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We study an intrinsic cohomology theory of Koszul–Vinberg algebras and their modules. Our results may be regarded as improvements of the attempt by Albert Nijenhuis in 1969. In particular, we completely solve a fundamental problem raised by M. Gerstenhaber. A short appendix is devoted to the relationships between our results and the pioneering work of Nijenhuis.

Introduction

According to M. Gerstenhaber [1964], "every restricted deformation theory generates its proper cohomology theory". The deformation theory of associative algebras and their modules involves the Hochschild cohomology theory of associative algebras, and that of Lie algebras involves the Chevalley–Eilenberg cohomology theory of Lie algebras. The first attempt to define a cohomology theory of Koszul–Vinberg algebras goes back to Albert Nijenhuis [1968]. Recently his pioneering results were rediscovered in [Chapoton and Livernet 2001; Dzhumadil'daev 1999]. The role played by Koszul–Vinberg algebras in differential geometry and in algebraic-analytic geometry is quite important [Koszul 1968; Milnor 1977; Vey 1968; Vinberg 1963; Vinberg and Kac 1967]. The deformation theory of these algebras is also related to Poisson manifolds [Nguiffo Boyom 2005] and to the theory of quantization deformation, to be discussed in a forthcoming work.

The main aim of the present work is to initiate an intrinsic cohomology theory of Koszul–Vinberg algebras and their modules, or KV-cohomology. We focus on relationships between this cohomology theory and some classical problems:

$$\begin{aligned}
H^0(\mathscr{A},\cdot) &\longleftrightarrow \mathscr{A}\text{-equivariant objects},\\
H^1(\mathscr{A},\cdot) &\longleftrightarrow \text{extensions of } \mathscr{A}\text{-modules},\\
H^2(\mathscr{A},\cdot) &\longleftrightarrow \text{extension classes of algebras},\\
H^2(\mathscr{A},\cdot) &\longleftrightarrow \text{deformation theory of algebraic structures},\\
H^3(\mathscr{A},\cdot) &\longleftrightarrow \text{formal deformations of algebraic structures}.
\end{aligned}$$

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Contents of the paper. This work consists of three parts. Part 1 is the theoretical part of the subject. Through Section 3 main definitions and examples are given. Section 4 is devoted to the intrinsic cohomology theory of Koszul–Vinberg algebras and their modules. Sections 5 and 6 are devoted to applications.

Part 2 covers the cohomology theory of Koszul–Vinberg algebras, which have their origin in differential geometry and in analytic-algebraic geometry; see [Vinberg 1963; Koszul 1968; Milnor 1977; Nguiffo Boyom 1968]. To every locally flat manifold (M, D) is attached a super-Koszul–Vinberg algebra and some KV-complexes. These complexes admit some natural filtrations leading to spectral sequences (which we not study deeply). The real KV-cohomology of locally flat manifolds is defined in Section 7, while in Section 8 we use real KV-cohomology to examine the rigidity of hyperbolic locally flat manifolds. We show that hyperbolic affine structures in $\mathbb{R}^+ \times \mathbb{R}$ admit nontrivial deformations. This is a particular case of a general nonrigidity theorem by Koszul [1968]. Section 9 is devoted to the relationships between the real KV-cohomology of a locally flat manifold (M, D) and the completeness of (M, D). We show that the volume class of a unimodular locally flat manifold (M, D) is a KV-cohomology obstruction to the completeness of (M, D).

Part 3 is also theoretical. It contains an introduction to the study of left-invariant locally flat structures on groups of diffeomorphisms. Every locally flat structure (M, D) on a compact manifold M gives rise to a Koszul–Vinberg algebra whose commutator Lie algebra is the Lie algebra $\mathscr{A}(M)$ of smooth vector fields on M. The question arises of determining whether every left-invariant locally flat structure on $\operatorname{Diff} M$ comes from a locally flat structure on the manifold M; this is discussed in Part 3. We show that its solution involves a special type of cohomology class (Theorem 10.6).

Part I. KV-cohomology theory

1. KV-algebras

Let $\mathbb{F}$ be a commutative field of characteristic zero. Let $\mathscr{A}$ be an algebra over $\mathbb{F}$. The product of two elements $a, b \in \mathscr{A}$ is denoted by ab. Given $a, b, c \in \mathscr{A}$ we will denote by (a, b, c) the associator of these elements, defined as

$$(a, b, c) = (ab)c - a(bc).$$

Definition [Nguiffo Boyom 1990; 1993]. An algebra $\mathscr{A}$ is called a *Koszul–Vinberg algebra*, or *KV-algebra*, if $(a, b, c) = (b, a, c)$ for all $a, b, c \in \mathscr{A}$.

Examples. (i) Every associative algebra is a Koszul–Vinberg algebra.

(ii) Consider the case when $\mathbb{F}$ is the field $\mathbb{R}$ of real numbers. The vector space $C^\infty(\mathbb{R}, \mathbb{R})$ of smooth functions is a KV-algebra whose multiplication is $fg = f(dg/dx)$.

(iii) (iii) Let M be a smooth manifold and let D be a torsion-free linear connection whose curvature tensor vanishes identically. Then $\mathcal{A} = (\mathcal{X}(M), D)$ is the algebra whose multiplication is defined by $ab = D_a b$. Here $\mathcal{X}(M)$ is the vector space of smooth vector fields on M. Actually $\mathcal{A}$ is a KV-algebra. The pair (M, D) is called a locally flat manifold and $\mathcal{A}$ is its KV-algebra.

Examples (ii) and (iii) are infinite-dimensional. Finite-dimensional KV-algebras are related to the geometry of bounded domains [Koszul 1968; Vinberg and Kac 1967], while Example (iii) is related to affine geometry. We are going to define a cochain complex that provides a good framework for investigating the topology of hyperbolic affine manifolds (M, D).

KV-algebras are called *left symmetric algebras* in [Dzhumadil'daev 1999] and [Nguiffo Boyom 1968], and *pre-Lie algebras* in [Chapoton and Livernet 2001].

Definition. The subspace $J(\mathcal{A})$ of Jacobi elements of a KV-algebra $\mathcal{A}$ is the subset of $\xi \in \mathcal{A}$ satisfying the identity $(a, b, \xi) = 0$ for all $a, b \in \mathcal{A}$.

Actually $J(\mathcal{A})$ is an associative subalgebra containing the center of $\mathcal{A}$.

2. Koszul–Vinberg modules

Let $\mathcal{A}$ be a KV-algebra. We consider a vector space W with two bilinear maps

$$\begin{aligned} \mathcal{A} \times W &\to W, \\ (a, w) &\mapsto aw \end{aligned} \quad \text{and} \quad \begin{aligned} W \times \mathcal{A} &\to W, \\ (w, a) &\mapsto wa. \end{aligned}$$

Given $a, b \in \mathcal{A}$ and $w \in W$ one sets $(a, b, w) = (ab)w - a(bw)$, $(a, w, b) = (aw)b - a(wb)$, and $(w, a, b) = (wa)b - w(ab)$.

Definition. A vector space W with bilinear maps as above is called a two-sided *$\mathcal{A}$-KV-module* if

$$(a, b, w) = (b, a, w) \quad \text{and} \quad (a, w, b) = (w, a, b).$$

A left (right) KV-module over $\mathcal{A}$ is a KV-module W whose right (left) $\mathcal{A}$-action is trivial, meaning that $wa = 0$ ($aw = 0$) for all $(w, a) \in W \times \mathcal{A}$.

The following claim is easily verified. Let W be a right KV-module over $\mathcal{A}$. Then W is a trivial module over the two-sided ideal $\mathcal{J}$ generated by the associators (a, b, c). So W becomes a right module over the associative algebra $\mathcal{A}/\mathcal{J}$.

Definition. The subspace $J(W)$ of Jacobi elements of a KV-module W consists of $w \in W$ satisfying $(a, b, w) = 0$ for all $a, b \in \mathcal{A}$.

Examples. (1) Consider the KV-algebra $\mathcal{A} = C^\infty(\mathbb{R}, \mathbb{R})$ whose multiplication is $fg = f(dg/dx)$. Then $J(\mathcal{A})$ is the subspace of affine functions.

(2) Let $\mathcal{A}$ be the KV-algebra of a locally flat manifold (M, D). Then $J(\mathcal{A})$ is the space of infinitesimal affine transformations of (M, D). The space $J_e(\mathcal{A})$ consisting of those $\xi \in J(\mathcal{A})$ which are complete vector fields is a finite-dimensional subalgebra of the Lie algebra $J(\mathcal{A})$. In particular, if M is compact then $J(\mathcal{A})$ is finite-dimensional. The simply connected Lie group $\mathcal{G}$ whose Lie algebra is $J(\mathcal{A})$ carries a two-sided invariant locally flat structure $(\mathcal{G}, \nabla)$. Moreover (M, D) admits $\mathcal{G}$ as an effective group of affine transformations [Palais 1957]. Concerning the relationships between the $\mathcal{G}$-geometry and the completeness of (M, D), see [Tsemo 1999]. Of course the notion of KV-morphism, sub-KV-module and KV-quotient module can be defined. In particular the image under a KV-morphism of a KV-module is a KV-submodule.

3. KV-module of linear maps

Let V and W be two-sided KV-modules over a KV-algebra $\mathcal{A}$. Let $L(W, V)$ be the vector space of linear maps from W to V. We shall consider the bilinear maps

$$\begin{aligned} \mathcal{A} \times L(W, V) &\to L(W, V), \\ (a, f) &\mapsto af \end{aligned} \quad \text{and} \quad \begin{aligned} L(W, V) \times \mathcal{A} &\to L(W, V), \\ (f, a) &\mapsto fa, \end{aligned}$$

where the linear maps af and fa are defined by

$$(af)(w) = a(f(w)) - f(aw) \quad \text{and} \quad (fa)(w) = (f(w))a.$$

One easily verifies that $(a, b, f) = (b, a, f)$ and $(a, f, b) = (f, a, b)$ for all $alla, b \in \mathcal{A}$ and all $f \in L(W, V)$. Thus $L(W, V)$ is a two-sided KV-module over the KV-algebra $\mathcal{A}$. If V is a left module then $L(W, V)$ is also a left module. If V and W are right modules then so is $L(W, V)$. Similarly, the vector space $L_q(W, V)$ of V-valued q-multilinear functions on W is a two-sided KV-module under the actions

$$\begin{aligned} (af)(w_1, \dots, w_q) &= a(f(w_1, \dots, w_q)) - \sum_{j=1}^{q} f(w_1, \dots, aw_j, \dots, w_q), \\ (fa)(w_1, \dots, w_q) &= (f(w_1, \dots, w_q))a. \end{aligned}$$

4. KV-Cohomology

Let $\mathcal{A}$ be an associative algebra or Lie algebra, and let W be a module over $\mathcal{A}$. Hochschild cohomology and Chevalley–Eilenberg cohomology provide powerful tools for bringing under control many problems such as deformations and extensions of algebraic structures [Piper 1967] and deformation quantization [Kontsevich 2003].

Suppose the algebra $\mathscr{A}$ to be neither an associative algebra nor a Lie algebra. In general it is easy to define the notion of two-sided module over $\mathscr{A}$. Nevertheless it is a nontrivial problem to define cohomologies of $\mathscr{A}$. Albert Nijenhuis [1968] tried to construct a cohomology theory for KV-algebras. The task turned out to be rather difficult, so he only initiated a theory which is closely related to the Chevalley–Eilenberg cohomology of Lie algebras. Here is what he did. Every KV-algebra $\mathscr{A}$ gives rise to a Lie algebra $\mathscr{A}_L$ whose bracket operation is defined by $[a, b] = ab - ba$. If W is a left KV-module over the KV-algebra $\mathscr{A}$, then the vector spaces $\mathscr{A}$, W and $L(\mathscr{A}, W) = \mathrm{Hom}(\mathscr{A}, W)$ are left $\mathscr{A}_L$-modules.

Definition [Nijenhuis 1968]. Given a KV-algebra $\mathscr{A}$ and a left $\mathscr{A}$-KV-module W, the q-th cohomology space $H^q_N(\mathscr{A}, W)$ of $\mathscr{A}$ with coefficients in W is defined to be the $(q-1)$-th cohomology space of the Chevalley–Eilenberg complex $C^\star(\mathscr{A}_L, L(\mathscr{A}, W))$. Namely, $H^q_N(\mathscr{A}, W) = H^{q-1}(\mathscr{A}_L, L(\mathscr{A}, W))$.

Remark. This definition collapses in degree zero. This fact is not without consequences.

Notation. Whenever there is no risk of confusion we will write $f(a_1..a_i..a_q)$ for $f(a_1, \dots, a_i, \dots, a_q)$.

Intrinsic KV-cohomology theory. We shall introduce an intrinsic cohomology theory for KV-algebras. We shall show that the theory is coherent. Furthermore the cohomology spaces of degrees zero, one and two will be interpreted as expected. We start by fixing a KV-algebra $\mathscr{A}$ and a two-sided KV-module over $\mathscr{A}$, denoted by W. Let q be a positive integer. Let $C_q(\mathscr{A}, W)$ be the vector space of q-linear maps from $\mathscr{A}$ to W. The space $C_q(\mathscr{A}, W)$ is a two-sided $\mathscr{A}$-KV-module under the actions of $\mathscr{A}$ on $C_q(\mathscr{A}, W)$ given by

$$\begin{aligned}(af)(a_1 \dots a_q) &= a(f(a_1 \dots a_q)) - \sum_{j=1}^{q} f(a_1 \dots aa_j, \dots, a_q),\\ (fa)(a_1 \dots a_q) &= (f(a_1 \dots a_q))a,\end{aligned}$$

where $a \in \mathscr{A}$ and $f \in C_q(\mathscr{A}, W)$. For each $\rho = 1, \dots, q$, denote by $e_\rho(a) : C_q(\mathscr{A}, W) \to C_{q-1}(\mathscr{A}, W)$ the linear map defined by

$$(e_\rho(a)f)(a_1..a_{q-1}) = f(a_1..a_{\rho-1}, a, a_\rho..a_{q-1}).$$

Next we define the coboundary operator $\delta : C_q(\mathscr{A}, W) \to C_{q+1}(\mathscr{A}, W)$. For $f \in C_q(\mathscr{A}, W)$ and $(a_1, \dots, a_{q+1}) \in \mathscr{A}^{q+1}$, The coboundary $\delta f \in C_{q+1}(\mathscr{A}, W)$ is given by

$$(\delta f)(a_1..a_{q+1}) = \sum_{1\le j\le q} (-1)^j \big((a_j f)(a_1..\hat{a}_j..a_{q+1}) + e_q(a_j)(f a_{q+1})\,(a_1..\hat{a}_j..\hat{a}_{q+1})\big). \tag{4–1}$$

Lemma 4.1. $\delta \circ \delta = 0$.

