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We study isolated singularities for a fully nonlinear elliptic PDE of subcritical type. This equation appears in conformal geometry when dealing with the k-curvature of a locally conformally-flat manifold. (The k-curvature generalizes scalar curvature.) We give a classification result: either the function is bounded near the singularity, or it has a specific asymptotic behavior.

1. Introduction

The study of singularities for the subcritical problem

$$-\Delta u = u^{\beta} \qquad \text{in } B\backslash\{0\}, \qquad \beta \in \Big(\frac{n}{n-2}, \frac{n+2}{n-2}\Big), \tag{1-1}$$

has received a lot of attention. In particular, Gidas and Spruck [1981] gave a classification result: a positive solution of (1-1) with a nonremovable singularity at zero must behave like

$$u(x) = \big(1+o(1)\big)\frac{c_0}{|x|^{2/(\beta-1)}} \qquad \text{near } x=0,$$

for some $c_0 = c_0(\beta, n)$. In this paper, we deal with a more general subcritical equation, of the form

$$\sigma_k(A^{g_v}) = v^{\alpha} \qquad \text{in } B\backslash\{0\}, \qquad \alpha > 0, \tag{1-2}$$

where $g_v = v^{-2}|dx|^2$ for $v > 0$ is a locally conformally-flat metric on the unit ball $B \subset \mathbb{R}^n$, with an isolated singularity at the origin. For a general metric g, the matrix A^g is given by $A^g = g^{-1}\tilde{A}^g$, where $\tilde{A}^g$ is the Schouten tensor

$$\tilde{A}^g_{ij} = \frac{1}{n-2}\Big(Ric_{ij} - \frac{1}{2(n-1)}R\, g_{ij}\Big),$$

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while *Ric* and R denote the Ricci tensor and the scalar curvature of g. In the metric g_v, the Schouten tensor becomes

$$A^{g_v} = v(D^2 v) - \tfrac{1}{2}|\nabla v|^2 I.$$

The curvatures σ_k are defined as symmetric functions of the eigenvalues $\lambda_1, \dots, \lambda_n$ of the $(1, 1)$-tensor A^g,

$$\sigma_k := \sigma_k(A^g) = \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

The scalar curvature is simply

$$\sigma_1 = \lambda_1 + \cdots + \lambda_n = \frac{1}{2(n-1)} R.$$

Problem (1-2) for $k = 1$ becomes the well known (1-1): if we write $u^{4/(n-2)} = v^{-2}$ and $1 + (n/2) - \beta(n-2)/2 = \alpha$, the two problems are equivalent. Note that the critical exponent is $\beta = (n+2)/(n-2)$, or $\alpha = 0$.

For a general k, we are dealing with a fully nonlinear equation of second order. The problem is elliptic in the positive cone

$$\Gamma_k^+ = \{v \mid \sigma_1(A^{g_v}), \dots, \sigma_k(A^{g_v}) > 0\},$$

but, in general, not uniformly elliptic. However, it still carries an "almost" divergence structure

$$m\,\sigma_m = v\,\partial_j\big(v_i T_{ij}^{m-1}\big) - n T_{ij}^{m-1} v_i v_j + \frac{n-m+1}{2}\sigma_{m-1}|\nabla v|^2,$$

where T_{ij}^m denotes the Newton tensor (2-1). This was explored in [González 2005b].

Our main result is a classification of the isolated singularities of (1-2):

Theorem 1.1. *Let $\alpha \in (0, k)$ and $n > 2(k+1)$. If v is a solution of*

$$\sigma_k(v) = v^\alpha \qquad \text{in } B\backslash\{0\}, \tag{1-3}$$

with $v > 0$, $v \in \Gamma_k^+$, and $v^{-1} \in \mathscr{C}^3(B\backslash\{0\})$, then

$$v^{-1}(x) \le \frac{C}{|x|^{2k/(2k-\alpha)}} \qquad \text{near } x = 0.$$

Theorem 1.2. *Let v be a solution of (1-3) for $\alpha \in \big(0, 2k/(k+1)\big)$ and $n > 2(k+1)$, with $v^{-1} \in \mathscr{C}^3(B\backslash\{0\})$. If the function v^{-1} is not bounded near the origin, then there exist $c_1, c_2 > 0$ such that*

$$\frac{c_1}{|x|^{2k/(2k-\alpha)}} \le v^{-1}(x) \le \frac{c_2}{|x|^{2k/(2k-\alpha)}} \qquad \text{near } x = 0.$$

The local behavior of singularities for the critical problem $\sigma_k(v) = 1$ has been addressed in [González 2005a]. There, we gave a sufficient condition for the function to be bounded near the singularity: the finiteness of the volume of the metric g_v (when $n > 2k$). The same result was obtained by Han [2004] for $n = 2k$. For the Laplacian problem ($k = 1$), a complete classification of solutions was obtained by Caffarelli, Gidas, and Spruck [1989].

