Pacific Journal of Mathematics

CIRCLE PACKINGS ON SURFACES WITH PROJECTIVE STRUCTURES AND UNIFORMIZATION

SADAYOSHI KOJIMA, SHIGERU MIZUSHIMA AND SER PEOW TAN

Volume 225 No. 2 June 2006

PACIFIC JOURNAL OF MATHEMATICS
Vol. 225, No. 2, 2006

CIRCLE PACKINGS ON SURFACES WITH PROJECTIVE STRUCTURES AND UNIFORMIZATION

SADAYOSHI KOJIMA, SHIGERU MIZUSHIMA AND SER PEOW TAN

Dedicated to Professor Yukio Matsumoto on his sixtieth birthday

Let Σ_g be a closed orientable surface of genus $g \geq 2$ and τ a graph on Σ_g with one vertex that lifts to a triangulation of the universal cover. We have shown before that the cross ratio parameter space $\mathscr{C}_\tau$ associated with τ, which can be identified with the set of all pairs of a projective structure and a circle packing on it with nerve isotopic to τ, is homeomorphic to $\mathbb{R}^{6g-6}$, and moreover that the forgetting map of $\mathscr{C}_\tau$ to the space of projective structures is injective. Here we show that the composition of the forgetting map with the uniformization from $\mathscr{C}_\tau$ to the Teichmüller space $\mathscr{T}_g$ is proper.

1. Introduction

In [Kojima et al. 2003], we initiated the study of circle packings on Riemann surfaces with complex projective structures. Since the topic is rather new, we recall the setting briefly.

A projective structure on a surface is, by definition, a geometric structure modeled on the pair of the Riemann sphere $\hat{\mathbb{C}}$ and the projective linear group $\mathrm{PGL}_2(\mathbb{C})$ acting on $\hat{\mathbb{C}}$ by projective transformations. Hence it is in particular a complex structure, but finer than the complex structure up to conformal equivalence. We call a surface with a projective structure a *projective Riemann surface* for short throughout this paper.

The key observation is that circles and disks are fundamental objects in one-dimensional complex projective geometry, without reference to a metric, since a projective transformation sends a circle in $\hat{\mathbb{C}}$ to another. This is despite the fact that $\mathrm{PGL}_2(\mathbb{C})$ does not preserve any metric on the Riemann sphere. Thus, circles on a projective Riemann surface are not metric circles in the usual sense, rather, they are homotopically trivial closed curves that are taken to circles in $\hat{\mathbb{C}}$ by the developing

MSC2000: 57M27,

Keywords: circle packing, projective structure, uniformization, Teichmüller space.

Tan gratefully acknowledges support from the National University of Singapore academic research grant R-146-000-031-112 and the hospitality of the Department of Mathematical and Computing Sciences, Tokyo Institute of Technology.

map. A circle in this article is a circle in this projective sense, and not a circle with respect to the hyperbolic metric conformally equivalent to the projective structure.

Suppose we are given a closed orientable surface Σ_g of genus $g \geq 2$ without any auxiliary structure, and a graph τ on Σ_g that lifts to an honest triangulation of the universal cover $\widetilde{\Sigma}_g$. We are interested in the moduli space of all pairs (S, P) consisting of a projective Riemann surface S with a reference homeomorphism $h : \Sigma_g \to S$ and a circle packing P on S whose nerve is isotopic to $h(\tau)$. In [Kojima et al. 2003], this moduli space is shown to be identifiable with what we call the *cross ratio parameter space* $\mathscr{C}_\tau$.

Let $\mathscr{P}_g$ be the space of all *marked projective structures on* Σ_g, that is, all projective Riemann surfaces homeomorphic to Σ_g up to marked projective equivalence. To each pair $(S, P) \in \mathscr{C}_\tau$, assign its first component to obtain the forgetting map

$$f : \mathscr{C}_\tau \to \mathscr{P}_g.$$

One of main results in [Kojima et al. 2003] is that when τ has exactly one vertex, $\mathscr{C}_\tau$ is homeomorphic to euclidean space of dimension $6g - 6$ and f is injective. Injectivity implies that each projective Riemann surface admits at most one circle packing dominated by τ. In other words, the packings are in fact rigid.

At the same time, by assigning the underlying complex structure to each projective Riemann surface, we obtain the uniformization map

$$u : \mathscr{P}_g \to \mathscr{T}_g$$

from $\mathscr{P}_g$ to the Teichmüller space $\mathscr{T}_g$, the space of all complex structures on Σ_g up to marked conformal equivalence. By taking the Schwarzian derivative of the developing map, we can identify a projective structure with a holomorphic quadratic differential on the underlying Riemann surface, so the uniformization map is a complex vector bundle of rank $3g - 3$ over $\mathscr{T}_g$.

