{X}'}$ is ample. $\hat{X}$ has only rational singularity. It is easy to see that $q(\hat{X}') = 2$ and hence $p_g(\hat{X}') = 2$. Therefore $\hat{X}'$ is a surface of general type with $p_g = q = 2$ and $K^2 = 5$.

Proof of Claim 1. We follow [Mumford 1970]. Let $H(M^\vee)$ be the kernel of $\phi_{M^\vee} : A \to \hat{A}$, i.e., the set of points $x \in A$ such that $T_x^* M^\vee \cong M^\vee$. Then $H(M) = H(M^\vee)$. Let $\mathcal{G}(M)$ be the set of pairs (x, φ) such that $x \in H(M)$ and φ is an isomorphism $\varphi : M \to T_x^* M$. Then $\mathcal{G}(M)$ is a group sitting in the exact sequence

$$0 \to k^* \to \mathcal{G}(M) \to H(M) \to 0.$$

There is an isomorphism of groups $\mathcal{G}(M) \cong \mathcal{G}(\delta)$. Under this identification, the representation of $\mathcal{G}(M)$ on $H^0(A, M)$ corresponds to the unique representation of $\mathcal{G}(\delta)$ on $V = V(\delta) := \operatorname{Hom}(\mathbb{Z}_2, k)$, which is defined by

$$((\alpha, t, l) f)(x) = \alpha \cdot l(x) \cdot f(t + x).$$

With respect to the ordered basis formed by the characteristic functions of 0 and 1, this representation is induced by

$$(1,1,1) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (1,1,0) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1,0,1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding $\mathscr{G}(\delta)$ representation on $S^3 V^\vee \otimes \bigwedge^2 V \otimes V$ can easily be computed. By [Mukai 1981, Proposition 3.11] we have

$$\phi_{M^\vee}^* \mathscr{F} \cong H^2(A, M) \otimes M \cong (H^0(A, M) \otimes M^\vee)^\vee \cong \phi_{M^\vee}^* \mathscr{E}^\vee.$$

One sees that

$$H^0(A, S^3 \phi_{M^\vee}^* \mathscr{E}^\vee \otimes \textstyle\bigwedge^2 \phi_{M^\vee}^* \mathscr{E}) \cong S^3 H^0(A, M)^\vee \otimes \textstyle\bigwedge^2 H^0(A, M) \otimes H^0(A, M).$$

This vector space is in turn isomorphic to $\bigoplus_{i=1}^4 H^0(A, M)$. We can now compute the corresponding $\mathscr{G}(M)$ representation in terms of the above $\mathscr{G}(\delta)$ representation.

Let p_i, $i = 1, \dots, 4$ denote the projection onto the i-th factor. With respect to the ordered basis

$$\begin{aligned}
\{e_1, e_2, \dots, e_8\}
&= \{p_1^* X, p_1^* Y, \dots, p_4^* X, p_4^* Y\} \\
&= \{\hat{X}^3 \otimes X \wedge Y \otimes X, \hat{X}^3 \otimes X \wedge Y \otimes Y, \hat{X}^2 \hat{Y} \otimes X \wedge Y \otimes X, \hat{X}^2 \hat{Y} \otimes X \wedge Y \otimes Y, \\
&\qquad \hat{X} \hat{Y}^2 \otimes X \wedge Y \otimes X, \hat{X} \hat{Y}^2 \otimes X \wedge Y \otimes Y, \hat{Y}^3 \otimes X \wedge Y \otimes X, \hat{Y}^3 \otimes X \wedge Y \otimes Y\},
\end{aligned}$$

the element $(1, 1, 1)$ maps to $R \in M_8(k)$ defined by $R_{i,j} = 0$ if $i + j \neq 8$ and $R_{i,8-i} = \{-1, 1, 1, -1, -1, 1, 1, -1\}$, and $(1, 1, 0)$ maps to $M \in M_8(k)$ defined by $M_{i,j} = 0$ if $i + j \neq 8$ and $M_{i,8-i} = \{1, 1, 1, 1, 1, 1, 1, 1\}$. In particular,

$$R^2 = M^2 = 1 \quad \text{and} \quad RM = MR.$$

There is an induced representation of $H(M^\vee) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-invariant elements form the subspace

$$\{s(e_1 - e_8) + t(e_4 - e_5) \mid s, t \in k\} = \{(sX, tY, -tX, -sY) \mid s, t \in k\}.$$

These invariant elements correspond to the subspace $\phi_{M^\vee}^* \operatorname{Hom}(S^3 \mathscr{E}, \bigwedge^2 \mathscr{E})$. □

