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We prove that the Cheng–Yau gradient estimate on positive harmonic functions on manifolds with nonnegative Ricci curvature is globally stable under certain perturbations of the metric. In some cases, one only needs the condition $\text{Ricci}(x) \geq -\varepsilon/(1 + d(x)^{2+\delta})$, with $\delta > 0$ and $\varepsilon > 0$ sufficiently small.

1. Introduction

One of the most useful inequalities in geometric analysis is the Cheng–Yau estimate on the gradient of positive harmonic functions.

Theorem (Cheng–Yau). *Let M be a complete manifold of dimension $n \geq 2$ and Ricci curvature bounded below by $-k$, with $k \geq 0$. Suppose u is any positive harmonic function in a geodesic ball $B(x_0, r) \subset M$. Then*

$$(1-1) \quad \sup_{B(x_0, r/2)} \frac{|\nabla u|}{u} \leq \frac{c_n}{r} + c_n \sqrt{k},$$

where c_n depends only on the dimension n .

When $k = 0$ (so M has nonnegative Ricci curvature), the Cheng–Yau estimate becomes

$$(1-2) \quad \sup_{B(x_0, r/2)} \frac{|\nabla u|}{u} \leq \frac{c_n}{r},$$

which is sharp, as we can see by considering the Euclidean case. However, even if M contains a small compact region where the Ricci curvature is not nonnegative, estimate (1-1) becomes very different from (1-2) when r is large, because of the \sqrt{k} term. Whether estimate (1-2) is stable under perturbation has been an open question for some time, in light of the known stability results on weaker properties of harmonic functions, such as the Harnack inequality.

Here we confirm that (1-2) is stable when the nonpositive part of the Ricci curvature is sufficiently small in an integral sense.

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Some smallness condition for the nonpositive part of the Ricci curvature is surely necessary for (1–2) to hold. For instance, if the nonpositive part of the Ricci curvature is so large that M admits a bounded nonconstant harmonic function, clearly (1–2) cannot hold.

Throughout the paper Δ is the Laplace–Beltrami operator, $d(x, y)$ is the distance between x and y , and $d(x)$ is the distance between x and a fixed reference point. $|B(x, r)|$ denotes the volume of the geodesic ball of radius r centered at x .

We now lay out the basic assumptions required. As will be explained later, these assumptions are stable under certain perturbations of the metric.

Assumption A. M is a complete noncompact Riemannian manifold of dimension $n \geq 2$ satisfying the volume doubling property

$$|B(x, 2r)| \leq 2^\nu |B(x, r)|$$

for all $x \in M$, $r > 0$ and some $\nu > 0$.

Assumption B. The heat kernel G of the Laplace–Beltrami operator satisfies a Gaussian upper bound

$$G(x, t; y, 0) \leq \frac{B_1}{|B(x, \sqrt{t})|} \exp \frac{-b_1 d^2(x, y)}{t},$$

for some $b_1, B_1 > 0$, and all $x, y \in M$ and $t > 0$.

Several conditions are known to be equivalent to assumptions A and B. For instance, it was proved in [Grigoryan 1991] that the two assumptions together are equivalent to:

Faber–Krahn inequality. For any $x \in M$, $r > 0$, and nonempty subset $\Omega \subset B(x, r)$,

$$\lambda_1(\Omega) \geq \frac{c}{r^2} \left(\frac{|B(x, r)|}{|\Omega|} \right)^{2/\nu}.$$

Here $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω and $c > 0$.

It is also known [Saloff-Coste 1992] that if Assumption A is satisfied, Assumption B is equivalent to:

Assumption B'. The following Sobolev inequality holds for all $\phi \in C_0^\infty(B(x, r))$, $x \in M$, $r > 0$, and a fixed $\alpha > 2$:

$$\left(\int \phi^{2\alpha/(\alpha-2)} dy \right)^{(\alpha-2)/\alpha} \leq S_0 |B(x, r)|^{-2/\alpha} \int (r^2 |\nabla \phi|^2 + \phi^2) dy.$$

Also under [Assumption A](#), it is known that [Assumption B](#) is implied by a certain Poincaré inequality [[Saloff-Coste 1992](#); [Grigoryan 1991](#)], and by a certain mean-value inequality [[Li and Wang 1999](#)]. There is an extensive literature on manifolds satisfying various global conditions including the ones mentioned above; see [[Grigor'yan 1999](#); [Saloff-Coste 2002](#)] and references therein.

There exist many manifolds satisfying assumptions [Assumption A](#) and [Assumption B](#), among them is the class of manifolds quasiisometric to manifolds with non-negative Ricci curvature, and connected sums of two copies of \mathbb{R}^n . See [[Coulhon and Saloff-Coste 1995](#)], for example.

Next we introduce the conditions on the nonpositive part of the Ricci curvature that will imply the global Cheng–Yau estimate (1–2). The conditions, in general integral form first, will be elucidated in the [Corollary](#) below by simple conditions. Essentially, the nonpositive part of the Ricci curvature is required to be small and decay sufficiently fast near infinity.

Let $\lambda = \lambda(x)$ be the lowest eigenvalue of $\text{Ric}(x)$, for $x \in M$. We use the notation

$$V(x) := \frac{1}{2}(|\lambda(x)| - \lambda(x))$$

Assumption C. *The function V lies in $L^\infty(M)$ and there exist $\varepsilon_0 > 0$ and $K > 0$ such that*

$$N(V) \equiv \sup_{x \in M} \int_0^\infty \int_M \frac{e^{-d(x,y)^2/t}}{|B(x, \sqrt{t})|} V(y) dy dt < \varepsilon_0$$

and

$$\int_M V(y)\phi^2(y) dy < \frac{1}{11n} \int_M |\nabla\phi(x)|^2 dx + \frac{K}{r^2} \int_M \phi^2(y) dy$$

for all $\phi \in C_0^\infty(B(x, r))$.

Here is the key result in this paper.