Proof. Formula (4–1) implies that $\delta f(a_1..a_{q+1})$ is the sum of q terms.

$$(-1)^j \big((a_j f)(a_1..\hat{a}_j..a_{q+1}) + (f(a_1..\hat{a}_j..a_q, a_j))\,a_{q+1}\big).$$

Fix i, j, k such that $1 \le i < j < k < q+2$. Since $\delta(\delta f)(a_1..a_{q+2})$ is the sum of the $q+1$ terms $(-1)^i \big((a_i \delta f)(a_1..\hat{a}_i..a_{q+2}) + (\delta f(a_1..\hat{a}_i..a_{q+1}, a_i))\,a_{q+2}\big)$, set

$$\begin{aligned}\tau_{ii} &= a_i(\delta f(a_1..\hat{a}_i..a_j..a_k..a_{q+2})),\\ \tau_{ij} &= \delta f(a_1..\hat{a}_i..a_i a_j..a_k..a_{q+2}),\\ \tau_{ik} &= \delta f(a_1..\hat{a}_i..a_j..a_i a_k..a_{q+2}),\\ \tau_{iq+2} &= \delta f(a_1..\hat{a}_i..a_j..a_k..a_i, a_{q+2}).\end{aligned}$$

Then $(a_i \delta f)(a_1..\hat{a}_i..a_{q+2}) = \tau_{ii} - \sum_{\rho\ne i} \tau_{i\rho}$. To calculate the summands τ_{ii} and τ_{ij} we adopt the same scheme. Thus we write

$$\tau_{ii} = a_i\Big(\sum_{\rho\le i}(-1)^\rho \Gamma_\rho + \sum_{i<\rho<q+2} (-1)^{\rho-1}\Gamma_\rho\Big),$$

with

$$\Gamma_\rho = a_\rho(f(a_1..\hat{a}_i..\hat{a}_\rho..a_{q+2})) - \sum_s f(a_1..\hat{a}_i..\hat{a}_\rho..a_\rho a_s..a_{q+2}) + (f(a_1..\hat{a}_i..\hat{a}_\rho..a_{q+1}, a_\rho))\,a_{q+2}.$$

We focus on indexes $i < j < k$, setting

$$\Gamma_j = a_j(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2})) + (f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))\,a_{q+2} - f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+2}) - f(a_1..\hat{a}_i..\hat{a}_j..a_j a_{q+2}) + \text{other summands}.$$

Therefore we see that τ_{ii} may be written as

$$\tau_{ii} = (-1)^{j-1} a_i \left\{\begin{array}{l} a_j(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2}))\\ \quad +\ (f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))\,a_{q+2}\\ \quad -\ f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+2})\\ \quad -\ f(a_1..\hat{a}_i..\hat{a}_j..a_j a_{q+2}) + \cdots \end{array}\right\} + \cdots, \tag{4–2}$$

where the ellipses inside the braces stand for other terms that do not concern us and those outside the braces stand for other Γ_ρ's. Now mutatis mutandis we get

$$\tau_{jj} = (-1)^i a_j \left\{ \begin{array}{l} a_i(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2})) \\ \quad + (f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..a_i a_k..a_{q+2}) \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..a_i a_{q+2}) + \cdots \end{array} \right\} + \cdots . \tag{4–3}$$

In $(\delta \circ \delta f)(a_1 \ldots a_{q+2})$ the sign of the summand τ_{ii} is $(-1)^i$. As we just did, we take fixed indexes $i < j < k$. We get

$$\tau_{ij} = (-1)^{j-1} \left\{ \begin{array}{l} a_i a_j(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2})) \\ \quad + (f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_i a_j))a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..(a_i a_j)a_k..a_{q+2}) - \cdots \end{array} \right\} + \cdots , \tag{4–4}$$

$$\tau_{ji} = (-1)^i \left\{ \begin{array}{l} (a_j a_i)(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2})) \\ \quad + (f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j a_i))a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..(a_j a_i)a_k..a_{q+2}) - \cdots \end{array} \right\} + \cdots . \tag{4–5}$$

The summands τ_{ij} and τ_{ji} in $\delta^2 f(a_1 \ldots a_{q+2})$ have signs $(-1)^{i+1}$ and $(-1)^{j+1}$ respectively.

Now we calculate the summands τ_{ik} and τ_{jk} by the same method:

$$\tau_{jk} = (-1)^i \left\{ \begin{array}{l} a_i(f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+2})) \\ \quad + (f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+1}, a_i))a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..a_i(a_j a_k)..a_{q+2}) - \cdots \end{array} \right\} + \cdots \tag{4–6}$$

$$\tau_{ik} = (-1)^{j-1} \left\{ \begin{array}{l} a_j(f(a_1..\hat{a}_i..\hat{a}_j..a_i a_k..a_{q+2})) \\ \quad + (f(a_1..\hat{a}_i..\hat{a}_j..a_i a_k..a_{q+1}, a_j))a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..a_j(a_i a_k)..a_{q+2}) - \cdots \end{array} \right\} + \cdots . \tag{4–7}$$

Actually for every $l \leq i$ the expression of τ_{jk} contains the summands

$$(-1)^l \left\{ \begin{array}{l} a_l(f(a_1..\hat{a}_l..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+2})) \\ \quad + (f(a_1..\hat{a}_l..\hat{a}_j..a_j a_k..a_{q+1}, a_l))a_{q+2} \\ \quad - f(a_1..\hat{a}_l..a_l a_i..\hat{a}_j..a_j a_k..a_{q+2}) - \cdots \end{array} \right\} . \tag{4–8}$$

It is useful to calculate $\tau_{\rho,q+2}$ for $\rho = i$ and $\rho = j$:

$$\tau_{i,q+2} = (-1)^{j-1} \left\{ \begin{array}{l} a_j(f(a_1..\hat{a}_i..\hat{a}_j..a_i a_{q+2})) \\ \quad + (f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))a_i a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+1}, a_i a_{q+2}) - \cdots \end{array} \right\} + \cdots , \tag{4–9}$$

$$\tau_{j,q+2} = (-1)^i \left\{ \begin{array}{l} a_i(f(a_1..\hat{a}_i..\hat{a}_j..a_j a_{q+2})) \\ \quad + (f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_i))a_j a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..a_i a_k..a_j a_{q+2}) - \cdots \end{array} \right\} + \cdots . \tag{4–10}$$

One applies the same ideas to express $(\delta f.a_{q+2})(..\hat{a}_\rho..a_{q+1}, a_\rho)$:

$$\begin{aligned} &(\delta f.a_{q+2})(a_1..\hat{a}_i..a_{q+1}, a_i) \\ &\quad = (-1)^{j-1} \left\{ \begin{array}{l} (a_j(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1})))a_{q+2} \\ \quad + ((f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))a_i)a_{q+2} \\ \quad - f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+1}, a_i))a_{q+2} - \cdots \end{array} \right\} + \cdots , \end{aligned} \tag{4–11}$$

$$\begin{aligned} &\big(\delta f.a_{q+2})(a_1..\hat{a}_i..a_{q+1}, a_j) \\ &\quad = (-1)^i \left\{ \begin{array}{l} (a_i(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j)))a_{q+2} \\ \quad + ((f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_i))a_j)a_{q+2} \\ \quad - (f(a_1..\hat{a}_i..\hat{a}_j..a_i a_k..a_{q+1}, a_j))a_{q+2} - \cdots \end{array} \right\} + \cdots . \end{aligned} \tag{4–12}$$

We are now poised to prove that $(\delta \circ \delta)f = 0$. Indeed, we see that the expression of $(\delta \circ \delta)f(a_1..a_1..a_{q+1})$ is

$$\sum_{1 \le \rho \le q+1} (-1)^\rho \big((a_\rho \Gamma f)(..\hat{a}_\rho..a_{q+2}) + (\delta f(a_1..\hat{a}_\rho..a_{q+1}, a\rho))a_{q+2}\big).$$

Keeping in mind the signs of $\tau_{\rho,s}$, we focus on indexes $i < j < k$. From (4–2), (4–3), (4–4) and (4–5) we deduce that $\delta \circ \delta f(a_1..a_{q+2})$ contains the following expression as its summand:

$$\begin{aligned} &(-1)^{i+j-1} a_i(a_j f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2})) \\ &\quad + (-1)^{i+j} a_j(a_i f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2})) \\ &\quad + (-1)^{i+j}(a_i a_j) f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2}) \\ &\quad + (-1)^{i+j-1}(a_j a_i) f(a_1..\hat{a}_i..\hat{a}_j..a_{q+2}). \end{aligned}$$

The identity $(a, b, w) = (b, a, w)$ for $w \in W$ and $a, b \in \mathcal{A}$ shows that this vanishes.

From (4–2), (4–9), (4–11) and (4–12) one easily sees that $\delta \circ \delta f(a_1..a_{q+2})$ contains the summand

$$\begin{aligned} &(-1)^{i+j}(a_i(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j)))a_{q+2} \\ &\quad + (-1)^{i+j-1} a_i((f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))a_{q+2}) \\ &\quad + (-1)^{i+j}(f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))a_i a_{q+2} \\ &\quad + (-1)^{i+j-1}((f(a_1..\hat{a}_i..\hat{a}_j..a_{q+1}, a_j))a_i)a_{q+2}. \end{aligned}$$

This expression is simply $(a_i, w, a_{q+2}) - (w, a_i, a_{q+2})$, and therefore vanishes.

Considering (4–4), (4–5), (4–6) and (4–7) we see that $\delta \circ \delta f(a_1..a_{q+2})$ contains the summand

$$\begin{aligned}(-1)^{i+j-1} & f(a_1..\hat{a}_i..\hat{a}_j..(a_i a_j)a_k..a_{q+2}) \\ &+ (-1)^{i+j} f(a_1..\hat{a}_i..\hat{a}_j..a_i(a_j a_k)..a_{q+2}) \\ &+ (-1)^{i+j} f(a_1..\hat{a}_i..\hat{a}_j..a_i(a_j a_k)..a_{q+2}) \\ &+ (-1)^{i+j-1} f(a_1..\hat{a}_i..\hat{a} j..a_j(a_i a_k)..a_{q+2}).\end{aligned}$$

This equals

$$(-1)^{i+j-1}\big(f(a_1..\hat{a}_i..\hat{a}_j..(a_i, a_j, a_k)..a_{q+2}) - f(a_1..\hat{a}_i..\hat{a}_j..(a_j, a_i, a_k)..a_{q+2})\big),$$

and so vanishes by the identity $(a_i, a_j, a_k) = (a_j, a_i, a_k)$.

From (4–2) and (4–6) we see that $\delta \circ \delta f(a_1..a_{q+2})$ contains twice the summand $a_i f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+2})$ with opposite signs; hence its contribution is reduced to zero.

From (4–6) and (4–11) one deduces that $\delta \circ \delta f(a_1..a_{q+2})$ contains twice the summand

$$\big(f(a_1..\hat{a}_i..\hat{a}_j..a_j a_k..a_{q+1}, a_i)\big)a_{q+2}$$

with opposite signs, so it too contributes nothing. To end the proof of Lemma 4.1 it remains to examine the summands of $\delta \circ \delta f(a_1..a_{q+2})$ having the following form $f(a_1..\hat{a}_l..a_l a_i..\hat{a}_j..a_j a_k..a_{q+2})$. These summands come from (4–11). Indeed, let $l < i < j < k$. By virtue of (4–1), $\delta \circ \delta f(a_1..a_{q+2})$ contains twice the expression

$$f(a_1..\hat{a}_l..a_l a_i..\hat{a}_j..a_j a_k..a_{q+2}). \tag{4–13}$$

First this appears as a summand of $-(-1)^j \delta f(a_1..a_l..a_i..\hat{a}_j..a_j a_k..a_{q+2})$ with the sign $(-1)^{j+l}$; then it appears as a summand of $-(-1)^l \delta f(a_1..\hat{a}_l..a_l a_i..a_j..a_k..a_{q+2})$ with the sign $(-1)^{j+l-1}$. Therefore $(\delta \circ \delta) f(a_1..a_{q+2})$ does not contain any nonzero summand of the form (4–13).

We have just examined all the types of summands in $(\delta \circ \delta) f(a_1..a_{q+2})$. We conclude that $\delta \circ \delta f(a_1..a_{q+2}) = 0$ for all $f \in C_q(\mathscr{A}, W)$ and all $a_1, \dots, a_{q+2} \in \mathscr{A}$. □

Set $C(\mathscr{A}, W) = \bigoplus_{q \ge 1} C_q(\mathscr{A}, W)$. By Lemma 4.1 the coboundary operator δ given by (4–1) endows $C(\mathscr{A}, W)$ with the structure of a graded cochain complex

$$\cdots \longrightarrow C_q(\mathscr{A}, W) \xrightarrow{\delta} C_{q+1}(\mathscr{A}, W) \xrightarrow{\delta} C_{q+2}(\mathscr{A}, W) \longrightarrow \cdots.$$

The q-th cohomology $H^q(\mathscr{A}, W)$ of this complex is well defined for $q > 1$.

Before proceeding we set $C_q(\mathscr{A}, W) = 0$ for $q < 0$. To complete the picture we must define $C_0(\mathscr{A}, W)$ and $\delta : C_0(\mathscr{A}, W) \to C_1(\mathscr{A}, W)$ such that $\delta \circ \delta(C_0(\mathscr{A}, W))$ vanishes. Once $\delta : C_0(\mathscr{A}, W) \to C_1(\mathscr{A}, W)$ is defined we shall be able to define $H^0(\mathscr{A}, W)$ and $H^1(\mathscr{A}, W)$.

Definition. Set $C_0(\mathscr{A}, W) = J(W)$ and $\delta w(a) = -aw + wa$ for all $a \in \mathscr{A}$ and all $w \in J(W)$.

Now let $w \in W$. It is easy to see that $\delta(\delta w) = 0$ if and only if $w \in J(W)$. Thus we get the total complex

$$C_\tau(\mathscr{A}, W) = J(W) \oplus C(\mathscr{A}, W).$$

Remark. If W is a right KV-module over $\mathscr{A}$ then $C_0(\mathscr{A}, W) = W$. In connection with a question raised by Gerstenhaber [1964], Lemma 4.1 allows us to point out two conclusions. First, the category of KV-algebras admits its proper cohomology theory. Secondly, the category of associative algebras admits an alternative cohomology theory different from the Hochschild cohomology theory. For instance, let h be the associative algebra of upper triangular 3×3 nilpotent matrices. Its KV-cohomology vector space $H^2(h, h)$ is four-dimensional while its second Hochschild cohomology space is three-dimensional.

