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In this paper we obtain a multivariable commutator lifting inequality, which extends to several variables a recent result of Foiaş, Frazho, and Kaashoek. This inequality yields a multivariable lifting theorem that generalizes the noncommutative commutant lifting theorem.

This is then used to solve new operator-valued interpolation problems of Schur–Carathéodory, Nevanlinna–Pick, and Sarason type on Fock spaces. Some consequences to norm constrained analytic interpolation in the unit ball of $\mathbb{C}^n$ are also considered.

Introduction

Foiaş, Frazho, and Kaashoek [Foiaş et al. 2002a] solved a problem proposed by B. Sz.-Nagy in 1968, extending the commutant lifting theorem to the case when the underlying operators do not intertwine. Their main result establishes minimal norm liftings of certain commutators. Our main goal is to obtain a multivariable version of their result.

Let $T = [T_1, \dots, T_n]$ with $T_i \in B(\mathcal{H})$ be a row contraction, that is,

$$T_1T_1^* + \cdots + T_nT_n^* \leq I,$$

and let $V = [V_1, \dots, V_n]$ with $V_i \in B(\mathcal{K})$ be an isometric lifting of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, that is,

$$V_i^* V_j = \delta_{ij} I \qquad \text{and} \qquad P_{\mathcal{H}} V_i = T_i P_{\mathcal{H}}$$

for any $i, j = 1, \dots, n$. Let $Y = [Y_1, \dots, Y_n]$ with $Y_i \in B(\mathcal{Y})$ be another row contraction and let $W = [W_1, \dots, W_n]$ with $W_i \in B(\mathcal{X})$ be an isometric lifting of Y on a Hilbert space $\mathcal{X} \supseteq \mathcal{Y}$. In Section 1 we will prove the following commutator lifting inequality:

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If $A \in B(\mathcal{Y}, \mathcal{H})$ is a contraction, then there is a contraction $B \in B(\mathcal{X}, \mathcal{K})$ such that $B^*|_{\mathcal{H}} = A^*$ and

$$\left\| V_i B - B W_i \right\| \le \sqrt{2} \left\| [T_1 A - A Y_1, \ldots, T_n A - A Y_n] \right\|^{1/2}$$

for any $i = 1, \ldots, n$. Moreover, $\sqrt{2}$ is the best possible constant.

In the particular case when $T_i A = A Y_i$ for every $i = 1, \ldots, n$, the inequality implies the noncommutative commutant lifting theorem for row contractions [Popescu 1989b, 1992] (for the classical case $n = 1$, see [Sz.-Nagy and Foiaş 1970; Foiaş and Frazho 1990]). When $n = 1$, we obtain the Foiaş–Frazho–Kaashoek result [2002a]. In Section 1, we obtain an improved version of the above-mentioned inequality (Theorem 1.1), which has as consequence a generalization of the noncommutative commutant lifting theorem (see Section 2) and of the lifting theorem obtained by Foiaş, Frazho, and Kaashoek [2002b].

In the last section of this paper, we use our new lifting theorem to solve the operator-valued interpolation problems of Schur–Carathéodory type [Schur 1918; Carathéodory 1907], Nevanlinna–Pick type [Nevanlinna 1919], and Sarason type [Sarason 1967] on Fock spaces.

To give the reader a flavor of our new interpolation results, we mention (as a particular case) the scalar Nevanlinna–Pick type interpolation problem for $F^2(H_n)$, the full Fock space with n generators. Let k, m be nonnegative integers and let

$$\mathbb{B}_n := \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : |\lambda_1|^2 + \cdots + |\lambda_n|^2 < 1 \right\}$$

be the open unit ball of $\mathbb{C}^n$. For $\{z_j\}_{j=1}^m \subset \mathbb{B}_n$ and $\{w_j\}_{j=1}^m \subset \mathbb{C}$, there exists an $f \in F^2(H_n)$ such that

$$\|f\|_{\mathcal{P}_k} \le 1 \qquad \text{and} \qquad f(z_j) = w_j, \quad j = 1, \ldots, m,$$

if and only if

$$\left[\frac{1}{1 - \langle z_j, z_q \rangle} \right]_{j,q=1}^m \ge \left[\frac{(1 - \langle z_j, z_q \rangle^{k+1}) w_j \overline{w}_q}{1 - \langle z_j, z_q \rangle} \right]_{j,q=1}^m .$$

Recall that $\|f\|_{\mathcal{P}_k}$ is defined by

$$\|f\|_{\mathcal{P}_k} := \sup\left\{ \|f \otimes p\| : p \in \mathcal{P}_k, \|p\| \le 1 \right\},$$

where $\mathcal{P}_k$ is the set of all polynomials of degree $\le k$ in the Fock space $F^2(H_n)$, and that $\left(F^2(H_n), \| \cdot \|_{\mathcal{P}_k} \right)$ is a Banach space. Moreover, if f lies in $F^2(H_n)$ and $\lim_{k\to\infty} \|f\|_{\mathcal{P}_k}$ exists, then f lies in the noncommutative analytic Toeplitz algebra F_n^∞ introduced in [Popescu 1991] (see also [1995a; 1995b]). In this case we have

$$\|f\|_\infty = \lim_{k\to\infty} \|f\|_{\mathcal{P}_k}.$$

We remark that, given $f \in F^2(H_n)$, the *evaluation* $z \mapsto f(z)$ on $\mathbb{B}_n$ is a holomorphic function on the unit ball of $\mathbb{C}^n$ (see Section 3). Furthermore, [Arveson 1998] proved that, when $f \in F_n^\infty$, the map $z \mapsto f(z)$ on $\mathbb{B}_n$ is a multiplier of the reproducing kernel Hilbert space, whose reproducing kernel $K_n : \mathbb{B}_n \times \mathbb{B}_n \to \mathbb{C}$ is defined by

$$K_n(z, w) := \frac{1}{1 - \langle z, w\rangle_{\mathbb{C}^n}}, \qquad z, w \in \mathbb{B}_n.$$

The above-mentioned interpolation problem is an $F^2(H_n)$-interpolation problem if $k = 0$. Setting $k \to \infty$, it implies the Nevanlinna–Pick interpolation problem for the noncommutative analytic Toeplitz algebra F_n^∞, which was solved in [Arias and Popescu 2000; Popescu 1998] and, independently, in [Davidson and Pitts 1998]. Recently, interpolation problems on the unit ball $\mathbb{B}_n$ were also considered in [Arias and Popescu 1999; Agler and McCarthy 2000; Ball and Bolotnikov 2002; Eschmeier and Putinar 2002; Ball et al. 2001; Popescu 2001a; 2001b; 2002a; 2002b; 2003].

In a future paper, we will provide an explicit solution (the central interpolant) for our multivariable lifting interpolation problem (see Theorem 2.2), show that the maximal entropy principle [Foiaş et al. 1994] is valid in this new setting, and obtain Kaftal–Larson–Weiss suboptimization type results [Kaftal et al. 1992; Popescu 2002a] on Fock spaces. We will also find explicit solutions for the operator-valued interpolation problems considered in the present paper.

1. Commutator lifting inequalities

Let H_n be an n-dimensional complex Hilbert space (n can be infinite) with an orthonormal basis $e_1, e_2, \ldots, e_n$. Consider the full Fock space of H_n, defined by

$$F^2(H_n) := \bigoplus_{k\geq 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}^1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. For $i = 1, \ldots, n$, define the left creation operators $S_i : F^2(H_n) \to F^2(H_n)$ by

$$S_i\psi := e_i \otimes \psi, \quad \psi \in F^2(H_n).$$

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := k$ if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, and $|\alpha| := 0$ if $\alpha = g_0$. We set $e_\alpha := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ and $e_{g_0} = 1$. It is clear that $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis of $F^2(H_n)$. If $T_1, \ldots, T_n \in B(\mathcal{H})$ (the algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$), we define $T_\alpha := T_{i_1} T_{i_2} \cdots T_{i_k}$ if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, and $T_{g_0} := I_{\mathcal{H}}$.

Let us recall from [Frazho 1982; Bunce 1984; Popescu 1989a; 1989b; 1989c] a few results concerning the noncommutative dilation theory for sequences of operators (for the classical case $n = 1$, see [Sz.-Nagy and Foiaș 1970]). A sequence of operators $T = [T_1, \ldots, T_n]$ with $T_i \in B(\mathcal{H})$ is called a *row contraction* if

$$T_1 T_1^* + \cdots + T_n T_n^* \le I_{\mathcal{H}}.$$

We say that a sequence of isometries $V = [V_1, \ldots, V_n]$ with $V_i \in B(\mathcal{K})$ is a *minimal isometric dilation* of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ if the following properties are satisfied:

(1) $V_i^* V_j = 0$ for all $i \neq j$, $i, j \in \{1, \ldots, n\}$;

(2) $V_j^*|_{\mathcal{H}} = T_j^*$ for all $j = 1, \ldots, n$;

(3) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

If V satisfies only the condition (1) and $P_{\mathcal{H}} V_i = T_i P_{\mathcal{H}}$ for $i = 1, \ldots, n$, then V is called an *isometric lifting* of T. The minimal isometric dilation of T is an isometric lifting and is uniquely determined up to an isomorphism [Popescu 1989b].