For the genus-one case, when τ has one vertex, it was shown in [Mizushima 2000] in a slightly different language that the composition of the forgetting map with the uniformization map is a homeomorphism. In [Kojima et al. 2003], we conjectured that this holds in general, regardless of the (positive) genus g and the graph τ. In this paper we take the first step towards solving that conjecture, by proving the following properness theorem for graphs τ having a single vertex:

Theorem 1.1. *Let τ be a one-vertex graph on Σ_g $(g \geq 2)$ that lifts to an honest triangulation of $\widetilde{\Sigma}_g$, and let $\mathscr{C}_\tau$ be the cross ratio parameter space associated with τ. The composition $u \circ f : \mathscr{C}_\tau \to \mathscr{T}_g$ of the forgetting map with the uniformization map is proper.*

To complete the proof of the conjecture for such graphs τ, since $\mathscr{C}_\tau$ in this case was shown to be homeomorphic to $\mathbb{R}^{6g-6}$, it suffices to show that the map is locally injective. This sort of question for the grafting map based on Tanigawa's properness theorem [1997] was settled in [Scannell and Wolf 2002]. See also [Faltings 1983; McMullen 1998] for earlier proofs of special cases. However, it is not clear that the proofs in these papers can be extended to our setting.

The rest of this paper is organized as follows. In Section 2 we recall the definition of the cross ratio parameter space and show that it is properly embedded in euclidean space. In Section 3, following the exposition in [Kamishima and Tan 1992], we briefly review the Thurston coordinates of $\mathcal{P}_g$ in terms of hyperbolic structures and measured laminations, and show that the projected image of $f(\mathscr{C}_\tau)$ to the space of measured laminations on Σ_g is bounded. The results up to that section are valid for any graph τ. In Section 4, under the assumption that τ has exactly one vertex, we show that the holonomy map from $f(\mathscr{C}_\tau)$ to the algebrogeometric quotient of the space of representations of $\pi_1(\Sigma_g)$ in $\mathrm{PGL}_2(\mathbb{C})$ up to conjugacy is proper, and deduce that the projection of $f(\mathscr{C}_\tau)$ to the space of the hyperbolic structures in the Thurston coordinates is also proper. In Section 5, we complete the proof of the theorem using Tanigawa's inequality [1997].

2. The cross ratio parameter space

Let Σ_g be a closed orientable surface of genus $g \geq 2$ and τ a graph on Σ_g that lifts to an honest triangulation of $\widetilde{\Sigma}_g$. Suppose that S is a projective Riemann surface lying in $f(\mathscr{C}_\tau)$ with a reference homeomorphism $h : \Sigma_g \to S$, and let P be a circle packing on S with nerve isotopic to $h(\tau)$. In [Kojima et al. 2003], we defined the *cross ratio parameter* for the pair (S, P), which is a function

$$\mathbf{c} : E_\tau \to \mathbb{R},$$

where E_τ denotes the set of edges of τ. To each edge e of τ is associated a configuration of four circles in the developed image surrounding a preimage $\tilde{e}$ of e; see Figure 1, left. The real number $\mathbf{c}(e)$ assigned to the edge e is obtained by taking the imaginary part of the cross ratio

$$(p_{14}, p_{23}, p_{12}, p_{13}) = \frac{(p_{14} - p_{12})(p_{23} - p_{13})}{(p_{14} - p_{13})(p_{23} - p_{12})}$$

of the four contact points $(p_{14}, p_{23}, p_{12}, p_{13})$ in the configuration chosen as in Figure 1, right, with an orientation convention. (For the definition of the cross ratio of four ordered points, see [Ahlfors 1953].) This number is the modulus of the rectangle obtained by normalizing the configuration by moving the contact point p_{13} to ∞; see Figure 2, where it can be seen that the cross ratio is purely imaginary. Moreover, unless the imaginary part is positive, neighboring triangular interstices overlap and the circles cannot correspond to a correct packing.

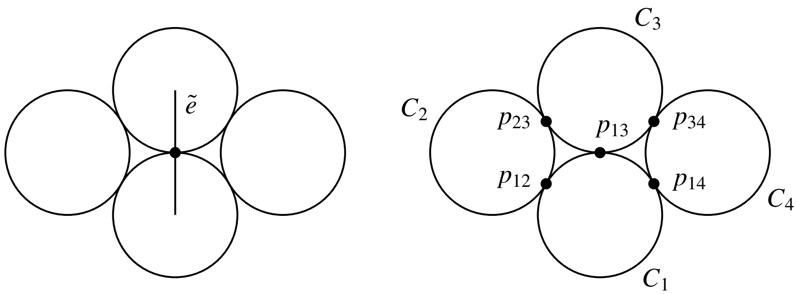

Figure 1

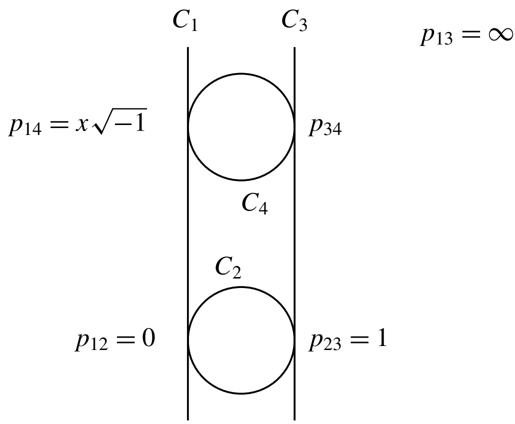

Figure 2

Because the developing map is a local homeomorphism, the cross ratio parameter must satisfy certain conditions, best expressed in terms of the associated matrices. If e is an edge of τ with real value $x = \mathbf{c}(e)$, we associate to e the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix},$$

where x is positive real. The matrix A represents a transformation sending the left triangular interstice to the right one in the normalized picture of Figure 1, which, as we recall, is obtained by sending p_{12}, p_{13} and p_{23} to ∞, 0 and $\sqrt{-1}$ respectively. Now, if v is a vertex of τ with valence m, we read off the edges $e_1, \dots, e_m$ incident to v in a clockwise direction to obtain a sequence of real values $x_1 = \mathbf{c}(e_1), \dots, x_m = \mathbf{c}(e_m)$ associated to v. Set

$$W_j = A_1 A_2 \cdots A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \quad \text{for } j = 1, \dots, m,$$