Proof of Claim 2. On a neighborhood of one of the base loci o, p, q, r we may assume that X, Y (or any two distinct sections of $H^0(A, M)$) are local coordinates. By [Harris 1992, p. 14, exercise 1.25], the 2×2 minors mentioned above define a twisted cubic if and only if for all $[u : v] \in \mathbb{P}^1$ the linear forms

$$u(Z + sX) - vtY, \; u(W + 2sY) - v(Z - 2sX), \; -utX - v(W - sY)$$

are linearly independent: in other words, if and only if the matrix

$$\begin{pmatrix} us & -vt & u & 0 \\ 2vs & 2us & -v & u \\ -ut & vs & 0 & -v \end{pmatrix}$$

has a nonzero 3×3 minor for every u, v. By inspection one sees that this is the case for general s, t (more precisely for $t \neq 0$ and $t^2 \neq 9s^2$). $\square$

3. Computation of $\mathrm{alb}_{X,*}(\omega_X)$

Using the techniques of [Hacon and Pardini 2002], we now find restrictions on the structure of the coherent sheaf $\mathrm{alb}_{X,*}(\omega_X)$. It was this computation that suggested to us the possibility of constructing the example of Section 2.

Proposition 3.1 [Ciliberto and Mendes Lopes 2002, Proposition 2.3]. *Let X be a minimal surface of general type with $p_g = q = 2$. Then $a := \mathrm{alb}_X : X \to \mathrm{Alb}(X) =: A$ is not surjective if and only if $B := a(X)$ is a curve of genus 2 and $a : X \to B$ has smooth connected fibers of genus 2 with constant modulus and $K_X^2 = 8$.*

We now therefore consider the situation where $a : X \to \mathrm{Alb}(X) =: A$ is surjective. For any coherent sheaf $\mathcal{F}$ on X, define

$$V^i(X, \mathcal{F}) := \{P \in \mathrm{Pic}^0(X) \mid h^i(X, \mathcal{F} \otimes P) \neq 0\}.$$

Since a is generically finite, $R^i a_* \omega_X = 0$ for all $i > 0$ and so $V^i(X, \omega_X) = V^i(A, a_*\omega_X)$ for all i.

Lemma 3.2. *Let X be a minimal surface with $p_g = q = 2$ and surjective Albanese map. If* $\dim V^1(X, \omega_X) \geq 1$, *there exists an elliptic pencil $X \to E$ with $g(E) = 1$.*

Proof. By the generic vanishing theorems of Green and Lazarsfeld, we have $\dim V^1(X, \omega_X) < 2$, and if T is a component of $V^1(X, \omega_X)$ of dimension 1, then T is a translate of an elliptic curve $T_0 \subset \mathrm{Pic}^0(X)$. The pencil $X \to E$ is induced by $a : X \to \mathrm{Alb}(X)$ composed with the dual map of abelian varieties $\mathrm{Alb}(X) \to E := T_0^\vee$. $\square$

Corollary 3.3. *Let X be a minimal surface of general type with $p_g = q = 2$ without irrational pencils. Then $a : X \to A$ is surjective with $V^1(A, a_*\omega_X)$ supported on finitely many points.* $\square$

A vector bundle U on an abelian variety A is unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

such that $U_i / U_{i-1} \cong \mathcal{O}_A$. A vector bundle is homogeneous if and only if it is isomorphic to $\bigoplus_{i=1}^n (P_i \otimes U_i)$ with $P_i \in \mathrm{Pic}^0(A)$ and U_i unipotent vector bundles. By [Mukai 1981], there is an one-to-one correspondence between sheaves supported on finitely many points and homogeneous vector bundles via the Fourier–Mukai transform.

Lemma 3.4. *Let $\mathcal{F}$ be a coherent sheaf on an abelian surface. Then*

$$R^i\mathcal{S}R^j\hat{\mathcal{S}}(\mathcal{F}) = 0 \quad \textit{for } (i, j) \in \{(1,2), (2,2), (0,0), (1,0)\}.$$

Moreover, there exist an injection $d : R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F}$ and a surjection $d' : R^0\mathcal{S}R^2\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}$). In particular, $R^0\hat{\mathcal{S}}\mathcal{F}$ and $R^2\hat{\mathcal{S}}\mathcal{F}$ satisfy the WIT of index 2 and 0, respectively.