Examples. (1) If $\mathscr{A}$ is the KV-algebra of a locally flat manifold (M, D), then $H^1(\mathscr{A}, \mathscr{A}) = 0$. Indeed, let ψ be a cocycle of degree one. Then $D_{a_1}\psi(a_2) + D_{\psi(a_1)}a_2 = \psi(D_{a_1}a_2)$ for all $a_1, a_2 \in \mathscr{A}$. Thus ψ is a derivation of the Lie algebra $\mathscr{A}(M)$ of smooth vector fields on M. By virtue of a classical theorem by Takens every derivation of $\mathscr{A}(M)$ is an inner derivation. Thus there exists a smooth vector field ξ such that $\psi(a) = [\xi, a] = -a\xi + \xi a$. The claim $\delta\psi = 0$ yields $\xi \in J(\mathscr{A})$.

(2) Let W be the vector space of real valued smooth functions on a locally flat smooth manifold (M, D). It is a left KV-module over $\mathscr{A}$ under the Lie derivation. Let $\psi : \mathscr{A} \mapsto W$ be a 1-cocycle, so that $a\psi(b) - \psi(ab) = 0$ for $a, b \in \mathscr{A}$. Therefore ψ is a D-parallel linear map. This implies $w(a\psi(b)) = (wa)\psi(b) = \psi((wa)b)$ for all $w \in W$. Every vector field may be locally written as D_ab. Then we see that $\psi(wa) = w\psi(a)$ for all $w \in W$ and all $a \in \mathscr{X}(M)$. Hence ψ is a usual differential 1-form on the manifold M. Since $D\psi = 0$, ψ is a de Rham cocycle. On the other hand $J(W)$ consists of affine functions. The subspace of 1-cocycles of $C_1(\mathscr{A}, W)$ consists of locally linear closed 1-forms, and $\delta C_0(\mathscr{A}, W) = \delta J(W)$ consists of the differentials of affine functions. Therefore

$$H^1(\mathscr{A}, W) = \frac{[\text{locally linear closed 1-forms}]}{d\,[\text{affine functions}]},$$

where $d : C^\infty(M, \mathbb{R}) \to \Omega^1(M, \mathbb{R})$ is the de Rham differential operator. So there is a canonical injective linear map $H^1(\mathscr{A}, W) \to H^1_{\text{de Rham}}(M, \mathbb{R})$.

(3) Our third example is a combination of the first two. Consider $\mathscr{A}$ and W as in (2). We equip $\mathscr{A} \oplus W \cong \mathscr{A} \times W$ with the multiplication defined by $(a, w)(a', w') = (aa', aw' + ww')$, where ww' is the usual product of two real-valued functions. If

$(a'', w'') \in J(\mathcal{A} \oplus W)$ then $w'' = 0$ and $a'' \in J(\mathcal{A})$. Thus $J(\mathcal{A} \oplus W) \simeq J(\mathcal{A})$. Let $(a'', 0) \in J(\mathcal{A} \oplus W) = C_0(\mathcal{A} \oplus W, \mathcal{A} \oplus W)$ then

$$(\delta(a'', 0))(a, w) = (-aa'' + a''a,\ a''w).$$

Thereby $\delta(a'', 0) = 0$ if and only if $a'' = 0$. Hence the boundary map $\delta : J(\mathcal{A}) \to C_1(\mathcal{A} \oplus W, \mathcal{A} \oplus W)$ is an injective map. Let $\theta : \mathcal{A} \times W \to \mathcal{A} \times W$ be an 1-cocycle. For $(a, w) \in \mathcal{A} \oplus w$, set $\theta(a, w) = (\phi(a, w), \psi(a, w))$. Then for (a, w) and (a', w') in $\mathcal{A} \times W$ one has

$$\begin{aligned} -\delta\theta((a, w), (a', w')) &= (a, w)(\phi(a', w'), \psi(a', w')) \\ &\quad - (\phi(aa', aw' + ww'), \psi(aa', aw' + ww')) + (\phi(a, w), \psi(a, w))(a', w'). \end{aligned}$$

The equation $\delta\theta = 0$ gives rise to the system

$$\begin{aligned} a\phi(a', w') - \phi(aa', aw' + ww') + \phi(a, w).a' &= 0, \\ a\psi(a'w') + w\psi(a'w') - \psi(aa', aw' + ww') + \phi(a, w).w' + \psi(a, w)w' &= 0. \end{aligned}$$

We may write $\theta(a, w) = \theta(a, 0) + \theta(0, w)$. Then the identity satisfied by ϕ shows that $\phi(a, w)$ does not depend on w. Thus by setting $\phi(a, 0) = \phi(a)$ we see that $a\phi(a') + \phi(a)a' = \phi(aa')$. Thus there exists $\xi \in \mathcal{A}$ such that $\phi(a) = [\xi, a] = -a\xi + \xi a$. Since θ is an 1-cocycle we have $\xi \in J(\mathcal{A})$. Now we examine the W-component of θ, namely $\psi(a, w)$. Set $\psi(a, w) = \psi(a, 0) + \psi(0, w) = \lambda(a) + \mu(w)$. We know that $\phi(a, w) = [\xi, a]$, with $\xi \in J(\mathcal{A})$. Hence $\delta\theta = 0$ yields

$$\begin{aligned} a(\lambda(a') + \mu(w')) + w(\lambda(a') + \mu(w')) - \lambda(aa') - \mu(aw' + ww') \\ + [\xi, a].w' + (\lambda(a) + \mu(w))w' = 0. \end{aligned}$$

Thus one sees that $w\mu(w') + \mu(w)w' - \mu(ww') = 0$. Consequently there is a smooth vector field $\zeta \in \mathcal{X}(M)$ such that $\mu(w) = \langle dw, \zeta \rangle$. We also have $a\lambda(a') - \lambda(aa') = 0$. We have already shown that such λ is a D-parallel closed 1-form on the locally flat manifold (M, D). The condition $\delta\theta(0, w), (a', 0) = 0$ yields the identity $w\lambda(a') = 0$, $w \in W$ and $a' \in \mathcal{X}(M)$. Hence $\lambda = 0$. So the 1-cocycle $\theta(a, w)$ has the form $\theta(a, w) = ([\xi, a], \langle dw, \zeta \rangle)$ for some fixed $(\xi, \zeta) \in J(\mathcal{A}) \times \mathcal{A}$. It is easy to verify that if a 1-chain θ has the form

$$(a, w) \to \big([\xi, a], \langle dw, \zeta \rangle\big)$$

with $\xi \in J(\mathcal{A})$ then it is a 1-cocycle of $C_1(\mathcal{A} \oplus W, \mathcal{A} \oplus W)$. What we have just done is the computation of $H^0(\mathcal{A} \oplus W, \mathcal{A} \oplus W)$ and $H^1(\mathcal{A} \oplus W, \mathcal{A} \oplus W)$:

$$H^0(\mathcal{A} \oplus W, \mathcal{A} \oplus W) = \{0\}, \qquad H^1(\mathcal{A} \oplus W; \mathcal{A} \oplus W) \simeq \mathcal{A}/J(\mathcal{A}).$$

If the manifold M is compact, the vector space $H^1(\mathcal{A} \oplus W,\ \mathcal{A} \oplus W)$ is infinite-dimensional. The same conclusion holds if (M, D) is geodesically complete.

KV-Cohomology with values in $L(W, V)$**.** Let W and V be two-sided KV-modules over a KV-algebra $\mathscr{A}$. The vector space W is endowed with the trivial algebra structure. We equip $\mathscr{A} \oplus W$ with the multiplication defined by $(a, w)(a', w') = (aa', aw' + wa')$. Then $\mathscr{A} \oplus W$ is a KV-algebra yielding the exact sequence of KV-algebras

$$0 \to W \hookrightarrow \mathscr{A} \oplus W \to \mathscr{A} \mapsto 0.$$

Furthermore the vector space V is a two-sided KV-module over the KV-algebra $\mathscr{A} \oplus W$ under the actions defined by $(a, w).v = av$, $v.(a, w) = va$.

Consider the KV-cochain complex $C_\tau(\mathscr{A} \oplus W, V)$. It is easy to check that $J_{\mathscr{A}\oplus W}(V) = J_{\mathscr{A}}(V)$. The space $C_\tau(\mathscr{A} \oplus W, V)$ is bigraded as $C_{p,q}(\mathscr{A} \oplus W, V) = L(\mathscr{A}^q \otimes W^p, V)$. So an element $\theta \in C_{p,q}(\mathscr{A} \oplus W, V)$ is a V-valued $(p+q)$-multilinear function on $\mathscr{A} \oplus W$ which is homogeneous of degree p with respect to elements of W and homogeneous of degree q with respect to elements of $\mathscr{A}$. By setting $k = p + q$ the vector space of V-valued k-cochains of $\mathscr{A} \oplus W$ is bigraded by the subspaces $C_{p,q}(\mathscr{A} \oplus W, V)$.

Lemma 4.2. *The coboundary operator* $\delta : C_k(\mathscr{A} \oplus W, V) \to C_{k+1}(\mathscr{A} \oplus W, V)$ *sends* $C_{p,q}(\mathscr{A} \oplus W, V)$ *to* $C_{p,q+1}(\mathscr{A} \oplus W, V)$.

Proof. For all $\theta \in C_{p,q}(\mathscr{A} \oplus W, V)$ we have $\theta(\xi_1, \dots, \xi_{p+q}) = 0$ if more than p arguments belong to W, or if more than q arguments belong to $\mathscr{A}$. □

The lemma implies that $C_\tau(\mathscr{A} \oplus W, V)$ can be equipped with two filtrations:

$$F^P(\mathscr{A} \oplus W, V) = \bigoplus_{q,s\geq p} C_{s,q}(\mathscr{A} \oplus W, V),$$
(4–14)
$$F_P(\mathscr{A} \oplus W, V) = \bigoplus_{q,s\leq p} C_{s,q}(\mathscr{A} \oplus W, V).$$

One has $F^{p+1}(\mathscr{A} \oplus W, V) \subset F^P(\mathscr{A} \oplus W, V)$ and $F_P(\mathscr{A} \oplus W, V) \subset F_{p+1}(\mathscr{A} \oplus W, V)$. Moreover, $\delta F^P(\mathscr{A} \oplus W, V) \subset F^P(\mathscr{A} \oplus W, V)$ and $\delta F_P(\mathscr{A} \oplus W, V) \subset F_P(\mathscr{A} \oplus W, V)$. Both filtrations will give rise to spectral sequences. The study of these spectral sequences is not the purpose of the present work. We shall mainly be interested in the subcomplex $\bigoplus_q C_{1,q}(\mathscr{A} \oplus W, V)$. The vector space $C_{1,q}(\mathscr{A} \oplus W, V)$ is regarded as the space of $L(W, V)$-valued q-linear functions on $\mathscr{A}$. Actually $L(W, V)$ is a two-sided KV-module over $\mathscr{A}$ under the actions

$$(a\theta)(w) = a_V(\theta(w)) - \theta(a_W w),$$
(4–15)
$$(\theta a)(w) = (\theta(w))a_V.$$

By identifying $L(W, V)$ with $C_{1,0}(\mathscr{A} \oplus W, V)$, we define $\delta : C_{1,0}(\mathscr{A} \oplus W, V) \to C_{1,1}(\mathscr{A} \oplus W, V)$ by putting

$$\delta\theta(a, w) = -a\theta(w) + \theta(aw),$$
$$\delta\theta(w, a) = \theta(wa) - (\theta(w))a.$$

It is easy to verify that $(\delta \circ \delta)\theta = 0$ for all $\theta \in C_{1,0}(\mathscr{A} \oplus W, V)$. At the same time, each $C_{1,q}(\mathscr{A} \oplus W, V)$ is a two-sided KV-module over $\mathscr{A} \oplus W$. Hence, extending the formula (4–1), we have to $\bigoplus_{q \geq 0} C_{1,q}(\mathscr{A} \oplus W, V)$ yields the complex

$$\cdots \xrightarrow[1,\, q]{C} (\mathscr{A} \oplus W, V) \xrightarrow{\delta} C_{1,q+1}(\mathscr{A} \oplus W, V) \xrightarrow{\delta}$$

Definition. We denote by $E_1^{1,q}(\mathscr{A} \oplus W, V)$ the cohomology space of the complex above at the level $C_{1,q}(\mathscr{A} \oplus W, V)$.

Remark. There is a canonical linear map $E_{1,}^{1,q}(\mathscr{A} \oplus W, V) \to H^q(\mathscr{A}, L(W, V))$. By virtue of (4–15) the total space of $L(W, V)$-valued cochains of $\mathscr{A} \oplus W$ is

$$J(L(W, V)) \oplus \sum_{q>0} C_q(\mathscr{A} \oplus W, L(W, V)).$$

The cohomology space of the complex $\bigoplus_{q>0} C_{1,q}(\mathscr{A} \oplus W, V)$ is related to the spectral sequence which is associated to the filtration of $C_\tau(\mathscr{A} \oplus W, V)$ by the subspaces $F^P(\mathscr{A} \oplus W, V)$.

Consistency. Let W and V be two-sided KV-modules over a KV-algebra $\mathscr{A}$. Let $\phi : W \to V$ be a morphism of KV-modules, so that

$$\phi(aw) = a\phi(w) \quad \text{and} \quad \phi(wa) = \phi(w)a$$

for all $w \in W$ and all $a \in \mathscr{A}$. Take $f \in C_q(\mathscr{A}, W)$; then $\phi^*(f) \in C_q(\mathscr{A}, V)$ is defined by $\phi^*(f) = \phi \circ f$. It is easy to see that

$$[a(\phi \circ f)](b) = [\phi(af)](b),$$
$$[(\phi \circ f)a](b) = [\phi(fa)](b).$$

These identities show that $\delta[\phi \circ f] = \phi \circ (\delta f)$ for all $f \in C_q(\mathscr{A}, W)$. Thus ϕ canonically induces the linear map $\tilde{\phi} : H^q(\mathscr{A}, W) \to H^q(\mathscr{A}, V)$.

To prove the consistency of the KV-cohomology theory derived from (4–1), we turn to the relevant long cohomology exact sequences.

