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We show that the Alexander–Conway polynomial is recoverable from the Links–Gould (LG) polynomial via a certain reduction, and hence that the LG polynomial is a generalization of the Alexander–Conway polynomial. Furthermore, the LG polynomial inherits some properties of the Alexander–Conway polynomial. For example, the LG polynomial is a Laurent polynomial in a particular pair of symmetric variables, and this is related to a symmetry of the Alexander–Conway polynomial.

1. Introduction

Since the discovery of the Jones polynomial [1985], many new link invariants have been defined, including the so-called quantum invariants. The *Links–Gould polynomial* [1992], commonly known as the LG polynomial, is derived from the one-parameter family of four-dimensional representations of the quantum superalgebra $U_q\big[\mathfrak{gl}(2|1)\big]$. It is a two-variable polynomial invariant of oriented links.

D. De Wit, L. H. Kauffman and J. R. Links [De Wit et al. 1999] gave the explicit form of its R-matrix. The LG polynomial is a complete invariant for all prime knots with up to 10 crossings [De Wit 2000], as well as for the Kanenobu knots, which include infinitely many knots with the same HOMFLY polynomial [Ishii 2003]. On the other hand, we have recently constructed arbitrarily many links with the same LG polynomial [Ishii and Kanenobu 2005].

Because of the size of its R-matrix, it is in general not easy to evaluate the LG polynomial without the aid of a computer. That said, we have given in [Ishii 2004a] two useful skein relations that lead to algorithms for the recursive calculation of the LG polynomial for various links, including Conway's algebraic links; this represents an improvement of the methods for evaluating LG.

In this paper, by supplying relationships between the LG polynomial and the Alexander–Conway polynomial, we give the LG polynomial a more concrete position among the many invariants, and thereby offer a potent motivation for its study.

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The LG polynomial was originally defined as a polynomial invariant in two variables p and q. In [Ishii 2003; 2004b; 2004a; Ishii and Kanenobu 2005] we used instead the symmetrical variables $t_0 = p^{-2}$ and $t_1 = p^2 q^2$, making the LG polynomial into a polynomial in $t_0^{1/2}$ and $t_1^{1/2}$. In this paper we will show that the LG polynomial is actually a Laurent polynomial in t_0 and t_1 (see Theorem 1).

In Theorem 5 we will prove that the Alexander–Conway polynomial $\Delta_L(t)$ is determined by the LG polynomial $\mathrm{LG}_L(t_0, t_1)$, as was conjectured in [Ishii and Kanenobu 2005]:

$$\mathrm{LG}_L(t, -t^{-1}) = \Delta_L(t^2) \quad \text{for every link } L.$$

The R-matrix R of the LG polynomial does *not* satisfy the equality:

$$R\Big|_{t_1=-t^{-1}}^{t_0=t,} - R^{-1}\Big|_{t_1=-t^{-1}}^{t_0=t,} = \big(t - t^{-1}\big)\, \mathrm{id}_{V\otimes V}\,.$$

If this equality held, the proof of the theorem would be clear. A key to the actual proof is the determination of the kernel of the quantum trace (Lemma 3).

The variables t_0 and t_1 are not only eigenvalues of the R-matrix, but their use is appropriate also from the viewpoint of the Alexander–Conway polynomial.

The symmetry $\Delta(t) \doteq \Delta(t^{-1})$ of the Alexander–Conway polynomial transforms into the symmetry $\mathrm{LG}_L(t_0, t_1) = \mathrm{LG}_L(t_1, t_0)$ of the LG polynomial (the symbol $\doteq$ is used to indicate equality up to a unit factor). We remark that, because of the symmetry $\Delta(t) \doteq \Delta(t^{-1})$, the Alexander–Conway polynomial does not detect the chirality of knots. In contrast, the LG polynomial does detect the chirality of knots, at least for all prime knots with up to 12 crossings [De Wit and Links 2005].

The LG polynomial inherits some properties from the Alexander–Conway polynomial. Corresponding to the property

$$\Delta_L(1) = \begin{cases} 1 & \text{if } L \text{ is a knot,} \\ 0 & \text{otherwise,} \end{cases}$$

we have the following result (see Theorem 7): for every link L

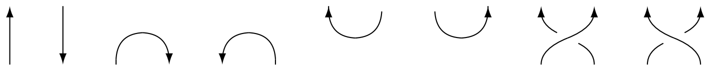
$$\mathrm{LG}_L(t_0, 1) = \mathrm{LG}_L(1, t_1) = \begin{cases} 1 & \text{if } L \text{ is a knot,} \\ 0 & \text{otherwise.} \end{cases}$$

The LG polynomial of an alternating knot seems to ‘alternate’ in the same way as the Alexander–Conway polynomial. In fact, this property holds for all prime knots with up to 10 crossings.

Since one of the biggest problems of quantum topology is finding a topological meaning for quantum invariants, the analogy between LG and Δ is interesting.

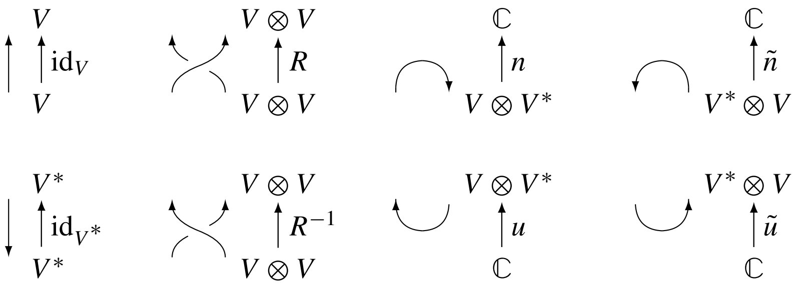
2. Preliminaries

Any oriented tangle diagram can be presented, up to isotopy, as a diagram composed from the elementary oriented tangle diagrams shown here:

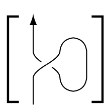


Furthermore, any oriented tangle diagram can be presented, up to isotopy, as a *sliced diagram*, which is such a diagram sliced by horizontal lines so that each domain between adjacent horizontal lines has either a single crossing or a single critical point.

