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EXISTENCE OF TIME-PERIODIC SOLUTIONS TO THE NAVIER–STOKES EQUATIONS AROUND A MOVING BODY

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We demonstrate the existence of time-periodic motions of an incompressible Navier–Stokes fluid subject to a time-periodic body force, occupying the region exterior to a body that performs a periodic rigid motion of same period.

1. Introduction

Consider a rigid body $\mathscr{B}$ moving through an infinitely extended Navier–Stokes liquid $\mathscr{L}$, which is subject to an external force f. If Ω is the three-dimensional region exterior to $\mathscr{B}$, with boundary Σ, the equations of motion of $\mathscr{L}$ with respect to a frame attached to $\mathscr{B}$ and with the origin at the center of mass of $\mathscr{B}$ are

$$
\text{(1–1)} \qquad \begin{cases} \left.\begin{aligned} &\partial_t u = \nu \Delta u - \nabla p + (V - u)\cdot\nabla u - \omega \times u + f \\ &\nabla \cdot u = 0 \end{aligned}\right\} \text{in } \Omega\times\mathbb{R}, \\ u = V \qquad \text{on } \Sigma\times\mathbb{R}, \\ \lim_{|x|\to\infty} u(x,t) = 0 \quad \text{for } t \in \mathbb{R}; \end{cases}
$$

see [Galdi 2002]. Here $u = u(x,t)$ is the velocity field of the liquid, ν is the kinematic viscosity coefficient of $\mathscr{L}$, and $p = p(x,t)$ is the pressure field divided by the (constant) density of $\mathscr{L}$, and ω is the angular velocity of $\mathscr{B}$. The velocity field associated with the rigid motion of $\mathscr{B}$ is

$$V(x,t) = \xi(t) + \omega(t)\times x,$$

where ξ is the velocity of the center of mass of $\mathscr{B}$.

The question we address is the following. Assume that $\mathscr{B}$ moves periodically with period T (that is, ξ and ω are periodic functions of time), and that f is also periodic with the same period. Then, does the fluid execute a time-periodic motion

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of period T? Though simple in its formulation and physically significant, this problem seemingly has been solved only when $\mathcal{B}$ is at rest [Maremonti 1991a; 1991b; Kozono and Nakao 1996; Maremonti and Padula 1996; Salvi 1995; Yamazaki 2000; Galdi and Sohr 2004]. (See also [Morimoto 1971/72] and the references therein for the case where Ω is a bounded domain.) The methods adopted in all these papers do not extend directly to the case when $\mathcal{B}$ undergoes periodic motion; they basically revolve around the properties of solutions of the *linearized* problem, $\mathcal{P}_L$, obtained by disregarding the nonlinear term $u \cdot \nabla u$ in equation (1–1)$_1$. If the body is at rest, $\mathcal{P}_L$ involves only the Stokes operator, $A = -P\Delta$ (where P is the Helmholtz projection), and it reduces to the well-known Stokes problem. If $\mathcal{B}$ is in motion, by contrast, $\mathcal{P}_L$ involves the linear operator

$$A + (\xi + \omega \times x) \cdot \nabla u - \omega \times u.$$

(In the appropriate function class, we have $P\big((\xi + \omega \times x) \cdot \nabla u - \omega \times u\big) = (\xi + \omega \times x) \cdot \nabla u - \omega \times u$.) Then, especially due to the presence of the *unbounded* coefficient $\omega \times x$, the linearized problem is much more complicated than the Stokes problem and its functional properties, to date, are not completely understood; see [Hishida 1999; Galdi 2003; Farwig et al. 2004]. One must therefore resort to other approaches. Note that exactly the same difficulty arises in the study of the initial-boundary and boundary value problems associated to (1–1), for whose results and corresponding methods we refer to [Hishida 1999; Galdi and Silvestre 2002; Galdi 2003; Silvestre 2004] and references therein.

To our knowledge, even for the simpler case when $\omega \equiv 0$ and $\xi \not\equiv 0$ no results are available.

In this paper we show the existence of weak and strong periodic solutions to problem (1–1) in the case when $\mathcal{B}$ moves by an arbitrary time-periodic motion and f is time-periodic with the same period. We prove these results by means of the classical Faedo–Galerkin approach suitably coupled with an "invading domains" technique [Ladyzhenskaya 1969; Heywood 1980]. Specifically, in each bounded domain Ω_k of an increasing sequence of domains covering Ω, we show the existence of a periodic solution $(u^{(k)}, p^{(k)})$. This solution is "weak", in the sense of Leray and Hopf, for ξ, ω and f of arbitrary size in a suitable function class, and for an arbitrary exterior domain Ω. Moreover, if Ω is of class C^2 and the size of ξ, ω and f is appropriately restricted, we prove the existence of more regular solutions such that $du^{(k)}/dt, u^{(k)}, \nabla u^{(k)}, D^2 u^{(k)} \in L^2(\Omega_k \times [0, T])$, and $p^{(k)}, \nabla p^{(k)} \in L^2(\Omega_k \times [0, T])$. Because the term $(\xi + \omega \times x) \cdot \nabla u^{(k)} - \omega \times u^{(k)}$ possesses nice functional properties on each *bounded* Ω_k [Galdi and Silvestre 2002; 2005; Silvestre 2004], we are able to obtain estimates for $u^{(k)}$, *uniformly* in k, that allow us to pass to the limit $k \to \infty$ and to prove that weak (Theorem 3.2) and strong (Theorem 4.1) periodic solutions to (1–1) exist in the whole of Ω. In the special case $\xi \equiv \omega \equiv 0$, our results improve those previously known, in that we require no regularity on Ω (versus the C^2-regularity needed in [Maremonti and Padula 1996]) in the case of weak solutions, and only C^2-regularity (versus the C^3-regularity needed in [Salvi 1995]) in the case of strong solutions.

The *uniqueness* problem is left open, even for strong solutions. As shown in [Galdi and Sohr 2004] for the simpler instance $\xi \equiv \omega \equiv 0$, uniqueness is not related to the local regularity of solutions but, rather, to their asymptotic behavior *in space*. The determination of this latter for the case at hand appears to be a challenging question that will be treated elsewhere.

The paper is organized as follows. After recalling in Section 2 some notation and preparatory results, in Section 3 we show the existence of weak periodic solution, while Section 4 is dedicated to the existence of strong periodic solutions.