Main Theorem. *Suppose M satisfies Assumptions A and B. There exists $\varepsilon_0 > 0$, depending only on the parameters in those assumptions, so that if Assumption C on the Ricci curvature holds, the following statement holds.*

Let u be a positive harmonic function in the ball $B(x, r)$. Then

$$\sup_{y \in B(x, r/2)} \frac{|\nabla u(y)|}{u(y)} \leq \frac{C_0}{r},$$

where C_0 depends only on the assumptions' parameters $\nu, b_1, B_1, \varepsilon_0, K$ and n .

Remark 1. In many cases, the condition on V in the theorem simply means that

$$\text{Ric}(x) \geq -\frac{\varepsilon}{1 + d(x)^{2+\delta}}$$

for some small enough $\varepsilon > 0$ and $\delta > 0$. This is indicated in the [Corollary](#) below. In general, the first equation in [Assumption C](#) is a Kato-type condition and the

second condition takes the form of Hardy’s inequality; that is, for $f \in C_0^\infty(\mathbb{R}^n)$ ($n \geq 3$), we have

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{1}{|x|^2} f(x)^2 dx \leq \int_{\mathbb{R}^n} |\nabla f|^2(x) dx.$$

For versions of Hardy’s inequality in manifolds, see [Carron 1997; Ancona 1990]. In many situations the first item implies the second, as we now see:

Corollary. *Suppose that M satisfies the Sobolev inequality Assumption B’ with $\alpha = n$ and that $|B(x, r)|$ is comparable with r^n , for $n > 2$, meaning that there exists $a > 0$ such that $a^{-1}r^n \leq |B(x, r)| \leq ar^n$ for all $x \in M$ and $r > 0$. Then the gradient bound (1–2) holds if*

$$\text{Ric}(x) \geq -\frac{\varepsilon}{1 + d(x)^{2+\delta}}$$

for a sufficiently small $\varepsilon > 0$ and $\delta > 0$. Here ε depends only on b_1, B_1, δ, a, n .

In particular, if M is a small compact perturbation of \mathbb{R}^n , where $n \geq 3$, Equation (1–2) holds.

Likewise, (1–2) holds if M is a small compact perturbation of a manifold of dimension at least 3 that has nonnegative Ricci curvature and maximum volume growth (that is, $|B(x, r)| \sim r^n$).

Proof. Since M satisfies the extra conditions in the volume of geodesic balls in the corollary, it is easy to see that

$$N(V) = \sup_{x \in M} \int_0^\infty \int_M \frac{e^{-d(x,y)^2/t}}{|B(x, \sqrt{t})|} V(y) dy dt \leq c \sup_{x \in M} \int_M \frac{d(x, y)^2}{|B(x, d(x, y))|} V(y) dy.$$

Write

$$K(V) \equiv \sup_{x \in M} \int_M \frac{d(x, y)^2}{|B(x, d(x, y))|} V(y) dy.$$

By direct calculation we see that $V(x) \leq \varepsilon/(1 + d(x)^{2+\delta})$ implies

$$\begin{aligned} K(V) &\leq \sup_{x \in M} \int_{d(x,y) \geq d(y)/2} \frac{d(x, y)^2}{|B(x, d(x, y))|} V(y) dy \\ &\quad + \sup_{x \in M} \int_{d(x,y) \leq d(y)/2} \frac{d(x, y)^2}{|B(x, d(x, y))|} V(y) dy \\ &\leq C\varepsilon. \end{aligned}$$

Next, given $\phi \in C_0^\infty(B(x_0, r))$,

$$\int_M V(x)\phi^2(x) dx \leq \left(\int_M V^{n/2}(x) dx \right)^{2/n} \left(\int_M \phi^{2n/(n-2)}(x) dx \right)^{(n-2)/n}.$$

By the Sobolev inequality,

$$\int_M V(x)\phi^2(x) dx \leq S_0 \left(\int_M V^{n/2}(x) dx \right)^{2/n} \left(\int_M |\nabla\phi|^2(x) dx + \frac{k}{r^2} \int_M \phi^2(x) dx \right).$$

A simple calculation then shows that

$$\int_M V(x)\phi^2(x) dx \leq C\varepsilon S_0 \left(\int_M |\nabla\phi|^2(x) dx + \frac{k}{r^2} \int_M \phi^2(x) dx \right).$$

Hence all the conditions of the [Main Theorem](#) are satisfied when ε is sufficiently small.

There are plenty of examples of such manifolds due the stability of $G(x, t; y, 0)$ under perturbation of the metric. For instance, let M be \mathbb{R}^n equipped with a metric coming from a small perturbation of the Euclidean metric. Here $n \geq 3$. Then by standard results B_1 and b_1 can be chosen to be close to $1/(2\sqrt{\pi})^n$ and $1/4$, the Euclidean constants. At the same time, the nonpositive part of the Ricci curvature can be arbitrarily small. Therefore the above quantity $N(V)$ can be arbitrarily small while ε_0 , depending only on B_1, b_1 and the doubling constant, is bounded away from zero. Thus $N(V) < \varepsilon_0$.

The last statement in the corollary is proved in the same manner. □

Remark 2. The constant ε_0 and C_0 in the [Main Theorem](#) can be estimated explicitly, as indicated in the proof. The assumption $V \in L^\infty(M)$ is not necessary. But we will not seek the full generality.

It is not clear whether the current method can show the Li–Yau gradient estimate on caloric functions [[1986](#)] is stable.

2. Proof of the [Main Theorem](#)

Let u be a positive solution of $\Delta u = 0$. The proof is carried out in several steps.

2.1. Derivation of an equation for $\log u$. In this initial step we follow [[Yau 1975](#)] and [[Cheng and Yau 1975](#)]. Set $f = \log u$; then $\Delta f = -|\nabla f|^2$. Define

$$F \equiv |\nabla f|^2.$$

By Bochner’s identity, F obeys

$$(2-1) \quad \Delta F \geq -2\nabla f \nabla F + \frac{2}{n} F^2 - 2VF.$$

Here $V = V(x)$ is the absolute value of the negative part of the lowest eigenvalue of $\text{Ric}(x)$.

2.2. Preparing the ground for a bound on F . From here on our proof is different from those in [Yau 1975] and [Cheng and Yau 1975], where the maximum principle was used. We will use certain integral estimates motivated by the De Giorgi–Nash–Moser theory on linear elliptic and parabolic equations.