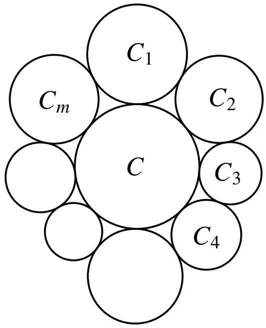

Figure 3

where A_i is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & x_i \end{pmatrix}$ associated to e_i. Then, for each vertex v of τ,

$$W_v = A_1 A_2 \cdots A_m = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and

$$a_j,\, c_j < 0 \text{ and } b_j,\, d_j > 0 \quad \text{for } 1 \le j \le m-1, \text{ except for } a_1 = d_{m-1} = 0. \tag{2}$$

Roughly, the first condition means that a fixed triangular interstice is mapped back to its original position by the composition of the transformations represented by the associated matrices, and this ensures the consistency of the chain of circles surrounding the circle corresponding to v; see Figure 3. The second condition eliminates the case where the chain surrounds the central circle more than once. It does not matter which edge we start out with in this construction.

Conversely, if an assignment to the edges of τ,

$$\mathbf{c} : E_\tau \to \mathbb{R},$$

satisfies conditions (1) and (2) for each vertex, the map is the cross ratio parameter for some packing P on a projective Riemann surface $S \in f(\mathscr{C}_\tau)$, and both S and P are determined by $\mathbf{c}$ [Kojima et al. 2003, Main lemma]. Hence, the set

$$\mathscr{C}_\tau = \{\mathbf{c} : E_\tau \to \mathbb{R} \mid \mathbf{c} \text{ satisfies (1) and (2) for each vertex}\},$$

called the *cross ratio parameter space*, can be identified with the space of pairs (S, P) of a projective Riemann surface S and a circle packing P on S, and $\mathbf{c}$ parametrizes the space of such pairs.

From the definition we see that $\mathscr{C}_\tau$ is a semialgebraic set in $\mathbb{R}^{E_\tau}$ defined by the equations (1) and the inequalities (2) for each vertex v. We first show that the strict inequalities of (2) can be replaced by nonstrict inequalities; namely, for each vertex

v and with the same notation as before, consider the set of conditions

$$a_j,\ c_j \le 0 \text{ and } b_j,\ d_j \ge 0 \quad \text{for } 1 \le j \le m-1, \text{ except for } a_1 = d_{m-1} = 0. \tag{3}$$

Lemma 2.1. *Conditions* (1) and (2) *are equivalent to* (1) and (3).

Proof. It is sufficient to prove that (2) follows from (1) and (3). We have the identity

$$\begin{pmatrix} a_{j+1} & b_{j+1} \\ c_{j+1} & d_{j+1} \end{pmatrix} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x_{j+1} \end{pmatrix} = \begin{pmatrix} -b_j & a_j + b_j x_{j+1} \\ -d_j & c_j + d_j x_{j+1} \end{pmatrix} \tag{4}$$

for $j = 1, \ldots, m-1$. Condition (2) is then shown to follow from (1) and (3) by induction as follows. We first show that $a_j < 0$ and $b_{j-1} > 0$ for $j = 2, \ldots, m-1$.

Set the inductive hypotheses to be $a_j < 0$ and $b_{j-1} > 0$. Since $a_2 = -b_1 = -1 < 0$, the hypotheses are true for $j = 2$. Assume that (1) and (3) hold and the hypotheses are true for $j = k$, where $2 \le k \le m-2$. Then $b_k \ge 0$, by (3). If $b_k = 0$, $b_{k+1} = a_k + b_k x_{k+1} = a_k < 0$ by (4) and the hypotheses, contradicting $b_{k+1} \ge 0$. Thus $b_k > 0$. Also $a_{k+1} = -b_k < 0$; hence the hypotheses are true for $j = k+1$. Combining this induction with the relations $b_{m-1} = -a_m = -1 < 0$ gives the strict inequalities for the a_j and b_j in (2).

A similar argument yields the strict inequalities for the c_j and d_j. □

This lemma tells us that condition (2) does not divide a connected component of the algebraic set determined by (1). It just chooses appropriate connected components. With this, we can easily prove the next result:

Lemma 2.2. *The inclusion map of $\mathscr{C}_\tau$ into $\mathbb{R}^{E_\tau}$ is proper, where E_τ is the set of edges of τ. This holds for general τ, with no restriction on the number of vertices.*

Proof. Let $\{\mathbf{c}_n\}$ be a sequence of points in the intersection of a compact set in $\mathbb{R}^{E_\tau}$ with $\mathscr{C}_\tau$. Then there exists a convergent subsequence. Let $\mathbf{c}_\infty$ denote the limit of the subsequence. Each $\mathbf{c}_n$ satisfies the conditions (1) and (2). Each entry of the product of matrices in conditions (1) and (2) is a polynomial in the x_i and is thus continuous. Hence, the limit $\mathbf{c}_\infty$ satisfies the conditions (1) and (3), and also lies in $\mathscr{C}_\tau$ by Lemma 2.1. □

3. Measured laminations

We recall Thurston's parametrization of $\mathscr{P}_g$ and how $f(\mathscr{C}_\tau)$ is located with respect to the parametrization.