Let us consider a canonical realization of such a dilation on Fock spaces. For convenience of notation, we will sometimes identify the n-tuple $T = [T_1, \ldots, T_n]$ with the row operator $T = [T_1 \cdots T_n]$. Define the operator

$$D_T : \bigoplus_{j=1}^n \mathcal{H} \to \bigoplus_{j=1}^n \mathcal{H} \qquad \text{by} \qquad D_T := \bigl(I_{\oplus_{j=1}^n \mathcal{H}} - T^* T\bigr)^{1/2},$$

and set

$$\mathcal{D} := \overline{D_T\bigl(\bigoplus_{j=1}^n \mathcal{H}\bigr)},$$

where $\bigoplus_{j=1}^n \mathcal{H}$ denotes the direct sum of n copies of $\mathcal{H}$. Let $D_i : \mathcal{H} \to 1 \otimes \mathcal{D} \subset F^2(H_n) \otimes \mathcal{D}$ be defined by

$$D_i h := 1 \otimes D_T\bigl(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, h, 0, \ldots\bigr), \quad i = 1, \ldots, n.$$

Consider the Hilbert space $\mathcal{K} := \mathcal{H} \oplus \bigl(F^2(H_n) \otimes \mathcal{D}\bigr)$ and define $V_i : \mathcal{K} \to \mathcal{K}$ by

$$V_i(h \oplus \xi) := T_i h \oplus \bigl(D_i h + (S_i \otimes I_{\mathcal{D}})\xi\bigr) \tag{1-1}$$

for any $h \in \mathcal{H}$ and $\xi \in F^2(H_n) \otimes \mathcal{D}$. We have

$$V_i = \begin{bmatrix} T_i & 0 \\ D_i & S_i \otimes I_{\mathcal{D}} \end{bmatrix} \tag{1-2}$$

with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \bigl(F^2(H_n) \otimes \mathcal{D}\bigr)$. In [Popescu 1989b] we proved that $V := [V_1, \ldots, V_n]$ is the minimal isometric dilation of T.

The main result of this section is the following lifting inequality in several variables:

Theorem 1.1. *Let $T = [T_1, \dots, T_n]$ with $T_i \in B(\mathcal{H})$ be a row contraction and let $V = [V_1, \dots, V_n]$ with $V_i \in B(\mathcal{K})$ be an isometric lifting of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. For $i = 1, \dots, n$, let $\mathcal{X}_i \subseteq \mathcal{X}$ be orthogonal subspaces and $R_i \in B(\mathcal{X}_i, \mathcal{X})$ be contractions. If $A \in B(\mathcal{X}, \mathcal{H})$ is a contraction, then there exists a contraction $B \in B(\mathcal{X}, \mathcal{K})$ such that $P_{\mathcal{H}} B = A$ and*

$$\|V_i B R_i - B|_{\mathcal{X}_i}\| \le \sqrt{2}\,\big\|\big[T_1 A R_1 - A|_{\mathcal{X}_1},\ \dots,\ T_n A R_n - A|_{\mathcal{X}_n}\big]\big\|^{1/2} \tag{1-3}$$

for any $i = 1, \dots, n$. Moreover, $\sqrt{2}$ is the best possible constant.

Proof. For $i = 1, \dots, n$, define the operators $X_i \in B(\mathcal{X}_i, \mathcal{H})$ by $X_i := T_i A R_i$ and let $X := [X_1, \dots, X_n]$. Since X is a row contraction, we have

$$\begin{aligned}\big\|D_X\big(\textstyle\bigoplus_{i=1}^n h_i\big)\big\|^2 &= \big\|D_T\big(\textstyle\bigoplus_{i=1}^n AR_i h_i\big)\big\|^2 + \big\|\textstyle\bigoplus_{i=1}^n D_A R_i h_i\big\|^2 + \big\|\textstyle\bigoplus_{i=1}^n D_{R_i} h_i\big\|^2\\ &\ge \big\|D_T\big(\textstyle\bigoplus_{i=1}^n AR_i h_i\big)\big\|^2 + \big\|\textstyle\bigoplus_{i=1}^n D_A R_i h_i\big\|^2\end{aligned}$$

for any $h_i \in \mathcal{X}_i,\ i = 1, \dots, n$. Since $A \in B(\mathcal{X}, \mathcal{H})$ is a contraction and the subspaces $\mathcal{X}_i$ are orthogonal, the operator $[A|_{\mathcal{X}_1}, \dots, A|_{\mathcal{X}_n}]$ is a contraction acting from the Hilbert space $\bigoplus_{i=1}^n \mathcal{X}_i$ to $\mathcal{H}$. For $i = 1, \dots, n$, define the operators $M_i \in B(\mathcal{X}_i, \mathcal{H})$ by $M_i := X_i - A|_{\mathcal{X}_i}$ and write $M := [M_1, \dots, M_n]$. Since

$$M = X - [A|_{\mathcal{X}_1}, \dots, A|_{\mathcal{X}_n}], \tag{1-4}$$

it is clear that $\|M\| \le 2$. Setting $\gamma := 2\|M\|$, we have $\|M^*M\| \le \gamma$ and it makes sense to define the *defect operator*

$$D_{M,\gamma} := \big(\gamma I - M^*M\big)^{1/2} \in B\big(\textstyle\bigoplus_{i=1}^n \mathcal{X}_i\big).$$

Note also that

$$\big\|[A|_{\mathcal{X}_1}, \dots, A|_{\mathcal{X}_n}]^* M \ +\ M^* [A|_{\mathcal{X}_1}, \dots, A|_{\mathcal{X}_n}]\big\| \ \le\ \gamma. \tag{1-5}$$

Taking into account the relations (1-4) and (1-5), we obtain

$$\begin{aligned}&\big\|D_X\big(\textstyle\bigoplus_{i=1}^n h_i\big)\big\|^2\\ &\qquad\le \big\langle\big(I - [A|_{\mathcal{X}_1}, \dots, A|_{\mathcal{X}_n}]^* [A|_{\mathcal{X}_1}, \dots, A|_{\mathcal{X}_n}]\big)\big(\textstyle\bigoplus_{i=1}^n h_i\big),\ \textstyle\bigoplus_{i=1}^n h_i\big\rangle\\ &\qquad\qquad + \big\langle\big(\gamma I - M^*M\big)\big(\textstyle\bigoplus_{i=1}^n h_i\big),\ \textstyle\bigoplus_{i=1}^n h_i\big\rangle\\ &\qquad= \big\|\textstyle\bigoplus_{i=1}^n h_i\big\|^2 - \big\|\textstyle\sum_{i=1}^n A h_i\big\|^2 + \big\|D_{M,\gamma}\big(\textstyle\bigoplus_{i=1}^n h_i\big)\big\|^2\\ &\qquad= \big\|D_A\big(\textstyle\sum_{i=1}^n h_i\big)\big\|^2 + \big\|D_{M,\gamma}\big(\textstyle\bigoplus_{i=1}^n h_i\big)\big\|^2\end{aligned}$$

for any $h_i \in \mathcal{X}_i,\ i = 1, \dots, n$. The latter equality is due to the fact that the subspaces $\mathcal{X}_i$ are orthogonal and A is a contraction. Now, putting together the two inequalities for D_X, we obtain

$$\left\| \begin{bmatrix} [D_A|_{\mathscr{X}_1}, \dots, D_A|_{\mathscr{X}_n}] \\ D_{M,\gamma} \end{bmatrix} \left(\bigoplus_{i=1}^n h_i\right) \right\| \geq \left\| \begin{bmatrix} D_T\left(\bigoplus_{i=1}^n AR_i\right) \\ \bigoplus_{i=1}^n D_A R_i \end{bmatrix} \left(\bigoplus_{i=1}^n h_i\right) \right\|$$

for any $h_i \in \mathscr{X}_i$, $i = 1, \dots, n$. Hence, by using Douglas' factorization theorem, we infer that there is a contraction

$$\begin{bmatrix} C & E \\ Z & F \end{bmatrix} : \mathscr{D}_A \oplus \left(\bigoplus_{i=1}^n \mathscr{X}_i\right) \to \mathscr{D}_T \oplus \left(\bigoplus_{i=1}^n \mathscr{D}_A\right)$$

such that

$$\begin{bmatrix} C & E \\ Z & F \end{bmatrix} \begin{bmatrix} [D_A|_{\mathscr{X}_1}, \dots, D_A|_{\mathscr{X}_n}] \\ D_{M,\gamma} \end{bmatrix} = \begin{bmatrix} D_T\left(\bigoplus_{i=1}^n AR_i\right) \\ \bigoplus_{i=1}^n D_A R_i \end{bmatrix}.$$

The operator $C \in B(\mathscr{D}_A, \mathscr{D}_T)$ satisfies the equation

$$C\left[D_A|_{\mathscr{X}_1}, \dots, D_A|_{\mathscr{X}_n}\right] + ED_{M,\gamma} = D_T\left(\bigoplus_{i=1}^n AR_i\right), \tag{1-6}$$

while the operator $Z \in B\left(\mathscr{D}_A, \bigoplus_{i=1}^n \mathscr{D}_A\right)$ satisfies

$$Z\left[D_A|_{\mathscr{X}_1}, \dots, D_A|_{\mathscr{X}_n}\right] + FD_{M,\gamma} = \bigoplus_{i=1}^n D_A R_i. \tag{1-7}$$

The equality (1-6) implies

$$CD_A h_i + ED_{M,\gamma,i} h_i = D_i AR_i h_i, \qquad h_i \in \mathscr{X}_i, \tag{1-8}$$

where $D_{M,\gamma,i} : \mathscr{X}_i \to \bigoplus_{i=1}^n \mathscr{X}_i$ is the i-th column of the operator matrix of $D_{M,\gamma}$.