Let W, V, T be two-sided $\mathscr{A}$-modules. Suppose we have a short exact sequence of KV-modules

$$0 \to V \to T \to W \to 0. \tag{4–16}$$

Regarded as an exact sequence of $\mathbb{F}$-vector spaces, this sequence is $\mathbb{F}$-splittable. The exact sequence

$$0 \to C_q(\mathscr{A}, V) \to C_q(\mathscr{A}, T) \to C_q(\mathscr{A}, W) \to 0.$$

is $\mathbb{F}$-splittable as well. This latter sequence yields the following $\mathbb{F}$-splittable exact sequence of cochain complexes:

$$0 \longrightarrow C_\tau(\mathscr{A}, V) \xrightarrow{i} C_\tau(\mathscr{A}, T) \xrightarrow{\mu} C_\tau(\mathscr{A}, W) \longrightarrow 0.$$

Therefore the exact sequence (4–16) gives rise to the long exact sequence

$$\cdots \xrightarrow{\delta} H^q(\mathscr{A}, W) \xrightarrow{i} H^q(\mathscr{A}, T) \xrightarrow{\mu} H^q(\mathscr{A}, W) \xrightarrow{\delta} H_{q+1}(\mathscr{A}, V) \xrightarrow{i} \cdots.$$

Now suppose the diagram below is a morphism exact sequences of two-sided KV-modules:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "\phi"'] & T \arrow[r, "P"] \arrow[d, "\phi"'] & W \arrow[r] \arrow[d, "\bar{\phi}"'] & 0 \\
0 \arrow[r] & V' \arrow[r, "i'"] & T' \arrow[r, "P'"] & W' \arrow[r] & 0
\end{tikzcd}$$

Then we deduce a morphism of cohomology exact sequences

$$\begin{tikzcd}
\arrow[r] & H^q(\mathscr{A}, V) \arrow[r, "i"] \arrow[d, "\phi"'] & H^q(\mathscr{A}, T) \arrow[r, "P"] \arrow[d, "\phi"'] & H^q(\mathscr{A}, W) \arrow[r, "\delta"] \arrow[d, "\bar{\phi}"'] & H^{q+1}(\mathscr{A}, V) \arrow[r] \arrow[d, "\tilde{\phi}"'] & {} \\
\arrow[r] & H^q(\mathscr{A}, V') \arrow[r, "i'"] & H^q(\mathscr{A}, T') \arrow[r, "P'"] & H^q(\mathscr{A}, W') \arrow[r, "\delta"] & H^{q+1}(\mathscr{A}, V) \arrow[r] & {}
\end{tikzcd}$$

The properties just pointed out prove the consistency of the KV-cohomology theory derived from formula (4–1).

5. Interpretation of some KV-cohomology spaces

We now turn to the interpretation of some KV-cohomology spaces. We show that these spaces play an essential role in some important questions.

Extensions of KV-algebras. Recall that cohomology classifies extensions

$$0 \to \mathscr{B} \to \mathscr{G} \to \mathscr{A} \to 0 \tag{5–1}$$

of associative algebras if $\mathscr{B}$ is a two-sided null-ideal of the associative algebra $\mathscr{G}$, and that it classifies extensions (5–1) of Lie algebras if $\mathscr{B}$ is an abelian ideal of the

The diagrams in this article were produced with Paul Taylor's diagrams.sty package.

Lie algebra $\mathcal{G}$. We consider a short exact sequence of two-sided KV-modules over the KV-algebra $\mathcal{A}$:

$$0 \longrightarrow W \xrightarrow{\;i\;} T \xrightarrow{\;\mu\;} \mathcal{A} \longrightarrow 0.$$

Regarding the sequence (5–1) as a sequence of vector spaces we use a section σ of μ to identify the vector space T with $W \oplus \mathcal{A}$. From now on we make the assumption that (5–1) is an exact sequence of KV-algebras. We suppose W to be a two-sided null-ideal of T. Therefore the multiplication map of T is given by

$$(a+w)(a'+w') = aa' + aw' + wa' + \omega_\sigma(a, a'),$$

where $\omega : \mathcal{A} \times \mathcal{A} \to W$ is a bilinear map. Let $a+w$, $a'+w'$ and $a''+w''$ be elements of $\mathcal{A} \oplus W$. From the identity $(a+w, a'+w', a''+w'') = (a'+w', a+w, a''+w'')$ one deduces that

$$\delta\omega_\sigma(a, a', a'') = 0$$

for all $a, a', a'' \in \mathcal{A}$. If one identifies T with $W \oplus \mathcal{A}$ using another section $\sigma' : \mathcal{A} \to T$, then the induced 2-cocycle $\omega_{\sigma'}$ will be related to the preceding one by

$$\omega_{\sigma'} = \omega_\sigma + \delta\psi,$$

where $\psi \in C_1(\mathcal{A}, W)$. Thus the cohomology class $[\omega_\sigma] \in H^2(\mathcal{A}, W)$ does not depend on the choice of the section $\sigma : \mathcal{A} \to T$.

Remark. Consider an exact sequence of KV-algebras

$$0 \to W \to T \to \mathcal{A} \to 0.$$

If the restriction to W of the multiplication map is nonzero, this sequence will not be an exact sequence of $\mathcal{A}$-KV-modules, and therefore not related to any 2-KV-cohomology class of $\mathcal{A}$. A similar failure is well known for extensions of Lie algebras with nonabelian kernel [Bourbaki 1971].

Digression. Many important examples of KV-algebras and KV-modules coming from differential geometry are infinite-dimensional topological vector spaces. In this case a closed vector subspace is not necessarily a direct summand, that is, it may not admit a complementary closed subspace. This observation should be kept in mind whenever one deals with continuous KV-cohomology of locally flat manifolds. For instance, the KV-algebra $\mathcal{A}$ of a locally flat manifold (M, D) is a topological KV-algebra and the space W of real-valued smooth functions on M is a topological left module over $\mathcal{A}$ when both $\mathcal{A}$ and W are endowed with the Whitney topology. A section of a short exact sequence of topological two-sided modules $0 \to V \to T \to W \to 0$ always means a continuous section.

$\mathbb{F}$-projective KV-modules. From now on the base field $\mathbb{F}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The notion of $\mathbb{F}$-projectivity to be introduced below is motivated by the digression just above. Let W be a two-sided KV-module over a KV-algebra $\mathscr{A}$. Then W is $\mathbb{F}$-projective if every short exact sequence of vector spaces $0 \to V \to T \to W \to 0$ is $\mathbb{F}$-splittable. Of course if we are dealing with abstract algebras and abstract modules, viz vector spaces are endowed with the discrete topology, then every vector space is $\mathbb{F}$-projective. Henceforth we simultaneously deal with the both topological case and abstract case.

Suppose that $\mathscr{A}$ is a $\mathbb{F}$-projective KV-algebra. Then for every KV-module W equipped with the trivial multiplication map $w.w' = 0$, every short exact sequence of two-sided KV-modules $0 \to W \to T \to \mathscr{A} \to 0$ is also an exact sequence of KV-algebras when $W \times \mathscr{A}$ is given the multiplication defined by $(w, a).(w', a') = (wa' + aw', aa')$. Mutatis mutandis the same property holds in the case of associative algebras and their two-sided modules.

Let V and W be two-sided KV-modules over a KV-algebra $\mathscr{A}$. The analogue to the bijective map: $H^1(\mathscr{G}, L(W, V)) \to$ {equivalence classes of extensions of W by V} fails. Our aim is to bring under control the classification problem for extensions of two-sided KV-modules. Consider a short exact sequence of two-sided KV-modules over $\mathscr{A}$, namely $0 \to V \to T \to W \to 0$. We shall prove that the classification problem for these extensions involves the term E_1 of the spectral sequence defined by the second filtration in (4–14).

Let W be a two-sided KV-module over a KV-algebra $\mathscr{A}$. Assume that W is $\mathbb{F}$-projective. We shall consider W as a trivial KV-algebra. Thus we get the semidirect product of KV-algebras $\mathscr{A} \times W$ whose multiplication is defined by

$$(a, w)(a', w') = (aa', aw' + wa')$$

for (a, w) and (a', w') in $\mathscr{A} \times W$. Of course in the topological setting this semidirect product is a toplogical KV-algebra as well.

Now suppose given an exact sequence of two-sided KV-modules,

$$0 \to V \to T \to W \to 0.$$

We may identify the vector space T with the direct sum $V \oplus W$. In the topological case both V and W are closed subspaces of the topological vector space T. We equip V with the structure of a two-sided KV-module over the semidirect product $\mathscr{A} \times W$ by putting $(a, w).v = av$ and $v.(a, w) = va$. Consider the cochain complex C_τ, namely $C_\tau(\mathscr{A} \times W, V) = \sum_{q \geq 0} C_{1,q}(\mathscr{A} \times W, V)$. Thus one has the sequence

$$\cdots \longrightarrow C_{1,q-1}(\mathscr{A} \times W, V) \xrightarrow{\delta} C_{1,q}(\mathscr{A} \times W, V) \xrightarrow{\delta} C_{1,q+1}(\mathscr{A} \times W, V) \longrightarrow \cdots$$

We can now state the classification theorem for the extensions of KV-algebras and the classification theorem for the extensions of two-sided KV-modules.

To begin with, consider a short exact sequence of two-sided $\mathcal{A}$-modules

$$0 \to W \to T \to \mathcal{A} \to 0$$

and suppose $\mathcal{A}$ to be $\mathbb{F}$-projective. Let $\sigma : \mathcal{A} \mapsto T$ be a section of the exact sequence of vector spaces

$$0 \to W \to T \to \mathcal{A} \to 0.$$

Let ω be the 2-cochain of $C_2(\mathcal{A}, W)$ defined by $\omega(a, a') = \sigma(a)\sigma(a') - \sigma(aa')$. Actually ω is a 2-cocycle. Define a multiplication on $\mathcal{A} \times W$ by

$$(a, w)(a', w') = \big(aa',\ aw' + wa' + \omega(a, a')\big).$$

The cohomology class $[\omega] \in H^2(\mathcal{A}, W)$ does not depend on the choice of σ. Furthermore two equivalent extensions yield the same cohomology class. Our discussion yields:

Theorem 5.1. *Let W be a trivial KV-algebra that is also a two-sided KV-module over the KV-algebra $\mathcal{A}$. Then there is a one to one map from $H^2(\mathcal{A}, W)$ onto the set of equivalence classes of splittable extensions of the KV-algebra $\mathcal{A}$ by the trivial algebra W.*

Corollary 5.2. *Under the assumptions of Theorem 5.1, if the KV-algebra $\mathcal{A}$ is $\mathbb{F}$-projective there is a bijective correspondence between $H^2(\mathcal{A}, W)$ and the set* $\mathrm{Ext}(\mathcal{A}, W)$ *of equivalent classes of extenstion of the KV-algebra $\mathcal{A}$ by the trivial algebra W.*

Extensions of KV-modules. Let $\mathcal{G}$ be an abstract Lie algebra and let V and W be abstract $\mathcal{G}$-modules. Then every exact sequence of $\mathcal{G}$-modules

$$0 \to V \to T \to W \to 0$$

gives rise to a Chevalley–Eilenberg cohomology class in $H^1(\mathcal{G}, L(W, V))$ that determines the equivalence class of this extension. This also holds for Hochschild cohomology of associative algebras with coefficients in their two-sided modules. The same feature is far from being true in the category of two-sided KV-modules and KV-cohomology.

Let V, W be two-sided modules over a KV-algebra $\mathcal{A}$. We shall show that the cohomology space responsible for equivalence classes of extensions of W by V is not $H^1(\mathcal{A}, L(W, V))$ but rather the cohomology space $E_1^{1,1}(\mathcal{A} \times W, V)$ at the level $C_{1,1}(\mathcal{A} \times W, V)$.

We fix some notation. Henceforth $\mathcal{A} \oplus W$ is endowed with the KV-algebra structure defined by $(a, w)(a', w') = (aa', aw' + wa')$. Take $f \in C_{1,1}(\mathcal{A} \times W, V)$.

Then δf belongs to $C_{1,2}(\mathscr{A}\oplus W, V)$. For every $(a,w)\in\mathscr{A}\times W$ we set

$$\theta(a,w)=f(a,w),\qquad \psi(a,w)=f(w,a). \tag{5–2}$$

Given $a,b\in\mathscr{A}$ and $w\in W$, one has

$$\begin{aligned}\delta f(a,b,w)&=\delta\theta(a,b,w)\\&=-a\theta(b,w)+\theta(ab,w)+\theta(b,aw)+b\theta(a,w)-\theta(ba,w)-\theta(a,bw)\end{aligned}$$

and

$$\begin{aligned}\delta f(a,w,b)={}&-af(w,b)+f(aw,b)+f(w,ab)-f(w,a)b\\&+wf(a,b)-f(wa,b)-f(a,wb)+f(a,w)b.\end{aligned}$$

So in view of (5–2) we see that θ and ψ are related by

$$\begin{aligned}\delta f(a,w,b)={}&-a\psi(b,w)+\psi(b,aw)+\psi(ab,w)-(\psi(a,w))b\\&-\psi(b,wa)-\theta(a,wb)+(\theta(a,w))b,\\ \delta f(a,b,w)={}&\delta\theta(a,b,w).\end{aligned}$$

If f is a cocycle we get $\delta\theta(a,b,w)=0$ and

$$a\psi(b,w)+\psi(b,aw)+\psi(ab,w)-(\psi(a,w))b-\psi(b,wa)=\theta(a,wb)-(\theta(a,w))b.$$

Consider an $\mathbb{F}$-splittable exact sequence

$$0\to V\to T\to W\to 0.$$

of two-sided KV-modules over $\mathscr{A}$. Fix a section $\sigma\in\mathrm{Hom}(W,T)$. Then σ will define $f_\sigma\in C_{1,1}(\mathscr{A}\times W,V)$ as follows:

$$\begin{aligned}f_\sigma(a,w)&=a\sigma(w)-\sigma(aw),\\ f_\sigma(w,a)&=\sigma(w)a-\sigma(wa).\end{aligned}\tag{5–3}$$

Lemma 5.3. $\delta f_\sigma=0$.