The space we are concerned with consists of isotopy classes of measured laminations on Σ_g, and is denoted by $\mathscr{ML}_g$. A measured lamination is defined to be a closed subset on Σ_g locally homeomorphic to a product of a totally disconnected subset of the interval with an interval, together with a transverse measure. Although a measured lamination is a topological concept, once we put a hyperbolic metric on Σ_g, its support is canonically realized as a disjoint union of simple geodesics which forms a closed subset on the surface. Such a lamination is called a *measured geodesic lamination*, and hence $\mathcal{ML}_g$ can be identified with the set of measured geodesic laminations on a hyperbolic surface homeomorphic to Σ_g. The space $\mathcal{ML}_g$ has a natural topology induced from the weak topology on measures on the set of undirected geodesics [Thurston 1979], and is known to be homeomorphic to euclidean space of dimension $6g-6$. A detailed account of the theory in terms of measured foliations (a notion equivalent to that of a measured geodesic lamination: see [Miller 1982]) can be found in [Fathi et al. 1979; Thurston 1988].

Thurston has shown that any projective Riemann surface corresponds uniquely to a hyperbolic surface pleated along geodesic laminations with a bending measure. Following [Kamishima and Tan 1992], we briefly review his parametrization.

Start with a projective Riemann surface S that is not a hyperbolic surface. Consider the set of maximal disks in the universal cover $\tilde{S}$. Each maximal disk is naturally endowed with the hyperbolic metric, the boundary of each disk intersects the ideal boundary of $\tilde{S}$ in two or more points, and we can take the convex hull of these ideal boundary points. It can be shown that this gives a stratification of $\tilde{S}$ by ideal polygons, and ideal bigons foliated by "parallel lines" joining the two ideal vertices of the bigons. The polygonal parts support a canonical hyperbolic metric. Collapsing each bigon foliated by the parallel lines in $\tilde{S}$ to a line and taking the quotient of the result by the action of the fundamental group, we obtain a hyperbolic surface H.

A hyperbolic surface is in particular a projective Riemann surface and there is a natural section

$$s : \mathcal{T}_g \to \mathcal{P}_g$$

to the uniformization map $u : \mathcal{P}_g \to \mathcal{T}_g$ obtained by assigning a hyperbolic surface to each conformal class of Riemann surfaces. Thus assigning H to each S, we obtain the *hyperbolization map*

$$\pi : \mathcal{P}_g \to s(\mathcal{T}_g).$$

Also the stratification defines a geodesic lamination λ on H by taking the union of collapsed lines. Moreover, using the convex hull of the ideal points of the maximal disk not in the disk but in 3-dimensional hyperbolic space $\mathbb{H}^3$, we can assign a transverse bending measure supported on λ. This defines the *pleating map*

$$\beta : \mathcal{P}_g \to \mathcal{ML}_g.$$

Thurston showed that π and β together form a homeomorphism parametrizing $\mathcal{P}_g$:

$$(\pi, \beta) : \mathcal{P}_g \xrightarrow{\cong} s(\mathcal{T}_g) \times \mathcal{ML}_g.$$

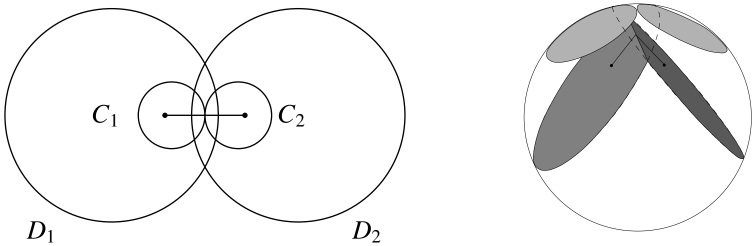

Figure 4

Lemma 3.1. *$\beta(f(\mathscr{C}_\tau))$ is bounded in $\mathscr{ML}_g$. This holds for a general τ.*

Proof. Let S be a surface in $f(\mathscr{C}_\tau) \subset \mathscr{P}_g$ and P a circle packing on S with nerve τ. Set $H = \pi(S)$ and $\lambda = \beta(S)$. The measured lamination λ can be pulled back to a lamination with a transverse measure by blowing up in a canonical way the atomic leaves on H to parallel leaves in S with stretched transverse measure. Hence we regard S as a surface with this measured lamination μ. To see the boundedness of $\beta(f(\mathscr{C}_\tau))$, it is sufficient to show that the measure along each edge of τ is uniformly bounded, since τ generates the fundamental group of Σ_g and μ collapses to λ.

We can choose a reference point for each circle in P to represent the vertex v of τ such that the supporting maximal disk of the point representing v contains the circle. We take these points' preimage in $\tilde{P}$ to get equivariant reference vertices. Let C_1 and C_2 be contact circles in $\tilde{P}$, and D_i, for $i = 1, 2$, the supporting maximal disk of the reference point v_i of C_i. Now D_1 contains C_1 and D_2 contains C_2, so the D_i form a π roof over the pleated hyperbolic surface in $\mathbb{H}^3$ locally corresponding to the developed image of $\tilde{S}$; see Figure 4. So the total transverse measure along a path $\tilde{e}$ between v_1 and v_2 contained in $D_1 \cup D_2$ is bounded above by π. □

4. Holonomy representations

We now restrict ourselves to the case where τ has one vertex, and see how the holonomy representations behave.