Let V be a vector space and let V^* be its dual. We consider an invertible endomorphism $R : V \otimes V \to V \otimes V$, as well as linear maps $n : V \otimes V^* \to \mathbb{C}$, $\tilde{n} : V^* \otimes V \to \mathbb{C}$, $u : \mathbb{C} \to V \otimes V^*$ and $\tilde{u} : \mathbb{C} \to V^* \otimes V$. We associate these maps to elementary oriented tangle diagrams as follows:



Corresponding to any oriented tangle diagram D, we obtain a linear map $[D]$ as the composition of tensor products of the linear maps associated with the oriented elementary tangle diagrams within D. For example,

(1)  $= (\mathrm{id}_V \otimes n)(R \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes u).$

Now let cl be the linear map $\mathrm{End}(V^{\otimes k+1}) \to \mathrm{End}(V^{\otimes k})$ that transforms A into $(\mathrm{id}_V^{\otimes k} \otimes n)(A \otimes \mathrm{id}_{V^*})(\mathrm{id}_V^{\otimes k} \otimes u)$; that is, the application of cl is the algebraic equivalent of closing the rightmost strand of an oriented $(k+1, k+1)$-tangle. Observe that (1) describes $\mathrm{cl}(R)$.

The LG polynomial is defined as follows: Let V be a four-dimensional vector space with basis $\{e_i\}_{i=1}^4$ and dual basis $\{e_i^*\}_{i=1}^4$. Denote by $e_{j_1\cdots j_n}^{i_1\cdots i_n}$ the linear map $V^{\otimes n} \to V^{\otimes n}$ defined by

$$e_{j_1\cdots j_n}^{i_1\cdots i_n}(e_{k_1} \otimes \cdots \otimes e_{k_n}) = \delta_{k_1}^{j_1} \cdots \delta_{k_n}^{j_n} e_{i_1} \otimes \cdots \otimes e_{i_n},$$

where δ^i_j is the Kronecker symbol. In a similar way, define linear maps $e_{i_1\cdots i_n}: W_1\otimes\cdots\otimes W_n\to\mathbb{C}$ and $e^{i_1\cdots i_n}:\mathbb{C}\to W_1\otimes\cdots\otimes W_n$, where each W_k is either V or V^*. For example, $e_{i_1i_2}: V\otimes V\to\mathbb{C}$ and $e^{i_1i_2}:\mathbb{C}\to V\otimes V^*$ are defined by $e_{i_1i_2}(e_{k_1}\otimes e_{k_2})=\delta^{i_1}_{k_1}\delta^{i_2}_{k_2}$ and $e^{i_1i_2}(1)=e_{i_1}\otimes e^*_{i_2}$.

We obtain the bracket [ ] by setting

$$\begin{aligned}R = t_0e^{11}_{11} &- \bigl(e^{22}_{22}+e^{33}_{33}\bigr)+t_1e^{44}_{44}+(t_0-1)\bigl(e^{21}_{21}+e^{31}_{31}\bigr)+(t_0-1)(1-t_1)e^{41}_{41}\\ &+(t_1-1)\bigl(e^{42}_{42}+e^{43}_{43}\bigr)+(t_0t_1-1)e^{23}_{23}+\bigl(e^{14}_{41}+e^{41}_{14}\bigr)-t_0^{1/2}t_1^{1/2}\bigl(e^{23}_{32}+e^{32}_{23}\bigr)\\ &+t_0^{1/2}\bigl(e^{12}_{21}+e^{21}_{12}+e^{13}_{31}+e^{31}_{13}\bigr)+t_1^{1/2}\bigl(e^{24}_{42}+e^{42}_{24}+e^{34}_{43}+e^{43}_{34}\bigr)\\ &-t_0^{1/2}t_1^{1/2}\bigl((t_0-1)(1-t_1)\bigr)^{1/2}\bigl(e^{23}_{41}+e^{41}_{23}\bigr)+\bigl((t_0-1)(1-t_1)\bigr)^{1/2}\bigl(e^{32}_{41}+e^{41}_{32}\bigr),\end{aligned}$$

$$n=e_{11}+e_{22}+e_{33}+e_{44},\qquad u=t_0^{-1}e^{11}-t_1e^{22}-t_0^{-1}e^{33}+t_1e^{44},$$
$$\tilde n=t_0e_{11}-t_1^{-1}e_{22}-t_0e_{33}+t_1^{-1}e_{44},\qquad \tilde u=e^{11}+e^{22}+e^{33}+e^{44}.$$

For any $(1,1)$-tangle T, the LG polynomial of the link $\widehat{T}$ (the closure of T) is defined by

$$[D_T]=\mathrm{LG}_{\widehat{T}}(t_0,t_1)\,\mathrm{id}_V,$$

where D_T is a tangle diagram of T. Note that $\mathrm{LG}_L(p^{-2},p^2q^2)$ with $p=q^\alpha$ coincides with the Links–Gould invariant from [De Wit et al. 1999], where α originates as a complex parameter of a family of $U_q\bigl[\mathfrak{gl}(2|1)\bigr]$-representations. For the details we refer the reader to [De Wit et al. 1999; Ohtsuki 2002].

3. Laurent polynomials

We now show that the LG polynomial, when expressed in terms of the variables t_0 and t_1, is indeed a Laurent polynomial. Observe that, after changing basis by

$$e_1\mapsto\bigl((t_0-1)(1-t_1)\bigr)^{1/2}e_1\quad\text{and}\quad e_i\mapsto e_i\ \text{ for } i\neq 1,$$

the coefficients of the R-matrix belong to the set $\mathbb{Z}\bigl[t_0^{\pm1/2},t_1^{\pm1/2}\bigr]$. Hence, we immediately see that $\mathrm{LG}_L(t_0,t_1)\in\mathbb{Z}\bigl[t_0^{\pm1/2},t_1^{\pm1/2}\bigr]$.