2. Notation and preparatory results

Let $\mathscr{A}$ be a domain of $\mathbb{R}^3$. We denote by $\delta(\mathscr{A})$ the diameter of $\mathscr{A}$ and, for $R > \delta(\mathscr{A})$, we set $\mathscr{A}_R = \mathscr{A} \cap B_R$ and $\mathscr{A}^R = \mathscr{A} \setminus \overline{\mathscr{A}_R}$, where $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$, and the bar denotes closure.

An *exterior domain* is the complement of the closure of a bounded domain in $\mathbb{R}^3$.

We shall use standard notation for function spaces [Adams 1975]. For instance, $L^q(\mathscr{A})$, $H^m(\mathscr{A}) := W^{m,2}(\mathscr{A})$, $H_0^m(\mathscr{A}) := W_0^{m,q}(\mathscr{A})$, etc., denote the usual Lebesgue and Sobolev spaces on the domain $\mathscr{A}$, with norms $\|\cdot\|_{q,\mathscr{A}}$ and $\|\cdot\|_{m,2,\mathscr{A}}$, respectively.

If G, H are second-order tensor fields and g, h are vector fields on $\mathscr{A}$, we set

$$(G, H)_{\mathscr{A}} = \int_{\mathscr{A}} G_{ij} H_{ij}, \quad (g, h)_{\mathscr{A}} = \int_{\mathscr{A}} g_i h_i,$$

whenever the integrals make sense. If there is no danger of confusion, we shall omit the subscript $\mathscr{A}$.

The trace space on $\partial\mathscr{A}$ for functions from $H^m(\mathscr{A})$ is denoted by $H^{m-1/2}(\partial\mathscr{A})$ and its norm by $\|\cdot\|_{m-1/2,\partial\mathscr{A}}$. Classical properties and results related to these spaces can be found in [Adams 1975; Galdi 1994a]. The following spaces of solenoidal functions will be needed:

$$\begin{aligned}
\mathfrak{D}(\mathscr{A}) &= \{\phi \in C_0^\infty(\mathscr{A}) : \nabla \cdot \phi = 0\}, \\
H(\mathscr{A}) &= \text{ completion of } \mathfrak{D}(\mathscr{A}) \text{ in the norm } \|\cdot\|_2, \\
V(\mathscr{A}) &= \text{ completion of } \mathfrak{D}(\mathscr{A}) \text{ in the norm } \|\nabla(\cdot)\|_2.
\end{aligned}$$

The dual space of $V(\mathscr{A})$ will be denoted by $V'(\mathscr{A})$ with norm $\|\cdot\|_{-1,\mathscr{A}}$ and the duality $\langle F, u\rangle_{\mathscr{A}}$ will indicate the value of $F \in V'(\mathscr{A})$ at $u \in V(\mathscr{A})$. If $\mathscr{A}$ is locally

Lipschitzian with outward unit normal N, we have

$$H(\mathscr{A}) = \{\Phi \in L^2(\mathscr{A}) : \nabla \cdot \Phi = 0 \text{ and } \Phi_{|\partial\mathscr{A}} \cdot N = 0\},$$
$$V(\mathscr{A}) = \{\Phi \in H^1_{loc}(\bar{\mathscr{A}}) : \nabla\Phi \in L^2(\mathscr{A}),\ \nabla \cdot \Phi = 0 \text{ and } \Phi_{|\partial\mathscr{A}} = 0\};$$

see [Temam 1984; Galdi 1994a]. The orthogonal complement of $H(\mathscr{A})$ in $L^2(\mathscr{A})$ is

$$G(\mathscr{A}) = \{\nabla\pi \in L^2(\mathscr{A}) : \pi \in H^1_{loc}(\bar{\mathscr{A}})\}.$$

The orthogonal projection of $L^2(\mathscr{A})$ onto $H(\mathscr{A})$ is denoted by P.

If X is a Banach space, we denote by $L^q(0, T; X)$ the space of all measurable functions from $[0, T]$ to X, such that $\int_0^T \|u(t)\|_X^p \, dt < \infty$, and by $C([0, T]; X)$ the space of continuous function from $[0, T]$ to X.

Lemma 2.1. *Let X_0, X_1, X be Hilbert spaces such that the injection of X_0 into X is compact and the injection of X into X_1 is continuous. Then the injection of the space*

$$\{v \in L^2(0, T; X_0) : \frac{dv}{dt} \in L^1(0, T; X_1)\}.$$

into $L^2(0, T; X)$ is compact.

Proof. See [Temam 1984]. □

The boundary velocity $u(x, t) = \xi(t) + \omega(t) \times x$ has a solenoidal extension:

Lemma 2.2. *Let Ω be an exterior domain of $\mathbb{R}^3$, and let $\xi, \omega \in H^1(0, T)$. Given $\epsilon > 0$, there exists a solenoidal function $\tilde{u} \in H^1(0, T; W^{m,q}(\Omega))$, $m \in \mathbb{N}$, $q \in [1, \infty]$, such that*

$$\|\tilde{u}\|_{H^1(0,T;W^{m,q}(\Omega))} \le C(\Sigma, m, q)(\|\xi\|_{H^1(0,T)} + \|\omega\|_{H^1(0,T)}),$$
$$\|\tilde{u}(t)\|_{m,q} \le C(\Sigma, T, m, q)(\|\xi\|_{H^1(0,T)} + \|\omega\|_{H^1(0,T)}) \quad \textit{for all } t \in [0, T].$$

Moreover,

$$\left| \int_{\Omega_R} v(x, t) \cdot \nabla\tilde{u}(x, t) \cdot v(x, t) dx \right| < \epsilon \|\nabla v(t)\|_2^2,$$

for all $v \in C([0, T]; V(\Omega_R))$, $t \in [0, T]$, $R > \delta(\mathscr{B})$, and $\tilde{u}$ is T-periodic if ξ and ω are T-periodic.

Proof. Using Lemma [Galdi 1994a, III.6.2], we consider a function $\eta_\alpha \in C_0^\infty(\overline{\Omega})$ such that $0 \le \eta_\alpha \le 1$, $\eta_\alpha = 1$ if $\operatorname{dist}(x, \Sigma) < e^{-1/\alpha}/2$, $\eta_\alpha = 0$ if $\operatorname{dist}(x, \Sigma) \ge 2e^{-1/\alpha}$, and $|\nabla\eta_\alpha(x)| \le \alpha \operatorname{dist}(x, \Sigma)$, for all $x \in \Omega$, with $\alpha > 0$. The extension is defined by

$$\tilde{u}(x, t) = -\nabla \times \left(\eta_\alpha(x)(\xi_i(t) x_{(i+1) \bmod 3} e_i + \tfrac{1}{2}|x|^2 \omega(t))\right).$$

Taking into account the properties of the function η_α [Galdi 1994b, Chapter IX], it is possible to choose α such that $\tilde{u}$ satisfies the desired properties for a given ϵ. □

We end this section with some fundamental estimates of suitable three-linear forms.