Our main concern is the space of representations of $\pi_1(\Sigma_g)$ in $\mathrm{PGL}_2(\mathbb{C})$ up to conjugacy, which is unfortunately non-Hausdorff. Hence we take further the algebrogeometric quotient, to obtain

$$\mathscr{X}_g = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}_2(\mathbb{C})) /\!/ \sim,$$

which can be identified with the space of characters. Assign to each projective Riemann surface its holonomy representation, to obtain the map

$$\mathrm{hol} : \mathscr{P}_g \to \mathscr{X}_g,$$

which is a local homeomorphism [Hejhal 1975].

Lemma 4.1. *Suppose that τ has only one vertex. There is a finite subset F of the fundamental group $\pi_1(\Sigma_g)$ such that, if $\{\mathbf{c}_n\}$ is a sequence of points in $\mathscr{C}_\tau$ that escapes away from the compact sets, there is an element $g \in F$ and a subsequence of $\{\mathbf{c}_n\}$ for which $|\operatorname{tr} \rho_n(g)| \to \infty$, where $\rho_n = \operatorname{hol}(f(\mathbf{c}_n))$. In particular, the composition* $\operatorname{hol} \circ f : \mathscr{C}_\tau \to \mathscr{X}_g$ *is proper.*

Remark. Although the image of the holonomy lies in $\mathrm{PGL}_2(\mathbb{C})$ and the trace makes sense only up to a sign, its absolute value is well defined.

Proof. Since τ has one vertex, each edge e_i of τ starts and ends at the same vertex v, and so corresponds to a pair of elements $g_i^{\pm 1} \in \pi_1(\Sigma_g)$, and the set $\{g_i^{\pm 1}\}$ forms a generating set for $\pi_1(\Sigma_g)$. Define F as the set of all words of length 2 in this generating set.

Passing to a subsequence we may assume by Lemma 2.2 that the real value $\mathbf{c}_n(e)$ of some fixed edge e of τ approaches either 0 or ∞ as $n \to \infty$. By using (1), (2) and (4), we may further assume that, in fact, it approaches ∞. For each $S_n = f(\mathbf{c}_n)$, consider the developed image of $\tilde{S}_n$ and in particular the configuration of 6 circles $C_1, \ldots, C_6$ in $\tilde{P}_n$ with corresponding vertices $v_1, \ldots, v_6$ of $\tilde{\tau}$ as given in Figure 5, left. Here the edge $e_3 = v_1 v_3$ is the one whose assigned real value $\mathbf{c}_n(e_3)$ approaches ∞ as $n \to \infty$. Note that the configuration of C_j and v_j depends on n.

For each n, we may normalize the developed image so that it becomes as in Figure 5, right, where the tangency point p_{13} between C_1 and C_3 is $\sqrt{-1}$. The concatenation of the two directed edges $e_2 = v_2 v_1$ and $e_4 = v_1 v_4$, which are the neighboring edges of e_3 about the vertex v_1, corresponds to an element $g = g_4 g_2^{-1} \in F$. Its holonomy image $\varphi_n := \rho_n(g) = \rho_n(g_4 g_2^{-1})$ is an element of $\mathrm{PGL}_2(\mathbb{C})$ mapping C_2 to C_4. In the normalized picture, the radius of C_4 approaches zero as $n \to \infty$, since $\mathbf{c}_n(e_3) \to \infty$. Hence we can represent the images of 0, 1 and ∞ under φ_n, all lying on C_2, by $\sqrt{-1} + \varepsilon_1$, $\sqrt{-1} + \varepsilon_2$ and $\sqrt{-1} + \varepsilon_3$ respectively, where ε_1, ε_2, ε_3 depend on n and they all approach 0 as n tends to infinity. Note that the point 1

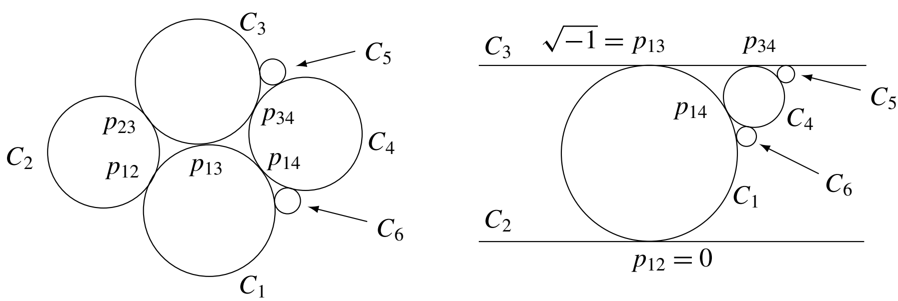

Figure 5

and its image are not necessarily contact points. Setting $K_n = (\varepsilon_2 - \varepsilon_3)/(\varepsilon_2 - \varepsilon_1)$, we can write

$$\varphi_n^{-1}(z) = K_n \frac{z - (\sqrt{-1} + \varepsilon_1)}{z - (\sqrt{-1} + \varepsilon_3)}.$$