Proof. As mentioned above, by [Mukai 1981], there is an isomorphism of functors

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-2].$$

In particular there is a spectral sequence $E_2^{p,q} = R^p\mathcal{S}R^q\hat{\mathcal{S}}\mathcal{F}$ with $E_\infty^{p,q} = 0$ if $p+q \neq 2$. The only possibly nonvanishing differentials d_2 are

$$d : R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \quad \textit{and} \quad d' : R^0\mathcal{S}R^2\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}.$$

One sees that $E_2^{p,q} = E_\infty^{p,q} = 0$ for $(p, q) \in \{(1,2), (2,2), (0,0), (1,0)\}$. Moreover, $\ker d = E_3^{0,1} = E_\infty^{0,1} = 0$, so d is an injection. Similarly d' is a surjection. □

Theorem 3.5. *Let X be a minimal surface of general type with $p_g = q = 2$ without any irrational pencil. Then there exist homogeneous vector bundles $\mathcal{H}$, and a negative definite line bundle L on $\hat{A} = \mathrm{Pic}^0(A)$ (i.e. $L^\vee$ is ample) such that $a_*\omega_X$ fits into the exact sequences*

$$0 \to \mathcal{O}_A \to a_*\omega_X \to \mathcal{F} \to 0,$$
$$0 \to \mathcal{H} \to \hat{L} \to (-1_A)^*\mathcal{F} \to 0.$$

Proof. Notice that $\omega_A = \mathcal{O}_A$. By assumption, X has no irrational pencils; thus $a : X \to A$ is surjective and $\dim V^1(X, \omega_X) = 0$, hence $V^1(X, \omega_X) = \{\mathcal{O}_X, P_1, \ldots, P_n\}$. Let $\mathcal{F}$ be the coherent sheaf defined by the short exact sequence

$$0 \to \mathcal{O}_A \to a_*\omega_X \to \mathcal{F} \to 0.$$

Since $R^i a_*\omega_X = 0$ for $i > 0$, one sees that for $i \geq 0$,

$$H^i(A, a_*\omega_X) \cong H^i(X, \omega_X) \cong H^i(A, \omega_A)$$

and therefore $h^1(\mathcal{F}) = h^2(\mathcal{F}) = 0$. Moreover, for all $\mathcal{O}_X \neq P \in \mathrm{Pic}^0(A)$, one has $h^i(A, \mathcal{F} \otimes P) = h^i(X, \omega_X \otimes P)$ for all i. In particular $V^2(A, \mathcal{F}) = \varnothing$ and $V^1(A, \mathcal{F}) = \{P_1, \ldots, P_n\}$. We have $R^2\hat{\mathcal{S}}\mathcal{F} = 0$ and $R^1\hat{\mathcal{S}}\mathcal{F} = \bigoplus B_i$, where the sheaves B_i are supported at the points P_i (and are artinian $\mathcal{O}_{\hat{A},P_i}$-modules; see [Mukai 1981, Example 2.9]). In particular, $R^1\hat{\mathcal{S}}\mathcal{F}$ satisfies the WIT of index 0.

Now consider the spectral sequence of the proof of Lemma 3.4. The only nonzero E_2 terms are $E_2^{0,1}$ and $E_2^{2,0}$. Therefore, one has the exact sequence

$$0 \to R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to (-1_A)^*\mathcal{F} \to 0.$$

First note that $R^1\hat{\mathcal{S}}\mathcal{F}$ is supported on finitely many points. It follows that $R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} = R\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}$ is a homogeneous vector bundle; call it $\mathcal{H}$. It suffices to show that $R^0\hat{\mathcal{S}}\mathcal{F}$ is a negative line bundle.

Let $U = \mathrm{Pic}^0(A) - \{\mathcal{O}_A, P_1, \ldots, P_n\}$. Then, for all $P \in U$,

$$h^0(A, \mathcal{F} \otimes P) = h^0(A, a_*\omega_X \otimes P) = \chi(X, \omega_X) = 1.$$

Thus $R^0\hat{\mathcal{S}}\mathcal{F}|_U$ is locally free of rank 1. Let $L = (R^0\hat{\mathcal{S}}\mathcal{F})^{\vee\vee}$. Then L is a reflexive sheaf of rank 1 on a nonsingular surface and hence a line bundle. Since $R^0\hat{\mathcal{S}}\mathcal{F} = R^0\hat{\mathcal{S}}a_*\omega_X$ is torsion-free, we have an exact sequence of coherent sheaves on $\hat{A}$:

$$0 \to R^0\hat{\mathcal{S}}\mathcal{F} \to L \to Q \to 0$$

where Q is supported at most on the points P_i.

We claim that $Q = 0$. Suppose on the contrary that $Q \neq 0$. By Lemma 3.4, $R^i\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} = 0$ for $i = 0, 1$, hence $R^0L \cong R^0Q$. So for general $P \in A = \mathrm{Pic}^0(\hat{A})$ one has

$$h^0(L \otimes P) = h^0(Q \otimes P) \neq 0,$$

since $Q \neq 0$ is supported on points. It follows that L is an ample line bundle and therefore satisfying I.T of index 0. In particular, $R^2\mathcal{S}L = 0$. On the other hand, since Q is supported on points, we have that $R^1\mathcal{S}Q = 0$. The exact sequence

$$R^1\mathcal{S}Q \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}L$$

yields $R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} = 0$. It follows that $\mathcal{F} = 0$, since there is a surjection

$$R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to (-1)^*\mathcal{F}.$$

One concludes that $\mathcal{O}_A = a_*\omega_X$, and so that $X \to A$ is birational, which is the required contradiction.