Lemma 2.3. *Let Ω be an exterior domain of $\mathbb{R}^3$, and let $v \in V(\Omega_R) \cap H^2(\Omega_R)$, $\omega \in H^1(0,T)$. Then, for any $\varepsilon > 0$ there is $C = C(\Omega, \varepsilon) > 0$ such that*

(i) $(v \cdot \nabla v, P\Delta v)_{\Omega_R} \le C\left(\|\nabla v\|_{2,\Omega_R}^4 + \|\nabla v\|_{2,\Omega_R}^6\right) + \varepsilon \|P\Delta v\|_{2,\Omega_R}^2$ *and*

(ii) $\big((\omega \times v - \omega \times x \cdot \nabla v), P\Delta v\big)_{\Omega_R} \le C\left(\|\omega\|_{H^1(0,T)} + \|\omega\|_{H^1(0,T)}^2\right)\|\nabla v\|_{2,\Omega_R}^2 + \varepsilon \|P\Delta v\|_{2,\Omega_R}^2$.

Proof. The inequality in (i) is well known; see [Heywood 1980], for example. The proof of (ii) is given in [Galdi and Silvestre 2005]. □

3. Existence of periodic weak solutions

Denote by $\mathcal{D}_{T,p}$ the class of functions Φ that are infinitely differentiable in $\Omega \times [0,T]$, of compact support in Ω, and satisfying $\operatorname{div}\Phi(x,t) = 0$ in $\Omega \times [0,T]$ and $\Phi(x,0) = \Phi(x,T)$ in Ω. If we formally multiply through both sides of $(1\text{–}1)_1$ by $\Phi \in \mathcal{D}_{T,p}$ and integrate by parts over $\Omega \times [0,T]$, we obtain

$$\big(u(T) - u(0), \Phi(0)\big) = \int_0^T \big((u, \Phi_t) - \nu(\nabla u, \nabla\Phi) + ((V-u)\cdot\nabla u, \Phi) - (\omega \times u, \Phi) + \langle f, \Phi\rangle\big).$$

Thus, if u is time-periodic of period T, the right-hand side of this equation vanishes. Conversely, if u is a sufficiently regular field (in space and time) for which the right-hand side of the relation above vanishes for all $\Phi \in \mathcal{D}_{T,P}$, it follows by standard arguments that u satisfies $(1\text{–}1)_{1,2}$ for some pressure field p and that $u(0) = u(T)$. We are thus led to:

Definition 3.1. A vector field u is a *periodic weak solution* to (1–1) if

(i) $u - \tilde{u} \in L^2(0,T; V(\Omega))$, where $\tilde{u}$ is the extension given in Lemma 2.2;

(ii) for all $\Phi \in \mathcal{D}_{T,p}$,

$$\int_0^T \big((u, \Phi_t) - \nu(\nabla u, \nabla\Phi) + ((V-u)\cdot\nabla u, \Phi) - (\omega \times u, \Phi) + \langle f, \Phi\rangle\big) = 0. \tag{3–1}$$

Remark. It is easy to show that, if $f \in L^1(0,T; V'(\Omega))$ and $\xi, \omega \in H^1(0,T)$, every periodic weak solution satisfies

$$\frac{du}{dt} \in L^1(0,T; V'(\Omega_R)) \quad \text{for all } R > \delta(\mathcal{B}),$$

and so, in particular,

$$u \in C([0,T]; V'(\Omega_R)) \quad \text{for all } R > \delta(\mathcal{B}).$$

In fact, set $\Phi = \varphi\psi$ in (3–1), where $\varphi \in \mathscr{D}(\Omega)$ and $\psi \in C_0^\infty(0,T)$. We obtain

$$\int_0^T (u(t),\varphi)\psi'(t) = -\int_0^T G_\varphi(t)\psi(t) \quad \text{for all } \psi \in C_0^\infty(0,T),$$

where

$$G_\varphi(t) = -\nu(\nabla u, \nabla\varphi) + ((V-u)\cdot\nabla u, \varphi) - (\omega\times u, \varphi) + \langle f, \varphi\rangle.$$

Using the inequality

$$\|u\|_{2,\Omega_R} \le C(\Omega_R)\big(\|\nabla u\|_2 + \|\tilde{u}\|_{1,2}\big),$$

along with the assumption on f, we obtain for a.a. $t \in [0,T]$

$$|G_\varphi(t)| \le C(\Omega_R,\nu) \\ \times \big((1+\|\xi\|_{H^1(0,T)}+\|\omega\|_{H^1(0,T)})\|\nabla u\|_{2,\Omega_R}+\|\nabla u_2\|_2^2+\|\tilde{u}\|_{1,2}^2+\|f\|_{-1}\big)\|\varphi\|_{1,2,\Omega_R}.$$

Thus, with the help of Definition 3.1(ii), we find $G_\varphi(t) = \langle g(t),\varphi\rangle$ with $g \in L^1(0,T;V'(\Omega_R))$ and

$$\frac{d}{dt}(u,\varphi) = \langle g,\varphi\rangle$$

in the sense of distributions on $[0,T]$. The desired property is then proved.

The objective of this section is to show:

Theorem 3.2. *Let Ω be an exterior domain of $\mathbb{R}^3$. Let $\xi,\omega \in H^1(0,T)$ with $\xi(0)=\xi(T)$, $\omega(0)=\omega(T)$, and let $f \in L^2(0,T;V'(\Omega))$ be periodic in time of period T. Then there exists at least one periodic weak solution to (1–1).*

For the proof we use the "invading domains" procedure (see [Heywood 1980], for instance), in two steps. In the first, using the method of [Prouse 1963], we show the existence of a periodic weak solution u_m on *bounded* domains Ω_{R_m}, $R_m > \delta(\mathscr{B})$, $m \in \mathbb{N}$, and establish suitable *a priori* estimates. In the second step, we let $R_m \to \infty$ and show that u_m converges, in a suitable sense, to a periodic weak solution to (1–1).

Step 1: Construction of approximating periodic weak solutions in $\boldsymbol{\Omega_{R_m}}$. Let $\mathscr{S} = \{R_m : m \in \mathbb{N}\}$ be an increasing and unbounded sequence of positive numbers with $R_1 > \delta(\mathscr{B})$ and let $\{\Omega_R : R \in \mathscr{S}\}$ with $\bigcup_{R\in\mathscr{S}}\Omega_R = \Omega$ be the corresponding sequence of bounded domains covering Ω.