Setting $Z = [Z_1 \ \cdots \ Z_n]^{\mathrm{tr}} : \mathscr{D}_A \to \bigoplus_{i=1}^n \mathscr{D}_A$, Equation (1-7) implies

$$Z_j D_A h_i + P_j FD_{M,\gamma,i} h_i = \delta_{ij} D_A R_i h_i, \qquad h_i \in \mathscr{X}_i,$$

for any $i, j = 1, \dots, n$, where P_j denotes the orthogonal projection of $\bigoplus_{i=1}^n \mathscr{D}_A$ onto the j-th component. In particular, if $i = j$, we get

$$Z_i D_A h_i + P_j FD_{M,\gamma,i} h_i = D_A R_i h_i, \qquad h_i \in \mathscr{X}_i. \tag{1-9}$$

Since $[C \ Z_1 \ \cdots \ Z_n]^{\mathrm{tr}} : \mathscr{D}_A \to \mathscr{D}_T \oplus \left(\bigoplus_{i=1}^n \mathscr{D}_A\right)$ is a contraction, one can prove that the operator $\Lambda : \mathscr{D}_A \to F^2(H_n) \otimes \mathscr{D}_T$ defined by

$$\Lambda h := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} e_\alpha \otimes CZ_{\tilde{\alpha}} h, \qquad h \in \mathscr{D}_A,$$

is also a contraction, where $\tilde{\alpha}$ stands for the reverse of $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbb{F}_n^+$, that is, $\tilde{\alpha} = g_{i_k} \cdots g_{i_2} g_{i_1}$. Indeed, since

$$\|Ch\|^2 + \sum_{i=1}^n \|Z_i h\|^2 \leq \|h\|^2, \qquad h \in \mathscr{D}_A, \tag{1-10}$$

we can replace h with $Z_j h$ in (1-10) and, summing up over $j = 1, \ldots, n$, we get

$$\sum_{j=1}^{n} \|CZ_j h\|^2 \le \sum_{j=1}^{n} \|Z_j h\|^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} \|Z_i Z_j h\|^2.$$

Similarly, we obtain

$$\sum_{|\alpha|=k} \|CZ_\alpha h\|^2 \le \sum_{|\alpha|=k} \|Z_\alpha h\|^2 - \sum_{|\beta|=k+1} \|Z_\beta h\|^2. \tag{1-11}$$

Summing up the inequalities (1-11) for $k = 0, 1, \ldots, m$, we obtain

$$\sum_{k=0}^{m}\sum_{|\alpha|=k} \|CZ_\alpha h\|^2 \le \|h\|^2 - \sum_{|\beta|=m+1} \|Z_\beta h\|^2 \le \|h\|^2,$$

which proves that Λ is a contraction. Now, we define the operator

$$B : \mathcal{H} \to \mathcal{H} \oplus \big(F^2(H_n) \otimes \mathcal{D}_T\big) \qquad \text{by} \qquad B := \begin{bmatrix} A \\ \Lambda D_A \end{bmatrix}.$$

We will prove that the contraction B has the required properties. Assume now that $[V_1, \ldots, V_n]$ is the minimal isometric dilation of $[T_1, \ldots, T_n]$. Since

$$Bh = Ah \oplus \big(\textstyle\sum_{\alpha\in\mathbb{F}_n^+} e_\alpha \otimes CZ_{\tilde{\alpha}} D_A h\big), \qquad h \in \mathcal{X}, \tag{1-12}$$

by taking into account the Fock space realization of the minimal isometric dilation of T, we obtain

$$\begin{aligned} V_i BR_i h_i &= V_i\big(AR_i h_i \oplus \textstyle\sum_{\alpha\in\mathbb{F}_n^+} e_\alpha \otimes CZ_{\tilde{\alpha}} D_A R_i h_i\big) \\ &= T_i AR_i h_i \oplus \big(1 \otimes D_i AR_i h_i + \textstyle\sum_{\alpha\in\mathbb{F}_n^+} e_{g_i\alpha} \otimes CZ_{\tilde{\alpha}} D_A R_i h_i\big) \end{aligned}$$

for any $h_i \in \mathcal{X}_i$, $i = 1, \ldots, n$. Hence, by using relation (1-12) again, we get

$$\begin{aligned} &V_i BR_i h_i - Bh_i = \\ &(T_i AR_i - A)h_i \oplus \Big(1 \otimes (D_i AR_i - CD_A)h_i + \sum_{\alpha\in\mathbb{F}_n^+} e_{g_i\alpha} \otimes (CZ_{\tilde{\alpha}} D_A R_i - CZ_{\tilde{\alpha}g_i} D_A)h_i\Big). \end{aligned}$$

Using relations (1-4), (1-8), and (1-9), we obtain

$$V_i BR_i h_i - Bh_i = M_i h_i \oplus \big(1 \otimes ED_{M,\gamma,i} h_i + \textstyle\sum_{\alpha\in\mathbb{F}_n^+} e_{g_i\alpha} \otimes CZ_{\tilde{\alpha}} P_i FD_{M,\gamma,i} h_i\big)$$

for any $h_i \in \mathcal{X}_i$, $i = 1, \ldots, n$. We deduce

$$\|V_i BR_i h_i - Bh_i\|^2 = \|M_i h_i\|^2 + \|ED_{M,\gamma,i} h_i\|^2 + \textstyle\sum_{\alpha\in\mathbb{F}_n^+} \|CZ_{\tilde{\alpha}} P_i FD_{M,\gamma,i} h_i\|^2$$

for any $h_i \in \mathcal{X}_i$, $i = 1, \dots, n$. Since Λ and $\left[\begin{smallmatrix} E \\ F \end{smallmatrix}\right]$ are contractions, we obtain

$$\begin{aligned} \|V_i B R_i h_i - B h_i\|^2 &\le \|M_i h_i\|^2 + \|E D_{M,\gamma,i} h_i\|^2 + \|P_i F D_{M,\gamma,i} h_i\|^2 \\ &\le \|M_i h_i\|^2 + \|E D_{M,\gamma,i} h_i\|^2 + \|F D_{M,\gamma,i} h_i\|^2 \\ &\le \|M_i h_i\|^2 + \|D_{M,\gamma,i} h_i\|^2 \\ &= \|[M_1, \dots, M_n]k\|^2 + \|D_{M,\gamma} k\|^2 \\ &= \gamma \|h_i\|^2 \end{aligned}$$

for any $h_i \in \mathcal{X}_i$. Here, $k := \bigoplus_{j=1}^n k_j$ with $k_i := h_i$ and $k_j := 0$ if $j \ne i$. Therefore,

$$\|V_i B R_i - B|_{\mathcal{X}_i}\| \le \sqrt{\gamma}$$

for any $i = 1, \dots, n$, which proves inequality (1-3).

Now, assume that $[V_1, \dots, V_n]$ is an arbitrary isometric lifting of $[T_1, \dots, T_n]$. The subspace $\mathcal{K}_0 := \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$ is reducing under each isometry $V_1, \dots, V_n$, and $[V_1|_{\mathcal{K}_0}, \dots, V_n|_{\mathcal{K}_0}]$ coincides with the minimal isometric dilation of $[T_1, \dots, T_n]$. Applying the first part of the proof, we find a contraction $B_0 \in B(\mathcal{X}, \mathcal{K}_0)$ such that $P_{\mathcal{H}} B_0 = A$ and

$$\left\|(V_i|_{\mathcal{K}_0}) B_0 R_i - B_0|_{\mathcal{X}_i}\right\| \le \sqrt{\gamma} \tag{1-13}$$

for any $i = 1, \dots, n$. Define $B \in B(\mathcal{X}, \mathcal{K})$ by setting $Bh := B_0 h$ for $h \in \mathcal{X}$. Observe that

$$V_i B R_i - B|_{\mathcal{X}_i} = (V_i|_{\mathcal{K}_0}) B_0 R_i - B_0|_{\mathcal{X}_i}, \qquad i = 1, \dots, n,$$

and use inequality (1-13). To complete the proof, notice that the constant $\sqrt{2}$ is the best possible in (1-3), since we get equality for some simple examples. $\square$

We can now prove the commutator lifting inequality announced in the introduction.