For all prime knots with up to 10 crossings, the LG polynomial is a Laurent polynomial in p^2 and q^2 (with $p=q^\alpha$), as was observed in [De Wit 2000, p. 322]. Since $t_0=p^{-2}$ and $t_1=p^2q^2$, the following theorem shows that this property actually holds for all links.

Theorem 1. *For any link L, we have*

$$\mathrm{LG}_L(t_0,t_1)\in\mathbb{Z}\bigl[t_0^{\pm1},t_1^{\pm1}\bigr].$$

We prepare a lemma for the proof of the theorem. Set $\mathcal{I}=\{1,2,3,4\}$. A *word* over $\mathcal{I}$ is a finite sequence of letters $a_1\cdots a_n$ with $a_1,\dots,a_n\in\mathcal{I}$. Denote the empty word by $\varnothing$. Let $\mathcal{W}$ be the set of all words over $\mathcal{I}$. Define the product of a word $u = a_1 \cdots a_n$ with a word $v = b_1 \cdots b_m$ as $uv = a_1 \cdots a_n b_1 \cdots b_m$. Then, with respect to this product, $\mathcal{W}$ is a monoid. A word w' is called a *subword* of w if w can be written in the form $uw'v$ for some $u, v \in \mathcal{W}$.

Let $\#_u(w)$ be the number of subwords of form u inside a word w, and let $\#_u^{f_i}\binom{w}{w'}$ be the difference

$$\#_u^{f_i}\binom{w}{w'} := \#_u\big(f_i(w)\big) - \#_u\big(f_i(w')\big),$$

for $i = 0, 1$. Here f_0 and f_1 are the endomorphisms of $\mathcal{W}$ given by

$$\begin{array}{llll} f_0(1) = 32, & f_0(2) = 2, & f_0(3) = 3, & f_0(4) = \varnothing, \\ f_1(1) = \varnothing, & f_1(2) = 2, & f_1(3) = 3, & f_1(4) = 32. \end{array}$$

For example,

$$\begin{aligned} \#_2^{f_0}\tbinom{1234}{4321} &= \#_2\big(f_0(1234)\big) - \#_2\big(f_0(4321)\big) \\ &= \#_2(3223) - \#_2(3232) = \#\{2, 2\} - \#\{2, 2\} = 2 - 2 = 0, \\ \#_{2\cdots3}^{f_0}\tbinom{1234}{4321} &= \#_{2\cdots3}(f_0(1234)) - \#_{2\cdots3}(f_0(4321)) \\ &= \#_{2\cdots3}(3223) - \#_{2\cdots3}(3232) = \#\{223, 23\} - \#\{23\} = 2 - 1 = 1. \end{aligned}$$

We remark that

$$\#_u^{f_i}\tbinom{w}{w''} = \#_u^{f_i}\tbinom{w}{w'} + \#_u^{f_i}\tbinom{w'}{w''}. \tag{2}$$

Set

$$\mathcal{I}_n := \Big\{\tbinom{i_1\cdots i_n}{j_1\cdots j_n} \;\Big|\; i_1, \ldots, i_n,\ j_1, \ldots, j_n \in \mathcal{I}, \#_j^{f_i}\tbinom{i_1\cdots i_n}{j_1\cdots j_n} = 0 \ \ (i = 0, 1;\ j = 2, 3)\Big\}.$$

By equality (2), we have the property

$$\tbinom{i_1\cdots i_n}{j_1\cdots j_n},\ \tbinom{j_1\cdots j_n}{k_1\cdots k_n} \in \mathcal{I}_n \implies \tbinom{i_1\cdots i_n}{k_1\cdots k_n} \in \mathcal{I}_n. \tag{3}$$

We also have the property

$$\tbinom{i_1\cdots i_n k}{j_1\cdots j_n k} \in \mathcal{I}_{n+1} \iff \tbinom{i_1\cdots i_n}{j_1\cdots j_n} \in \mathcal{I}_n \iff \tbinom{k i_1\cdots i_n}{k j_1\cdots j_n} \in \mathcal{I}_{n+1}, \tag{4}$$

which follows since

$$\begin{aligned} \#_j^{f_i}\tbinom{i_1\cdots i_n k}{j_1\cdots j_n k} &= \#_j\big(f_i(i_1 \cdots i_n k)\big) - \#_j\big(f_i(i_1 \cdots i_n k)\big) \\ &= \#_j\big(f_i(i_1 \cdots i_n) f_i(k)\big) - \#_j\big(f_i(i_1 \cdots i_n) f_i(k)\big) \\ &= \#_j\big(f_i(i_1 \cdots i_n)\big) + \#_j\big(f_i(k)\big) - \#_j\big(f_i(i_1 \cdots i_n)\big) - \#_j\big(f_i(k)\big) \\ &= \#_j\big(f_i(i_1 \cdots i_n)\big) - \#_j\big(f_i(j_1 \cdots j_n)\big) \\ &= \#_j^{f_i}\tbinom{i_1\cdots i_n}{j_1\cdots j_n}, \end{aligned}$$

and since, similarly, $\#_j^{f_i}\binom{k i_1\cdots i_n}{k j_1\cdots j_n} = \#_j^{f_i}\binom{i_1\cdots i_n}{j_1\cdots j_n}$.