In each Ω_R, $R \in \mathscr{S}$, we shall look for a T-periodic solution u_R in the form $u_R = v_R + \tilde{u}$, with $\tilde{u}$ given by Lemma 2.2, and v_R (in the appropriate functional class) satisfying the identity

$$\frac{d}{dt}(v_R,\varphi)_{\Omega_R} = -\nu(\nabla v_R,\nabla\varphi)_{\Omega_R} + ((V-v_R)\cdot\nabla v_R,\varphi)_{\Omega_R} - (v_R\cdot\nabla\tilde{u},\varphi)_{\Omega_R} - (\tilde{u}\cdot\nabla v_R,\varphi)_{\Omega_R} - (\omega\times v_R,\varphi)_{\Omega_R} + \langle\tilde{f},\varphi\rangle_{\Omega_R} \tag{3–2}$$

for all $\varphi\in\mathfrak{D}(\Omega_R)$, and a.a. $t\in\,]0,T[$, where

$$\tilde{f} = f + \nu\Delta\tilde{u} - \partial_t\tilde{u} - \tilde{u}\cdot\nabla\tilde{u} + V\cdot\nabla\tilde{u} - \omega\times\tilde{u}. \tag{3–3}$$

Under the hypotheses of Theorem 3.2 and with the help of Lemma 2.2, we deduce that $\tilde{f}$ is periodic of period T, $\tilde{f}\in L^2(0,T;V'(\Omega))$, and that

$$\begin{aligned}\|\tilde{f}\|_{L^2(0,T;V'(\Omega_R))} &\le \|\tilde{f}\|_{L^2(0,T;V'(\Omega))}\\ &\le C(\Sigma)(\|\xi\|_{H^1(0,T)}+\|\omega\|_{H^1(0,T)}) + \|f\|_{L^2(0,T;V'(\Omega))}.\end{aligned} \tag{3–4}$$

For each $R\in\mathcal{S}$, we consider a base $\{w_{Ri}\}_{i\in\mathbb{N}}$ of $V(\Omega_R)$ orthonormal in $H(\Omega_R)$. We let

$$v_{Rk}(x,t) = \sum_{i=1}^{k} c_{Rki}(t)w_{Ri}(x),$$

where the coefficients $c_{Rk}=\{c_{Rk1},\dots,c_{Rkk}\}$ are required to solve the system of ordinary differential equations

$$\frac{dc_{Rkj}}{dt} = \sum_{i=1}^{k} A_{ij}(t)c_{Rki} + \sum_{i,l=1}^{k} B_{ilj}c_{Rki}c_{Rkl} + C_j(t), \qquad j=1,\dots,k, \tag{3–5}$$

where

$$\begin{aligned}A_{ij} &= -\nu(\nabla w_{Ri},\nabla w_{Rj})_{\Omega_R} - (\omega\times w_{Ri},w_{Rj})_{\Omega_R} + (V\cdot\nabla w_{Ri},w_{Rj})_{\Omega_R}\\ &\qquad - (w_{Ri}\cdot\nabla\tilde{u},w_{Rj})_{\Omega_R} - (\tilde{u}\cdot\nabla w_{Ri},w_{Rj})_{\Omega_R},\\ B_{ilj} &= -(w_{Ri}\cdot\nabla w_{Rl},w_{Rj})_{\Omega_R},\\ C_j &= \langle\tilde{f},w_{Rj}\rangle_{\Omega_R}.\end{aligned}$$

Following [Prouse 1963], we begin to show the existence of a T-periodic solution to the system (3–5).

Lemma 3.1. *System (3–5) has a solution $c_{Rk}\in H^1(0,T)$, such that $c_{Rk}(0)=c_{Rk}(T)$.*

Proof. For each $R\in\mathcal{S}$ and each $k\in\mathbb{N}$, we choose an initial velocity $v_{0Rk}\in$ *span*$\{w_{R1},\dots,w_{Rk}\}$ and set $c_{Rkj}(0)=c_{0Rkj}:=(w_{Rj},v_{0Rk})_{\Omega_R}$. Since $\xi,\omega\in H^1(0,T)$ and $f\in L^2(0,T,V'(\Omega))$, the system (3–5) has a unique solution $c_{Rk}\in H^1(0,T_{Rk})$ for some $T_{Rk}\le T$. Multiplying (3–5) by c_{Rkj}, summing over j, integrating by parts, and recalling that

$$((V_{Rk}-v_{Rk})\cdot\nabla v_{Rk},v_{Rk}) = (\omega\times v_{Rk},v_{Rk}) = (\tilde{u}\cdot\nabla v_{Rk},v_{Rk}) = 0, \tag{3–6}$$

(where we have omit the subscript Ω_R for simplicity), we see that v_{Rk} satisfies the equation

$$\frac{1}{2}\frac{d}{dt}\|v_{Rk}\|_2^2+\nu\|\nabla v_{Rk}\|_2^2=\langle\tilde f, v_{Rk}\rangle-(v_{Rk}\cdot\nabla\tilde u, v_{Rk}). \tag{3–7}$$

Using Lemma 2.2 with $\epsilon=\frac{1}{4}\nu$, we get

$$-(v_{Rk}\cdot\nabla\tilde u, v_{Rk})\le\tfrac{1}{4}\nu\|\nabla v_{Rk}\|_2^2$$

and since

$$\langle\tilde f, v_{Rk}\rangle_{\Omega_R}\le\|\tilde f\|_{-1}\|\nabla v_{Rk}\|_2\le\tfrac{1}{4}\nu\|\nabla v_{Rk}\|_2^2+C(\nu)\|\tilde f\|_{-1}^2,$$

we obtain

$$\frac{d}{dt}\|v_{Rk}\|_2^2+\nu\|\nabla v_{Rk}\|_2^2\le C(\nu)\|\tilde f\|_{-1}^2. \tag{3–8}$$

Using the Poincaré inequality

$$\|\nabla w\|_2\ge\frac{C}{R}\|w\|_2,\quad w\in H_0^1(\Omega_R), \tag{3–9}$$

with C a positive, absolute constant, we get

$$\frac{d}{dt}\|v_{Rk}\|_2^2+\frac{\nu C_1}{R^2}\|v_{Rk}\|_2^2\le C_2(\nu)\|\tilde f\|_{-1}^2.$$

Consequently,

$$e^{\nu C_1 t/R^2}\|v_{Rk}(t)\|_2^2\le\|v_{0Rk}\|_2^2+C_2(\nu)\int_0^{T_{Rk}}e^{\nu C_1\tau/R^2}\|\tilde f(\tau)\|_{-1}^2d\tau \tag{3–10}$$

for all $t\in[0,T_{Rk}]$. From this inequality it follows that

$$\|v_{Rk}(t)\|_2^2\le\|v_{0Rk}\|_2^2+C_2(\nu)\int_0^{T}e^{\nu C_1\tau/R^2}\|f(\tau)\|_{-1}^2d\tau \tag{3–11}$$

for all $t\in[0,T_{Rk}]$. Using the orthogonality properties of $\{w_{R1},\dots,w_{Rk}\}$ we have $|c_{Rk}(t)|=\|v_{Rk}(t)\|_2$, from which we conclude that $T_{Rk}=T$.