Theorem 1.2. *Let $T = [T_1, \dots, T_n]$ with $T_i \in B(\mathcal{H})$ be a row contraction and let $V = [V_1, \dots, V_n]$ with $V_i \in B(\mathcal{K})$ be an isometric lifting of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $Y = [Y_1, \dots, Y_n]$ with $Y_i \in B(\mathcal{Y})$ be another row contraction and let $W = [W_1, \dots, W_n]$ with $W_i \in B(\mathcal{X})$ be an isometric lifting of Y on a Hilbert space $\mathcal{X} \supseteq \mathcal{Y}$. If $A \in B(\mathcal{Y}, \mathcal{H})$ is a contraction, then there is a contraction $B \in B(\mathcal{X}, \mathcal{K})$ such that $B^*|_{\mathcal{H}} = A^*$ and*

$$\|V_i B - B W_i\| \le \sqrt{2} \left\|[T_1 A - A Y_1, \ \dots \ , T_n A - A Y_n]\right\|^{1/2} \tag{1-14}$$

for any $i = 1, \dots, n$. Moreover, $\sqrt{2}$ is the best possible constant.

Proof. Define the contraction $\tilde{A} : \mathcal{X} \to \mathcal{H}$ by setting

$$\tilde{A}|_{\mathcal{Y}} = A \qquad \text{and} \qquad \tilde{A}|_{\mathcal{X} \ominus \mathcal{Y}} = 0.$$

For each $i = 1, \dots, n$, set $\mathcal{X}_i := W_i\mathcal{X}$ and define the operator $R_i : \mathcal{X}_i \to \mathcal{X}$ by $R_i := W_i^*|_{\mathcal{X}_i}$. Since $W_1, \dots, W_n$ are isometries with orthogonal subspaces, it is clear that the subspaces $\mathcal{X}_i$ are pairwise orthogonal and that R_i are contractions. Applying Theorem 1.1 to the contraction $\tilde{A}$, we find a contraction $B : \mathcal{X} \to \mathcal{K}$ such that $P_{\mathcal{H}}B = \tilde{A}$ and

$$\begin{aligned}\|V_iB - BW_i\| &= \|V_iBR_iW_i - BW_i\| \le \|V_iBR_i - B|_{\mathcal{X}_i}\| \\ &\le \sqrt{2}\,\big\|[T_1\tilde{A}R_1 - \tilde{A}|_{\mathcal{X}_1}, \dots, T_n\tilde{A}R_n - \tilde{A}|_{\mathcal{X}_n}]\big\|^{1/2} \\ &= \sqrt{2}\,\big\|[T_1\tilde{A} - \tilde{A}W_1, \dots, T_n\tilde{A} - \tilde{A}W_n]\big\|^{1/2}\end{aligned}$$

for any $i = 1, \dots, n$. The latter equality is due to the fact that $[W_1, \dots, W_n]$ is a row isometry. Therefore, we have

$$\|V_iB - BW_i\| \le \sqrt{2}\,\big\|[T_1\tilde{A} - \tilde{A}W_1, \dots, T_n\tilde{A} - \tilde{A}W_n]\big\|^{1/2} \tag{1-15}$$

for any $i = 1, \dots, n$. Notice that $B^*|_{\mathcal{H}} = A^*$. Moreover, since $[W_1, \dots, W_n]$ is an isometric lifting of $[Y_1, \dots, Y_n]$ and $\tilde{A}|_{\mathcal{X}\ominus\mathcal{Y}} = 0$, we have

$$\begin{aligned}(T_i\tilde{A} - \tilde{A}W_i)y &= T_iAy - \tilde{A}(P_{\mathcal{Y}} + P_{\mathcal{X}\ominus\mathcal{Y}})W_iy = T_iAy - \tilde{A}P_{\mathcal{Y}}W_iy \\ &= (T_iA - \tilde{A}Y_i)y\end{aligned}$$

for any $y \in \mathcal{Y}$ and $i = 1, \dots, n$. On the other hand, we have $(T_i\tilde{A} - \tilde{A}W_i)x = 0$ for any $x \in \mathcal{X}\ominus\mathcal{Y}$. Therefore

$$T_i\tilde{A} - \tilde{A}W_i = [T_iA - AY_i,\ 0], \qquad i = 1, \dots, n,$$

with respect to the orthogonal decomposition $\mathcal{X} = \mathcal{Y} \oplus (\mathcal{X}\ominus\mathcal{Y})$. Using (1-15), we deduce the inequality (1-14). The proof is complete. ☐

We remark that if one does not require the operator A to be a contraction in Theorem 1.2, then we can find an operator B with the properties that $\|B\| = \|A\|$, that $B^*|_{\mathcal{H}} = A^*$, and that

$$\|V_iB - BW_i\| \le \sqrt{2}\,\|A\|^{1/2}\,\big\|[T_1A - AY_1, \dots, T_nA - AY_n]\big\|^{1/2}$$

for any $i = 1, \dots, n$.

2. New lifting theorems in several variables

For $i = 1, \dots, n$, let $Y_i \in B(\mathcal{Y})$ and $T_i \in B(\mathcal{H})$ be operators such that $Y := [Y_1, \dots, Y_n]$ and $T := [T_1, \dots, T_n]$ are row contractions. Let $W = [W_1, \dots, W_n]$ be an isometric lifting of Y on a Hilbert space $\mathcal{X} \supseteq \mathcal{Y}$, and $V = [V_1, \dots, V_n]$ be an isometric lifting of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $A \in B(\mathcal{Y}, \mathcal{H})$ be an operator satisfying $AY_i = T_i A$ for $i = 1, \ldots, n$. An *intertwining lifting* of A is an operator $B \in B(\mathcal{X}, \mathcal{K})$ satisfying $P_{\mathcal{H}} B = A P_{\mathcal{Y}}$ and $BW_i = V_i B$ for $i = 1, \ldots, n$.

The noncommutative commutant lifting theorem for row contractions [Popescu 1989b; 1992] (for the classical case $n = 1$, see [Sz.-Nagy and Foiaș 1968; Foiaș and Frazho 1990]) states that, if $A \in B(\mathcal{Y}, \mathcal{H})$ is an operator satisfying

$$AY_i = T_i A, \qquad i = 1, \ldots, n,$$

then there exists an operator $B \in B(\mathcal{X}, \mathcal{K})$ with the following properties:

(1) $BW_i = V_i B$ for any $i = 1, \ldots, n$;

(2) $B^*|_{\mathcal{H}} = A^*$;

(3) $\|B\| = \|A\|$.

The noncommutative commutant lifting theorem for row contractions is a consequence of the commutator lifting inequality obtained in Theorem 1.2.

We present a new multivariable lifting theorem, which is a simple consequence of Theorem 1.1.

Theorem 2.1. *Let $T = [T_1, \ldots, T_n]$ with $T_i \in B(\mathcal{H})$ be a row contraction and let $V = [V_1, \ldots, V_n]$ with $V_i \in B(\mathcal{K})$ be an isometric lifting of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $\mathcal{X}_i \subseteq \mathcal{X}$ with $i = 1, \ldots, n$ be orthogonal subspaces and let $R_i \in B(\mathcal{X}_i, \mathcal{X})$ be contractions. If $A \in B(\mathcal{X}, \mathcal{H})$ is such that*

$$T_i A R_i = A|_{\mathcal{X}_i}$$

for $i = 1, \ldots, n$, then there exists an operator $B \in B(\mathcal{X}, \mathcal{K})$ such that $P_{\mathcal{H}} B = A$, $\|B\| = \|A\|$, and

$$V_i B R_i = B|_{\mathcal{X}_i}$$

for any $i = 1, \ldots, n$.

A very useful equivalent form of Theorem 2.1 is

Theorem 2.2. *Let $T = [T_1, \ldots, T_n]$ with $T_i \in B(\mathcal{H})$ be a row contraction and let $V = [V_1, \ldots, V_n]$ with $V_i \in B(\mathcal{K})$ be an isometric lifting of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $Q_i \in B(\mathcal{G}_i, \mathcal{X})$ be operators with orthogonal ranges and let $C_i \in B(\mathcal{G}_i, \mathcal{X})$ be such that $C_i^* C_i \leq Q_i^* Q_i$ for $i = 1, \ldots, n$. If $A \in B(\mathcal{X}, \mathcal{H})$ is such that, for any $i = 1, \ldots, n$,*

$$T_i A C_i = A Q_i \tag{2-1}$$

then there is an operator $B \in B(\mathcal{X}, \mathcal{K})$ such that $P_{\mathcal{H}} B = A$, $\|B\| = \|A\|$, and

$$V_i B C_i = B Q_i \tag{2-2}$$

for any $i = 1, \ldots, n$.