For $\binom{i_1\cdots i_n}{j_1\cdots j_n} \in \mathcal{I}_n$, set

$$C\left[\begin{smallmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{smallmatrix}\right] := t_0^{\frac{1}{2}\#^{f_0}_{2\cdots 3}\binom{i_1\cdots i_n}{j_1\cdots j_n}} \, t_1^{\frac{1}{2}\#^{f_1}_{2\cdots 3}\binom{i_1\cdots i_n}{j_1\cdots j_n}} \, \mathbb{Z}\left[t_0^{\pm 1},\, t_1^{\pm 1},\, \big((t_0-1)(1-t_1)\big)^{1/2}\right].$$

We have the following properties: for $\binom{i_1\cdots i_n}{j_1\cdots j_n}$, $\binom{j_1\cdots j_n}{k_1\cdots k_n} \in \mathcal{I}_n$ and $k \in \mathcal{I}$,

$$a, b \in C\left[\begin{smallmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{smallmatrix}\right] \implies a+b \in C\left[\begin{smallmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{smallmatrix}\right], \tag{5}$$

$$a \in C\left[\begin{smallmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{smallmatrix}\right],\ b \in C\left[\begin{smallmatrix} j_1\cdots j_n \\ k_1\cdots k_n \end{smallmatrix}\right] \implies ab \in C\left[\begin{smallmatrix} i_1\cdots i_n \\ k_1\cdots k_n \end{smallmatrix}\right], \tag{6}$$

$$C\left[\begin{smallmatrix} i_1\cdots i_n k \\ j_1\cdots j_n k \end{smallmatrix}\right] = C\left[\begin{smallmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{smallmatrix}\right] = C\left[\begin{smallmatrix} k i_1\cdots i_n \\ k j_1\cdots j_n \end{smallmatrix}\right]. \tag{7}$$

Property (5) is easily checked. From equality (2), we have property (6). The first equality in property (7) follows from

$$\begin{aligned}
\#^{f_i}_{2\cdots 3}\tbinom{i_1\cdots i_n k}{j_1\cdots j_n k} &= \#_{2\cdots 3}(f_i(i_1\cdots i_n k)) - \#_{2\cdots 3}(f_i(j_1\cdots j_n k)) \\
&= \#_{2\cdots 3}(f_i(i_1\cdots i_n) f_i(k)) - \#_{2\cdots 3}(f_i(j_1\cdots j_n) f_i(k)) \\
&= \#_{2\cdots 3}(f_i(i_1\cdots i_n)) + \#_2(f_i(i_1\cdots i_n))\#_3(f_i(k)) \\
&\qquad\qquad + \#_{2\cdots 3}(f_i(k)) - \#_{2\cdots 3}(f_i(j_1\cdots j_n)) \\
&\qquad\qquad - \#_2(f_i(j_1\cdots j_n))\#_3(f_i(k)) - \#_{2\cdots 3}(f_i(k)) \\
&= \#_{2\cdots 3}(f_i(i_1\cdots i_n)) - \#_{2\cdots 3}(f_i(j_1\cdots j_n)) \\
&\qquad\qquad + \big(\#_2(f_i(i_1\cdots i_n)) - \#_2(f_i(j_1\cdots j_n))\big)\#_3(f_i(k)) \\
&= \#^{f_i}_{2\cdots 3}\tbinom{i_1\cdots i_n}{j_1\cdots j_n} + \#^{f_i}_2\tbinom{i_1\cdots i_n}{j_1\cdots j_n}\#_3(f_i(k)) \\
&= \#^{f_i}_{2\cdots 3}\tbinom{i_1\cdots i_n}{j_1\cdots j_n},
\end{aligned}$$

where we noticed that $\#_3\big(f_i(k)\big) \in \{0, 1\}$ and that $\#^{f_i}_2\binom{i_1\cdots i_n}{j_1\cdots j_n} = 0$. The second equality in property (7) follows similarly from $\#^{f_i}_{2\cdots 3}\binom{k i_1\cdots i_n}{k j_1\cdots j_n} = \#^{f_i}_{2\cdots 3}\binom{i_1\cdots i_n}{j_1\cdots j_n}$.

We define the subset $\mathcal{A}_n \subset \mathrm{End}(V^{\otimes n})$ by

$$\mathcal{A}_n := \left\{ \sum_{\binom{i_1\cdots i_n}{j_1\cdots j_n} \in \mathcal{I}_n} a^{i_1\cdots i_n}_{j_1\cdots j_n} e^{i_1\cdots i_n}_{j_1\cdots j_n} \;\middle|\; a^{i_1\cdots i_n}_{j_1\cdots j_n} \in C\left[\begin{smallmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{smallmatrix}\right] \right\}.$$

Lemma 2. *For every $x, y \in \mathcal{A}_n$, we have*

$$xy \in \mathcal{A}_n, \quad x \otimes \mathrm{id}_V \in \mathcal{A}_{n+1}, \quad \mathrm{id}_V \otimes x \in \mathcal{A}_{n+1}, \quad \mathrm{cl}(x) \in \mathcal{A}_{n-1}.$$

Proof. Since by property (5) we have $x + y \in \mathcal{A}_n$ for every $x, y \in \mathcal{A}_n$, it is enough to consider $x = a^{i_1\cdots i_n}_{j_1\cdots j_n} e^{i_1\cdots i_n}_{j_1\cdots j_n}$ and $y = b^{s_1\cdots s_n}_{t_1\cdots t_n} e^{s_1\cdots s_n}_{t_1\cdots t_n}$ from $\mathcal{A}_n$.

If $j_k \neq s_k$ for some $k \in \{1, \ldots, n\}$, then $xy = 0 \in \mathcal{A}_n$. Thus, we can assume that $j_k = s_k$ for all k. Properties (3) and (6) then imply that $\binom{i_1\cdots i_n}{t_1\cdots t_n} \in \mathcal{I}_n$ and that $a^{i_1\cdots i_n}_{s_1\cdots s_n} b^{s_1\cdots s_n}_{t_1\cdots t_n} \in C\left[\begin{smallmatrix} i_1\cdots i_n \\ t_1\cdots t_n \end{smallmatrix}\right]$, respectively. Then $xy = a^{i_1\cdots i_n}_{s_1\cdots s_n} b^{s_1\cdots s_n}_{t_1\cdots t_n} e^{i_1\cdots i_n}_{t_1\cdots t_n} \in \mathcal{A}_n$.