Let ϱ be such that

$$\varrho^2\ge\frac{C_2(\nu)\int_0^T e^{\nu C_1\tau/R^2}\|\tilde f(\tau)\|_{-1}^2d\tau}{1-e^{-\nu C_1 T/R^2}} \tag{3–12}$$

and let $\mathbb{B}_\varrho^k$ be the ball of radius ϱ in $\mathbb{R}^k$. In view of (3–10) and (3–12), if $|c_{Rk}(0)|=\|v_{0Rk}\|_2\le\varrho$ then

$$|c_{Rk}(T)|=\|v_{Rk}(T)\|_2\le\varrho,$$

and thus the map $\mathcal{T}:\mathbb{B}_\varrho^k\to\mathbb{B}_\varrho^k$ such that $\mathcal{T}(c_{0Rk})=c_{Rk}(T)$ is well defined. By the same procedure used in [Prouse 1963], we can show that the map $\mathcal{T}$ is continuous, and therefore has a fixed point; that is, there exists a solution to (3–5) such that $c_{Rk}(0) = c_{Rk}(T)$. □

Here are some useful estimates for the approximating solution in Ω_R.

Lemma 3.2. *There exists a positive constant* $C = C(\nu, \Sigma)$ *such that*

$$\int_0^T \|\nabla v_{Rk}(\tau)\|_2^2 d\tau \le C\big(\|f\|^2_{L^2(0,T;V'(\Omega))} + \|\xi\|^2_{H^1(0,T)} + \|\omega\|^2_{H^1(0,T)}\big). \tag{3–13}$$

Moreover, there exists a positive constant C *independent of* $k \in \mathbb{N}$ *such that*

$$\|v_{Rk}(t)\|_2 \le C \quad \text{for all } t \in [0, T] \text{ and all } k \in \mathbb{N}. \tag{3–14}$$

Proof. Since, by Lemma 3.1, v_{Rk} is T-periodic, integrating (3–8) over $[0, T]$ we find

$$\int_0^T \|\nabla v_{Rk}(\tau)\|_2^2\, d\tau \le C(\nu) \int_0^T \|\tilde{f}(\tau)\|_{-1}^2\, d\tau \tag{3–15}$$

and then we use (3–4). This proves (3–13).

Equation (3–14) is an immediate consequence of (3–11). □

We can now easily show the existence of a periodic weak solution on each Ω_R, for $R \in \mathcal{S}$. Actually, using Lemma 3.2 and well-known procedures (see [Galdi 2000], for example), we prove the existence of a field v_R and of a subsequence, again denoted by $\{v_{Rk}\}_{k\in\mathbb{N}}$, such that

$$\begin{aligned}
&v_R \in L^2(0, T; V(\Omega_R)) \cap L^\infty(0, T; H(\Omega_R)),\\
&v_{Rk} \to v_R \text{ weakly in } L^2(0, T; V(\Omega_R)),\\
&v_{Rk} \to v_R \text{ strongly in } L^2(0, T; H(\Omega_R)),\\
&v_{Rk}(t) \to v_R(t) \text{ weakly in } L^2(\Omega_R), \text{ for all } t \in [0, T].
\end{aligned} \tag{3–16}$$

Recalling that $v_{Rk}(0) = v_{Rk}(T)$, for $k \in \mathbb{N}$, the last condition in (3–16) implies that $v_R(0) = v_R(T)$, namely, that v_R is T-periodic. Moreover, in view of $(3–16)_2$ and of (3–13), we find

$$\int_0^T \|\nabla v_R(\tau)\|^2_{2,\Omega_R} d\tau \le C(\|f\|^2_{L^2(0,T;V'(\Omega))} + \|\xi\|^2_{H^1(0,T)} + \|\omega\|^2_{H^1(0,T)}), \tag{3–17}$$

with $C = C(\nu, \Omega) > 0$. Finally, coupling (3–16) with classical arguments, we can prove that, for all $R \in \mathcal{S}$, v_R satisfies condition (3–2). Using $(3–16)_1$ along with a standard procedure (see [Temam 1984], for instance), we can also show that the right-hand side of (3–2), with $v \equiv v_R$, defines a continuous (linear) functional on $V(\Omega_R)$, and that

$$\frac{d}{dt}(v_R, \varphi)_{\Omega_R} = \Big\langle \frac{dv_R}{dt}, \varphi \Big\rangle_{\Omega_R} \quad \text{for all } \varphi \in V(\Omega_R) \text{ and a.a. } t \in\,]0, T[,$$

where

$$\frac{dv_R}{dt} \in L^{4/3}(0,T;V'(\Omega_R)).$$

Consequently, from (3–2) we deduce that

$$\left\langle \frac{dv_R}{dt}, \varphi \right\rangle_{\Omega_R} = -\nu(\nabla v_R, \nabla\varphi)_{\Omega_R} + ((V - v_R)\cdot\nabla v_R, \varphi)_{\Omega_R} - (v_R\cdot\nabla\tilde{u}, \varphi)_{\Omega_R} - (\tilde{u}\cdot\nabla v_R, \varphi)_{\Omega_R} - (\omega\times v_R\cdot\varphi)_{\Omega_R} + \langle\tilde{f}, \varphi\rangle_{\Omega_R}, \tag{3–18}$$

for all $\varphi \in V(\Omega_R)$ and a.a. $t \in]0,T[$.

***Step 2: Convergence of the sequence $\{v_R + \tilde{u}\}_{R\in\mathcal{S}}$ to a periodic weak solution to* (1–1).** We extend v_R by zero outside Ω_R, for $R \in \mathcal{S}$, and continue to denote the extension by v_R. Clearly, the extended fields satisfy (3–17) and (3–18). We shall next prove some appropriate estimates for them.