Proof. Since $C_i^*C_i \le Q_i^*Q_i$ for any $i = 1, \dots, n$, there exist some contractions $R_i : \overline{Q_i\mathcal{G}_i} \to \mathcal{X}$ such that $C_i = R_i Q_i$, $i = 1, \dots, n$. Set $\mathcal{X}_i := \overline{Q_i\mathcal{G}_i}$ and notice that $\mathcal{X}_i \perp \mathcal{X}_j$ if $i \neq j$. Relation (2-1) is equivalent to $T_i A R_i = A|_{\mathcal{X}_i}$ for any $i = 1, \dots, n$, while relation (2-2) is equivalent to $V_i B R_i = B|_{\mathcal{X}_i}$ for any $i = 1, \dots, n$. Using Theorem 2.1, we can complete the proof. □

Let us show that Theorem 2.2 implies Theorem 2.1. Indeed, write $\mathcal{G}_i := \mathcal{X}_i$, $R_i := C_i$, and $Q_i := I_{\mathcal{X}}|_{\mathcal{X}_i}$, $i = 1, \dots, n$. Applying Theorem 2.2, the implication follows.

As in the classical case, the general setting of the noncommutative commutant lifting theorem can be reduced to the case when $Y = [Y_1, \dots, Y_n]$ is a row isometry (see [Popescu 2003]). Notice that Theorem 2.2 implies the noncommutative commutant lifting theorem. Indeed, it is enough to consider $\mathcal{G}_i := \mathcal{X}$, $C_i := I_{\mathcal{X}}$, and $Q_i := Y_i \in B(\mathcal{X})$ for each $i = 1, \dots, n$, where $Y = [Y_1, \dots, Y_n]$ is a row isometry.

Applications of Theorem 2.2 to interpolation on Fock spaces and the unit ball of $\mathbb{C}^n$ will be considered in the next section.

3. Norm constrained interpolation problems on Fock spaces

We say that a bounded linear operator $M \in B\big(F^2(H_n) \otimes \mathcal{K}, F^2(H_n) \otimes \mathcal{K}'\big)$ is *multianalytic* if

$$M(S_i \otimes I_{\mathcal{K}}) = (S_i \otimes I_{\mathcal{K}'})M \qquad \text{for any } i = 1, \dots, n. \tag{3-1}$$

Note that M is uniquely determined by the operator $\theta : \mathcal{K} \to F^2(H_n) \otimes \mathcal{K}'$ defined by $\theta h := M(1 \otimes h)$ for $h \in \mathcal{K}$. The operator θ is called the *symbol* of M. We denote M by M_θ. Moreover, M_θ is uniquely determined by the "coefficients" $\theta_{(\alpha)} \in B(\mathcal{K}, \mathcal{K}')$ given by

$$\langle \theta_{(\tilde\alpha)} h, h' \rangle := \langle \theta h, e_\alpha \otimes h' \rangle = \langle M_\theta(1 \otimes h), e_\alpha \otimes h' \rangle, \tag{3-2}$$

where $h \in \mathcal{K}$, $h' \in \mathcal{K}'$, $\alpha \in \mathbb{F}_n^+$, and $\tilde\alpha$ is the reverse of α. Note that

$$\sum_{\alpha \in \mathbb{F}_n^+} \theta_{(\alpha)}^* \theta_{(\alpha)} \le \|M_\theta\|^2 I_{\mathcal{K}}.$$

We can associate to M_θ a unique formal Fourier expansion

$$M_\theta \sim \sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes \theta_{(\alpha)}, \tag{3-3}$$

where R_i, for $i = 1, \dots, n$, are the right creation operators on the full Fock space $F^2(H_n)$. For simplicity, since M_θ acts like its Fourier representation on "polynomials", we will identify the two. The set of all multianalytic operators acting from $F^2(H_n) \otimes \mathcal{K}$ to $F^2(H_n) \otimes \mathcal{K}'$ coincides with $R_n^\infty \mathbin{\overline{\otimes}} B(\mathcal{K}, \mathcal{K}')$, where R_n^∞ is the weakly closed algebra generated by the identity and the right creation operators on the full Fock space. A multianalytic operator M_θ (respectively, its symbol θ) is called *inner* if M_θ is an isometry. More about multianalytic operators on Fock spaces can be found in [Popescu 1989a; 1989d; 1991; 1995a; 1995b].

We remark that, in general, if $\theta : \mathcal{K} \to F^2(H_n) \otimes \mathcal{K}'$ is a bounded operator (which is equivalent to the weak convergence of the series $\sum_{\alpha \in \mathbb{F}_n^+} \theta_{(\alpha)}^* \theta_{(\alpha)}$), then the linear map M_θ uniquely determined by relations (3-2) and (3-1) is not a bounded operator. However, for each $k = 0, 1, \ldots$, the restriction of M_θ to $\mathcal{P}_k \otimes \mathcal{K}$, that is, to the set of all polynomials of degree $\le k$, is a bounded operator acting from $\mathcal{P}_k \otimes \mathcal{K}$ to $F^2(H_n) \otimes \mathcal{K}'$. We define the $\mathcal{P}_k$-norm of M_θ by

$$\|M_\theta\|_{\mathcal{P}_k} := \sup\bigl\{\|M_\theta q\| : \ q \in \mathcal{P}_k \otimes \mathcal{K} \text{ and } \|q\| \le 1\bigr\}.$$

It is easy to see that $\|M_\theta\|_{\mathcal{P}_k} \le \|M_\theta\|_{\mathcal{P}_{k+1}}$. Observe that M_θ is a multianalytic operator if and only if $\theta \in B\bigl(\mathcal{K}, F^2(H_n) \otimes \mathcal{K}'\bigr)$ and the sequence $\bigl\{\|M_\theta\|_{\mathcal{P}_k}\bigr\}_{k=0}^\infty$ converges as $k \to \infty$. In this case, we have

$$\|M_\theta\| = \lim_{k \to \infty} \|M_\theta\|_{\mathcal{P}_k}.$$

For $i = 1, \ldots, n$, define the operators C_i and Q_i from $\mathcal{P}_{k-1} \otimes \mathcal{K}$ to $\mathcal{P}_k \otimes \mathcal{K}$ by

$$C_i := I_{\mathcal{P}_k \otimes \mathcal{K}}|_{\mathcal{P}_{k-1} \otimes \mathcal{K}}, \qquad \text{and} \qquad Q_i := P_{\mathcal{P}_k \otimes \mathcal{K}}(S_i \otimes I_{\mathcal{K}})|_{\mathcal{P}_{k-1} \otimes \mathcal{K}}, \tag{3-4}$$

where $P_{\mathcal{P}_k \otimes \mathcal{K}}$ is the orthogonal projection from $F^2(H_n) \otimes \mathcal{K}$ onto $\mathcal{P}_k \otimes \mathcal{K}$.

We recall that the invariant subspaces under the operators $S_1 \otimes I_{\mathcal{K}'}, \ldots, S_n \otimes I_{\mathcal{K}'}$ ($\mathcal{K}'$ is a Hilbert space) were characterized in [Popescu 1989a]. The next lifting theorem will play an important role in our investigation.

Theorem 3.1. *Let $\mathcal{H} \subset F^2(H_n) \otimes \mathcal{K}'$ be an invariant subspace under each operator $S_i^* \otimes I_{\mathcal{K}'}$, for $i = 1, \ldots, n$, and let $A : \mathcal{P}_k \otimes \mathcal{K} \to \mathcal{H}$ be a bounded operator. Let*

$$T_i := P_{\mathcal{H}}(S_i \otimes I_{\mathcal{K}'})|_{\mathcal{H}}, \qquad i = 1, \ldots, n, \tag{3-5}$$

and let C_i, Q_i be the operators defined by relation (3-4). There exists an operator $\theta \in B\bigl(\mathcal{K}, F^2(H_n) \otimes \mathcal{K}'\bigr)$ such that

$$P_{\mathcal{H}} M_\theta|_{\mathcal{P}_k \otimes \mathcal{K}} = A \quad \textit{and} \quad \|M_\theta\|_{\mathcal{P}_k} \le 1 \tag{3-6}$$

if and only if $\|A\| \le 1$ and $T_i A C_i = A Q_i$ for any $i = 1, \ldots, n$.