The idea is to convert (2–1) into a linear inequality and prove that the heat kernel of the corresponding operator satisfies a global Gaussian upper bound when $N(V)$, defined in Assumption C, is sufficiently small. Then one can use the local representation formula for solutions and the Hölder inequality to derive an L^∞ bound for F . However, it is not clear that the linear operator

$$-\Delta - 2\nabla f \nabla + \frac{2}{n} F - 2V$$

associated with (2–1) is positive definite. Hence there is no hope of proving a global Gaussian upper bound for the corresponding heat kernel.

To overcome the difficulty, we consider the function

$$w = F^m, \quad \text{where } m = 5n.$$

By direct calculation, one easily finds using (2–1) that

$$\Delta w + 2\nabla f \nabla w - 10Fw + 10nVw \geq 0.$$

We define the operators L_1 and L_2 by

$$(2-2) \quad \begin{aligned} L_1 &= \Delta + 2\nabla f \nabla - 10F, \\ L_2 &= \Delta + 2\nabla f \nabla - 10F + 10nV, \end{aligned}$$

and their corresponding heat kernels by G_1 and G_2 . We will eventually prove that G_2 has a global Gaussian upper bound when $N(V)$ is small. More importantly, the coefficients in the Gaussian upper bound should be independent of f , which is not a fixed function. This is achieved by exploiting the special structure of the operator L_1 .

First, we have to show that G_1 satisfies a global Gaussian upper bound. This is the longest part of the proof, and we do it in the next two sections.

2.3. A mean-value inequality. Set

$$L_3 = \Delta + 2\nabla f \nabla - 8F.$$

We will prove an L^2 mean-value inequality for positive solutions of

$$(2-3) \quad L_3 w - w_t \geq 0.$$

Let $Q_r(x, t)$ (or simply Q_r) denote $B(x, r) \times [t - r^2, t]$. The inequality to be proved is

$$(2-4) \quad \sup_{Q_{r/2}(x,t)} w^2 \leq \frac{C_7}{B(x, r)r^2} \|w\|_{L^2(Q_r(x,t))}^2.$$

Notice that the order-zero term in L_3 is $-8Fw$ instead of $-10Fw$ in L_1 . This makes L_3 a “bigger” operator than L_1 .

Fix $\sigma > 1$. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a cutoff function supported in $(t - (\sigma r)^2, t)$ and such that $\eta(s) = 1$ for $s \in [t - r^2, t]$ and $|\eta'| \leq 2/((\sigma - 1)r)^2$ for $0 \leq \eta \leq 1$. Similarly, let $\phi : \mathbb{R}^n \rightarrow [0, 1]$ be a function supported in $B(x, \sigma r)$ and such that $\phi(y) = 1$ for $y \in B(x, r)$ and

$$|\nabla\phi| \leq \frac{A}{(\sigma - 1)r}, \quad \text{with } A > 0.$$

Then set $\psi(y, s) = \phi(y)\eta(s)$. Using $w\psi^2$ as a test function in (2-3), one obtains

$$\int_{Q_{\sigma r}} (\Delta w - 2\nabla f \nabla w - 8Fw - \partial_s w) w \psi^2 dy ds \geq 0.$$

Integrating by parts, one deduces

$$(2-5) \quad \int_{Q_{\sigma r}} \nabla(w\psi^2)\nabla w dy ds \leq \int_{Q_{\sigma r}} 2\nabla f \nabla w(w\psi^2) dy ds - \int_{Q_{\sigma r}} 8Fw^2 dy ds - \int_{Q_{\sigma r}} (\partial_s w) w \psi^2 dy ds.$$

By direct calculation,

$$\begin{aligned} \int_{Q_{\sigma r}} \nabla(w\psi^2)\nabla w dy ds &= \int_{Q_{\sigma r}} \nabla((w\psi)\psi)\nabla w dy ds \\ &= \int_{Q_{\sigma r}} (\nabla(w\psi)(\nabla(w\psi) - (\nabla\psi)w) + w\psi\nabla\psi\nabla w) dy ds \\ &= \int_{Q_{\sigma r}} (|\nabla(w\psi)|^2 - |\nabla\psi|^2 w^2) dy ds. \end{aligned}$$

Substituting this in (2-5), we obtain

$$(2-6) \quad \begin{aligned} \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds &\leq \int_{Q_{\sigma r}} 2\nabla f \nabla w(w\psi^2) dy ds - \int_{Q_{\sigma r}} 8Fw^2 dy ds \\ &\quad - \int_{Q_{\sigma r}} (\partial_s w) w \psi^2 dy ds + \int_{Q_{\sigma r}} |\nabla\psi|^2 w^2 dy ds. \end{aligned}$$

Next, notice that

$$\begin{aligned} \int_{Q_{\sigma r}} (\partial_s w) w \psi^2 dy ds &= \frac{1}{2} \int_{Q_{\sigma r}} (\partial_s w^2) \psi^2 dy ds \\ &= - \int_{Q_{\sigma r}} w^2 \phi^2 \eta \partial_s \eta dy ds + \frac{1}{2} \int_{B(x, \sigma r)} w^2(y, t) \phi^2(y) dy. \end{aligned}$$

Combining this with (2-6), we see that

$$\begin{aligned} (2-7) \quad & \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds + \frac{1}{2} \int_{B(x, \sigma r)} w^2(y, t) \phi^2(y) dy \\ & \leq \int_{Q_{\sigma r}} (|\nabla\psi|^2 + \eta \partial_s \eta) w^2 dy ds + \int_{Q_{\sigma r}} 2\nabla f \nabla w (w\psi^2) dy ds \\ & \quad - \int_{Q_{\sigma r}} 8Fw^2 dy ds. \end{aligned}$$