Lemma 3.3. *Let $R_0 > \delta(\mathcal{B})$. There exists a positive constant C depending only on the data and R_0 such that*

$$\int_0^T \|v_R(t)\|^2_{2,\Omega_{R_0}}\,dt + \int_0^T \left\|\frac{dv_R}{dt}(t)\right\|_{V'(\Omega_{R_0})} dt \le C,$$

for all $R > R_0$, $R \in \mathcal{S}$.

Proof. The estimate on v_R is an obvious consequence of (3–9) and of (3–17). Let φ be any function in $\mathcal{D}(\Omega_{R_0})$. From Hölder's inequality, Lemma 2.2, (3–9) and the Sobolev inequalities

$$\|w\|_6 \le C\|\nabla w\|_2, \quad \|w\|_3 \le C\,R\,\|\nabla w\|_2, \quad w \in V(\Omega_R)$$

with C a positive, absolute constant, we find

$$\begin{aligned}
-(\nabla v_R, \nabla\varphi) &\le \|\nabla v_R\|_2\|\nabla\varphi\|_2,\\
(V\cdot\nabla v_R, \varphi) &\le \|V\|_{3,\Omega_{R_0}}\|\nabla v_R\|_2\|\varphi\|_6\\
&\qquad\le C(\Sigma,T)(R_0\|\xi\|_{H^1(0,T)} + R_0^2\|\omega\|_{H^1(0,T)})\|\nabla v_R\|_2\|\nabla\varphi\|_2,\\
-(v_R\cdot\nabla v_R, \nabla\varphi) &\le \|v_R\|_6\|\nabla v_R\|_2\|\varphi\|_3 \le C\,R_0\|\nabla v_R\|_2^2\|\nabla\varphi\|_2,\\
-(v_R\cdot\nabla\tilde{u}, \varphi) &\le \|\nabla\tilde{u}\|_{3/2}\|v_R\|_6\|\varphi\|_6\\
&\qquad\le C\,(\Sigma,T)(\|\xi\|_{H^1(0,T)} + \|\omega\|_{H^1(0,T)})\|\nabla v_R\|_2\|\nabla\varphi\|_2,\\
-(\tilde{u}\cdot\nabla v_R, \varphi) &\le \|\tilde{u}\|_3\|\nabla v_R\|_2\|\varphi\|_6\\
&\qquad\le C\,(\Sigma,T)(\|\xi\|_{H^1(0,T)} + \|\omega\|_{H^1(0,T)})\|\nabla v_R\|_2\|\nabla\varphi\|_2,\\
-(\omega\times v_{Rk}, \varphi) &\le |\omega|\;\|v\|_2\|\varphi\|_2 \le C(\Sigma,T)\,R_0^2\|\omega\|_{H^1(0,T)}\|\nabla v_R\|_2\|\nabla\varphi\|_2.
\end{aligned}$$

The lemma follows from these inequalities and from (3–18), (3–17) and (3–4). □

Lemma 3.4. *There exists a field v and a sequence $\{v_\rho : \rho \in \mathcal{S}' \subset \mathcal{S}\}$ such that, for all $R_0 > \delta(\mathcal{B})$,*

$$
\begin{aligned}
&v \in L^2(0,T;V(\Omega)) \cap L^2(0,T;L^2(\Omega_{R_0}))\\
&v_\rho \to v \quad \textit{weakly in } L^2(0,T;V(\Omega)),\\
&v_\rho \to v \quad \textit{strongly in } L^2(0,T;L^2(\Omega_{R_0})).
\end{aligned}
\tag{3–19}
$$

Proof. From the bound (3–17), we deduce that there is a subsequence of $\{v_R\}$, again denoted by $\{v_R\}$, and a field $v \in L^2(0,T;V(\Omega))$ for which $(3\text{–}19)_2$ holds. Fix $R_0 > \delta(\mathcal{B})$ and apply Lemma 2.1 with $X_0 = H^1(\Omega_{R_0})$, $X = L^2(\Omega_{R_0})$ and $X_1 = V'(\Omega_{R_0})$. There follows the existence of a subsequence, still denoted by $\{v_R\}$, satisfying conditions $(3\text{–}19)_3$. This latter subsequence may depend on R_0. However, covering Ω, with an increasing sequence of bounded domains and using Cantor diagonalization method, we may select a subsequence $\{v_\rho\}$ for which the property $(3\text{–}19)_3$ holds for all R_0. The lemma is therefore proved. □

In conclusion to this section, we shall prove that $u \equiv v + \tilde{u}$ is a periodic weak solution to (1–1). In view of Lemmas 2.2 and 3.4, we have only to show that u satisfies (3–1). To this end, set $u_\rho = v_\rho - \tilde{u}$ in (3–2), where $\{v_\rho\}$ is the sequence of Lemma 3.4. Multiplying both sides of the resulting equation by an arbitrary $\psi \in C^1[0,T]$ such that $\psi(0) = \psi(T)$, integrating in time between 0 and T and recalling that $u_\rho(0) = u_\rho(T)$, we obtain

$$
\int_0^T \left\{(u_\rho, \phi_t) - \nu(\nabla u_\rho, \nabla \phi) + ((V-u)\cdot\nabla u_\rho, \phi) - (\omega \times u_\rho, \phi) + \langle f, \phi\rangle\right\},
\tag{3–20}
$$

with $\phi = \psi\varphi$, for any fixed $\varphi \in \mathcal{D}(\Omega)$ and all sufficiently large ρ. We then pass to the limit $\rho \to \infty$ in this relation and use the convergence properties stated in (3–19). It is routine to show (see [Galdi 2000], for example) that u satisfies (3–1), with $\Phi = \phi$. However, any $\Phi \in \mathcal{D}_{T,p}$ can be approximated, together with its first derivatives, uniformly pointwise by suitable linear combinations of such a ϕ [Galdi 2000], and so the proof of Theorem 3.2 is completed.

4. Existence of Periodic Strong Solutions

We now show that if Ω and the data are more regular and if these latter are sufficiently small, then a periodic strong solution exists.