Proof. Note first that relation (3-5) implies that

$$P_{\mathcal{H}}(S_i \otimes I_{\mathcal{K}'}) = T_i P_{\mathcal{H}}, \qquad i = 1, \ldots, n, \tag{3-7}$$

This shows that $[S_1 \otimes I_{\mathcal{K}'}, \ldots, S_n \otimes I_{\mathcal{K}'}]$ is an isometric lifting of $[T_1, \ldots, T_n]$. Assume that relation (3-6) holds. Because of the definitions of the operators M_θ, C_i

and Q_i, we have

$$(S_i \otimes I_{\mathcal{K}'})(M_\theta|_{\mathcal{P}_k \otimes \mathcal{K}})C_i = (M_\theta|_{\mathcal{P}_k \otimes \mathcal{K}})Q_i, \qquad i = 1, \dots, n. \tag{3-8}$$

Using relations (3-7) and (3-8), we obtain

$$\begin{aligned} T_i A C_i &= T_i (P_{\mathcal{H}} M_\theta|_{\mathcal{P}_k \otimes \mathcal{K}}) C_i = P_{\mathcal{H}} (S_i \otimes I_{\mathcal{K}'})(M_\theta|_{\mathcal{P}_k \otimes \mathcal{K}}) C_i \\ &= P_{\mathcal{H}} (M_\theta|_{\mathcal{P}_k \otimes \mathcal{K}}) Q_i = A Q_i \end{aligned}$$

for any $i = 1, \dots, n$. It is clear that $\|A\| \le 1$.

Conversely, assume that $A : \mathcal{P}_k \otimes \mathcal{K} \to \mathcal{H}$ is a contraction such that $T_i A C_i = A Q_i$ for any $i = 1, \dots, n$. According to Theorem 2.2, there exists an operator $B : \mathcal{P}_k \otimes \mathcal{K} \to F^2(H_n) \otimes \mathcal{K}'$ such that $\|B\| = \|A\|$, $P_{\mathcal{H}} B = A$, and

$$(S_i \otimes I_{\mathcal{K}'}) B C_i = B Q_i, \quad i = 1, \dots, n. \tag{3-9}$$

If $B : \mathcal{P}_k \otimes \mathcal{K} \to F^2(H_n) \otimes \mathcal{K}'$ is a bounded operator, then there is an operator $\theta \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{K}'\big)$ such that we have $B = M_\theta|_{\mathcal{P}_k \otimes \mathcal{K}}$ if and only if relation (3-9) holds. This completes the proof. □

The next result is a Sarason-type interpolation theorem [Sarason 1967] on Fock spaces, which generalizes the corresponding result for the noncommutative analytic Toeplitz algebra F_n^∞ that was obtained by Arias and the author in [Arias and Popescu 2000; Popescu 1998], as well as in [Davidson and Pitts 1998].

Theorem 3.2. *Let $\varphi \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{K}'\big)$ and let $M_\theta \in R_n^\infty \,\overline{\otimes}\, B(\mathcal{E}, \mathcal{K}')$ be an inner multianalytic operator. There exists $\psi \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{E}\big)$ such that*

$$\|M_\varphi - M_\theta M_\psi\|_{\mathcal{P}_k} \le 1$$

if and only if the operator $A := P_{\mathcal{H}} M_\varphi|_{\mathcal{P}_k \otimes \mathcal{K}}$ is a contraction, where $\mathcal{H}$ is the subspace

$$\mathcal{H} := \big(F^2(H_n) \otimes \mathcal{K}'\big) \ominus M_\theta\big(F^2(H_n) \otimes \mathcal{E}\big).$$

Proof. First, note that if $f = \varphi - M_\theta \psi$ for some $\psi \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{E}\big)$, then $M_\varphi p - M_f p = M_\theta M_\psi p$ for any polynomial $p \in \mathcal{P}_k \otimes \mathcal{K}$. Hence,

$$A = P_{\mathcal{H}} M_\varphi|_{\mathcal{P}_k \otimes \mathcal{K}} = P_{\mathcal{H}} M_f|_{\mathcal{P}_k \otimes \mathcal{K}}$$

and

$$\|A\| \le \|M_f\|_{\mathcal{P}_k} = \|M_\varphi - M_\theta M_\psi\|_{\mathcal{P}_k}.$$

Therefore, we have

$$\|A\| \le \inf\big\{\|M_\varphi - M_\theta M_\psi\|_{\mathcal{P}_k} : \psi \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{E}\big)\big\}. \tag{3-10}$$

Let us prove that we have equality in (3-10). According to [Popescu 1989a], the subspace $\mathcal{H}$ is invariant under each operator $S_i^* \otimes I_{\mathcal{K}'}$, for $i = 1, \dots, n$. For $i =$

$1, \ldots, n$, let $T_i := P_{\mathcal{H}}(S_i \otimes I_{\mathcal{K}'})|_{\mathcal{H}}$ and note that $[S_1 \otimes I_{\mathcal{K}'}, \ldots, S_n \otimes I_{\mathcal{K}'}]$ is an isometric lifting of $[T_1, \ldots, T_n]$. A straightforward calculation shows that

$$\begin{aligned} T_i A C_i &= T_i P_{\mathcal{H}} M_\varphi C_i = P_{\mathcal{H}}(S_i \otimes I_{\mathcal{K}'}) M_\varphi C_i \\ &= P_{\mathcal{H}} M_\varphi (S_i \otimes I_{\mathcal{K}'}) C_i = A(S_i \otimes I_{\mathcal{K}'}) C_i \\ &= A Q_i \end{aligned}$$

for any $i = 1, \ldots, n$. Now, using Theorem 3.1, we find $f \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{K}'\big)$ such that $A = P_{\mathcal{H}} M_f|_{\mathcal{P}_k \otimes \mathcal{K}}$ and $\|M_f\|_{\mathcal{P}_k} = \|A\|$. Since $P_{\mathcal{H}}(M_\varphi - M_f)|_{\mathcal{P}_k \otimes \mathcal{K}} = 0$, there exists $\psi \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{E}\big)$ such that $\varphi - f = M_\theta \psi$. Hence,

$$\|A\| = \|M_f\|_{\mathcal{P}_k} = \|M_\varphi - M_\theta \psi\|_{\mathcal{P}_k},$$

which proves that equality holds in (3-10). This completes the proof. □

Corollary 3.3. *Under the hypotheses of Theorem 3.2, we have*

$$\min\big\{\|M_\varphi - M_\theta M_\psi\|_{\mathcal{P}_k} : \psi \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{E}\big)\big\} = \|A\|,$$

where $A := P_{\mathcal{H}} M_\varphi|_{\mathcal{P}_k \otimes \mathcal{K}}$.

We can now extend the Schur–Carathéodory interpolation result [Carathéodory 1907; Schur 1918; Popescu 1995b] to Fock spaces.

Theorem 3.4. *Let k, m be nonnegative integers and let $\Theta := \sum_{|\alpha| \le m} R_\alpha \otimes \theta_{(\alpha)}$ with $\theta_{(\alpha)} \in B(\mathcal{K}, \mathcal{K}')$. There exists an operator $\phi \in B\big(\mathcal{K}, F^2(H_n) \otimes \mathcal{K}'\big)$ such that*

$$\|M_\phi\|_{\mathcal{P}_k} \le 1 \qquad \text{and} \qquad \theta_{(\alpha)} = \phi_{(\alpha)} \ \text{ whenever } |\alpha| \le m \tag{3-11}$$

if and only if

$$\begin{cases} \|P_{\mathcal{P}_m \otimes \mathcal{K}'} \Theta|_{\mathcal{P}_k \otimes \mathcal{K}}\| \le 1 & \text{if } k \le m, \\ \|P_{\mathcal{P}_m \otimes \mathcal{K}'} \Theta|_{\mathcal{P}_m \otimes \mathcal{K}}\| \le 1 & \text{if } k > m. \end{cases}$$

Proof. Let $\mathcal{H} := \mathcal{P}_k \otimes \mathcal{K}'$. The subspace $\mathcal{H}$ is invariant under each operator $S_i^* \otimes I_{\mathcal{K}'}$ for $i = 1, \ldots, n$. Let

$$T_i := P_{\mathcal{H}}(S_i \otimes I_{\mathcal{K}'})|_{\mathcal{H}}, \qquad i = 1, \ldots, n.$$

Since $\Theta(S_i \otimes I_{\mathcal{K}}) = (S_i \otimes I_{\mathcal{K}'})\Theta$ for any $i = 1, \ldots, n$, a straightforward calculation shows that

$$T_i A C_i = A Q_i, \qquad i = 1, \ldots, n, \tag{3-12}$$

where the operator $A : \mathcal{P}_k \otimes \mathcal{K} \to \mathcal{H}$ is defined by

$$A := \begin{cases} P_{\mathcal{P}_m \otimes \mathcal{K}'} \Theta|_{\mathcal{P}_k \otimes \mathcal{K}} & \text{if } k \le m \\ P_{\mathcal{P}_m \otimes \mathcal{K}'} \Theta|_{\mathcal{P}_m \otimes \mathcal{K}} & \text{if } k > m. \end{cases} \tag{3-13}$$

Note that (3-11) is equivalent to the existence of an operator $\phi \in B\big(\mathcal{K}, F^2(H_n)\otimes\mathcal{K}'\big)$ such that

$$P_{\mathcal{H}} M_\phi|_{\mathcal{P}_k\otimes\mathcal{K}} = A \qquad \text{and} \qquad \big\|M_\phi|_{\mathcal{P}_k\otimes\mathcal{K}}\big\| \le 1.$$

One can now apply Theorem 3.1 and complete the proof. □

Corollary 3.5. *Let k, m be nonnegative integers and let $\Theta = \sum_{|\alpha|\le m} R_\alpha\otimes\theta_{(\alpha)}$ with $\theta_{(\alpha)} \in B(\mathcal{K},\mathcal{K}')$. We have*

$$\min\big\{\|M_\phi\|_{\mathcal{P}_k} : \phi \in B\big(\mathcal{K}, F^2(H_n)\otimes\mathcal{K}'\big),\ \theta_{(\alpha)} = \phi_{(\alpha)} \text{ if } |\alpha| \le m\big\} = \|A\|,$$

where the operator A is defined by (3-13).