The first term on the right in (2-7) is already in good shape. We estimate the second as follows:

$$\begin{aligned} & \int_{Q_{\sigma r}} 2\nabla f (\nabla w) (w\psi^2) dy ds \\ & = 2 \int_{Q_{\sigma r}} \nabla f (\nabla(w\psi) - w\nabla\psi) w\psi dy ds \\ & = 2 \int_{Q_{\sigma r}} \nabla f \nabla(w\psi) w\psi dy ds - 2 \int_{Q_{\sigma r}} (w\psi \nabla f) w\nabla\psi dy ds \\ & \leq \frac{1}{2} \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds + 4 \int_{Q_{\sigma r}} |\nabla f|^2 (w\psi)^2 dy ds \\ & \quad + \int_{Q_{\sigma r}} |\nabla f|^2 (w\psi)^2 dy ds + \int_{Q_{\sigma r}} w^2 |\nabla\psi|^2 dy ds. \end{aligned}$$

Recall that $|\nabla f|^2 = F$. Hence this inequality becomes

$$\begin{aligned} & \int_{Q_{\sigma r}} 2\nabla f (\nabla w) (w\psi^2) dy ds \\ & \leq \frac{1}{2} \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds + 5 \int_{Q_{\sigma r}} F (w\psi)^2 dy ds + \int_{Q_{\sigma r}} w^2 |\nabla\psi|^2 dy ds. \end{aligned}$$

Substituting this on the right-hand side of (2-7), we deduce that

$$\begin{aligned} \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds + \frac{1}{2} \int_{B(x, \sigma r)} w^2(y, t) \phi^2(y) dy \\ \leq 2 \int_{Q_{\sigma r}} (2|\nabla\psi|^2 + \eta \partial_s \eta) w^2 dy ds. \end{aligned}$$

Thus the terms containing F drop out. For later use we remark that if w satisfies $L_3 w - w_t + h \geq 0$ for some function h , we obtain, using exactly the same calculation,

$$(2-8) \quad \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds + \frac{1}{2} \int_{B(x, \sigma r)} w^2(y, t) \phi^2(y) dy \\ \leq 2 \int_{Q_{\sigma r}} (2|\nabla\psi|^2 + \eta\partial_s\eta) w^2 dy ds + \int_{Q_{\sigma r}} h w \psi^2 dy ds.$$

By direct calculation it is easy to see that, for any $p > 1$,

$$L_3 w^p = \Delta w^p + 2\nabla f \nabla w^p - 8F w^p - (w^p)_t \\ \geq p(p-1)|\nabla w|^2 w^{p-2} + 8F(p-1)w^p \geq 0.$$

Hence, by repeating the argument above, we obtain, for any $p > 1$,

$$\int_{Q_{\sigma r}} |\nabla(w^p\psi)|^2 dy ds + \frac{1}{2} \int_{B(x, \sigma r)} (w^p(y, t))^2 \phi^2(y) dy \\ \leq 2 \int_{Q_{\sigma r}} (2|\nabla\psi|^2 + \eta\partial_s\eta) (w^p)^2 dy ds.$$

Therefore

$$(2-9) \quad \int_{Q_{\sigma r}} |\nabla(w^p\psi)|^2 dy ds + \frac{1}{2} \int_{B(x, \sigma r)} (w^p(y, t))^2 \phi^2(y) dy \\ \leq \frac{C}{r^2} \int_{Q_{\sigma r}} (w^p)^2 dy ds.$$

Equation (2-9) and the Sobolev inequality lead to the following mean-value inequality via Moser iteration:

$$(2-10) \quad \sup_{Q_r} w^2 \leq \frac{B}{|Q_r|} \int_{Q_{2r}} w^2 dy ds.$$

By keeping track of the constants in the computation, we can show that the constant B is independent of f or F . For the sake of completeness we give a sketch of the proof. Hölder’s inequality,

$$\iint (w^p\psi)^{2(1+(2/\alpha))} dy ds \leq \int \left(\int (w^p\psi)^{2\alpha/(\alpha-2)} dy \right)^{(\alpha-2)/\alpha} \left(\int (w^p\psi)^2 \right)^{2/\alpha} ds.$$

Using the Sobolev inequality of Assumption B', one obtains

$$\iint (w^p\psi)^{2(1+(2/\alpha))} dy ds \leq \\ S_0 |B(x, r)|^{-2/\alpha} \sup_{s \in [t-\sigma r^2, t]} \left(\int (w^p\psi)^2 dy \right)^{2/\alpha} \iint (r^2 |\nabla(w^p\psi)|^2 + (w^p\psi)^2) dy ds.$$

This, together with (2–9), implies that

$$(2-11) \quad \int_{Q_{\sigma' r}(x,t)} w^{2p\theta} \leq \left(C_5 S_0 \lambda(r)^{-1} \int_{Q_{\sigma r}(x,t)} w^{2p} \right)^\theta,$$

where $\theta = 1 + 2/\alpha$, $\tau = \sigma - \sigma'$ and

$$\lambda(r) = |B(x, r)|^{2/(2+\alpha)} (r\tau)^{4/(2+\alpha)}.$$

We now set

$$\tau_i = 2^{-i-1}, \quad \sigma_0 = 1, \quad \sigma_i = \sigma_{i-1} - \tau_i = 1 - \sum_1^i \tau_j, \quad p = \theta^i.$$

Inequality (2–11) then yields

$$\int_{Q_{\sigma_{i+1}}(x,t)} w^{2\theta^{i+1}} \leq C \left(C_6^{i+1} \lambda(r)^{-1} \int_{Q_{\sigma_i r}(x,t)} w^{2\theta^i} \right)^\theta.$$

After iteration, this implies

$$\left(\int_{Q_{\sigma_{i+1}}(x,t)} w^{2\theta^{i+1}} \right)^{\theta^{-i-1}} \leq C^{\Sigma\theta^{-j-1}} C_6^{-\Sigma(j+1)\theta^{-j-1}} (\lambda(r)^{-1})^{\Sigma\theta^{-j}} \int_{Q_r(x,t)} w^2,$$

where j ranges from 0 to i . Letting i go to ∞ and noticing that $\sum_{j=0}^\infty \theta^{-j} = (\alpha + 2)/2$, we arrive at

$$\sup_{Q_{r/2}(x,t)} w^2 \leq \frac{C_7}{B(x, r)r^2} \|w\|_{L^2(Q_r(x,t))}^2.$$

This proves the mean-value inequality for w satisfying $L_3 w - w_t \geq 0$.