Proof. Take $a,b\in\mathscr{A}$ and $w\in W$. Then

$$\begin{aligned}\delta f_\sigma(a,b,w)&=-a(b\sigma(w)-\sigma(bx))+(ab)\sigma(w)-\sigma((ab)w)\\&\qquad+\ b\sigma(aw)-\sigma(b(aw))-b(a\sigma(w))+b\sigma(aw)\\&\qquad-\ (ba)\sigma(w)+\sigma((ba)w)-a\sigma(bw)+\sigma(a(bw))\\&=(b,a,\sigma(w))-(a,b,\sigma(w))+\sigma((a,b,w)-(b,a,w))=0,\\ \delta f_\sigma(a,w,b)&=-a(\sigma(w)b)-a\sigma(wb)+\sigma(aw)b-\sigma((aw)b)\\&\qquad+\ \sigma(w)ab-\sigma(w(ab))-(\sigma(w)a)b+\sigma(wa)b\\&\qquad-\ \sigma(wa)b+\sigma((wa)b)-a\sigma(wb)+\sigma(a(wb))\\&\qquad+\ (a\sigma(w))b-(\sigma(aw))b\\&=(a,\sigma(w),b)-(\sigma(w),a,b)+\sigma((a,w,b)-(w,a,b))=0.\end{aligned}$$

□

Consider the cochains θ, ψ defined by

$$\theta(a, w) = f_\sigma(a, w) \quad \text{and} \quad \psi(a, w) = f_\sigma(w, a). \tag{5–4}$$

They are related as in (5–3). By Lemma 5.3 the section σ gives rise to $[f_\sigma] \in E_1^{1,1}(\mathcal{A} \times W, V)$. If $\sigma' \in \mathrm{Hom}(W, T)$ is another section then $f_{\sigma'} \in [f_\sigma]$. Thus the cohomology class $[f_\sigma]$ depends only on the equivalence class of the extension of W by V. Conversely given a class $[f] \in E_1^{1,1}$, every $f \in [f]$ yields the extension of W by V which is determined by the pair (θ, ψ) which is defined by $\theta(a, w) = f(a, w)$, $\psi(a, w) = f(w, a)$ for all $(a, w) \in \mathcal{A} \times W$. The actions of $\mathcal{A}$ on $T = V \oplus W$ is defined by $a(v, w) = (av + \theta(a, w), aw)$ and $(v, w)a = (va + \psi(a, w),\ wa)$. Now we can state the classification theorem for extensions of two-sided KV-modules.

Theorem 5.4. *Let W and V be two-sided KV-modules over a KV-algebra $\mathcal{A}$. Then there is a bijective correspondence between $E_1^{1,1}(\mathcal{A} \times W, V)$ and the set* $\mathrm{Ext}(W, V)$ *of equivalence classes of splittable exact sequences of two-sided KV-modules*

$$0 \to V \to T \to W \to 0.$$

Corollary 5.5. *Under the assumptions of Theorem 5.4, assume W is $\mathbb{F}$-projective. There is a bijective correspondence between the set* $\mathrm{Ext}(W, V)$ *of equivalence classes of extensions of W by V and the set $E_1^{1,1}(\mathcal{A} \times W, V)$.*

6. Deformations of algebraic structures

The set (V, μ) of associative algebra structures (or of Lie algebra structures) on a vector space V is a singular algebraic variety with a natural action of the linear group $\mathrm{GL}(V)$. The study of that variety involves the Hochschild cohomology (in the case of associative algebras) or the Chevalley–Eillenberg cohomology (in the case of Lie algebras); see [Gerstenhaber 1964; Nijenhuis 1968; Piper 1967; Koszul 1968]. The second cohomology space may be regarded either as obstruction to rigidity or as a tool to understand how wide is a $\mathrm{GL}(V)$-orbit in the ambient variety. (In fact it measures the transverse structure of $\mathrm{GL}(V)$-orbits). Another interesting question is whether (V, μ) admits nontrivial formal deformations. That is, does there exist a formal series $\mu(t)$ with a given starting point? In both associative and Lie theories obstructions exist and live in the third cohomology space. Our aim is to investigate what happens in the KV-cohomology theory. We denote by $|\mathcal{A}|$ the underlying vector space of a KV-algebra $\mathcal{A}$. Let $KV(|\mathcal{A}|)$ be the variety of KV-algebra structures on $|\mathcal{A}|$. Let μ, ν be $\mathcal{A}$-valued bilinear functions on $\mathcal{A}$. We denote by $d_\mu \nu$ the trilinear function defined by

$$
\begin{aligned}
(6\text{–}1)\quad d_\mu \nu(a,b,c) = -\mu(a,\nu(b,c)) + \nu(\mu(a,b),c) \\
+ \nu(b,\mu(a,c)) - \mu(\nu(b,a),c) + \mu(b,\nu(a,c)) \\
- \nu(\mu(b,a),c) - \nu(a,\mu(b,c)) + \mu(\nu(a,b),c).
\end{aligned}
$$

Deformations of KV-algebras. Let $\mathscr{A}$ be a KV-algebra whose multiplication is denoted by aa'. One is given a one parameter power series of KV-algebra structures on $|\mathscr{A}|$, namely

$$\mu_t = \sum_{j=0}^{\infty} \mu_j t^j, \qquad \text{with } \mu_0(a,a') = aa'.$$

Henceforth $(a,a',a'')_t$ stands for the associator map of μ_t. One has

$$(6\text{–}2)\qquad (a,b,c)_t - (b,a,c)_t = 0$$

for all $a,b,c \in |\mathscr{A}|$. According to (6–1) we set $d_i = d_{\mu_i}$. Identity (6–2) is equivalent to the system

$$(6\text{–}3)\qquad \delta\mu_1 = 0, \qquad \delta\mu_k = \sum_{\substack{i+j=k\\ i>0,\ j>0}} d_i \mu_j.$$

Thus we can identify the Zariski tangent space of $KV(|\mathscr{A}|)$ at $\mu_0 \in KV(|\mathscr{A}|)$ with the space $Z_2(\mathscr{A},\mathscr{A})$ of 2-cocycles. So the family μ_t gives rise to the cohomology class $[\mu_1] \in H^2(\mathscr{A},\mathscr{A})$. We consider a one-parameter power series $\phi_t \in \mathrm{GL}(|\mathscr{A}|)$ with $\phi_0(a) = a$ for all $a \in \mathscr{A}$. Roughly speaking one has $\phi_t(a) = a + t\theta_1(a) + \cdots + t^j\theta_j(a) + \cdots$, where the θ_j are linear endomorphisms of $|\mathscr{A}|$. Now by setting

$$\mu_t(a,a') = \phi_t(\phi_t^{-1}(a)\phi_t^{-1}(a'))$$

we define a one parameter family $\mathscr{A}(t) = (|\mathscr{A}|, \mu_t)$ of KV-algebra structures all of which are isomorphic to $\mathscr{A}$. Take the MacLaurin expansion of $\mu_t(a,a')$, namely $\mu_t(a,a') = aa' + \sum_{k>0} t^k \mu_k(a,a')$. Relation (6–2) yields $\mu_1(a,a') = \delta\theta_1(a,a')$. Thereby the space $B_2(\mathscr{A},\mathscr{A})$ of 2-coboundaries of $\mathscr{A}$ is the Zariski tangent space of the orbit of $\mathscr{A}$ under the action of $\mathrm{GL}(|\mathscr{A}|)$. Relation (6–3) also highlights how formal deformations will involve the third cohomology space. We summarize our discussion:

Theorem 6.1 (rigidity). (i) *A necessary condition for $\mathscr{A}$ to be rigid is the vanishing of $H^2(\mathscr{A},\mathscr{A})$.*

(ii) *Conversely, if $H^2(\mathscr{A},\mathscr{A})$ vanishes, $\mathscr{A}$ is formally rigid.*

(iii) *If $H^3(\mathscr{A},\mathscr{A})$ vanishes, the set of nontrivial formal deformations of $\mathscr{A}$ is parametrized by the first cohomology space $H^1(\mathscr{A},\mathscr{A})$.*

The space $H^2(\mathscr{A}, \mathscr{A})$ can be viewed as the set of nontrivial infinitesimal deformations of $\mathscr{A}$. In some special situations, $H^2(\mathscr{A}, \mathscr{A}) \neq 0$ will imply that $\mathscr{A}$ admits nontrivial deformations; we turn to this in Part 2.

Part II. Differential geometry

Besides the conjecture of M. Gerstenhaber, there are many geometric problems which also motivate the need of the cohomology theory of KV-algebras; see for example [Koszul 1968; Milnor 1977; Vinberg and Kac 1967; Vey 1968]. In this Part 2 we intend to raise some of those geometric problems. We start with some observations.

(1) Let $\mathscr{A}$ be an associative algebra. Let $C_{\text{ass}}(\mathscr{A}, \mathscr{A})$ be the Hochschild complex of $\mathscr{A}$. Every Hochschild cocycle $\omega \in C^2_{\text{ass}}(\mathscr{A}, \mathscr{A})$ is also a KV-cocycle of the KV-algebra $\mathscr{A}$. The inverse is not true. Thus one gets a canonical injective map $H^2(\mathscr{A}, \mathscr{A})_{\text{ass}} \to H^2(\mathscr{A}, \mathscr{A})_{KV}$. This map is not surjective. For instance, let $\mathscr{A}$ be $\mathbb{R}^2$ with the multiplication defined by $(x, y).(x', y') = (xx', 0)$. The set $Z^2_{KV}(\mathscr{A}, \mathscr{A})$ of 2-KV-cocycles consists of pairs (f, h) of linear maps defined by $(f(x, y), h(x, y)) = ((ax + by, bx), (ux + vy, wy))$ while the set $Z^2_{\text{ass}}(\mathscr{A}, \mathscr{A})$ consists of pairs $(f(x, y), h(x, y)) = ((ax + by, bx), (ux, vy))$. On the other hand the space $B^2_{KV}(\mathscr{A}, \mathscr{A}) = B^2_{\text{ass}}(\mathscr{A}, \mathscr{A})$ consists of pairs $(f(x, y), h(x, y)) = ((ax + by, bx), (ux, 0))$.

(2) The graded space $C^\star_N(\mathscr{A}, W) = \operatorname{Hom}\big(\bigwedge^{\star-1}\mathscr{A}, L(\mathscr{A}, \mathscr{A})\big)$ defined in [Nijenhuis 1968] is actually a subcomplex of the KV-complex $C^\star(\mathscr{A}, W)$ but the derived cohomology (by Nijenhuis) differs from the KV-cohomology. Nevertheless, if W is a left KV-module then the cohomology defined by Nijenhuis coincides with the KV-cohomology (see the Appendix). We return to differential geometry.

7. KV-Cohomology of locally flat manifolds

Let (M, D) be a locally flat manifold whose KV-algebra is denoted by $\mathscr{A}$. The space $T(M)$ of smooth tensors on M is an infinite-dimensional bigraded vector space. Thus $T(M) = \sum_{p,q} T^{p,q}(M)$ where $T^{p,q}(M)$ is the vector space of smooth sections of the vector bundle

$$T^{p,q}M = \Big(\bigotimes_p TM\Big) \otimes \Big(\bigotimes_q T^*M\Big).$$

The vector space $T(M)$ is a two-sided KV-module over the KV-algebra $\mathscr{A}$. To see this it is sufficient to write the two-sided actions of $\mathscr{A}$ on $T^{1,0}(M)$ and on $T^{0,1}(M)$ respectively. Take $a \in \mathscr{A}$, $X \in T^{1,0}(M)$ and $\theta \in T^{0,1}(M)$, and set $aX = D_aX$, $Xa = D_Xa$, $a\theta = D_a\theta$, $\theta a = 0$. The differential 1-form $D_a\theta$ is defined by

$\langle (D_a\theta), X\rangle = \langle d(\theta(X)), a\rangle - \langle\theta, aX\rangle$. Of course $T^{p,0}(M)$ is defined for $p > 0$ while $T^{0,q}$ is defined for nonnegative integers under the convention that

$$T^{0,0}(M) = W = C^\infty(M, \mathbb{R}).$$

Denote the $T(M)$-valued complex of $\mathscr{A}$ by $C(\mathscr{A}, T(M)) = \bigoplus_\rho C_\rho(\mathscr{A}, T(M))$. Its coboundary operator $\delta : C_\rho(\mathscr{A}, T(M)) \to C_{\rho+1}(\mathscr{A}, T(M))$ is defined by formula (4–1). More explicitly, for $f \in C_\rho(\mathscr{A}, T(M))$ we have

$$\delta f(a_1..a_{\rho+1}) = \sum_{i\leq\rho}(-1)^i\{(a_i f)(..\hat{a}_i..a_{\rho+1}) + (e_\rho(a_i).(f a_{\rho+1})(..\hat{a}_i..a_\rho)\}.$$

The cohomology space at the level $C_\rho(\mathscr{A}, T(M)$ is written $H^\rho(\mathscr{A}, T(M))$. Note that not all of the cochains of $C_\rho(\mathscr{A}, T(M))$ are tensorial. Actually $C(\mathscr{A}, T(M))$ admits a triple grading $C(\mathscr{A}, T(M)) = \sum C_\rho^{p,q}(\mathscr{A}, T(M))$. The action of $\mathscr{A}$ on $T(M) = \oplus_{p,q} T^{p,q}(M)$ is of zero degree. Thereby every subspace $T^{p,q}(M)$ is a two-sided KV-module over $\mathscr{A}$. Let one set $C_\rho^{p,q}(\mathscr{A}, T(M)) = C_\rho(\mathscr{A}, T^{p,q}(M))$. The subspace $\oplus_\rho C_\rho^{p,q}(\mathscr{A}, T(M)$ is a subcomplex of $C(\mathscr{A}, T(M))$. The subcomplex $C^{0,0}(\mathscr{A}, T(M))$ is precisely the complex $C^\star(\mathscr{A}, W)$. The subspace $\tau(M, \mathbb{R})$ of tensorial cochains is a KV-subcomplex of $C(\mathscr{A}, W)$. It is graded by the subspaces $\tau^\rho(M, \mathbb{R}) = \tau(M, \mathbb{R}) \cap C_\rho(\mathscr{A}, T(M))$.

Definition. The KV-complex $\tau(M, \mathbb{R})$ is called the scalar KV-complex of the locally flat manifold (M, D).