Theorem 4.1. *Let Ω be an exterior domain of $\mathbb{R}^3$ of class C^2. Let $\xi, \omega \in H^1(0,T)$ with $\xi(0) = \xi(T)$, $\omega(0) = \omega(T)$, and $f \in L^2(0,T;L^2(\Omega) \cap V'(\Omega))$. There is a positive constant $C_0 = C_0(\Omega, \nu)$ such that if*

$$
D_\alpha \equiv \alpha\,\|f\|_{L^2(0,T;V'(\Omega))} + \|f\|_{L^2(0,T;L^2(\Omega))} + \alpha\,(\|\xi\|_{H^1(0,T)} + \|\omega\|_{H^1(0,T)}) \le C_0,
\tag{4–1}
$$

where $\alpha = \max\{1, 1/T\}$, *then there exists* (u, p) *satisfying conditions* (i) *and* (ii) *of Definition 3.1 and the following properties for all* $R > \delta(\mathcal{B})$:

$$
\begin{aligned}
&u \in C([0,T]; L^2(\Omega_R)), \quad \frac{du}{dt} \in L^2(0,T; L^2(\Omega_R)) \\
&\nabla u \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)), \\
&\nabla p \in L^2(0,T; L^2(\Omega)).
\end{aligned} \tag{4–2}
$$

If, in particular, $\omega(t) \equiv 0$, *we have, in addition,*

$$\frac{du}{dt} \in L^2(0,T; L^2(\Omega)).$$

In any case, $u(0) = u(T)$ *and* (u, p) *satisfies* $(1–1)_1$ *a.e. in* $\Omega \times [0,T]$.

In order to construct such a solution we choose the basis functions w_{Rj} in (3–5) as the eigenfunctions of the Stokes problem:

$$P\Delta w_{Rj} = -\lambda_{Rj} w_{Rj} \quad w_{Rj} \in V(\Omega_R) \cap H^2(\Omega_R). \tag{4–3}$$

Clearly, by the same argument of Step 1 of Section 3, we show the existence of a periodic approximating solution v_{Rk} to (3–5) that satisfies the estimates of Lemma 3.2. Under the assumptions (4–1), we shall now establish further suitable estimates on the first and second spatial derivatives of the approximating solutions.

Lemma 4.1. *Let condition (4–1) hold. Then*

$$\|\nabla v_{Rk}(t)\|_{2,\Omega_R} + \int_0^T \|D^2 v_{Rk}(\tau)\|^2_{L^2(\Omega_R)} d\tau \le C, \tag{4–4}$$

with C depending only upon the data.

Proof. Multiplying both sides of (3–5) by $\lambda_{Rkj} c_{Rkj}$, summing over j and taking into account (4–3), we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v_{Rk}\|_2^2 + \|P\Delta v_{Rk}\|_2^2 \\
&\quad = -(\xi \cdot \nabla v_{Rk}, P\Delta v_{Rk}) + (\tilde{u} \cdot \nabla v_{Rk}, P\Delta v_{Rk}) + (v_{Rk} \cdot \nabla \tilde{u}, P\Delta v_{Rk}) \\
&\qquad -(\tilde{f}, P\Delta v_{Rk}) + (v_{Rk} \cdot \nabla v_{Rk}, P\Delta v_{Rk}) + ((\omega \times v_{Rk} - \omega \times x \cdot \nabla v_{Rk}), P\Delta v_{Rk}),
\end{aligned} \tag{4–5}
$$

where all integrals are taken over Ω_R. Using the Schwarz inequality along with Lemmas 2.2 and 2.3, we easily obtain from (4–5)

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla v_{Rk}\|_2^2 + \nu \|P\Delta v_{Rk}\|_2^2 \\
&\qquad \le C(\Omega, \nu)\left((\gamma + \gamma^2)\|\nabla v_{Rk}\|_2^2 + \|\nabla v_{Rk}\|_2^4 + \|\nabla v_{Rk}\|_2^6\right) + C(\nu)\|\tilde{f}\|_2^2,
\end{aligned} \tag{4–6}
$$

where $\gamma = \|\xi\|_{H^1(0,T)} + \|\omega\|_{H^1(0,T)}$. In order to obtain the desired estimate from this differential inequality, we need an upper bound for the quantity $\|\nabla v_{Rk}(0)\|_2$. It is clear that

$$\|\nabla v_{Rk}(0)\|_2 = \|\nabla v_{Rk}(T)\|_2. \tag{4–7}$$

From (3–15) and the mean value theorem for continuous functions, we deduce the existence of $\bar{t} \in [0, T]$ such that

$$\|\nabla v_{Rk}(\bar{t})\|_2^2 = \frac{1}{T}\int_0^T \|\nabla v_{Rk}(\tau)\|_2^2 \, d\tau \le \frac{C(\nu)}{T}\int_0^T \|\tilde{f}(\tau)\|_{-1}^2 \, d\tau.$$

Hence, from (4–6), we deduce that there exists a positive constant $C' = C'(\Omega, \nu)$ such that $D_\alpha \le C'$ implies $\|\nabla v_{Rk}(t)\|_2 \le C(\nu, \Omega, D_\alpha)$ for all $t \in [\bar{t}, T]$. Taking into account (4–7), we then get

$$\|\nabla v_{Rk}(0)\|_2 \le C(\nu, \Omega, D_\alpha)$$

and again, making suitable assumptions about the smallness of D_α, we conclude

$$\|\nabla v_{Rk}(t)\|_2 \le C(\nu, \Omega, D_\alpha) \quad \text{for all } t \in [0, T]. \tag{4–8}$$

Having established this, we go back to (4–6) and obtain the uniform estimate

$$\int_0^T \|P\Delta v_{Rk}(\tau)\|_{2,\Omega_R}^2 d\tau \le C(\nu, \Omega, D_\alpha), \tag{4–9}$$

and since $\|D^2 v_{Rk}\|_{2,\Omega_R} \le C\left(\|P\Delta v_{Rk}\|_{2,\Omega_R} + \|\nabla v_{Rk}\|_{2,\Omega_R}\right)$ (see [Heywood 1980], for example), with C independent of k and R, from (4–9) we infer $(4–4)_2$. □

We next provide an estimate for the time derivative of v_{Rk}.

Lemma 4.2. *Suppose the assumptions of the Lemma 4.1 hold. There exists a positive constant C depending on the data and on R such that*

$$\int_0^T \|\partial_\tau v_{Rk}(\tau)\|_{2,\Omega_R}^2 d\tau \le C. \tag{4–10}$$

Proof. Multiplying (3–5) by $\dfrac{dc_{Rkj}}{dt}$ and summing over j we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v_{Rk}\|_2^2 + \nu\|\partial_t v_{Rk}\|_2^2 \\
&\quad = -(v_{Rk}\cdot\nabla v_{Rk}, \partial_t v_{Rk})_{\Omega_R} + (\xi\cdot\nabla v_{Rk}, \partial_t v_{Rk})_{\Omega_R} + (\omega\times x\cdot\nabla v_{Rk}, \partial_t v_{Rk})_{\Omega_R} \\
&\qquad -(\omega\times v_{Rk}, \partial_t v_{Rk})_{\Omega_R} - (v_{Rk}\cdot\nabla\tilde{u}, \partial_t v_{Rk})_{\Omega_R} - (\tilde{u}\cdot\nabla v_{Rk}, \partial_t v_{Rk})_{\Omega_R} + (\tilde{f}, \partial_t v_{Rk})_{\Omega_R}.
\end{aligned}$$