In what follows, we present the left tangential Nevanlinna–Pick interpolation problem with operatorial argument for $B\big(\mathcal{H}, F^2(H_n)\otimes\mathcal{K}\big)$.

As in [Popescu 1989c], the spectral radius associated with a sequence $Z = (Z_1,\dots,Z_n)$ of operators $Z_i \in B(\mathcal{Y})$ is given by

$$r(Z) := \lim_{k\to\infty}\big\|\textstyle\sum_{|\alpha|=k} Z_\alpha Z_\alpha^*\big\|^{1/2k} = \inf_{k\to\infty}\big\|\textstyle\sum_{|\alpha|=k} Z_\alpha Z_\alpha^*\big\|^{1/2k}.$$

Note that if $Z_1Z_1^*+\dots+Z_nZ_n^* < rI_{\mathcal{Y}}$ with $0<r<1$, then $r(Z)<1$. Any element $\psi \in B\big(\mathcal{H}, F^2(H_n)\otimes\mathcal{Y}\big)$ has a unique representation $\psi h := \sum_{\alpha\in\mathbb{F}_n^+} e_\alpha\otimes A_{(\alpha)}h$ for some operators $A_{(\alpha)} \in B(\mathcal{H},\mathcal{Y})$. Therefore,

$$M_\psi \sim \sum_{\alpha\in\mathbb{F}_n^+} R_{\tilde\alpha}\otimes A_{(\alpha)}$$

and $\|\psi\|^2 = \big\|\sum_{\alpha\in\mathbb{F}_n^+} A_{(\alpha)}^*A_{(\alpha)}\big\|$. If $r(Z)<1$, it makes sense to define the *evaluation* of ψ at $(Z_1,\dots,Z_n)$ by setting

$$\psi(Z_1,\dots,Z_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} Z_{\tilde\alpha}A_{(\alpha)}, \tag{3-14}$$

where $\tilde\alpha$ is the reverse of α. Using the fact that the spectral radius of Z is strictly less than 1, one can prove the norm convergence of the series (3-14). Indeed, it is enough to observe that

$$\begin{aligned}\big\|\textstyle\sum_{|\alpha|=k} Z_{\tilde\alpha}A_{(\alpha)}\big\| &\le \big\|\textstyle\sum_{|\alpha|=k} Z_\alpha Z_\alpha^*\big\|^{1/2}\big\|\textstyle\sum_{|\alpha|=k} A_{(\alpha)}^*A_{(\alpha)}\big\|^{1/2}\\ &\le \|f\|\big\|\textstyle\sum_{|\alpha|=k} Z_\alpha Z_\alpha^*\big\|^{1/2}.\end{aligned}$$

Given $C \in B(\mathcal{H},\mathcal{Y})$, we define $W_{\{Z,C\}} : F^2(H_n)\otimes\mathcal{H} \to \mathcal{Y}$, the *controllability operator* associated with $\{Z,C\}$, by setting

$$W_{\{Z,C\}}\big(\textstyle\sum_{\alpha\in\mathbb{F}_n^+} e_\alpha\otimes h_\alpha\big) := \sum_{k=0}^\infty\sum_{|\alpha|=k} Z_\alpha Ch_\alpha.$$

Since $r(Z) < 1$, note that $W_{\{Z,C\}}$ is a well-defined bounded operator. We call the positive operator $G_{\{Z,C\}} := W_{\{Z,C\}} W^*_{\{Z,C\}}$ the *controllability grammian* for $\{Z, C\}$. It is easy to see that

$$G_{\{Z,C\}} = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_\alpha C C^* Z^*_\alpha, \tag{3-15}$$

where the series converges in norm. As in the classical case ($n = 1$), we say that the pair $\{Z, C\}$ is controllable if its grammian $G_{\{Z,C\}}$ is strictly positive. We remark that $X = G_{\{Z,C\}}$ is the unique positive solution of the Lyapunov equation

$$X = \sum_{i=1}^{n} Z_i X Z_i^* + CC^*. \tag{3-16}$$

For any nonnegative integer k, we define $W_{\{Z,C,k\}} : \mathcal{P}_k \otimes \mathcal{H} \to \mathcal{Y}$, the *k-controllability operator* associated with $\{Z, C\}$, by setting

$$W_{\{Z,C,k\}}\Big(\sum_{|\alpha|\le k} e_\alpha \otimes h_\alpha\Big) := \sum_{p=0}^{k} \sum_{|\alpha|=p} Z_\alpha C h_\alpha.$$

The corresponding Grammian is $G_{\{Z,C,k\}} := W_{\{Z,C,k\}} W^*_{\{Z,C,k\}}$.

Let $\mathcal{H}, \mathcal{K}$, and $\mathcal{Y}_i$, with $i = 1, \ldots, m$, be Hilbert spaces and consider the operators

$$B_j : \mathcal{K} \to \mathcal{Y}_j, \qquad C_j : \mathcal{H} \to \mathcal{Y}_j, \quad j = 1, \ldots, m \tag{3-17}$$

$$Z_j := [Z_{j,1}, \ldots, Z_{j,n}] : \textstyle\bigoplus_{i=1}^{n} \mathcal{Y}_j \to \mathcal{Y}_j, \quad j = 1, \ldots, m, \tag{3-18}$$

such that $r(Z_j) < 1$ for any $j = 1, \ldots, m$. Given a nonnegative integer k, the left tangential Nevanlinna–Pick interpolation problem with operatorial argument for $B\big(\mathcal{H}, F^2(H_n) \otimes \mathcal{K}\big)$ is to find $\phi \in B\big(\mathcal{H}, F^2(H_n) \otimes \mathcal{K}\big)$ such that $\|M_\phi\|_{\mathcal{P}_k} \le 1$ and

$$[I \otimes B_j)\phi](Z_j) = C_j, \qquad j = 1, \ldots, m. \tag{3-19}$$

Theorem 3.6. *Given two nonnegative integers k, m, the left tangential Nevanlinna–Pick interpolation problem, with operatorial argument and data Z_j, B_j, and C_j, $j = 1, \ldots, m$, has a solution in $B\big(\mathcal{H}, F^2(H_n) \otimes \mathcal{K}\big)$ if and only if*

$$\left[\sum_{p=0}^{\infty} \sum_{|\alpha|=p} Z_{j,\alpha} B_j B_i^* Z^*_{i,\alpha}\right]_{i,j=1}^{m} \ge \left[\sum_{|\alpha|\le k} Z_{j,\alpha} C_j C_i^* Z^*_{i,\alpha}\right]_{i,j=1}^{m}. \tag{3-20}$$

Proof. Define the following operators

$$B := \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} : \mathcal{K} \to \textstyle\bigoplus_{j=1}^{m} \mathcal{Y}_j, \qquad C := \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} : \mathcal{H} \to \textstyle\bigoplus_{j=1}^{m} \mathcal{Y}_j,$$

and $Y := [Y_1, \dots, Y_n]$, where Y_i is the diagonal operator defined by

$$Y_i := \begin{bmatrix} Z_{1,i} & 0 & 0 \\ 0 & Z_{2,i} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & Z_{m,i} \end{bmatrix} : \bigoplus_{j=1}^m \mathcal{Y}_j \to \bigoplus_{j=1}^m \mathcal{Y}_j,$$

for each $i = 1, \dots, n$. Since $r(Y) < 1$ and $M_\phi \sim \sum_{\alpha \in \mathbb{F}_n^+} R_{\tilde{\alpha}} \otimes A_{(\alpha)}$, note that

$$[I \otimes B)\phi](Y) = C \tag{3-21}$$

if and only if

$$\sum_{p=0}^{\infty} \sum_{|\alpha|=p} Z_{j,\tilde{\alpha}} B_j A_{(\alpha)} = C_j,$$

for $j = 1, \dots, m$. Therefore, relation (3-19) is equivalent to relation (3-21). On the other hand, a straightforward computation on the elements of the form $e_\beta \otimes h$, $h \in \mathcal{H}$ and $\beta \in \mathbb{F}_n^+$ shows that relation (3-21) holds if and only if

$$W_{\{Y,B\}} M_\phi|_{\mathcal{P}_k \otimes \mathcal{H}} = W_{\{Y,C,k\}}. \tag{3-22}$$