At the time this paper was submitted, it was conjectured that a similar classification result was true also for σ_k, where $n > 2k$. This has now been proved [Li 2006]. In the case $n < 2k$, all the singularities are removable [Gursky and Viaclovsky 2005].

One of the motivations for the study of (1-1) is that it appears in the resolution of the Yamabe problem (for a very good survey, see [Lee and Parker 1987]). We can establish an analogous k-Yamabe problem: find the infimum over all the metrics $g_v = v^{-2} g_0$ with $v > 0$ of the functional

$$\mathcal{F}_k(g) = \frac{1}{\operatorname{vol}(g)^{(n-2k)/n}} \int_M \sigma_k(A^g)\, d\mathrm{vol}_g. \tag{1-4}$$

This functional was first introduced by Viaclovsky [2000], and it generalizes the Yamabe functional. Its Euler equation is precisely $\sigma_k(v) = 1$.

The global subcritical problem has been understood by Li and Li [2003]. Indeed, if v is a positive solution of

$$\sigma_k(v) = v^\alpha \quad \text{in } \mathbb{R}^n, \quad \alpha \geq 0,$$

with $v^{-1} \in \mathscr{C}^2(\mathbb{R}^n)$, then either v is constant, or $\alpha = 0$ and

$$v^{-1}(x) = \frac{a}{1 + b^2 |x - \bar{x}|^2}$$

for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b.

The methods of Gidas and Spruck [1981] for the problem with $k = 1$ can be generalized to our case. The key ingredient in the present paper is to understand the structure of σ_k and, in particular, to replace the traceless Ricci tensor by the traceless k-Newton tensor (2-2).

The paper is structured as follows: in Section 2 we give some properties of σ_k that will be crucial in the proofs. We use the divergence structure of σ_k (2-5), an inductive process (2-7), and the properties of the traceless Newton tensor (2-2).

In Section 3 we establish the expression that will allow us to obtain the necessary L^p estimates, through a generalization of an argument due to Obata and very successfully used by Chang, Gursky, and Yang [2002] and by Li and Li [2002]. In particular, we give a more refined formula (3-1) that is precisely the missing ingredient for the critical problem. The L^p estimates are in Section 4, while in the last two sections we prove the theorems.

Remark 1.3. We believe that the theorems are also true for $n = 2k+1$, but, as in the case of [Gidas and Spruck 1981], one needs different estimates in (4-12).

Remark 1.4. We make the regularity assumption $v^{-1} \in \mathscr{C}^3(B\backslash\{0\})$. However, many of the arguments use integral estimates and only require that v^{-1} is in some suitable Sobolev space; for instance, the whole of Section 4.

2. Algebraic properties of σ_k

For a general $n \times n$ matrix A, take its eigenvalues $\lambda_1, \dots, \lambda_n$ and construct the symmetric functions σ_k, as well as the *k-th Newton tensor*

$$T^k := \sigma_k - \sigma_{k-1}A + \cdots + (-1)^k A^k = \sigma_k I - T^{k-1}A \tag{2-1}$$

and the *traceless Newton tensor*

$$L^k := \frac{n-k}{n}\sigma_k I - T^k. \tag{2-2}$$

Remark 2.1. Take $\sigma_0 := 1$ and $T^0_{ij} := \delta_{ij}$. Although the standard notation for a (1,1)-tensor is A_i^j, we write both indices as subindices without risking confusion.

Lemma 2.2 [Gårding 1959; Reilly 1973].

(1) $(n-k)\sigma_k = \operatorname{trace} T^k$;

(2) $(k+1)\sigma_{k+1} = \operatorname{trace}(AT^k)$;

(3) $\operatorname{trace} L^k = 0$;

(4) *if* $\sigma_1, \dots, \sigma_k > 0$, *then* T^m *is positive definite for* $m = 1, \dots k-1$;

(5) *if* $\sigma_1, \dots, \sigma_k > 0$, *then* $\sigma_k \le C_{n,k}(\sigma_1)^k$.