2.4. A Gaussian upper bound for G_1 . The proof of the upper bound is done by modifying the standard method due to E. B. Davies [1989]. To prove a bound that is independent of f or F , we have to use the special structure of the operator L_1 .

For a fixed $\lambda \in \mathbb{R}$ and a fixed bounded function ψ such that $|\nabla\psi| \leq 1$, we write

$$q(y) = e^{\lambda\psi(y)} \int G_1(y, s; z, 0) e^{-\lambda\psi(z)} h(z) dz,$$

where h is a smooth compactly supported function. Then

$$\begin{aligned}
 (2-12) \quad \partial_s \|q\|_2^2 &= 2 \int_M q(y, s) \partial_s q(y, s) \\
 &= 2 \int_M e^{\lambda\psi(y)} q(y, s) \int_M \partial_s G_1(y, s; z, 0) e^{-\lambda\psi(z)} h(z) dz dy \\
 &= 2 \int_M e^{\lambda\psi(y)} q(y, s) \int_M (\Delta_y G_1 + 2\nabla_y f \nabla_y G_1 - 10FG_1) \\
 &\qquad\qquad\qquad \times e^{-\lambda\psi(z)} h(z) dz dy \\
 &= 2I_1 + 4I_2 - 20I_3,
 \end{aligned}$$

where we have set

$$\begin{aligned}
 I_1 &= \int_M \int_M e^{\lambda\psi(y)} q(y, s) \Delta_y G_1 e^{-\lambda\psi(z)} h(z) dz dy, \\
 I_2 &= \int_M \int_M e^{\lambda\psi(y)} q(y, s) \nabla_y f \nabla_y G_1 e^{-\lambda\psi(z)} h(z) dz dy, \\
 I_3 &= \int_M \int_M e^{\lambda\psi(y)} q(y, s) FG_1 e^{-\lambda\psi(z)} h(z) dz dy.
 \end{aligned}$$

Integrating by parts and employing standard arguments, we get

$$(2-13) \quad I_1 \leq - \int_M |\nabla q(y, s)|^2 dy + c\lambda^2 \int_M q^2(y, s) dy.$$

Next observe that

$$\begin{aligned}
 I_2 &= \int_M e^{\lambda\psi(y)} q(y, s) \nabla_y f \int_M \nabla_y G_1 e^{-\lambda\psi(z)} h(z) dz dy \\
 &= \int_M e^{\lambda\psi(y)} q(y, s) \nabla_y f \nabla_y \int_M G_1(y, s; z, 0) e^{-\lambda\psi(z)} h(z) dz dy \\
 &= \int_M e^{\lambda\psi(y)} q(y, s) \nabla_y f \nabla_y (e^{-\lambda\psi(y)} q(y, s)) dy \\
 &= \frac{1}{2} \int_M e^{2\lambda\psi(y)} \nabla_y f \nabla_y ((e^{-\lambda\psi(y)} q(y, s))^2) dy \\
 &= -\lambda \int_M q^2(y, s) \nabla_y \psi \nabla_y f dy - \frac{1}{2} \int_M q^2(y, s) \Delta_y f dy.
 \end{aligned}$$

Since $|\nabla\psi| \leq 1$ and $\Delta_y f = -|\nabla f|^2 = -F$, it follows that

$$I_2 \leq \frac{\lambda^2}{2} \int_M q^2(y, s) dy + \frac{1}{2} \int_M q^2(y, s) |\nabla_y f|^2 dy + \frac{1}{2} \int_M q^2(y, s) |\Delta_y f| dy;$$

that is,

$$I_2 \leq \frac{\lambda^2}{2} \int_M q^2(y, s) dy + \int_M q^2(y, s) F dy.$$

Notice also that

$$I_3 = \int_M q^2(y, s) F dy.$$

Substituting (2–13) and the last two relations in (2–12), we see that the terms containing F are negative. Hence

$$\partial_s \|q(\cdot, s)\|_2^2 \leq c_0 \lambda^2 \|q(\cdot, s)\|_2^2,$$

which implies

$$\|q(\cdot, s)\|_2^2 \leq e^{c_0 \lambda^2 s} \|h\|_2^2.$$

Now consider the function

$$u(y, s) = e^{-\lambda \psi(y)} q(y, s),$$

which is a solution to $L_1 u - u_s = 0$ in $M \times (0, \infty)$. Hence

$$L_3 u - u_s = L_1 u - u_s + 2F u \geq 0,$$

where L_3 is defined at the start of Section 2.3. By the mean-value inequality of (2–10) with $Q_{\sqrt{t}/2}(x, t) = B(x, \sqrt{t}/2) \times (3t/4, t)$, we obtain

$$u(x, t)^2 \leq \frac{C}{|Q_{\sqrt{t}/2}(x, t)|} \int_{3t/4}^t \int_{B(x, \sqrt{t}/2)} u^2.$$

It follows that

$$\begin{aligned} e^{2\lambda \psi(x)} u(x, t)^2 &\leq e^{2\lambda \psi(x)} \frac{C}{|Q_{\sqrt{t}/2}(x, t)|} \int_{3t/4}^t \int_{B(x, \sqrt{t}/2)} u^2 \\ &= \frac{C}{|Q_{\sqrt{t}/2}(x, t)|} \int_{3t/4}^t \int_{B(x, \sqrt{t}/2)} e^{2\lambda(\psi(x) - \psi(z))} q^2 \\ &\leq e^{2\lambda \sqrt{t}} \frac{C}{|B(x, \sqrt{t})|} e^{c_0 \lambda^2 t} \|h\|_2^2. \end{aligned}$$

Taking the supremum over all $h \in L^2(B(y, \sqrt{t}))$ with $\|h\| = 1$, we find that

$$e^{2\lambda(\psi(x) - \psi(y))} \int_{B(y, \sqrt{t}/2)} G_1(x, t; z, 0)^2 dz \leq C e^{4\lambda \sqrt{t} + c_0 \lambda^2 t} \frac{1}{|B(x, \sqrt{t})|}.$$