Properties (4) and (7) imply that $\binom{i_1\cdots i_n k}{j_1\cdots j_n k}, \binom{k i_1\cdots i_n}{k j_1\cdots j_n} \in \mathcal{I}_{n+1}$ and that $a^{i_1\cdots i_n}_{j_1\cdots j_n} \in C\left[\begin{smallmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{smallmatrix}\right] = C\left[\begin{smallmatrix} i_1\cdots i_n k \\ j_1\cdots j_n k \end{smallmatrix}\right] = C\left[\begin{smallmatrix} k i_1\cdots i_n \\ k j_1\cdots j_n \end{smallmatrix}\right]$, respectively. Then

$$x \otimes \mathrm{id}_V = \textstyle\sum_{k\in\mathcal{I}} a^{i_1\cdots i_n}_{j_1\cdots j_n}\, e^{i_1\cdots i_n k}_{j_1\cdots j_n k}, \quad \mathrm{id}_V \otimes x = \textstyle\sum_{k\in\mathcal{I}} a^{i_1\cdots i_n}_{j_1\cdots j_n}\, e^{k i_1\cdots i_n}_{k j_1\cdots j_n} \;\in\; \mathcal{A}_{n+1}.$$

Since $\mathrm{cl}(x) = 0 \in \mathcal{A}_{n-1}$ for $i_n \neq j_n$, we can assume that $i_n = j_n$. Then

$$\mathrm{cl}(x) \;=\; c_{i_n} a^{i_1\cdots i_{n-1} i_n}_{j_1\cdots j_{n-1} i_n}\, e^{i_1\cdots i_{n-1}}_{j_1\cdots j_{n-1}},$$

where $(c_1, c_2, c_3, c_4) = (t_0^{-1}, -t_1, -t_0^{-1}, t_1)$. Properties (4) and (7) imply that $\binom{i_1\cdots i_{n-1}}{j_1\cdots j_{n-1}} \in \mathcal{I}_{n-1}$ and that $c_{i_n} a^{i_1\cdots i_{n-1} i_n}_{j_1\cdots j_{n-1} i_n} \in c_{i_n} C\left[\begin{smallmatrix} i_1\cdots i_{n-1} \\ j_1\cdots j_{n-1} \end{smallmatrix}\right] = C\left[\begin{smallmatrix} i_1\cdots i_{n-1} \\ j_1\cdots j_{n-1} \end{smallmatrix}\right]$, respectively. Then $\mathrm{cl}(x) \in \mathcal{A}_{n-1}$. □

Proof of Theorem 1. Let B_n be the n-string braid group and let $\sigma_1, \ldots, \sigma_{n-1}$ be its standard generators. There is a unique homomorphism $B_n \to \mathrm{Aut}(V^{\otimes n})$ which transforms σ_i into $\mathrm{id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V^{\otimes(n-i-1)}$ for $i = 1, \ldots, n-1$. Denote this homomorphism by b_R. For a closed braid $\widehat{\theta}$ (with $\theta \in B_n$),

$$\mathrm{cl}^{n-1}\big(b_R(\theta)\big) \;=\; \mathrm{cl}^{n-1}\big([\theta]\big) \;=\; \mathrm{LG}_{\widehat{\theta}}(t_0, t_1)\,\mathrm{id}_V \;=\; \textstyle\sum_{k=1}^4 \mathrm{LG}_{\widehat{\theta}}(t_0, t_1)\, e^k_k, \tag{8}$$

where $\mathrm{cl}^{n-1} = \mathrm{cl} \circ \cdots \circ \mathrm{cl}$ $(n-1$ times$) : \mathrm{End}(V^{\otimes n}) \to \mathrm{End}(V)$.

By definition, $R, R^{-1} \in \mathcal{A}_2$:

$$\begin{aligned} R \;=\; & t_0 e^{11}_{11} - \big(e^{22}_{22} + e^{33}_{33}\big) + t_1 e^{44}_{44} + (t_0-1)\big(e^{21}_{21} + e^{31}_{31}\big) + (t_0-1)(1-t_1) e^{41}_{41} \\ & + (t_1-1)\big(e^{42}_{42} + e^{43}_{43}\big) + (t_0 t_1 - 1) e^{23}_{23} + \big(e^{14}_{41} + e^{41}_{14}\big) - t_0^{1/2} t_1^{1/2}\big(e^{23}_{32} + e^{32}_{23}\big) \\ & + t_0^{1/2}\big(e^{12}_{21} + e^{21}_{12} + e^{13}_{31} + e^{31}_{13}\big) + t_1^{1/2}\big(e^{24}_{42} + e^{42}_{24} + e^{34}_{43} + e^{43}_{34}\big) \\ & - t_0^{1/2} t_1^{1/2}\big((t_0-1)(1-t_1)\big)^{1/2}\big(e^{23}_{41} + e^{41}_{23}\big) + \big((t_0-1)(1-t_1)\big)^{1/2}\big(e^{32}_{41} + e^{41}_{32}\big), \end{aligned}$$

$$\begin{aligned} R^{-1} \;=\; & t_0^{-1} e^{11}_{11} - \big(e^{22}_{22} + e^{33}_{33}\big) + t_1^{-1} e^{44}_{44} + (t_0^{-1}-1)\big(e^{12}_{12} + e^{13}_{13}\big) \\ & + (t_0^{-1}-1)(1-t_1^{-1}) e^{14}_{14} + (t_1^{-1}-1)\big(e^{24}_{24} + e^{34}_{34}\big) + (t_0^{-1} t_1^{-1} - 1) e^{32}_{32} \\ & + \big(e^{14}_{41} + e^{41}_{14}\big) - t_0^{-1/2} t_1^{-1/2}\big(e^{23}_{32} + e^{32}_{23}\big) + t_0^{-1/2}\big(e^{12}_{21} + e^{21}_{12} + e^{13}_{31} + e^{31}_{13}\big) \\ & + t_1^{-1/2}\big(e^{24}_{42} + e^{42}_{24} + e^{34}_{43} + e^{43}_{34}\big) + t_0^{-1/2} t_1^{-1/2}\big((t_0-1)(1-t_1)\big)^{1/2}\big(e^{23}_{14} + e^{14}_{23}\big) \\ & - t_0^{-1} t_1^{-1}\big((t_0-1)(1-t_1)\big)^{1/2}\big(e^{32}_{14} + e^{14}_{32}\big). \end{aligned}$$