In particular, if $A = A^{g_v}$ for $g_v = v^{-2}|dx|^2$, then the Schouten tensor becomes

$$A_{ij} = v_{ij}v - \frac{1}{2}|\nabla v|^2\delta_{ij}, \tag{2-3}$$

while the traceless Ricci tensor (strictly speaking, a constant multiple of the actual traceless Ricci tensor) is now

$$E_{ij} := L^1_{ij} = v v_{ij} - \frac{1}{n} v\,\Delta v\,\delta_{ij}. \tag{2-4}$$

Lemma 2.3 [Viaclovsky 2000]. *Let* $g_v = v^{-2}|dx|^2$. *The Newton tensor* T^m *for* $m \le n-1$ *is divergence-free with respect to this metric*; *that is*,

$$\sum_j \tilde{\partial}_j T^m_{ij} = 0, \qquad \textit{for all } i.$$

As a consequence,

$$\sum_j \tilde{\partial}_j L^m_{ij} = \frac{n-m}{n}\,\partial_i \sigma_m(A^{g_v}),$$

where $\tilde{\partial}_j$ *is the* j*-th covariant derivative with respect to the metric* g_v, *while* ∂_j *denotes the usual Euclidean derivative.*

The following two lemmas were proved in [González 2005b]. Expression (2-6) shows the 'almost' divergence structure of σ_m, while (2-7) is an inductive formula allowing us to handle the nondivergence terms (of order $m-1$) that appear in (2-6).

Lemma 2.4. *In this setting*,

$$\sum_j \partial_j T^m_{ij} = -(n-m)\sigma_m v_i v^{-1} + n\sum_i T^m_{ij} v_i v^{-1} \qquad \textit{for each } i; \tag{2-5}$$

$$m\,\sigma_m(A^{g_v}) = v\sum_{i,j}\partial_j(v_i T^{m-1}_{ij}) - n\sum_{i,j} T^{m-1}_{ij} v_i v_j + \frac{n-m+1}{2}\sigma_{m-1}|\nabla v|^2. \tag{2-6}$$

Lemma 2.5. *Let* U *be a domain in* $\mathbb{R}^n$, $v^{-1}\in\mathscr{C}^\infty(U)$, *and* $\varphi\in\mathscr{C}^\infty_0(U)$ *a smooth cutoff function. For any integers* $1\le s\le k\le n$ *and real number* γ,

$$\begin{aligned}\int_U \sum_{i,j} T^{k-s}_{ij} v_i v_j |\nabla v|^{2(s-1)}\varphi^{2k} v^{-\gamma}\,dx &= \Bigl(1+\frac{k-s}{2s}\Bigr)\int_U \sigma_{k-s}|\nabla v|^{2s}\varphi^{2k}v^{-\gamma}\,dx \\ &\quad + \frac{s+n+1-\gamma}{2s}\int_U\sum_{i,j} T^{k-s-1}_{ij} v_i v_j|\nabla v|^{2s}\varphi^{2k}v^{-\gamma}\,dx \\ &\quad - \frac{n-k+s+1}{4s}\int_U \sigma_{k-s-1}|\nabla v|^{2(s+1)}\varphi^{2k}v^{-\gamma}\,dx \\ &\quad + \frac{k}{s}\int_U\sum_{i,j} T^{k-s}_{ij} v_j\varphi_i|\nabla v|^{2(s-1)}\varphi^{2k-1}v^{1-\gamma}\,dx.\end{aligned} \tag{2-7}$$

In Section 3 we will need a similar formula for the traceless Newton tensor:

Corollary 2.6. *For any fixed* i,

$$\sum_j \partial_j(L^m_{ij}) = \frac{n-m}{n}\partial_i\sigma_k + n\sum_j L^m_{ij} v_i v^{-1} \tag{2-8}$$

Proof. Follows easily from (2-5) and (2-2). □

Lemma 2.7. *If* $\sigma_1,\dots,\sigma_m>0$ *and* $m\le n$, *then*

$$\|T^{m-1}_{ij}\| \le C_{m,n}\sigma_{m-1}.$$

Proof. Because of Lemma 2.2, T^{m-1} is positive definite. To estimate its norm we just need to look at its biggest eigenvalue. We are done, because

$$\operatorname{trace} T^{m-1} = (n-m)\sigma_{m-1}.$$ □

Lemma 2.8. *For any* $1 \le k \le n-1$, *if we have a metric* $g = v^{-2}|dx|^2$ *in the positive cone* Γ_k^+, *then*

$$\sum_{i,j} L_{ij}^k E_{ij} \ge 0,$$

with equality if and only if $E = 0$.