Using the mean-value inequality on the second entries of the heat kernel G_1 backward in time in the cube $B(y, \sqrt{t}/2) \times [0, t/4]$, we have

$$\begin{aligned} G_1(x, t; y, 0)^2 &\leq \frac{C}{|Q_{\sqrt{t}/2}(y, t)|} \int_0^{t/4} \int_{B(y, \sqrt{t}/2)} G_1(x, t; z, s)^2 dz ds \\ &\leq \frac{C}{|B(x, \sqrt{t})| |B(y, \sqrt{t})|} \exp(4\lambda\sqrt{t} + c_0\lambda^2 t - 2\lambda(\psi(x) - \psi(y))) \end{aligned}$$

Here we remark that the second entries of G_1 satisfies the conjugate equation of $L_1 u - u_s = 0$. That is, if $v(z, s) = G_1(x, t; z, s)$, then

$$\Delta v - 2\nabla f \nabla v - 2\Delta f v - 10Fv + v_s = 0.$$

Recall that $\Delta f = -F$. Hence v satisfies

$$L_3 v + v_s = \Delta v - 2\nabla f \nabla v - 8Fv + v_s = 0.$$

From Section 2.3 it is clear that the mean-value inequality still holds on the backward parabolic cube.

Choosing $\lambda = d(x, y)/(c_0 t)$ and ψ such that $\psi(x) - \psi(y) = d(x, y)$, we reach

$$G_1(x, t; y, 0)^2 \leq \frac{C}{|B(x, \sqrt{t})| |B(y, \sqrt{t})|} \exp\left(-\frac{d(x, y)^2}{2c_0 t}\right).$$

This proves the Gaussian upper bound for G_1 .

2.5. A Gaussian upper bound of G_2 by perturbation. Using the bound for G_1 and a perturbation argument in [Zhang 1997], we will prove a Gaussian upper bound for G_2 when $N(V)$ is small.

By Duhamel’s formula,

$$G_2(x, t; y, 0) = G_1(x, t; y, 0) + 10n \int_0^t \int_M G_1(x, t; z, \tau) V(z) G_2(z, \tau; y, 0) dz d\tau.$$

From Section 2.4 we know that, for some $c_1, C_1 > 0$,

$$G_1(x, t; z, \tau) \leq \frac{C_1}{|B(x, \sqrt{t-\tau})|} \exp\left(-\frac{c_1 d(x, z)^2}{t-\tau}\right).$$

Also, by standard perturbation argument (since V is bounded), there is $M_\tau > 0$ such that

$$G_2(z, \tau; y, 0) \leq \frac{M_\tau}{|B(z, \sqrt{\tau})|} \exp\left(-\frac{c_1 d(z, y)^2}{\tau}\right) \leq \frac{M_\tau}{|B(z, \sqrt{\tau})|} \exp\left(-\frac{c_1 d(z, y)^2}{2\tau}\right).$$

We need to prove that M_τ can be chosen independent of time.

Therefore

$$G_2(x, t; y, 0) \leq \frac{C_1 e^{-c_1 d(x,y)^2/t}}{|B(x, \sqrt{t})|} + C_1 \int_0^t \int_M \frac{e^{-c_1 d(x,z)^2/(t-\tau)}}{|B(x, \sqrt{t-\tau})|} V(z) \frac{M_\tau e^{-c_1 d(z,y)^2/(2\tau)}}{|B(z, \sqrt{\tau})|} dz d\tau.$$

Let m_τ be the minimum of the constants M_τ such that

$$G_2(z, \tau; y, 0) \leq \frac{m_\tau}{|B(z, \sqrt{\tau})|} e^{-c_1 d(z,y)^2/(2\tau)}$$

for all $\tau \in (0, t]$ and $y, z \in M$. Then

$$G_2(x, t; y, 0) \leq \frac{C_1 e^{-c_1 d(x,y)^2/t}}{|B(x, \sqrt{t})|} + C_1 m_t \int_0^t \int_M \frac{e^{-c_1 d(x,z)^2/(t-\tau)}}{|B(x, \sqrt{t-\tau})|} V(z) \frac{e^{-c_1 d(z,y)^2/(2\tau)}}{|B(z, \sqrt{\tau})|} dz d\tau.$$

By [Zhang 1997, Lemma 4.1, p. 1003], there exists a constant c_5 , depending only on the doubling constant ν , such that

$$\int_0^t \int_M \frac{e^{-c_1 d(x,z)^2/(t-\tau)}}{|B(x, \sqrt{t-\tau})|} V(z) \frac{e^{-c_1 d(z,y)^2/(2\tau)}}{|B(z, \sqrt{\tau})|} dz d\tau \leq c_5 M(V) \frac{1}{|B(x, \sqrt{t})|} e^{-c_1 d(x,y)^2/(2t)},$$

where

$$M(V) \equiv \sup_{x \in M} \int_0^\infty \int_M \frac{e^{-c_1 d(x,z)^2/(2t)}}{|B(x, \sqrt{t})|} V(z) dz dt.$$

In the lemma quoted above, the constant in the exponential term of $M(V)$ was not given explicitly. However, by tracking the proof, one immediately concludes that the coefficient $-c_1/2$ above works. We caution that it is not clear that one can choose the original constant $-c_1$, except in the Euclidean case. We mention that the bound above for the integral is the parabolic counterpart of the basic inequality, for $n \geq 3$,

$$\int_{\mathbb{R}^n} \frac{1}{|x-z|^{n-2}} |V(z)| \frac{1}{|z-y|^{n-2}} dz \leq C \sup_w \int_{\mathbb{R}^n} \frac{|V(y)|}{|y-w|^{n-2}} \frac{1}{|x-y|^{n-2}} dy,$$

which can be found in many places, including, for instance, [Simon 1982].