Remark. Given a locally flat manifold (M, D), the connection D yields an exterior differential operator d_D on $T^\star M$-valued differential forms $\Omega(M, T^\star M)$. The aim of [Koszul 1974] was the study of the D-complex $(\Omega(M, T^\star M), d_D)$. However this D-complex studied by Koszul is the Nijenhuis complex $C_N(\mathscr{A}, W)$ of the KV-algebra $\mathscr{A}$ [Nijenhuis 1968]. On the other hand the graded space $\Omega(M, T^\star M) = \Sigma_p \Omega^p(M, T^\star M)$ is a subcomplex of the KV-complex $C(\mathscr{A}, W)$. Since W is a left KV-module over $\mathscr{A}$ the Koszul–Nijenhuis cohomology of the D-complex

$$\Omega((M, T^\star M), d_D)$$

coincides with the KV-cohomology of the KV-complex $(\Omega(M, T^\star M), \delta)$ (see the Appendix). This is a relevant aspect of the KV-cohomology of $\mathscr{A}$. We have pointed out the KV-complex inclusions $\tau(M, \mathbb{R}) \subset C(\mathscr{A}, W) \subset C(\mathscr{A}, T(M))$. The KV-cohomology spaces of these complexes are affine invariants, and indeed geometric invariants of the Lie group of D-preserving diffeomorphisms of the smooth manifold M.

8. Rigidity problem for hyperbolic locally flat manifolds

A relevant question is to give geometrical meaning to some cohomology classes of the KV-complexes $\tau(M, \mathbb{R})$, $C(\mathscr{A}, W)$,$C(\mathscr{A}, T(M))$. The next subsections are devoted to this question. Domains in euclidean space are endowed with the canonical euclidean connection D_0.

Hyperbolic affine manifolds. A locally flat manifold (M, D) is called *hyperbolic* if its universal covering $(\tilde{M}, \tilde{D})$ is isomorphic to a convex domain not containing any straight line. This definition is the affine analogue to hyperbolic holomorphic manifolds following W. Kaup (see [Vey 1968; Koszul 1968]). Let M be a compact manifold admitting hyperbolic locally flat structures. Let $\mathscr{F}(M)$ be the set of locally flat linear connections on M. The subset $\mathscr{F}c(M)$ consists of $D \in \mathscr{F}(M)$ satisfying the following condition: the universal covering of (M, D) is isomorphic to a convex cone not containing any straight line.

Theorem 8.1 [Koszul 1968, Theorem 3]. *If M is a compact manifold, $\mathscr{F}c(M)$ is an open subset of $\mathscr{F}(M)$.*

Koszul's proof involves locally Hessian Riemannian metrics. As a remarkable consequence of this theorem we have the nonrigidity of hyperbolic affine manifold structures defined by elements $D \in \mathscr{F}c(M)$. Koszul proved that every locally flat structure (M, D) defined by $D \in \mathscr{F}c(M)$ always admits nontrivial deformations. Roughly speaking, every neighborhood of D contains a $D' \in \mathscr{F}c(M)$ which is not isomorphic to D. The rigidity problem for (M, D) is related to the KV-cohomology space $H^2(\mathscr{A}, \mathscr{A})$. From this viewpoint the compactness of M may be unnecessary.

Example. Consider $\mathbb{R}^2 = \{\lambda\delta/\delta x + \mu\delta/\delta y : \lambda \in \mathbb{R},\ \mu \in \mathbb{R}\}$. By endowing this space with the bracket $[\delta/\delta x\ , \delta/\delta y] = \delta/\delta y$, we obtain the Lie algebra of infinitesimal affine transformations of $\mathbb{R}$. We define a left-invariant locally flat structure on the Lie group Aff $\mathbb{R}$ by setting

$$D_{\lambda,\mu}\lambda', \mu' = (\lambda\mu')\frac{\delta}{\delta y} = (0, \lambda\mu'). \tag{8–1}$$

Let $\mathscr{A} = (\mathscr{X}(\text{Aff } \mathbb{R}, D))$ be the corresponding KV-algebra. To $(\alpha, \beta) \in \mathbb{R}^2$ we assign $S_{\alpha,\beta} \in T^{1,2}(\text{Aff } \mathbb{R})$ defined by

$$S_{\alpha,\beta}((\lambda, \mu), (\lambda', \mu')) = (\alpha\lambda\lambda',\ \beta\lambda\lambda' + \alpha(\lambda\mu' + \lambda'\mu)),$$

with $\lambda, \lambda', \mu, \mu' \in C^\infty(\text{Aff } \mathbb{R}, \mathbb{R})$. Keeping in mind relation (6–1), we get:

Lemma 8.2. *For every pair $(\alpha, \beta) \in \mathbb{R}^\star \times \mathbb{R}$ of real numbers the cochain $S_{\alpha\beta} \in C_2(\mathscr{A}, \mathscr{A})$ is a nonexact KV-cocycle satisfying $d_{S_{\alpha\beta}} S_{\alpha\beta} = 0$.*

Corollary 8.3. *The locally flat manifold* (Aff $\mathbb{R}$, D) *defined by* (8–1) *admits nontrivial deformations.*

Proof. Define the one-parameter family $D_t \in \mathcal{F}(\mathrm{Aff}\,\mathbb{R})$ of linear connections by $D_t = D + tS_{\alpha\beta}$. By virtue of Lemma 8.2, D_t is locally flat. The corresponding KV-algebra is denoted by $\mathcal{A}(t)$. Since $S_{\alpha\beta}$ is nonexact the family $\mathcal{A}(t)$ is a nontrivial deformation of $\mathcal{A}$. To be convinced, fix $(\lambda_0, \mu_0) \in \mathbb{R}^2$. Then one considers the family ψ_t of linear maps defined by $\psi_t(\lambda, \mu) \to D_{(\lambda,\mu)}(\lambda_0, \mu_0) + tS_{\alpha,beta}((\lambda, \mu), (\lambda_0, \mu_0)) + (\lambda, \mu)$. Then ψ_0 is injective. Thereby $(\mathrm{Aff}\,\mathbb{R}, D_0)$ is complete. We have assumed that $\alpha \neq 0$. If $t \neq 0$ then $(\mathrm{Aff}\,\mathbb{R}, D_t)$ is not complete. Consequently none of the D_t is isomorphic to D_0. □

A locally flat manifold (M, D) is complete if and only if D is geodesically complete. Let one examine what do geodesics of $D_1 = D + S_{\alpha\beta}$ look like. Consider a geodesic of D_1, namely $c(t) = (x(t), y(t))$. We have $D_{(\dot{x},\dot{y})}(\dot{x}, \dot{y}) = (0, 0)$, where $(\dot{x}, \dot{y}) = (dx/dt)\,(\delta/\delta x) + (dy/dt)\,(\delta/\delta y)$. The geodesics of D_1 are the solutions of the system of differential equations

$$2\frac{d^2x}{dt^2} + \alpha\Big(\frac{dx}{dt}\Big)^2 = 0, \quad 2\frac{d^2y}{dt^2} + \beta\Big(\frac{dx}{dt}\Big)^2 + (1+2\alpha)\frac{dx}{dt}\frac{dy}{dt} = 0.$$

The first of these equations admits the solutions

$$x(t) = \frac{2}{\alpha}\log\Big|\frac{\alpha}{2}t + u\Big| + v,$$

where u and v are arbitrary constants. It becomes easy to integrate the second equation. Its solutions are

$$y(t) = \frac{\beta}{\alpha(1-2\alpha)}\log\Big|\frac{\alpha}{2}t + u\Big| - \frac{c}{1+\alpha}\exp\Big(-\frac{1+\alpha}{\alpha}\log\Big|\frac{\alpha}{2}t + u\Big|\Big) + d,$$

where c and d are arbitrary constants. Thus the geodesics of D_1 are defined only on the half-lines $]\gamma, \infty[$ or $]-\infty, \gamma[$, where γ is an arbitrary real number. So D_1 does not admit any complete geodesic. Of course the existence of solutions $(x(t), y(t))$ depends on other extra conditions such as $1 + \alpha \neq 0$. Our purpose is to show that $(\mathrm{Aff}\,\mathbb{R}, D_1)$ is isomorphic to a convex cone not containing any straight line. We just saw that $(\mathrm{Aff}\,\mathbb{R}, D_1)$ has no complete geodesic. We show that it is isomorphic to a convex cone. Since $[\delta/\delta x, \delta/\delta y] = \delta/\delta y$, the element $\delta/\delta x$ commutes with $e^{-x}\delta/\delta y$. These two elements are affine vector fields on the locally flat Lie group $(\mathrm{Aff}\,\mathbb{R}, D_1)$. If $\alpha(1+\alpha) \neq 0$, the functions $g(x, y) = \exp((1+\alpha)x + y)$ and $h(x, y) = \exp((1+\alpha)x + 2y + y^3)$ are affine. We identify the connected component of the identity of $\mathrm{Aff}\,\mathbb{R}$ with $\{(x, y) \in \mathbb{R}^2 : x > 0\}$. Then the map $\phi(x, y) = (g(x, y), h(x, y))$ is an affine isomorphism from $\mathrm{Aff}\,\mathbb{R}_0$ onto the open convex cone $\{(u, v) \in \mathbb{R}^2 : u > 0,\ v > 0\}$. This cone contains no straight lines.

Proposition 8.4. *Let $\mathscr{A}$ be the KV-algebra of a compact locally flat manifold (M, D). If $H^2(\mathscr{A}, \mathscr{A}) = 0$, every smooth one-parameter family of deformations of (M, D) is trivial.*

Proof. Let (M, D_t) be a smooth deformation of (M, D), with $D_0 = D$. One sets $S = (d/dt)\, D_{t/t=0}$. Then $S \in C_2(\mathscr{A}, \mathscr{A})$ is a 2-cocycle. Since $H^2(\mathscr{A}, \mathscr{A})$ vanishes, there exists a 1-chain $\phi \in C_1(\mathscr{A}, \mathscr{A})$ such that $S(a, b) = \delta\phi(a, b)$. The symmetry property of S implies that ϕ is a derivation of the Lie algebra of smooth vector fields on M. So there is a $\xi \in \mathscr{A}(M)$ such that $\phi(a) = [\xi, a]$ for all $a \in \mathscr{A}$. The 2-coboundary $S = \delta\phi$ takes the form $S(a, b) = (a, b, \xi)$. Let ϕ_t be the local flow of ξ. Then D and D_t are related by $D_t = \phi_{(t)} D$. □

9. Complex of superorder forms

The reference for this section is [Koszul 1968].

Differential forms of order $\le k$. Let $\mathscr{A}(M)$ be the Lie algebra of smooth vector fields on a manifold M. Let $T(M) = \bigoplus_{p,q} T^{p,q}(M)$ be the space of smooth tensors. We are concerned with the vector space $\bigoplus^q \operatorname{Hom}(T^{q,0}(M), T(M))$ of $T(M)$-valued $\mathbb{R}$-multilinear functions on $\mathscr{A}(M)$.

Definition [Koszul 1974]. Let k be a nonnegative integer. The function

$$\theta \in \operatorname{Hom}(\mathscr{A}(M), T(M))$$

is of order $\le k$ if at every point $x \in M$ the value $(\theta(X_1, \dots X_q))(x) \in T_x^{r,s}(M)$ depends on the k-jets $j_x^k(X_1), \dots, j_x^k(X_q)$.

Let $E \to M$ be a tensor vector bundle and let $\mathscr{E}$ be the vector space of smooth sections of E. Then $\mathscr{A}(M)$ acts on $\mathscr{E}$ by Lie derivation. Let $C(\mathscr{A}(M), \mathscr{E})$ be the Chevalley–Eilenberg complex of $\mathscr{A}(M)$. Koszul observed that in many situations the cohomology of $C(\mathscr{A}(M), \mathscr{E})$ contains some canonical nonvanishing cohomology classes.

Examples. (1) Let v be a volume form on a compact manifold M. The divergence class of (M, v) does not vanish.

(2) To every linear connection D on M one assigns the linear map $\theta(X) = L_X D$. This is a $T^{1,2}(M)$-valued nontrivial 1-cocycle of order ≤ 1.

We now deal with KV-complexes on a locally flat manifold (M, D). Then $\mathscr{A}(M)$ is the commutator Lie algebra of the KV-algebra $\mathscr{A}$ of (M, D). Set $W = C^\infty(M, \mathbb{R})$. Let $\tau(M, \mathbb{R}) \subset C(\mathscr{A}, W)$ be the subcomplex consisting of cochains of order ≤ 0. For instance a (pseudo)riemannian tensor is a 2-cochain of order ≤ 0 while its Levi-Civita connection $\triangledown \in C(\mathscr{A}, \mathscr{A})$ is a 2-cochain of order ≤ 1. The Levi-Civita connection $\triangledown$ is defined by the symmetric 2-cochain $S_\triangledown(a, a') = \triangledown_a^{a'} - aa'$. One has $\delta \bigtriangledown (a, a', a'') = \delta S_{\bigtriangledown}(a, a', a'')$. Whenever $\delta\bigtriangledown$ vanishes, the curvature tensor $R_{\bigtriangledown}$ is given by

$$R_{\bigtriangledown}(X, Y) = \big[S_{\bigtriangledown}(X, \cdot), S\bigtriangledown(Y, \cdot)\big].$$

This formula means that $R_{\bigtriangledown}(X, Y)(Z) = S_{\bigtriangledown}(X, S_{\bigtriangledown}(Y, Z)) - S_{\bigtriangledown}(Y, S_{\bigtriangledown}(X, Z))$. Boundaries of cochains of order $\leq k$ are cochains of order $\leq (k+1)$. For instance:

(1) Let ω be a de Rham closed differential 2-form. We regard it as a 2-chain of order ≤ 0 of the KV-complex $C(\mathscr{A}, W)$. Then we get $\delta\omega(a, b, c) = (c.\omega)(a, b) = c(\omega(a, b)) - \omega(ca, b) - \omega(a, cb)$. So that $\delta\omega = 0$ if and only if ω is a parallel 2-form with respect to the linear connection D. Now assume ω to be δ-exact and let $\theta \in C_1(\mathscr{A}, W)$ such that $\omega(X, Y) = -X\theta(Y) + \theta(XY)$. We may conclude that $-\omega(X, Y) = \frac{1}{2}\big(X\theta(Y) - Y\theta(X) + \theta([X, Y])\big)$. So a parallel closed 2-form is δ-exact if and only if it is de Rham exact.