Now, another calculation reveals that

$$W_{\{Y,B\}} W_{\{Y,B\}}^* - W_{\{Y,C,k\}} W_{\{Y,C,k\}}^* = \sum_{p=0}^{\infty} \sum_{|\alpha|=p} Y_\alpha B B^* Y_\alpha^* - \sum_{p=0}^{k} \sum_{|\alpha|=p} Y_\alpha C C^* Y_\alpha^*,$$

where $W_{\{Y,B\}}$ and $W_{\{Y,C,k\}}$ are the controllability operators associated with $\{Y, B\}$ and $\{Y, C\}$, respectively. The inequality (3-20) holds if and only if

$$W_{\{Y,B\}} W_{\{Y,B\}}^* - W_{\{Y,C,k\}} W_{\{Y,C,k\}}^* \ge 0. \tag{3-23}$$

Using the definitions of the controlability operators, we deduce that

$$W_{\{Y,B\}}(S_i \otimes I_{\mathcal{H}}) = Y_i W_{\{Y,B\}}, \tag{3-24}$$

$$W_{\{Y,C,k\}}(S_i \otimes I_{\mathcal{H}})|_{\mathcal{P}_{k-1} \otimes \mathcal{H}} = Y_i W_{\{Y,C,k\}}|_{\mathcal{P}_{k-1} \otimes \mathcal{H}}, \tag{3-25}$$

for $i = 1, \dots, n$. It is easy to see that if relation (3-22) holds and $\left\| M_\phi|_{\mathcal{P}_k \otimes \mathcal{H}} \right\| \le 1$, then the inequality (3-23) holds.

Conversely, assume that inequality (3-23) holds. Then there exists a contraction

$$\Lambda : \overline{\text{range}\, W_{\{Y,B\}}^*} \to \mathcal{P}_k \otimes \mathcal{H}$$

such that $\Lambda W_{\{Y,B\}}^* = W_{\{Y,C,k\}}^*$. Since $W_{\{Y,B\}}^* Y_i^* = (S_i^* \otimes I_{\mathcal{H}}) W_{\{Y,B\}}^*$ for $i = 1, \dots, n$, it is clear that the subspace $\mathcal{H}' := \overline{\text{range}\, W_{\{Y,B\}}^*}$ is invariant under each operator

$S_i^* \otimes I_{\mathcal{K}}$, $i = 1, \dots, n$. Let

$$T_i := P_{\mathcal{H}'}(S_i \otimes I_{\mathcal{K}})|_{\mathcal{H}'}, \quad i = 1, \dots, n, \tag{3-26}$$

and denote $A := \Lambda^* : \mathcal{P}_k \otimes \mathcal{H} \to \mathcal{H}$. Note that

$$W_{\{Y,B\}} A = W_{\{Y,C,k\}}. \tag{3-27}$$

We claim that

$$T_i A C_i = A Q_i, \; i = 1, \dots, n, \tag{3-28}$$

where

$$C_i := I_{\mathcal{P}_k \otimes \mathcal{H}}|_{\mathcal{P}_{k-1} \otimes \mathcal{H}} \qquad \text{and} \qquad Q_i := P_{\mathcal{P}_k \otimes \mathcal{H}}(S_i \otimes I_{\mathcal{H}})|_{\mathcal{P}_{k-1} \otimes \mathcal{H}}$$

for $i = 1, \dots, n$. Indeed, using relations (3-26), (3-27), (3-24), and (3-25), we obtain

$$\begin{aligned} W_{\{Y,B\}} T_i A C_i &= W_{\{Y,B\}} P_{\mathcal{H}'}(S_i \otimes I_{\mathcal{K}}) A C_i = W_{\{Y,B\}}(S_i \otimes I_{\mathcal{K}}) A C_i \\ &= Y_i W_{\{Y,B\}} A C_i = Y_i W_{\{Y,C,k\}} C_i \\ &= W_{\{Y,C,k\}}(S_i \otimes I_{\mathcal{H}})|_{\mathcal{P}_{k-1} \otimes \mathcal{H}} = W_{\{Y,B\}} A Q_i \end{aligned}$$

for $i = 1, \dots, n$. Since $W_{\{Y,B\}}|_{\mathcal{H}'}$ is one-to-one, we get relation (3-28). According to Theorem 3.1, there exists $\phi \in B\big(\mathcal{H}, F^2(H_n) \otimes \mathcal{K}\big)$ such that $\|M_\phi\|_{\mathcal{P}_k} \le 1$ and

$$P_{\mathcal{H}'} M_\phi|_{\mathcal{P}_k \otimes \mathcal{H}} = A. \tag{3-29}$$

Using (3-27), it is easy to see that (3-29) implies (3-22). This completes the proof. □

Now, assume that $\{Z, B\}$ is controlable, that is, its grammian $G_{\{Z,B\}}$ is strictly positive. It easy to see that the operator A in the proof of Theorem 3.6 has an explicit formula given by

$$A := W^*_{\{Z,B\}} G^{-1}_{\{Z,B\}} G_{\{Z,C,k\}}. \tag{3-30}$$

Corollary 3.7. *Under the conditions of Theorem 3.6 and assuming that $\{Z, B\}$ is controlable, we have*

$$\min\big\{\|M_\phi\|_{\mathcal{P}_k} : \phi \in B\big(\mathcal{H}, F^2(H_n) \otimes \mathcal{K}\big), \; [I \otimes B_j)\phi](Z_j) = C_j\big\} = \|A\|,$$

where the operator A is defined by Equation (3-30).

As a consequence of Theorem 3.6, we can obtain the following left tangential Nevanlinna–Pick interpolation problem in the unit ball of $\mathbb{C}^n$, which extends the corresponding results for the noncommutative analytic Toeplitz algebra F_n^∞ (see [Arias and Popescu 2000; Popescu 1998; 2003; Davidson and Pitts 1998]).

Corollary 3.8. *Let $z_j := (z_{j,1}, \dots, z_{j,n})$ for $j = 1, \dots, m$ be distinct points in $\mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$, and let $B_j \in B(\mathcal{H}, \mathcal{Y}_j)$ and $C_j \in B(\mathcal{H}, \mathcal{Y}_j)$, $j = 1, \dots, m$, be bounded operators. Given a nonnegative integer k, there exists an operator $\theta \in B\big(\mathcal{H}, F^2(H_n) \otimes \mathcal{K}\big)$ such that*

$$\|M_\theta\|_{\mathcal{P}_k} \le 1 \qquad \textit{and} \qquad B_j\theta(z_j) = C_j, \quad j = 1, \dots, m,$$

if and only if

$$\left[\frac{B_j B_q^*}{1 - \langle z_j, z_q\rangle}\right]_{j,q=1}^m \ge \left[\frac{(1 - \langle z_j, z_q\rangle^{k+1}) C_j C_q^*}{1 - \langle z_j, z_q\rangle}\right]_{j,q=1}^m .$$

Proof. For any $j, q = 1, \dots, m$, we have

$$\sum_{\alpha \in \mathbb{F}_n^+} z_{j,\alpha}\, \bar{z}_{q,\alpha} = \frac{1}{1 - \langle z_j, z_q\rangle}.$$

In Theorem 3.6, consider the particular case when $Z_{j,i} := z_{j,i} I_{\mathcal{Y}_j}$ for $j = 1, \dots, m$ and $i = 1, \dots, n$. A simple computation shows that

$$G_{\{Z,B\}} = \left[\sum_{\alpha \in \mathbb{F}_n^+} z_{j,\alpha} \bar{z}_{q,\alpha} B_j B_q^*\right]_{j,q=1}^m = \left[\frac{B_j B_q^*}{1 - \langle z_j, z_q\rangle}\right]_{j,q=1}^m .$$

Hence, we have

$$\begin{aligned} G_{\{Z,B\}} - G_{\{Z,C,k\}} &= \left[\sum_{p=0}^{\infty} \sum_{|\alpha|=p} Z_{j,\alpha} B_j B_q^* Z_{q,\alpha}^*\right]_{j,q=1}^m - \left[\sum_{|\alpha| \le k} Z_{j,\alpha} C_j C_q^* Z_{q,\alpha}^*\right]_{j,q=1}^m \\ &= \left[\frac{B_j B_q^*}{1 - \langle z_j, z_q\rangle}\right]_{j,q=1}^m - \left[\frac{(1 - \langle z_j, z_q\rangle^{k+1}) C_j C_q^*}{1 - \langle z_j, z_q\rangle}\right]_{j,q=1}^m . \end{aligned}$$

Now, applying Theorem 3.6, we complete the proof. □

We remark that the *evaluation* $z \mapsto \theta(z)$ on $\mathbb{B}_n$ is an operator-valued holomorphic function on the unit ball of $\mathbb{C}^n$.

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