We may therefore assume that $Q = 0$ and hence $L = R^0\hat{\mathcal{S}}\mathcal{F}$ is a line bundle. By Lemma 3.4, L satisfies the WIT of index 2, hence it is a negative definite line bundle. □

Remark. It follows that if $X \to A$ has degree 2, then $\mathcal{F}$ has rank 1. The only possibility is that $a_*\omega_X = \mathcal{O}_A \oplus \mathcal{O}_A(-\Theta)$, where Θ is a principal polarization. This is a 2 : 1 covering branched along a divisor $D \in |2\Theta|$.

We have given an example with $a_*\omega_X = \mathcal{O}_A \oplus \hat{L}$, where $L^\vee$ is an ample line bundle of type $(1, 2)$. We have not been able to rule out the cases in which $\mathcal{H} \neq 0$. For example, is it possible to have $a_*\omega_X = \mathcal{O}_A \oplus \mathcal{F}$, with $\mathcal{F}$ as follows?

Example. Let (A, L) be a general polarized abelian surface of type $(1, 3)$ and $x \in A$ a closed point. Then $h^i(A, L \otimes \mathcal{I}_x \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$. Let $\mathcal{E} = \widehat{L \otimes \mathcal{I}_x}$ and $\mathcal{F} = \mathcal{E}^\vee$. Then we have an exact sequence

$$0 \to P_x^\vee \to \hat{L}^\vee \to \mathcal{F} \to 0.$$

Acknowledgments

We are indebted to Rita Pardini for useful conversations on this subject. This work was completed during visits of the first author to the University of California, Riverside and the University of Utah. He thanks both universities for their hospitality.

References

[Barth 1987] W. Barth, "Abelian surfaces with $(1, 2)$-polarization", pp. 41–84 in *Algebraic geometry* (Sendai, 1985), edited by T. Oda, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987. MR 89i:14038 Zbl 0639.14023

[Ciliberto and Mendes Lopes 2002] C. Ciliberto and M. Mendes Lopes, "On surfaces with $p_g = q = 2$ and non-birational bicanonical maps", *Adv. Geom.* **2**:3 (2002), 281–300. MR 2004d:14053 Zbl 1027.14018

[Debarre 1982] O. Debarre, "Inégalités numériques pour les surfaces de type général", *Bull. Soc. Math. France* **110**:3 (1982), 319–346. MR 84f:14026 Zbl 0543.14026

[Hacon and Pardini 2002] C. D. Hacon and R. Pardini, "Surfaces with $p_g = q = 3$", *Trans. Amer. Math. Soc.* **354**:7 (2002), 2631–2638. MR 2003a:14057 Zbl 1009.14004

[Harris 1992] J. Harris, *Algebraic geometry*, Graduate Texts in Mathematics **133**, Springer, New York, 1992. MR 93j:14001 Zbl 0779.14001

[Manetti 2003] M. Manetti, "Surfaces of Albanese general type and the Severi conjecture", *Math. Nachr.* **261/262** (2003), 105–122. MR 2004m:14081 Zbl 1044.14017

[Miranda 1985] R. Miranda, "Triple covers in algebraic geometry", *Amer. J. Math.* **107**:5 (1985), 1123–1158. MR 86k:14008 Zbl 0611.14011

[Mukai 1981] S. Mukai, "Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves", *Nagoya Math. J.* **81** (1981), 153–175. MR 82f:14036 Zbl 0417.14036

[Mumford 1970] D. Mumford, *Abelian varieties*, Tata Studies in Math. **5**, Tata Institute of Fundamental Research, Bombay, and Oxford Univ. Press, London, 1970. MR 44 #219 Zbl 0223.14022

[Pirola 2002] G. P. Pirola, "Surfaces with $p_g = q = 3$", *Manuscripta Math.* **108**:2 (2002), 163–170. MR 2003f:14043 Zbl 0997.14009

[Zucconi 2003] F. Zucconi, "Surfaces with $p_g = q = 2$ and an irrational pencil", *Canad. J. Math.* **55**:3 (2003), 649–672. MR 2004d:14055 Zbl 1053.14042

Received May 13, 2004.

JUNGKAI A. CHEN
DEPARTMENT OF MATHEMATICS
NATIONAL TAIWAN UNIVERSITY
1 SEC. 4, ROOSEVELT RD.
TAIPEI, 106
TAIWAN

jkchen@math.ntu.edu.tw

CHRISTOPHER D. HACON
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
155 SOUTH 1400 EAST, JWB 233
SALT LAKE CITY, UTAH 84112-0090
UNITED STATES

hacon@math.utah.edu