(2) There are situations where the only (de Rham) closed 2-form ω satisfying $D\omega = 0$ and $\omega = \delta\beta$ is $\omega = 0$. For instance we consider the Lie group $\mathcal{M} = H_3 \times \mathbb{R}$ where H_3 is the 3-Heisenberg group. We fix a basis e_1, e_2, e_3 of the Lie algebra h_3 of H_3 such that $[e_1, e_2] = e_3$. Complete this basis to get a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathscr{H}_3 \oplus \mathbb{R}$; let $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ be its dual basis. Then $\omega = \epsilon_1 \wedge \epsilon_4 + \epsilon_3 \wedge \epsilon_2$ is a left invariant symplectic form on $\mathcal{M}$. Now N_1 and N_2 are Lie subgroups whose Lie algebras are $\mathcal{N}_1 = \mathrm{span}(e_1, e_3)$ and $\mathcal{N}_2 = \mathrm{span}(e_2, e_4)$ respectively. These subgroups define a pair $(\mathscr{L}_1, \mathscr{L}_2)$ of left-invariant lagrangian foliations which are transverse. Let Γ be a cocompact lattice in $\mathcal{M}$. The pair $(\mathscr{L}_1, \mathscr{L}_2)$ gives rise to the pair (l_1, l_2) of lagrangian foliations in the compact symplectic manifold $(\Gamma \setminus \mathcal{M}, \omega)$. Thereby $\Gamma \setminus \mathcal{M}$ admits a unique (l_1, l_2)-preserving torsion-free symplectic connection D; see [Nguiffo Boyom 1995]. Now let (x, y, z, t) be the euclidean coordinates of $\mathcal{M}$ in the basis (e_1, e_2, e_3, e_4). These coordinate functions give rise to local Darboux coordinate functions on $(\Gamma \setminus \mathcal{M}, \omega)$. Then x, z are characteristic functions of l_1 and y, t are characteristic functions of l_2. Therefore the curvature tensor of D vanishes, again by [Nguiffo Boyom 1995]. One gets the locally flat manifold $(\Gamma \setminus \mathcal{M}, D)$. Since $\Gamma \setminus \mathcal{M}$ is a closed manifold ω is not de Rham exact. Thereby ω is not δ-exact as well. This situation is the analogue to observations by Koszul.

Proposition 9.1. *Let (M, D) be a hyperbolic locally flat manifold. If the universal covering $(\tilde{M}, \tilde{D})$ of (M, D) is isomorphic to a convex cone then every D-parallel 2-cochain of $C_2(\mathscr{A}, W)$ is δ-exact.*

Proof. By [Koszul 1968, Lemma 3], there exists $H \in \mathscr{A}$ such that $aH = a$ for all $a \in \mathscr{A}$. If g is a D-parallel 2-chain we define the 1-chain θ by setting $\theta(a) = g(H, a)$. Since $Dg = 0$ we see that

$$0 = ag(H, b) - g(aH, b) - g(H, ab) = -\delta\theta(a, b) - g(a, b) \text{ for } a, b \in \mathscr{A}. \qquad \square$$

Let (M, D) be a locally flat manifold whose universal covering is $(\tilde{M}, \tilde{D})$. Let Γ be the fundamental group of M. The linear holonomy of (M, D) is denoted by $l(\Gamma)$, as in [Fried et al. 1981; Carrière 1989]. If $l(\Gamma)$ is unimodular then $\tilde{M}$ carries a $\tilde{D}$-parallel volume form $\tilde{v}$. Actually $\tilde{v}$ is a cocycle of $C(\mathcal{A}(\tilde{M}), C^\infty(\tilde{M}, \mathbb{R}))$.

Before proceeding, we make some explicit constructions. Given a locally flat manifold (M, D) let $c : [0, 1] \to M$ be a smooth curve. For $s \in [0, 1]$ let $\tau_s : T_{c(0)}M \to T_{c(s)}M$ be the D-parallel transport (along c). Set

$$q(c) = \int_0^1 \tau_s^{-1}\Big(\frac{dc}{ds}\Big)\,ds.$$

Actually $q(c)$ does not depend on the choice of c in its fixed ends homotopy class [Koszul 1968]. Now one identifies the universal covering $\tilde{M}$ with the set $[[0, 1], M]_{x_0}$ of homotopy classes of smooth curves c with $c(0) = x_0$. Thus one gets the development map $q : \tilde{M} \to T_{x_0}M$ which is a local diffeomorphism. The locally flat manifold (M, D) is called complete if its development map is a diffeomorphism, [Fried et al. 1981; Carrière 1989; Milnor 1977]. This is equivalent to the geodesic completeness of D.

Proposition 9.2. *If a locally flat manifold (M, D) is complete, the cohomology class $[\tilde{v}] \in H_m(\tilde{\mathcal{A}}, \tilde{W})$ vanishes.*

The proof is based on the fact that (M, D) is complete if and only if $(\tilde{M}, \tilde{D})$ is isomorphic to euclidean space $(\mathbb{R}^n, D_0)$.

Corollary 9.3. *Let (M, D) be a locally flat compact manifold whose universal covering is denoted by $(\tilde{M}, \tilde{D})$. Suppose the linear holonomy group of (M, D) to be either nilpotent or lorentzian. Then every $\tilde{D}$-parallel volume form is $\tilde{\delta}$-exact, where $\tilde{\delta}$ is the KV-coboundary operator of $C(\tilde{\mathcal{A}}, \tilde{W})$.*

Idea of proof. The existence of a $\tilde{D}$-parallel volume form implies that $l(\Gamma)$ is unimodular. Under the hypothesis of the corollary, (M, D) is complete; see [Carrière 1989; Fried et al. 1981]. □

Regarding the conjecture of Markus [Carrière 1989; Fried et al. 1981], Proposition 9.2 says the following. Given a locally flat compact manifold (M, D), if its universal covering $(\tilde{M}, \tilde{D})$ carries a $\tilde{D}$-parallel volume form $\tilde{v}$ whose cohomology class $[\tilde{v}] \in H_m(\tilde{\mathcal{A}}, \tilde{W})$ is nonzero, then (M, D) is not complete.

KV-cohomology has other interesting applications, such as the study of singular affine foliations [Nguiffo Boyom and Wolak 2002], the study local forms of KV-algebroids [Nguiffo Boyom and Wolak 2004], and the geometry of Poisson manifolds [Nguiffo Boyom 2001; 2005]. On the other hand the cotangent bundle $T^\star M$ carries the canonical Liouville symplectic form. The vertical foliation is a lagrangian foliation of the Liouville symplectic form. The KV-complex of this foliation gives rise to a spectral sequence $E_t^{r,s}$ with the following relevant features. The term $E_1^{r,0}$ vanishes for all of the positive integers r. The set of Poisson structures on M is isomorphic to a subset of $E_1^{0,2}$; see [Nguiffo Boyom 2005]. The last statement is the KV-analogue to the following result of Kontsevich [2003]: *There is a one-to-one correspondence between the Hochschild cohomology space* $H^2(C^\infty(M,\mathbb{R}), C^\infty(M,\mathbb{R}))$ *and the set of Poisson tensors on* M.

Part III. Groups of diffeomorphisms

This part is rather a sketch of a forthcoming research program. We outline the study of left-invariant affine structures on groups of diffeomorphisms. We show that this study involves some KV-cohomology complexes.

10. Lie groups of diffeomorphisms

Generalities. Let $\mathcal{M}$ be a smooth manifold. We denote by $\operatorname{Diff}\mathcal{M}$ the group of diffeomorphisms of $\mathcal{M}$, and by $\operatorname{Diff}_0\mathcal{M}$ the subgroup of those that are isotopic to the identity). In some situations $\operatorname{Diff}_0\mathcal{M}$ and many of its subgroups carry a structure of infinite-dimensional Lie group; see [Banyaga 1997; Leslie 1967; Rybicki 1997]. The Lie algebra $\mathcal{A}_0(M)$ of $\operatorname{Diff}_0\mathcal{M}$ is the algebra of compactly supported smooth vector fields.

Definition. A KV-structure $X, Y \to X.Y$ on the $\mathbb{R}$-vector space $\mathcal{A}_0(M)$ is called a left-invariant KV-structure on the Lie group $\operatorname{Diff}_0 M$ if its commutator bracket $[X, Y] = X.Y - Y.X$ is the Lie bracket of vector fields on M.

We are interested in the relationship between KV-structures on $\operatorname{Diff}_0 M$ and locally flat structures on M. There is no loss of generality in dealing with KV-structures on $\mathcal{A}(M)$. Given a manifold M let $KV_c(\mathcal{A}(M))$ be the set of KV-algebra structures whose commutator Lie algebra is $\mathcal{A}(M)$. From the KV-cohomology viewpoint it is natural to ask:

Question 10.1. *What does* $KV_c(\mathcal{A}(M))$ *look like*?

We begin by pointing out some useful tools.

Deformations of $\mathbb{Z}_2$-graded KV-algebras. We fix some notation. Let $\mathcal{A}$ be a KV-algebra. For every KV-module W we consider the semidirect product $\mathcal{A} \times W$. It is the KV-algebra whose multiplication is $(a + w).(a' + w') = aa' + aw' + wa'$. We equip the vector space $\mathcal{G} = \mathcal{A} \oplus W$ with a $\mathbb{Z}_2$-grading by setting $\mathcal{G}^0 = A$, $\mathcal{G}^1 = W$. Thus $\mathcal{G}$ is a $\mathbb{Z}_2$-graded KV-algebra. We are interested in the KV-complex $C(\mathcal{G}, \mathcal{G})$. The vector space $C_q(\mathcal{G}, \mathcal{G})$ of q-chains is $\mathbb{Z}_2$-graded as

$$C_q(\mathcal{G}, \mathcal{G}) = C_{q,0}(\mathcal{G}, \mathcal{G}) \oplus C_{q,1}(\mathcal{G}, \mathcal{G}),$$

where $C_{q,j}(\mathcal{G},\mathcal{G})$ is the subspace of $\mathcal{G}^j$-valued q-multilinear maps defined on $\mathcal{G}$. On the other hand, every $C_{q,j}(\mathcal{G},\mathcal{G})$ is bigraded by the subspaces $C^{r,s}_{q,j}(\mathcal{G},\mathcal{G}) = \mathrm{Hom}\big(\bigotimes^r |\mathcal{A}| \otimes \bigotimes^s, \mathcal{G}^j\big)$ with $r+s=q$.

Suppose that V is a two-sided KV-module over $\mathcal{A}$. Then V is a KV-module over $\mathcal{G}$. The boundary operator $\delta : C_q(\mathcal{G}, V) \to C_{q+1}(\mathcal{G}, V)$ sends $C^{r,s}(\mathcal{G}, V)$ to $C^{r+1,s}(\mathcal{G}, V)$. Consider the case where V is the space W of real-valued smooth functions. Then δ sends $C_{q,1}(\mathcal{G}, W)$ to $C_{q+1,1}(\mathcal{G}, W)$. Thus the complex

$$C_{\star,1}(\mathcal{G},\mathcal{G}) = C_{\star,1}(\mathcal{G}, W)$$

is just the KV-complex $C(\mathcal{G}, W)$. Set $a=(a,0)\in\mathcal{A}\times W$ and $w=(0,w)\in\mathcal{A}\times W$. If $(a_0,w_0)\in J(\mathcal{G})$ then $((a,w),(a',w'),(a_0,w_0))=(0,0)$ for all $(a,w),(a'w')\in\mathcal{G}$. This is equivalent to the following conditions:

(i) $a_0 \in J(\mathcal{A})$, $w_0 \in J(W)$.

(ii) $(a, w, a_0) = 0$ for all $(a, w) \in \mathcal{G}$.

Thus $J(\mathcal{G})$ is a homogeneous subalgebra of the Z_2-graded KV-algebra $\mathcal{G}$. We now focus on the subcomplex

$$\cdots \longrightarrow C^{r,s}_{q,1}(\mathcal{G}) \xrightarrow{\delta} C^{r+1,s}_{q+1,1}(\mathcal{G}) \longrightarrow \cdots .$$

Let $\theta \in C^{1,1}_{2,1}(\mathcal{G}) = \mathrm{Hom}(\mathcal{A}\otimes W, W)$. Then $\delta\theta \in C^{2,1}_{3,1}(\mathcal{G})$. If $\delta\theta = 0$ we get from (5–4) the following equalities:

$$\begin{aligned}&-a\theta(b,w)+\theta(ab,w)+\theta(b,aw) = -b\theta(a,w)+\theta(ba,w)+\theta(a,bw),\\ &-a\theta(w,b)+\theta(aw,b)+\theta(w,ab) = (\theta(w,a))b+\theta(wa,b)\\ &\hspace{20em}+\theta(a,wb)-(\theta(a,w))b,\end{aligned}$$

for all $a, b\in\mathcal{A}$, $w\in W$.

Suppose θ is a cocycle of $\in C^{0,2}_{2,1}(\mathcal{G})$. The identity $\delta\theta=0$ yields

$$\begin{gathered}-a\theta(w,w')+\theta(aw,w')+\theta(w,aw')-\theta(wa,w')=0,\\ \theta(w',wa)-\theta(w',w)a=\theta(w,w'a)-\theta(w,w')a.\end{gathered}$$

for all $a\in\mathcal{A}$ and $w, w'\in W$. Actually the subspace of $\mathcal{G}^0$-valued cochains is not a sub-complex of $C(\mathcal{G},\mathcal{G})$. Nevertheless one has to pay attention to the subspace $\mathrm{Sym}_{a,w}$ consisting of $\theta\in C^{1,1}_{2,0}(\mathcal{G})$ satisfying $\theta(a,w)=\theta(w,a)$ for all $(a,w)\in\mathcal{A}\times W$. Actually one has $\delta(\mathrm{Sym}_{a,w})\subset C^{2,1}_3(\mathcal{G},\mathcal{G})\oplus C^{1,2}_3(\mathcal{G},\mathcal{G})$. Take $\theta\in\mathrm{Sym}_{a,w}$

such that $\delta\theta = 0$. The following equalities ensue:

$$0 = \delta\theta(a, b, w) \Rightarrow -a\theta(b, w) + \theta(ab, w) + \theta(b, aw) \\ = -b\theta(a, w) + \theta(ba, w) + \theta(a, bw),$$

$$0 = \delta\theta(a, w, b) \Rightarrow -a\theta(w, b) + \theta(aw, b) + \theta(w, ab) - \theta(wa, b) - \theta(a, wb) = 0,$$

$$0 = \delta\theta(a, w, w') \Rightarrow 0 = -\theta(w, a)w' - w\theta(a, w') + \theta(a, w)w'.$$