Thus, Lemma 2 implies that $\mathrm{cl}^{n-1}\big(b_R(\theta)\big) \in \mathcal{A}_1$. By the equalities (8), we have

$$\mathrm{LG}_{\widehat{\theta}}(t_0, t_1) \;\in\; C\left[\begin{smallmatrix} k \\ k \end{smallmatrix}\right] = \mathbb{Z}\big[t_0^{\pm 1},\ t_1^{\pm 1},\ \big((t_0-1)(1-t_1)\big)^{1/2}\big].$$

Since $\mathrm{LG}_{\widehat{\theta}}(t_0, t_1)$ is in $\mathbb{Z}\big[t_0^{\pm 1/2},\ t_1^{\pm 1/2}\big]$, we have $\mathrm{LG}_{\widehat{\theta}}(t_0, t_1) \in \mathbb{Z}\big[t_0^{\pm 1},\ t_1^{\pm 1}\big]$. □

4. A relation between the LG polynomial and the Alexander–Conway polynomial

In this section we show the following relation:

$$\mathrm{LG}_L(t,-t^{-1}) = \Delta_L(t^2).$$

$\Delta_L(t)$ is the Alexander–Conway polynomial, defined by the relations

$$\Delta_{\bigcirc}(t) = 1 \quad \text{and} \quad \Delta(t) - \Delta(t) = \left(t^{1/2} - t^{-1/2}\right) \Delta(t).$$

If $V \equiv V(\alpha)$ is the vector space underlying the original definition of the LG polynomial, then, for generic values of α, the tensor-product module has the irreducible decomposition

$$V \otimes V = V_0 \oplus V_1 \oplus V_2$$

with respect to the coproduct [De Wit et al. 1999]. If P_i is the projector $V \otimes V \to V_i$, we have

$$P_i P_j = \delta^i_j P_i \quad \text{and} \quad P_0 + P_1 + P_2 = \mathrm{id}_{V\otimes V}, \tag{9}$$

From [De Wit et al. 1999, p. 170] we also have

$$R = t_0 P_0 + t_1 P_1 - P_2, \quad \text{and} \quad R^{-1} = t_0^{-1} P_0 + t_1^{-1} P_1 - P_2. \tag{10}$$

We denote $X\big|^{t_0=t,}_{t_1=-t^{-1}}$ by $\overline{X}$.

Lemma 3. *The following identities hold*:

$$\begin{aligned}
\mathrm{cl}(P_0) &= t_0(1-t_1)\big(t_0-t_1\big)^{-1}\big(t_0+1\big)^{-1}\,\mathrm{id}_V,\\
\mathrm{cl}(P_1) &= t_1(1-t_0)\big(t_1-t_0\big)^{-1}\big(t_1+1\big)^{-1}\,\mathrm{id}_V,\\
\mathrm{cl}(P_2) &= -(t_0t_1+1)\big(t_0+1\big)^{-1}\big(t_1+1\big)^{-1}\,\mathrm{id}_V\,.
\end{aligned}$$

In particular,

$$\begin{aligned}
\overline{\mathrm{cl}(P_0)} &= \big(t+t^{-1}\big)^{-1}\,\mathrm{id}_V,\\
\overline{\mathrm{cl}(P_1)} &= -\big(t+t^{-1}\big)^{-1}\,\mathrm{id}_V,\\
\overline{\mathrm{cl}(P_2)} &= 0\,\mathrm{id}_V\,.
\end{aligned}$$

Proof. From the equalities (9) and (10), we have

$$\begin{aligned}
(t_0-t_1)(t_0+1)P_0 &= t_0R + t_0(1-t_1)\,\mathrm{id}_{V\otimes V} - t_0t_1R^{-1},\\
(t_1-t_0)(t_1+1)P_1 &= t_1R + t_1(1-t_0)\,\mathrm{id}_{V\otimes V} - t_0t_1R^{-1},\\
(t_0+1)(t_1+1)P_2 &= -R + (t_0+t_1)\,\mathrm{id}_{V\otimes V} - t_0t_1R^{-1}.
\end{aligned}$$

Since $\mathrm{cl}(R) = \mathrm{cl}(R^{-1}) = \mathrm{LG}_{\bigcirc}(t_0, t_1)\,\mathrm{id}_V = \mathrm{id}_V$ and

$$\mathrm{cl}\big(\mathrm{id}_{V\otimes V}\big) = \mathrm{LG}_{\bigcirc\bigcirc}(t_0, t_1)\,\mathrm{id}_V = 0\,\mathrm{id}_V,$$

the results follow. $\square$

For any oriented $(2,2)$-tangle T, since $[T]$ is a product of intertwiners and $\{P_0, P_1, P_2\}$ is a basis for the space of intertwiners, we have

$$[T] = a_0^T(t_0,t_1)P_0 + a_1^T(t_0,t_1)P_1 + a_2^T(t_0,t_1)P_2, \tag{11}$$

with $a_i^T(t_0,t_1) \in \mathbb{Q}\big(t_0^{\pm 1/2},\ t_1^{\pm 1/2},\ ((t_0-1)(1-t_1))^{1/2}\big)$. The following lemma shows that $a_i^T(t_0,t_1) \in \mathbb{Z}\big[t_0^{\pm 1/2},\ t_1^{\pm 1/2},\ \big((t_0-1)(1-t_1)\big)^{1/2}\big]$, which then guarantees that $\overline{a_i^T(t_0,t_1)}$ is well defined, that is,

$$a_i^T(t,-t^{-1}) \in \mathbb{Z}\big[t^{\pm 1/2},\ (-t)^{\pm 1/2},\ \big(t-t^{-1}\big)^{1/2}\big].$$

Lemma 4. *For any oriented* $(2,2)$*-tangle* T,

$$a_i^T(t_0,t_1) \in \mathbb{Z}\big[t_0^{\pm 1/2},\ t_1^{\pm 1/2},\ \big((t_0-1)(1-t_1)\big)^{1/2}\big].$$