Further,

$$M_n = \frac{1}{\sqrt{K_n(\varepsilon_1 - \varepsilon_3)}} \begin{pmatrix} K_n & -K_n(\sqrt{-1} + \varepsilon_1) \\ 1 & -(\sqrt{-1} + \varepsilon_3) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

is a matrix representative of φ_n^{-1}. We have

$$\operatorname{tr} M_n = \frac{1}{\sqrt{\varepsilon_1 - \varepsilon_3}} \left(\sqrt{K_n} - \frac{1}{\sqrt{K_n}} (\sqrt{-1} + \varepsilon_3) \right)$$

and $|\operatorname{tr} \varphi_n| = |\operatorname{tr} \varphi_n^{-1}| = |\operatorname{tr} M_n|$. If $|\operatorname{tr} M_n| \to \infty$, we are done; otherwise, we must have

$$\sqrt{K_n} - \frac{1}{\sqrt{K_n}} (\sqrt{-1} + \varepsilon_3) \to 0,$$

since $(\varepsilon_1 - \varepsilon_3) \to 0$ as $n \to \infty$. This implies that K_n approaches $\sqrt{-1}$, since $\varepsilon_3 \to 0$ does. Geometrically, this means that the angle formed by $\sqrt{-1} + \varepsilon_1$, $\sqrt{-1} + \varepsilon_2$ and $\sqrt{-1} + \varepsilon_3$ approaches $\pi/2$, and hence $\sqrt{-1} + \varepsilon_1$ and $\sqrt{-1} + \varepsilon_3$ approach diametrically opposite positions on the circle C_4. In other words, the arc $\alpha_1 = \varphi_n([-\infty, 0])$ occupies half of the circle C_4 in the limit.

On the other hand, the minor arc α_2 on C_4 connecting p_{14} and p_{34} (this is the arc on the circle joining p_{14} to p_{34} which does not contain p_{45} and p_{46}) also occupies half of C_4 in the limit, since the radius of C_4, which we denote by $\operatorname{rad} C_4$, approaches 0, where p_{jk} is the tangency point between C_j and C_k.

For each circle, the contact points with neighboring circles cuts it into arcs meeting only at the endpoints.

Claim. *α_1 and α_2 are the images under φ_n of two distinct, nonadjacent arcs of C_2 and thus are distinct nonadjacent arcs on C_4.*

Proof. It suffices to show that the image of $[-\infty, 0]$ on C_1 under $\rho_n(g_2^{-1})$ and the image of the arc α_2 under $\rho_n(g_4^{-1})$ are nonadjacent arcs of C_1.

We recall from [Kojima et al. 2003, Section 4] the notion of intersecting and nonintersecting triples. For a packing with one circle, an edge of nerve τ corresponds to the self-contact points of the unique circle on Σ_g, and hence to the pair of contact points on a circle in the developed image, where the contact points are developed images of the prescribed contact point on Σ_g. A triangle of the nerve τ appears as a triple of such pairs of contact points, the six contact points consisting of three pairs of neighboring contact points at the same time. There are two possible cases, as shown in Figure 6.

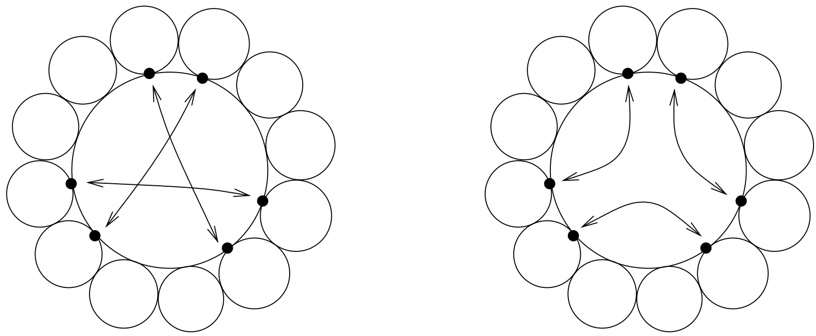

Figure 6. Intersecting and nonintersecting triples.

The pair of edges (e_2, e_3) are either part of an intersecting or a nonintersecting triple of edges, since they are adjacent. Similarly, the pair of edges (e_3, e_4) are also part of an intersecting or nonintersecting triple. For all possibilities, $\rho_n(g_2^{-1})[-\infty, 0]$ and $\rho_n(g_4^{-1})(\alpha_2)$ are nonadjacent arcs of C_1; see Figure 7. $\square$