Since $\theta \in \mathrm{Sym}_{a,w}$, the closedness hypothesis $\delta\theta = 0$ implies $w\theta(a, w') = 0$ for all $a \in \mathcal{A}$ and all $w, w' \in W$. This last implication has motivated our interest in well understanding the $\mathbb{Z}_2$-graded KV-algebra structure $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1$ whose subspace $\mathcal{G}^1$ is a left $\mathcal{G}^0$-module. Henceforth we deal with a $\mathbb{Z}_2$-graded KV-algebra $\mathcal{G} = \mathcal{A} \oplus W = \mathcal{G}^0 \oplus \mathcal{G}^1$ with $W.\mathcal{A} = (0)$. We suppose that $\theta \in C^{0,2}_{2,1}$ is a cocycle; then the identity $0 = \delta\theta(a, w, w')$ implies that

$$a\theta(w, w') = \theta(aw, w') + \theta(w, aw'). \tag{10–1}$$

We equip $W = \mathcal{G}^1$ with the multiplication defined by $w.w' = \theta(w, w')$. We get an algebra structure $\mathcal{W}_\theta = (W, \theta)$. The left action of $\mathcal{G}^0$ on $\mathcal{G}^1$ look like infinitesimal automorphisms of the algebra $\mathcal{W}_\theta$. On the other hand suppose $\psi \in \mathrm{Sym}_{a,w}$ to be a cocycle, then

$$0 = -a\psi(w, b) + \psi(aw, b) + \psi(w, ab) = 0 \text{ for all } a, b \in \mathcal{G}^0 \text{ and } w \in \mathcal{G}^1. \tag{10–2}$$

In regard to Question 10.1, relations (10–2) and (10–3) below provide useful information. We recall that $\mathcal{G}^1 \subset \mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1$ is a left $\mathcal{G}^0$-module. So the multiplication map of $\mathcal{G}$ is

$$(a, w)(a', w') = (aa', aw'). \tag{10–3}$$

***Special deformations of* (10–3).** Take $\theta \in C^{0,2}_{2,1}(\mathcal{G})$. We study the multiplication map on $\mathcal{G}$, defined by

$$(a, w)(a', w') = \big(aa',\ aw' + \theta(w, w')\big). \tag{10–4}$$

Definition. Let $\mathcal{G}^0 \oplus \mathcal{G}^1$ be a $\mathbb{Z}_2$-graded KV-algebra such that $\mathcal{G}^1$ is a left module over $\mathcal{G}^0$. A cochain $\theta \in C^{0,2}_{2,1}(\mathcal{G})$ is called a KV-cochain if

$$KV_\theta(w, w', w\text{”}) = (w, w', w'')_\theta - (w', w, w'')_\theta = 0$$

for all $w, w', w'' \in \mathcal{G}^1$. Here KV_θ is the Koszul–Vinberg anomaly of θ; the notation $(w, w', w'')_\theta$ stands for $\theta(w, \theta(w', w'')) - \theta(\theta(w, w'), w'')$.

Proposition 10.2. *The following statements are equivalent.*

(1) *The multiplication map* (10–4) *defines a KV-algebra structure in* $\mathcal{G}$.

(2) $\theta \in C^{0,2}_{2,1}(\mathcal{G})$ *is a KV-cocycle.*

The proof is a straightforward consequence of (10–1).

A KV-cochain $\theta \in C^{0,2}_{2,1}(\mathcal{G})$ is called *associative* or *commutative* if the algebra $\mathcal{W}_\theta$ is associative or commutative, respectively. It is easy to verify that a commutative KV-cochain is associative. Formulas (10–1) and (10–2) are useful to give a partial answer to Question 10.1.

Indeed, given a $\mathbb{Z}_2$-graded KV-algebra in which $\mathcal{G}^1$ is a left module over $\mathcal{G}^0$, every cocycles $\theta \in C^{0,2}_{2,1}(\mathcal{G})$ satisfies (10–1). On the other hand a cocycle $\psi \in C^{1,1}_{2,0}(\mathcal{G})$ satisfies the (10–2) if and only if $\psi \in \mathrm{Sym}_{a,w}$.

Henceforth we assume that $\mathcal{G}^0$ acts faithfully in $\mathcal{G}^1$.

Definition. A pair $(\theta, \psi) \in C^{0,2}_{2,1}(\mathcal{G}) \times C^{1,1}_{2,0}(\mathcal{G})$ of cocycles is called a connection-like pair if the following properties hold:

(c_1) $\psi \in \mathrm{Sym}_{a,w}$.

(c_2) θ is a commutative KV-cocycle.

(c_3) $\psi(\theta(w, w'), a) = \psi(w, \psi(w', a))$ for all $a \in \mathcal{G}^0$ and all $w, w' in \mathcal{G}^1$.

Theorem 10.3. *Every connection-like pair (θ, ψ) defines a new KV-algebra structure $\mathcal{G}_{\theta,\psi}$ whose multiplication is defined by*

$$(a, w)(a' w') = (aa' + \psi(w, a') + \psi(a, w'), aw' + \theta(w, w'))$$

for all $(a, w), (a', w') \in \mathcal{G}^0 \times \mathcal{G}^1$.

Idea of proof. The vanishing of the KV-anomaly results from a direct calculation. □

Example. Let (M, D) be a locally flat manifold whose KV-algebra is $\mathcal{A}$. The vector space $W = C^\infty(M, \mathbb{R})$ is a left KV-module over $\mathcal{A}$ under the Lie derivation; W is regarded as a KV-algebra with zero multiplication. Then $\mathcal{G} = \mathcal{A} \oplus W$ is a Z_2-graded KV-algebra whose multiplication map is $(a, w).(a', w') = (D_a a', L_a w') = (aa', D_a w')$. Consider the pair $(\theta, \psi) \in C^{0,2}_{2,1}(\mathcal{G}) \times C^{1,1}_{2,0}(\mathcal{G})$ defined by $\theta(w, w') = ww'$ and $\psi(w, a) = \psi(a, w) = wa$. Both θ and ψ are cocycles of the cochain complex $C(\mathcal{G}, \mathcal{G})$. They satisfy the conditions of the preceding definition.

In fact, a connection-like pair (θ, ψ) may be regarded as a deformation of the algebra $\mathcal{G}$ whose multiplication is $(a, w)(a', w') = (aa', aw')$. One starts with a $\mathbb{Z}_2$-graded KV-algebra $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1$ whose multiplication is defined by $(a, w).(a', w') = (aa', aw')$. Then every connection-like pair (θ, ψ) is called a connection-like deformation of the $\mathbb{Z}_2$-graded KV-algebra $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1$.

Analysis from a formal viewpoint. Suppose (θ, ψ) is a connection-like deformation of a $\mathbb{Z}_2$-graded KV-algebra $\mathcal{G} = \mathcal{G}^0 + \mathcal{G}^1$. Then:

$(\star_1)$ $\mathcal{G}^0$ stands for the space of vector fields on some manifold $\mathcal{M}$.

($\star_2$) $\mathcal{G}^1$ stands for the space of smooth functions on $\mathcal{M}$.

($\star_3$) The multiplication on $\mathcal{G}^0$ is taken to be a linear connection on $\mathcal{M}$.

($\star_4$) The left action of $\mathcal{G}^0$ on $\mathcal{G}^1$ stands for the Lie derivation by elements of the commutator Lie algebra $\mathcal{G}_L$.

($\star_5$) The cocycle ψ stands for the multiplication of vector fields by smooth functions. Formula (10–1) and (10–2) are then well understood.

Theorem 10.4. *Let (θ, ψ) be a connection-like deformation of the $\mathbb{Z}_2$-graded KV-algebra $\mathcal{G}^0 \oplus \mathcal{G}^1$ such that $\theta \neq 0$. Then the one parameter family $(t\theta, \psi)$ is an one parameter family of nontrivial connection-like deformations of $\mathcal{G}^0 \oplus \mathcal{G}^1$.*

Idea of proof. Consider the KV-cocycle $t\theta$, $t \in \mathbb{R}$. The pair $(t\theta, \psi)$ satisfies conditions (c_1), (c_2) and (c_3) from the definition on the preceding page. The cocycle θ is not exact. Therefore the deformation $\mathcal{G}_t$ is nontrivial. □

To wind up our study of Question 10.1, we observe that connection-like deformations of $\mathbb{Z}_2$-graded KV-algebras are controlled by rather special type of 2-cocycles. Indeed, every 2-cochain $\theta \in C_2(\mathcal{G}, \mathcal{G})$ has six components, namely $\theta_{2,j}^{r,s}$ with $r, s \in \{0, 1, 2\}$ and $j \in \mathbb{Z}_2$. In contrast to this general picture, a connection-like pair consists of cocycles having at most one nonnull component.

Theorem 10.5. *There is a bijective correspondence between the set of connection-like pairs of $C_2(\mathcal{G}, \mathcal{G})$ and the set of connection-like deformations of the $\mathbb{Z}_2$-graded KV-algebra $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1$.*

Sketch of proof. Let $c \in C_2(\mathcal{G}, \mathcal{G})$ be a connection-like cocycle of the $\mathbb{Z}_2$-graded KV-algebra $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1$. It has two components only, namely $c = (\theta, \psi) \in C_{2,1}^{0,2} \oplus C_{2,0}^{1,1}$. Taking into account that $\delta\theta \in C_{3,1}^{1,2}(\mathcal{G}, \mathcal{G})$ and $\delta\psi \in C_{3,0}^{2,1}(\mathcal{G}, \mathcal{G}) + C_{3,0}^{1,2}(\mathcal{G})$, one gets $c((a, w), (a', w')) = (\psi(a, w') + \psi(w, a'), \theta(w, w'))$ for all $(a, w), (a', w') \in \mathcal{G}$.

Since $\mathcal{G}^1$ is a left $\mathcal{G}^0$-KV-module, the closeness condition $\delta c = 0$ implies that

$$a\theta(w', w'') - \theta(aw', w'') - \theta(w', aw'') = 0,$$

$$a\psi(a', w'') - \psi(aa', w'') - \psi(a', aw'') = 0,$$

$$\psi(a, \theta(w', w'')) - \psi(\psi(a, w'), w'') = 0.$$

The first two of these identities say that θ and ψ are 2-cocycles satisfying conditions (10–1) and (10–2) respectively. The third identity says that θ and ψ satisfies the defining conditions (c_1)–(c_3). Since c is assumed to be a KV-cocycle, we have $KV_c(w, w', w'') = 0$ for all $w, w', w'' \in \mathcal{G}^1$. Thus (θ, ψ) defines a connection-like KV-algebra $\mathcal{G}_{\theta,\psi}$ with multiplication map given by

$$(a, w)(a', w') = \big(aa' + \psi(a, w') + \psi(w, a'),\ aw' + \theta(w, w')\big).$$

The converse can be easily proved by direct calculation. □

Without extra assumptions there is no reason why the cochains θ and ψ must be tensorial.

Theorem 10.6. *Let (θ, ψ) be a connection-like pair whose commutator Lie algebra is the Lie algebra $\mathscr{A}(M)$ of smooth vector fields on M. The following statements are equivalent.*

(1) *θ and ψ are of order ≤ 0.*

(2) *The KV-algebra structure $(\mathscr{G}_{\theta,\psi})$ gives rise to a locally flat structure on the manifold M.*

11. Left invariant KV-structures on $\mathrm{Diff}_0 M$

Let G be a finite-dimensional Lie group whose Lie algebra is $\mathscr{G}$. Let $KV(|\mathscr{G}|)$ be the set of KV-algebra structures on $|\mathscr{G}|$. To $\mu \in KV(|\mathscr{G}|)$ is assigned the commutator Lie algebra $\mathscr{G}_\mu$ whose bracket is $[a, a']\mu = \mu(a, a') - \mu(a', a)$. We denote by $KV(\mathscr{G})$ the subset consisting of $\mu \in KV(|\mathscr{G}|)$ such that $\mathscr{G}_\mu = \mathscr{G}$. Thus $KV(\mathscr{G})$ is the set of connection-like KV-algebra structures on $|\mathscr{G}|$. Replacing $\mathscr{G}$ by the Lie algebra $\mathscr{A}_0(M)$ of compactly supported vector fields, what we have dealt with in Section 10 is the (formal) analogue to $KV(|\mathscr{G}|)$. Considering Theorem 10.6, the Lie group $\mathrm{Diff}_0 M$ may carry left-invariant locally flat structures which are not related to locally flat linear connections on M. This is the sense of the definition on page 146.

We conclude with the following remark. Every locally flat structure (M, D) on a manifold M gives rise to a left-invariant affine structure on the group $\mathrm{Diff}_0 M$. On the other hand whenever M is a closed manifold $\mathrm{Diff}_0 M$ is a simple Lie group which may carry a left-invariant affine structure. For instance if M is the flat torus T^m then the group $\mathrm{Diff}_0 T^m$ is a simple Lie group admitting a left-invariant affine structure. This is in contrast with the case of finite-dimensional Lie groups; see [Nguiffo Boyom 1968].

Appendix

Here we relate KV-cohomology to the pioneering work of Albert Nijenhuis [1968]. Let W be a two-sided module over a Koszul–Vinberg algebra $\mathscr{A}$. Let $\mathscr{A}_L$ be the commutator Lie algebra of $\mathscr{A}$, then W is a left module over the Lie algebra $\mathscr{A}_L$. The linear space $\mathrm{Hom}(\mathscr{A}, W)$ is a (left) $\mathscr{A}_L$-module. Let $C_N(\mathscr{A}_L, \mathrm{Hom}(\mathscr{A}, W))$ be the Chevalley–Eilenberg complex of $\mathscr{A}_L$. We call it the *Nijenhuis complex* of the Koszul–Vinberg algebra $\mathscr{A}$. Its cohomology space is denoted by $H_N(\mathscr{A}, W)$. According to [Nijenhuis 1968], the W-valued k-th cohomology space of the Koszul–Vinberg algebra $\mathscr{A}$ is the $(k-1)$-th Chevalley–Eilenberg cohomology space of

$C_N(\mathscr{A}_L, \mathrm{Hom}(\mathscr{A}, W))$. At the same time, $C_N^k(\mathscr{A}_L, \mathrm{Hom}(\mathscr{A}, W))$ is a subspace of $C^{k+1}(\mathscr{A}, W)$.

Theorem. *The subspace $C_N(\mathscr{A}, W)$ of the KV-complex $C(\mathscr{A}, W)$ graded by the vector spaces $C_N^k(\mathscr{A}_L, \mathrm{Hom}(\mathscr{A}, W))$ is a subcomplex of the KV-complex $C(\mathscr{A}, W)$. If W is a left module over the Koszul–Vinberg algebra $\mathscr{A}$ then for every positive integer k the k-th KV-cohomology space $H_N^k(\mathscr{A}, W)$ of $C_N(\mathscr{A}, W)$ coincides with the $(k-1)$-th Chevalley–Eilenberg cohomology space $H^{k-1}(\mathscr{A}_L, \mathrm{Hom}(\mathscr{A}, W))$ of $C_N(\mathscr{A}_L, \mathrm{Hom}(\mathscr{A}, W))$.*

This is precisely the pioneering result of Nijenhuis.

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