Scaling the time variable suitably and use the volume doubling property, we see that

$$M(V) \leq cN(V) = c \sup_{x \in M} \int_0^\infty \int_M \frac{e^{-d(x,z)^2/t}}{|B(x, \sqrt{t})|} V(z) dz dt.$$

Hence

$$G_2(x, t; y, 0) \leq \frac{C_1 e^{-c_1 d(x,y)^2/t}}{|B(x, \sqrt{t})|} + C_1 c_5 m_t N(V) \frac{1}{|B(x, \sqrt{t})|} e^{-c_1 d(x,y)^2/(2t)}.$$

Now it follows that

$$G_2(x, t; y, 0) \leq (C_1 + C_1 c_5 m_t N(V)) \frac{1}{|B(x, \sqrt{t})|} e^{-c_1 d(x,y)^2/(2t)}.$$

By the definition of m_t we have $m_t \leq C_1 + C_1 c_5 m_t N(V)$. Hence, if $N(V) < 1/(C_1 c_5)$, then

$$m_t \leq \frac{C_1}{1 - C_1 c_5 N(V)},$$

for all $t > 0$. Therefore

$$G_2(x, t; y, 0) \leq \frac{C_1}{1 - C_1 c_5 N(V)} \frac{1}{|B(z, \sqrt{t})|} e^{-c_1 d(x,y)^2/(2t)}$$

This proves the global upper bound for $G_2(x, t; y, 0)$.

2.6. An L^2 estimate of $F = |\nabla \log u|^2$. Let $\phi = \phi(x)$ be a smooth cutoff function defined in $B(x, 2r)$ such that $0 \leq \phi \leq 1$, $\phi(y) = 1$ in $B(x, r)$ and $\phi(y) = 0$ in $B(x, 2r)^c$. Then

$$\begin{aligned} \int_{B(x,2r)} F \phi^2 dy &= \int_{B(x,2r)} \frac{\nabla u \nabla u}{u^2} \phi^2 dy = - \int_{B(x,2r)} u \operatorname{div} \left(\frac{\nabla u}{u^2} \phi^2 \right) dy \\ &= - \int_{B(x,2r)} u \frac{\Delta u}{u^2} \phi^2 dy - \int_{B(x,2r)} u \nabla u \nabla \left(\frac{\phi^2}{u^2} \right) dy \\ &= 2 \int_{B(x,2r)} \frac{u \nabla u \nabla u}{u^3} \phi^2 dy - 2 \int_{B(x,2r)} u \nabla u \frac{\phi \nabla \phi}{u^2} dy \\ &= 2 \int_{B(x,2r)} F \phi^2 dy - 2 \int_{B(x,2r)} u \nabla u \frac{\phi \nabla \phi}{u^2} dy. \end{aligned}$$

Therefore

$$\int_{B(x,2r)} \frac{|\nabla u|^2}{u^2} \phi^2 dy \leq 2 \int_{B(x,2r)} \frac{|\nabla u|}{u} \phi |\nabla \phi| dy.$$

Hence

$$\int_{B(x,2r)} \frac{|\nabla u|^2}{u^2} \phi^2 dy \leq 4 \int_{B(x,2r)} |\nabla \phi|^2 dy.$$

This implies that

$$(2-14) \quad \int_{B(x,r)} F(y) dy = \int_{B(x,r)} \frac{|\nabla u|^2}{u^2} dy \leq 4 \frac{|B(x, 2r)|}{r^2}.$$

2.7. A mean-value inequality for solutions of $L_2 w - w_t \geq 0$. (See (2–2) for the definition of L_2 .)

Let ψ be the smooth cutoff function defined on page 385, with $\sigma = 2$. Take r there to be $r/2$. Since ψ is supported in $Q_r(x, t)$, we know that

$$\begin{aligned} \Delta(w\psi) + 2\nabla f \nabla(w\psi) - 10F(w\psi) + 10nV(w\psi) - (w\psi)_t \\ \geq 2(\nabla f \nabla \psi)w + (\Delta\psi)w - w\psi_t + 2\nabla\psi \nabla w. \end{aligned}$$

Since G_2 is the fundamental solution of the left-hand side, we have

$$\begin{aligned} w(x, t) \leq -2 \int_{Q_r(x, t)} G_2(x, t; y, s) (\nabla f \nabla \psi) w \, dy \, ds \\ - \int_{Q_r(x, t)} G_2(x, t; y, s) ((\Delta\psi)w - w\psi_t) \, dy \, ds \\ - \int_{Q_r(x, t)} G_2(x, t; y, s) 2\nabla\psi \nabla w \, dy \, ds. \end{aligned}$$

After integration by parts, this becomes, setting $v = v(y, s) = G_2(x, t; y, s)$:

$$\begin{aligned} (2-15) \quad w(x, t) \leq -2 \int_{Q_r(x, t)} v \nabla f \nabla \psi w \, dy \, ds + \int_{Q_r(x, t)} v w \psi_t \, dy \, ds \\ + \int_{Q_r(x, t)} \nabla_y v \nabla \psi w \, dy \, ds - \int_{Q_r(x, t)} v \nabla \psi \nabla w \, dy \, ds \\ \equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We estimate each integral separately.

Since (x, t) is bounded away from the supports of the functions $\nabla\psi$ and ψ_t by a parabolic distance of r , we have

$$(2-16) \quad v \leq \frac{C}{|B(x, r)|}$$

in all the integrals on the right-hand side of (2–15). Hence (2–15) implies

$$\begin{aligned} |J_1 + J_2| \leq \frac{C}{|B(x, r)|} \left(\int_{Q_r(x, t)} |\nabla f|^2 \, dy \, ds \right)^{1/2} \left(\int_{Q_r(x, t)} w^2 \, dy \, ds \right)^{1/2} \\ + \frac{C}{r^2 |B(x, r)|} \int_{Q_r(x, t)} w \, dy \, ds. \end{aligned}$$

Using $|\nabla f|^2 = F = |\nabla u|^2/u^2$, by (2–14), we deduce

$$\begin{aligned} w(x, t) \\ \leq \left(\frac{C}{r^2 |B(x, r)|} \int_{Q_r(x, t)} w^2 \, dy \, ds \right)^{1/2} + \frac{C}{r^2 |B(x, r)|} \int_{Q_r(x, t)} w \, dy \, ds + |J_3| + |J_4|. \end{aligned}$$

This shows, by Hölder’s inequality,

$$(2-17) \quad w(x, t) \leq \left(\frac{C}{r^2 |B(x, r)|} \int_{Q_r(x, t)} w^2 dy ds \right)^{1/2} + |J_3| + |J_4|.$$

In the next two steps we will find a bound for the last integrals in (2-17).