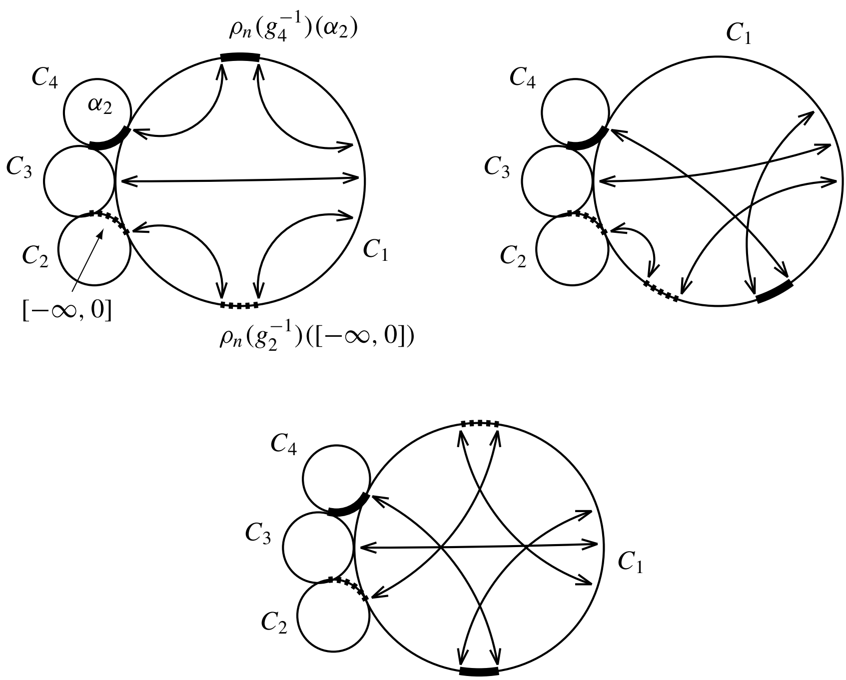

Figure 7. Top left: two nonintersecting triples; bottom: two intersecting triples; top right: one of each.

From the claim, we see that the angle subtended on C_4 between p_{45} and p_{34} in Figure 5 (right) approaches zero, as does the angle between p_{46} and p_{14}. Thus, the radii ratios $\operatorname{rad} C_5/\operatorname{rad} C_4$ and $\operatorname{rad} C_6/\operatorname{rad} C_4$ approach zero as $n \to \infty$. This in turn implies that the values of $\mathbf{c}_n$ at the two edges v_1v_4 and v_3v_4 both approach ∞. This can be seen by scaling the same figure so that $\operatorname{rad} C_4 = 1$ for each n; then C_1 tends to a straight line parallel to C_3 and the radii of C_5 and C_6 tend to 0, so the values of $\mathbf{c}_n$ at the two edges v_1v_4 and v_3v_4 both approach ∞ [Kojima et al. 2003, Proposition 2.8]. Repeating the argument with the roles of C_2 and C_4 reversed, we see that either $|\operatorname{tr} \varphi_n| \to \infty$ or the values of $\mathbf{c}_n$ at v_1v_2 and v_2v_3 both approach ∞. In other words, either $|\operatorname{tr} \varphi_n| \to \infty$ or the values of $\mathbf{c}_n$ at all neighboring edges approach ∞. By induction, using the edge e_4 instead of e_3 and repeating the argument, we see that either some element $g \in F$ has holonomy with diverging trace, or the values of $\mathbf{c}_n$ at all the edges of τ diverge to ∞. However, in the latter case, the $(2, 2)$ term d_m of the matrix corresponding to the word W_m defined in Section 2 has a dominant term $x_1x_2 \cdots x_m$ and diverges to ∞. This contradicts the fact that $d_m = -1$ for all points of the sequence by condition (1) in Section 2. □

Lemma 4.2. *Suppose that τ has only one vertex. The composition*

$$\pi \circ f : \mathscr{C}_\tau \to s(\mathscr{T}_g)$$

of the forgetting map with the collapsing map $\pi : \mathscr{P}_g \to s(\mathscr{T}_g)$ in the Thurston parametrization is proper. In particular, the restriction of π to $f(\mathscr{C}_\tau)$ is proper.

Proof. Suppose that $\{\mathbf{c}_n\}$ is a sequence in $\mathscr{C}_\tau$ that escapes away from compact sets. By passing to a subsequence, we may assume by Lemma 4.1 that there is an element $g \in \pi_1(\Sigma_g)$ with holonomy $\varphi_n = \rho_n(g)$ for which $|\operatorname{tr} \varphi_n| \to \infty$. This g corresponds to a closed curve γ on Σ_g and the length $l_n(\gamma)$ of the geodesic representative of γ on the collapsed surface $H_n = \pi(S_n)$ satisfies the inequality

$$l_n(\gamma) \geq d_{\mathbb{H}^3}(z_n, \varphi_n(z_n)),$$

where z_n is any point on the axis of φ_n in $\mathbb{H}^3$ and $d_{\mathbb{H}^3}(z_n, \varphi_n(z_n))$ is the hyperbolic distance between z_n and $\varphi_n(z_n)$. Since $|\operatorname{tr} \varphi_n| \to \infty$, we have $d_{\mathbb{H}^3}(z_n, \varphi_n(z_n)) \to \infty$ and hence $l_n(\gamma) \to \infty$. It follows that $\{H_n\}$ escapes away from compact sets in $s(\mathscr{T}_g)$. □

5. Proof of the main theorem

Theorem 5.1 [Tanigawa 1997, Theorem 3.4]. *Let $S = (H, \lambda)$ be a projective Riemann surface homeomorphic to Σ_g $(g \geq 2)$, where $H = \pi(S) \in s(\mathscr{T}_g)$ and $\lambda = \beta(S) \in \mathscr{ML}_g$. Let X be the underlying Riemann surface and $h : X \to H$ a* *harmonic map with respect to the hyperbolic metric on H. Let $\mathscr{E}(h)$ be its energy. Then*

$$l_H(\lambda) \le \frac{l_H(\lambda)^2}{E_X(\lambda)} \le 2\mathscr{E}(h) \le l_H(\lambda) + 8\pi(g-1), \tag{5}$$

where $l_H(\lambda)$ is the hyperbolic length of λ on H and $E_X(\lambda)$ is the extremal length of λ on X.