In particular,

$$a_i^T(t,-t^{-1}) \in \mathbb{Z}\big[t^{\pm 1/2},\ (-t)^{\pm 1/2},\ \big(t-t^{-1}\big)^{1/2}\big].$$

Proof. For any linear map A, we denote by $A^{i_1\cdots i_n}_{j_1\cdots j_m}$ the coefficient of $e^{i_1\cdots i_n}_{j_1\cdots j_m}$ in A:

$$A = \sum A^{i_1\cdots i_n}_{j_1\cdots j_m} e^{i_1\cdots i_n}_{j_1\cdots j_m}.$$

Since the coefficients of R, n, $\tilde{n}$, u, and $\tilde{u}$ are in $\mathbb{Z}\big[t_0^{\pm 1/2}, t_1^{\pm 1/2}, \big((t_0-1)(1-t_1)\big)^{1/2}\big]$ (see the definitions), so too are the coefficients of $[T]$. By [De Wit et al. 1999, p. 169], we have

$$(P_i)^{11}_{11} = \delta_i^0, \quad (P_i)^{44}_{44} = \delta_i^1, \quad (P_i)^{22}_{22} = \delta_i^2.$$

By comparing the coefficients in both sides of the equality (11), we get

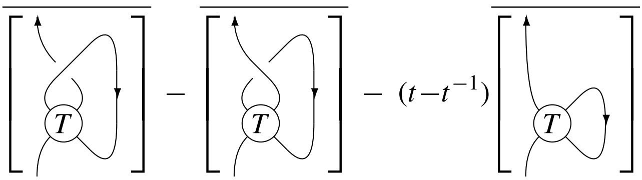
$$[T]^{11}_{11} = a_0^T(t_0,t_1), \quad [T]^{44}_{44} = a_1^T(t_0,t_1), \quad [T]^{22}_{22} = a_2^T(t_0,t_1).$$

Then $[T]^{11}_{11}$, $[T]^{22}_{22}$, $[T]^{44}_{44} \in \mathbb{Z}\big[t_0^{\pm 1/2},\ t_1^{\pm 1/2},\ \big((t_0-1)(1-t_1)\big)^{1/2}\big]$, which implies that $a_i^T(t_0,t_1) \in \mathbb{Z}\big[t_0^{\pm 1/2},\ t_1^{\pm 1/2},\ \big((t_0-1)(1-t_1)\big)^{1/2}\big]$. $\square$

Theorem 5. *For any link* L,

$$\mathrm{LG}_L(t,-t^{-1}) = \Delta_L(t^2).$$

Proof. For any oriented (2, 2)-tangle T, we have



$$
\begin{aligned}
&= \overline{\mathrm{cl}\big(R \circ [T]\big)} - \overline{\mathrm{cl}\big(R^{-1} \circ [T]\big)} - (t - t^{-1})\overline{\mathrm{cl}\big([T]\big)} \\
&= \overline{t_0 a_0^T(t_0, t_1)\,\mathrm{cl}(P_0) + t_1 a_1^T(t_0, t_1)\,\mathrm{cl}(P_1) - a_2^T(t_0, t_1)\,\mathrm{cl}(P_2)} \\
&\quad - \overline{t_0^{-1} a_0^T(t_0, t_1)\,\mathrm{cl}(P_0) + t_1^{-1} a_1^T(t_0, t_1)\,\mathrm{cl}(P_1) - a_2^T(t_0, t_1)\,\mathrm{cl}(P_2)} \\
&\quad - (t - t^{-1})\overline{\big(a_0^T(t_0, t_1)\,\mathrm{cl}(P_0) + a_1^T(t_0, t_1)\,\mathrm{cl}(P_1) + a_2^T(t_0, t_1)\,\mathrm{cl}(P_2)\big)} \\
&= t a_0^T(t, -t^{-1})\overline{\mathrm{cl}(P_0)} - t^{-1} a_1^T(t, -t^{-1})\overline{\mathrm{cl}(P_1)} - a_2^T(t, -t^{-1})\overline{\mathrm{cl}(P_2)} \\
&\quad - t^{-1} a_0^T(t, -t^{-1})\overline{\mathrm{cl}(P_0)} + t a_1^T(t, -t^{-1})\overline{\mathrm{cl}(P_1)} + a_2^T(t, -t^{-1})\overline{\mathrm{cl}(P_2)} \\
&\quad - (t - t^{-1})\big(a_0^T(t, -t^{-1})\overline{\mathrm{cl}(P_0)} + a_1^T(t, -t^{-1})\overline{\mathrm{cl}(P_1)} + a_2^T(t, -t^{-1})\overline{\mathrm{cl}(P_2)}\big) \\
&= 0\,\mathrm{id}_V\,.
\end{aligned}
$$

The second equality follows from (9)–(11), the third from Lemmas 3 and 4, and the last one from Lemma 3. Thus, we have the following skein relation:

$$\mathrm{LG}_{L_+}(t, -t^{-1}) - \mathrm{LG}_{L_-}(t, -t^{-1}) = \big(t - t^{-1}\big)\,\mathrm{LG}_{L_0}(t, -t^{-1}),$$

which implies that $\mathrm{LG}_L(t, -t^{-1}) = \Delta_L(t^2)$. $\square$

Problem. *More generally, is there a relation between each invariant* $\mathrm{LG}^{m,n}(p, q)$ *and the Alexander–Conway polynomial?*

In the case of $n = 1$, we have proved the following theorem in joint work with D. De Wit and J. R. Links.

Theorem 6 [De Wit et al. 2005]. *For any link* L,

$$\mathrm{LG}_L^{m,1}\big(p, e^{\pi\sqrt{-1}/m}\big) = \Delta_L\big(p^{-2m}\big),$$

with $p = q^\alpha$ *on the left hand side.*

5. Some properties of the polynomials

By setting $t_0 = t$ and $t_1 = -t^{-1}$, we obtain the Alexander–Conway polynomial from the LG polynomial (Theorem 5). That is to say, the LG polynomial may be regarded as a two-variable generalization of the Alexander–Conway polynomial. Indeed, the LG polynomial inherits some properties of the Alexander–Conway polynomial.