2.8. Controlling the remaining integrals. There remains to control the terms

$$J_3 = \int_{Q_r(x, t)} \nabla_y v \nabla \psi w dy ds$$

and

$$(2-18) \quad J_4 = \int_{Q_r(x, t)} v \nabla \psi \nabla w dy ds.$$

We deal with J_3 first. Using (2-16) and Hölder’s inequality, we get

$$(2-19) \quad |J_3| \leq \frac{C}{r} \left(\int_{Q_r(x, t) - Q_{r/2}(x, t)} |\nabla_y v|^2 dy ds \right)^{1/2} \left(\int_{Q_r(x, t)} w^2 dy ds \right)^{1/2}.$$

What remains is to estimate the first factor on the right-hand side.

Since G_2 is the heat kernel of L_2 , in other words the fundamental solution of the operator

$$\Delta + 2\nabla f \nabla - 10F + 10nV - \partial_s,$$

we know that $v = G_2(x, t; y, s)$ is a solution of the conjugate of $L_2 - \partial_s$, except at (x, t) :

$$\Delta v - 2\nabla f \nabla v - 2\Delta f v - 10Fv + 10nVv + v_s = 0.$$

Since $\Delta f = -F$, this becomes

$$(2-20) \quad \Delta v - 2\nabla f \nabla v - 8Fv + 10nVv + v_s = 0,$$

that is, $L_3 v + v_s + 10nVv = 0$.

Take a suitable cutoff function ψ_1 and use $\psi_1^2 v$ as a test function on (2-20) and $h = 10nVv$ in (2-8). We can follow the argument between (2-3) and (2-8) verbatim to obtain

$$\begin{aligned} & \int_{Q_r(x, t) - Q_{r/2}(x, t)} |\nabla_y v|^2 dy ds \\ & \leq \frac{C}{r^2} \int_{Q_{2r}(x, t) - Q_{r/4}(x, t)} v^2 dy ds + 10n \int_{Q_{4r}(x, t) - Q_{r/4}(x, t)} Vv^2 dy ds. \end{aligned}$$

It follows that (recalling that $v = G_2(x, t; y, s)$)

$$\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla_y v|^2 dy ds \leq \frac{C}{r^2} \int_{Q_{2r}(x,t)-Q_{r/4}(x,t)} v^2 dy ds + 10n \sup_{\substack{(y,s) \in \\ Q_{4r}(x,t)-Q_{r/4}(x,t)}} G_2(x, t; y, s) \int_{Q_{4r}(x,t)-Q_{r/4}(x,t)} V v dy ds.$$

Using the Gaussian bound on G_2 and the assumption on V we get

$$\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla_y G_2(x, t; y, s)|^2 dy ds \leq C \frac{1 + N(V)}{|B(x, r)|},$$

where we have used the inequality

$$\int_{Q_{4r}(x,t)-Q_{r/4}(x,t)} V G_2(x, t; y, s) dy ds \leq N(V),$$

which comes from the Gaussian upper bound of G_2 and rescaling in time; see [Assumption C](#). Inserting the L^2 estimate for the gradient of G_2 in (2–19) we obtain

$$(2-21) \quad |J_3| \leq \left(\frac{C}{r^2 |B(x, r)|} \int_{Q_{4r}(x,t)} w^2 dy ds \right)^{1/2}.$$

Next we turn our attention to the last remaining integral, defined in (2–18). Using (2–16) and Hölder’s inequality we reach

$$(2-22) \quad |J_4| \leq \frac{C}{r} \left(\int_{Q_r(x,t)-Q_{r/2}(x,t)} G_2(x, t; y, s)^2 dy ds \right)^{1/2} \left(\int_{Q_r(x,t)} |\nabla w|^2 dy ds \right)^{1/2} \leq C \left(\frac{1}{|B(x, r)|} \int_{Q_r(x,t)} |\nabla w|^2 dy ds \right)^{1/2}.$$

Recall that $L_2 w - w_s \geq 0$, and hence

$$L_3 w - w_s + 10n V w \geq 0,$$

where L_3 is defined at the start of [Section 2.3](#).

Taking $h = 10n V w$ and $\sigma = 2$ in (2–8), we obtain

$$\int_{Q_{2r}(x,t)} |\nabla(\psi_2 w)|^2 dy ds \leq \frac{C}{r^2} \int_{Q_{2r}} w^2 dy ds + 10n \int_{Q_{\sigma r}} V (w \psi_2)^2 dy ds.$$

Here ψ_2 is the cutoff function in (2–8) with $\sigma = 2$. By Assumption C for V we have, for a constant C' ,

$$\int_{Q_r(x,t)} |\nabla w|^2 dy ds \leq \int_{Q_{2r}(x,t)} |\nabla(\psi_2 w)|^2 dy ds \leq \frac{C'}{r^2} \int_{Q_{2r}} w^2 dy ds.$$

From this and (2–22), we get

$$(2-23) \quad J_4 \leq C \left(\frac{1}{r^2 |B(x,r)|} \int_{Q_{2r}(x,t)} w^2 dy ds \right)^{1/2}.$$

Substituting this and (2–21) to (2–17), we reach

$$w(x,t) \leq \left(\frac{C}{r^2 |B(x,r)|} \int_{Q_{4r}(x,t)} w^2 dy ds \right)^{1/2}.$$

2.9. Completion of the proof. Recall that $w = F^{5n}$ and that w is independent of time. Hence (2–21) becomes

$$(2-24) \quad F(x) \leq \left(\frac{C}{|B(x,r)|} \int_{B(x,2r)} F^{10n} dy \right)^{1/(10n)}$$

By a modification of a well known trick [Li and Schoen 1984], this inequality implies

$$(2-25) \quad F(x) \leq \frac{C}{|B(x,r)|} \int_{B(x,2r)} F dy$$

(In that paper it was shown that an L^2 mean-value inequality implies an L^1 mean-value inequality. Applying the same method, one can deduce (2–25) from (2–24) without difficulty.)

Combining (2–25) with (2–14), we have

$$\frac{|\nabla u|^2}{u^2} = F(x) \leq \frac{C}{|B(x,r)|} \int_{B(x,2r)} F dy \leq \frac{C}{r^2}.$$

This finishes the proof of the global gradient bound, and of the theorem.

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