We have

$$\left|(\omega\times v_{Rk},\partial_t v_{Rk})_{\Omega_R}\right|+\left|(\omega\times x\cdot\nabla v_{Rk},\partial_t v_{Rk})_{\Omega_R}\right| \\ \le C(\Sigma,T)\|\omega\|_{H^1(0,T)}R\|\partial_t v_{Rk}\|_2\|\nabla v_{Rk}\|_2,$$

and the remaining terms are estimated as in the previous lemma. □

Using the estimates of Lemmas 4.1 and 4.2, and proceeding as in Step 1 of Section 3, we show the existence of a T-periodic field v_R such that

$$\begin{aligned} &v_R\in L^\infty(0,T;H^1(\Omega_R))\cap L^2(0,T;H^2(\Omega_R)),\\ &\frac{dv_R}{dt}\in L^2(0,T;L^2(\Omega_R)),\end{aligned} \tag{4–11}$$

and satisfying, in addition, the estimate

$$\|\nabla v_R(t)\|_{2,\Omega_R}+\int_0^T\|D^2v_R(\tau)\|^2_{L^2(\Omega_R)}d\tau\le C, \tag{4–12}$$

with C independent of R. Furthermore, v_R satisfies the equation (3–2). Actually, by well known arguments, (3–2) and (4–11) imply the existence of a scalar field $p_R\in L^2(0,T;W^{1,2}(\Omega_R))$ such that the pair (v_R,p_R) satisfies the following equations a.a. in $\Omega\times[0,T]$:

$$\begin{aligned} &\frac{\partial v_R}{\partial t}=\nu\Delta v_R+(V-v_R)\cdot\nabla v_R-v_R\cdot\nabla\tilde u-\tilde u\cdot\nabla v_R-\omega\times v_R-\nabla p_R+\tilde f,\\ &\nabla\cdot v_R=0.\end{aligned} \tag{4–13}$$

From (4–13) we can now obtain an estimate on $\partial v_R/\partial t$, uniformly in R on any fixed compact set, for sufficiently large R.

Lemma 4.3. *Let the assumptions of Lemma 4.1 hold, and let $R_0>\delta(\mathscr{B})$. The following estimates hold*:

$$\begin{aligned} &\int_0^T\|\partial_t v_R(t)\|^2_{2,\Omega_{R_0}}dt\le C_1\quad\textit{for all } R>R_0,\ R\in\mathscr{S},\\ &\int_0^T\|\nabla p_R(t)\|^2_{2,\Omega_R}dt\le C_2\quad\textit{for all } R\in\mathscr{S},\end{aligned} \tag{4–14}$$

where C_1 depends only on the data and R_0, while C_2 depends only on the data. If, in particular, $\omega(t)\equiv 0$, we have the stronger estimate

$$\int_0^T\|\partial_t v_R(t)\|^2_{2,\Omega_R}dt\le C, \tag{4–15}$$

with C depending only on the data.

Proof. By the Helmholtz decomposition, we may write

$$\tilde{f} = P\tilde{f} + \nabla\tilde{p}, \quad \tilde{p} \in L^2(0,T;G(\Omega)). \tag{4–16}$$

Recalling that for all $v \in V(\Omega_R)\cap H^2(\Omega_R)$ we have, by [Galdi and Silvestre 2005, Lemma 3(i)],

$$((\xi+\omega\times x)\cdot\nabla v - \omega\times v) \in H(\Omega_R),$$

we find from (4–13) that $P_R \equiv p_R - \tilde{p}$ satisfies the following Neumann problem (in the sense of distributions):

$$\begin{aligned} \Delta P_R &= \nabla\cdot F \quad \text{in } \Omega_R,\\ \frac{\partial P_R}{\partial n} &= F\cdot n \quad \text{at } \partial\Omega_R, \end{aligned} \tag{4–17}$$

where $F = -v_R\cdot\nabla v_R - v_R\cdot\nabla\tilde{u} - \tilde{u}\cdot\nabla v_R + \nu\Delta v_R$. Formally multiplying both sides of (4–17)$_1$ by P_R, integrating by parts over Ω_R and using (4–17)$_2$ we deduce

$$\|\nabla P_R\|^2_{2,\Omega_R} = (F, \nabla P_R).$$

Thus, from the Schwarz inequality, from Lemma 4.1 and from the Sobolev-like inequality

$$\|w\|_{\infty,\Omega_R} \le C\,(\|\nabla w\|_{2,\Omega_R} + \|P\Delta w\|_{2,\Omega_R}), \quad w \in H^1_0(\Omega_R)\cap H^2(\Omega_R), \tag{4–18}$$

(for which see [Galdi 1994a]) with $C = C(\Omega)$, we deduce that

$$\int_0^T \|\nabla P_R(t)\|^2_{2,\Omega_R}\,dt \le C,$$

where C depends only the data. Plugging this information back in (4–13)$_1$ and using again Lemma 4.1, (4–16) and (4–18) we show (4–14).

We next observe that the dependence of the constant C on R_0 in (4–14) is due to the presence of the term $\omega\times x\cdot\nabla v_R - \omega\times v_R$ in (4–13)$_1$. Therefore, if $\omega(t) = 0$, for all $t \in [0,T]$, the constant C becomes independent of R_0 and we obtain the stronger estimate (4–15). □

The proof of Theorem 4.1 is now achieved as follows. We multiply (4–13) by $\Phi \in \mathcal{D}_{T,p}$ and integrate over $\Omega\times[0,T]$. We then let, in the resulting equation, $R\to\infty$, along a suitable sequence, and take into account (4–12), Lemma 4.3, and the embedding $H^1(0,T;L^2(\Omega_{R_0})) \subset C([0,T];L^2(\Omega_{R_0}))$, for all $R_0 > \delta(\mathcal{B})$.

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GIOVANNI P. GALDI
DEPARTMENT OF MECHANICAL ENGINEERING
630 BENEDUM HALL
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PA 15261
UNITED STATES

galdi@engr.pitt.edu

ANA L. SILVESTRE
CENTRO DE MATEMÁTICA E APLICAÇÕES
DEPARTAMENTO DE MATEMÁTICA
INSTITUTO SUPERIOR TÉCNICO
AV. ROVISCO PAIS
1049-001 LISBOA
PORTUGAL

Ana.Silvestre@math.ist.utl.pt