For example, the LG polynomial of a split link vanishes, as does the Alexander–Conway polynomial.

By Theorem 5, the symmetry $\Delta(t) \doteq \Delta(t^{-1})$ (where the symbol $\doteq$ indicates equality up to a unit factor) transforms into the symmetry $\mathrm{LG}_L(t_0, t_1) = \mathrm{LG}_L(t_1, t_0)$. Under chirality change, these polynomials behave as follows:

$$\Delta_{L^*}(t) = \Delta_L(t^{-1}), \quad \mathrm{LG}_{L^*}(t_0, t_1) = \mathrm{LG}_L(t_0^{-1}, t_1^{-1}).$$

Because of the symmetry $\Delta(t) \doteq \Delta(t^{-1})$, the Alexander–Conway polynomial does not detect the chirality of knots. In contrast, the LG polynomial does detect it, at least for knots of small crossing number. So we can think of the LG polynomial as the Alexander–Conway polynomial, "expanded" with respect to chirality.

It is well-known that $\Delta_L(1)$ equals 1 if L is a knot, and 0 otherwise. Corresponding to this property, we have:

Theorem 7. *For any link L,* $\mathrm{LG}_L(t_0, 1) = \mathrm{LG}_L(1, t_1) = \begin{cases} 1 & \textit{if } L \textit{ is a knot,} \\ 0 & \textit{otherwise.} \end{cases}$

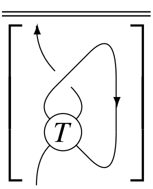
Proof. The first equality follows from the symmetry $\mathrm{LG}_L(t_0, t_1) = \mathrm{LG}_L(t_1, t_0)$. We show the result for $\mathrm{LG}_L(t_0, 1)$. Since the LG polynomial of a split link vanishes, it suffices to show that $\mathrm{LG}_L(t_0, 1)$ is a Vassiliev invariant of type 0; equivalently, that

LG$(t_0, 1)$ = LG$(t_0, 1)$.

This equality is verified in the same way as a similar one in the proof of Theorem 5. Denoting $X\big|_{t_1=1}$ by $\overline{\overline{X}}$, we get from Lemmas 3 and 4

$$\overline{\overline{\mathrm{cl}(P_0)}} = 0, \qquad \overline{\overline{\mathrm{cl}(P_1)}} = 1/2, \qquad \overline{\overline{\mathrm{cl}(P_2)}} = -1/2, \qquad a_i^T(t_0, 1) \in \mathbb{Z}\big[t_0^{\pm 1/2}\big].$$

Thus, for any oriented (2, 2)-tangle T, we have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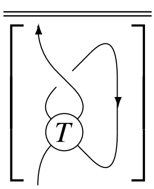

$$\begin{aligned}
&= \overline{\overline{\mathrm{cl}(R \circ [T])}} - \overline{\overline{\mathrm{cl}(R^{-1} \circ [T])}} \\
&= \overline{\overline{t_0 a_0^T(t_0, t_1)\,\mathrm{cl}(P_0) + t_1 a_1^T(t_0, t_1)\,\mathrm{cl}(P_1) - a_2^T(t_0, t_1)\,\mathrm{cl}(P_2)}} \\
&\qquad - \overline{\overline{t_0^{-1} a_0^T(t_0, t_1)\,\mathrm{cl}(P_0) + t_1^{-1} a_1^T(t_0, t_1)\,\mathrm{cl}(P_1) - a_2^T(t_0, t_1)\,\mathrm{cl}(P_2)}} \\
&= t_0 a_0^T(t_0, 1)\,\overline{\overline{\mathrm{cl}(P_0)}} + a_1^T(t_0, 1)\,\overline{\overline{\mathrm{cl}(P_1)}} - a_2^T(t_0, 1)\,\overline{\overline{\mathrm{cl}(P_2)}} \\
&\qquad - t_0^{-1} a_0^T(t_0, 1)\,\overline{\overline{\mathrm{cl}(P_0)}} - a_1^T(t_0, 1)\,\overline{\overline{\mathrm{cl}(P_1)}} + a_2^T(t_0, 1)\,\overline{\overline{\mathrm{cl}(P_2)}} \\
&= 0\,\mathrm{id}_V,
\end{aligned}$$

which implies that $\mathrm{LG}_{\times}(t_0, 1) = \mathrm{LG}_{\times}(t_0, 1)$. □

For any Laurent polynomial $A(t) \in \mathbb{Z}[t^{\pm 1}]$ with $A(1) = \pm 1$ and $A(t) \doteq A(t^{-1})$, there exists a knot K such that $\Delta_K(t) \doteq A(t)$ (see, for example, [Rolfsen 1976]). Then we ask:

Problem. *Let* $A(t_0, t_1) \in \mathbb{Z}\left[t_0^{\pm 1}, t_1^{\pm 1}\right]$ *be a Laurent polynomial satisfying the equalities* $A(t_0, 1) = A(1, t_1) = 1$ *and* $A(t_0, t_1) = A(t_1, t_0)$. *Does there exist a knot* K *such that* $\mathrm{LG}_K(t_0, t_1) = A(t_0, t_1)$?

The Alexander–Conway polynomial $\Delta_K(t) = \sum_i a_i t^i$ of an alternating knot K is "alternating", in the sense that $a_i a_{i'} \geq 0$ if $i - i'$ is even, and $a_i a_{i'} \leq 0$ otherwise [Murasugi 1958a; 1958b; Crowell 1959].

Conjecture. *The LG polynomial* $\mathrm{LG}_K(t_0, t_1) = \sum_{i,j} a_{ij} t_0^i t_1^j$ *of an alternating knot* K *is "alternating":* $a_{ij} a_{i'j'} \geq 0$ *if* $i + j - i' - j'$ *is even, and* $a_{ij} a_{i'j'} \leq 0$ *otherwise.*

This conjecture is true for all prime knots with up to 10 crossings.

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