Proof of Theorem 1.1. Assuming that τ has one vertex, we follow the argument in [Tanigawa 1997]. Let $\{\mathbf{c}_n\}$ be a sequence of points in $\mathscr{C}$ that escapes away from compact sets. Write $f(\mathbf{c}_n) = S_n = (H_n, \lambda_n) \in s(\mathscr{T}_g) \times \mathscr{ML}_g$ and let $X_n = u(S_n)$ be the corresponding underlying Riemann surface. By Lemma 4.2, we may assume that $\{H_n\}$ escapes away from compact sets in $\mathscr{T}$. By Lemma 3.1, since $\{\lambda_n\}$ lies in a compact subset of $\mathscr{ML}_g$, we may assume by taking a subsequence if necessary that $\lambda_n \to \lambda$ for some fixed measured lamination λ. Let h_n be the harmonic map from X_n to H_n with respect to a hyperbolic metric on H_n.

If $\mathscr{E}(h_n)$ is bounded, $\{X_n\}$ escapes away from compact sets as desired, since otherwise we have a contradiction with a result of M. Wolf [1989]. If not, the diverging rate of all terms in the inequalities (5) is the same as that of $l_{H_n}(\lambda_n)$ and hence

$$\lim_{n\to\infty} E_{X_n}(\lambda) = \lim_{n\to\infty} E_{X_n}(\lambda_n) = \lim_{n\to\infty} (l_{H_n}(\lambda_n) + O(1)) = \infty.$$

This implies that $\{X_n\}$ escapes away from compact sets as well. □

References

[Ahlfors 1953] L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, McGraw-Hill, New York, 1953. MR 14,857a Zbl 0052.07002

[Faltings 1983] G. Faltings, "Real projective structures on Riemann surfaces", *Compositio Math.* **48**:2 (1983), 223–269. MR 85d:32046 Zbl 0511.30034

[Fathi et al. 1979] A. Fathi, F. Laudenbach, V. Poenaru, et al., *Travaux de Thurston sur les surfaces*, Astérisque **66**, Société Mathématique de France, Paris, 1979. MR 82m:57003 Zbl 0406.00016

[Hejhal 1975] D. A. Hejhal, "Monodromy groups and linearly polymorphic functions", *Acta Math.* **135**:1 (1975), 1–55. MR 57 #3380 Zbl 0333.34002

[Kamishima and Tan 1992] Y. Kamishima and S. P. Tan, "Deformation spaces on geometric structures", pp. 263–299 in *Aspects of low dimensional manifolds*, edited by Y. Matsumoto and S. Morita, Adv. Stud. Pure Math. **20**, Kinokuniya, Tokyo, 1992. MR 94k:57023 Zbl 0798.53030

[Kojima et al. 2003] S. Kojima, S. Mizushima, and S. P. Tan, "Circle packings on surfaces with projective structures", *J. Differential Geom.* **63**:3 (2003), 349–397. MR 2004k:52024 Zbl 1075.52509

[McMullen 1998] C. T. McMullen, "Complex earthquakes and Teichmüller theory", *J. Amer. Math. Soc.* **11**:2 (1998), 283–320. MR 98i:32030 Zbl 0890.30031

[Miller 1982] R. T. Miller, "Geodesic laminations from Nielsen's viewpoint", *Adv. in Math.* **45**:2 (1982), 189–212. MR 83j:57004 Zbl 0496.57003

[Mizushima 2000] S. Mizushima, "Circle packings on complex affine tori", *Osaka J. Math.* **37**:4 (2000), 873–881. MR 2003b:52011 Zbl 1005.52009

[Scannell and Wolf 2002] K. P. Scannell and M. Wolf, "The grafting map of Teichmüller space", *J. Amer. Math. Soc.* **15**:4 (2002), 893–927. MR 2003d:32011 Zbl 1011.32009

[Tanigawa 1997] H. Tanigawa, "Grafting, harmonic maps and projective structures on surfaces", *J. Differential Geom.* **47**:3 (1997), 399–419. MR 99c:32029 Zbl 0955.32012

[Thurston 1979] W. P. Thurston, "The geometry and topology of three-manifolds", lecture notes, Princeton University, 1979, Available at http://msri.org/publications/books/gt3m.

[Thurston 1988] W. P. Thurston, "On the geometry and dynamics of diffeomorphisms of surfaces", *Bull. Amer. Math. Soc.* (*N.S.*) **19**:2 (1988), 417–431. MR 89k:57023 Zbl 0674.57008

[Wolf 1989] M. Wolf, "The Teichmüller theory of harmonic maps", *J. Differential Geom.* **29**:2 (1989), 449–479. MR 90h:58023 Zbl 0655.58009

Received August 18, 2004. Revised December 14, 2004.

SADAYOSHI KOJIMA
DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCES
TOKYO INSTITUTE OF TECHNOLOGY
OHOKAYAMA, MEGURO
TOKYO 152-8552
JAPAN

sadayosi@is.titech.ac.jp

SHIGERU MIZUSHIMA
DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCES
TOKYO INSTITUTE OF TECHNOLOGY
OHOKAYAMA, MEGURO
TOKYO 152-8552
JAPAN

mizusima@is.titech.ac.jp

SER PEOW TAN
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE 117543
SINGAPORE

mattansp@nus.edu.sg