Proof. Because E_{ij} is traceless,

$$\sum_{i,j} L_{ij}^k E_{ij} = -\sum_{i,j} T_{ij}^k E_{ij}.$$

Using

$$E_{ij} = -\frac{1}{n}\sigma_1 \delta_{ij} + A_{ij}, \qquad (k+1)\sigma_{k+1} = T_{ij}^k A_{ij}, \qquad T_{ij}^k \delta_{ij} = (n-k)\sigma_k,$$

we see that

$$\sum_{i,j} T_{ij}^k E_{ij} = -\frac{n-k}{n}\sigma_k \sigma_1 + (k+1)\sigma_{k+1}.$$

The result follows by the general inequality for matrices in the positive cone Γ_k^+:

$$\sigma_{k+1} \le \frac{n-k}{n(k+1)}\sigma_1 \sigma_k,$$

with equality if and only if $E \equiv 0$. □

3. An Obata-type formula

Obata's original result [1962] states that, if we have a metric g on the unit sphere $\mathbb{S}^n$ that is conformal to the standard metric g_c and of constant scalar curvature, then $E \equiv 0$; that is, g is the standard metric g_c or is obtained from it by a conformal diffeomorphism of the sphere. His method uses crucially the traceless Ricci tensor $E_{ij} = v v_{ij} - (1/n)\, v \Delta v \delta_{ij}$ and the Bianchi identity $\nabla^i E_{ij} = \nabla^j R$. Indeed, his main step is to prove that

$$\int_{\mathbb{S}^n} \sum_{i,j} E_{ij} E_{ij} v^{-1} d\mathrm{vol}_{g_c} = 0,$$

and thus establish that g is an Einstein metric on $\mathbb{S}^n$.

This same argument was generalized for constant σ_k (instead of constant R) by Viaclovsky [2000], with the role of E played now by L^k and the Bianchi identity replaced by (2-8). If the metric is defined on $\mathbb{R}^n$ instead of $\mathbb{S}^n$, an analogous argument works; however, a cutoff function η is introduced and, in order to get the same conclusion, a careful estimate of the error terms is needed. We should also mention the work of Chang, Gursky, and Yang [2002; 2003] and of Li and Li [2002].

However, we are interested in the subcritical-problem approach of Gidas and Spruck [1981]; they have refined the computation of

$$0 \le \int_B \sum_{i,j} E_{ij} E_{ij} v^{-\delta} \eta \, dx = \cdots$$

for any $\delta \in \mathbb{R}$. The main result of this section is the corresponding refinement for σ_k:

Proposition 3.1. *Let $\alpha > 0$ and $n > 2k$. Take a solution v of $\sigma_k(v) = v^\alpha$ in U, with $v \in \Gamma_k^+$, $v > 0$, and $v^{-1} \in \mathscr{C}^3(U)$, where U is a domain in $\mathbb{R}^n$. Pick $\eta \in \mathscr{C}_0^\infty(U)$ and a big positive integer θ. There exist constants d_{k-s} such that*

$$\int_U \sum_{i,j} L_{ij}^k E_{ij} v^{-\delta} \eta^\theta + \Big(\frac{n-k}{n}\alpha - (1+n-\delta)\frac{k(n+2)}{2n}\Big)\int_U v^\alpha |\nabla v|^2 v^{-\delta}\eta^\theta + (1+n-\delta)\sum_{s=1}^{k} d_{k-s}\int_U \sigma_{k-s}|\nabla v|^{2(s+1)} v^{-\delta}\eta^\theta = E_1(\eta), \tag{3-1}$$

where

$$E_1(\eta) \lesssim \Big|\int_U \sum_{i,j} L_{ij}^k v_i \eta_j v^{1-\delta}\eta^{\theta-1}\Big| + \sum_{s=1}^{k}\Big|\int_U \sum_{i,j} T_{ij}^{k-s} v_j \eta_j |\nabla v|^{2s} v^{1-\delta}\eta^{\theta-1}\Big|. \tag{3-2}$$

In addition, if δ is smaller than but close enough to $n+1$, all the coefficients in front of the integrals in (3-1) are positive.

Proof. One uses the inductive method developed in [González 2005b; 2005a] and the properties of L^k. In view of (2-4), integrate over U to get

$$\int \sum_{i,j} L_{ij}^k E_{ij} v^{-\delta}\eta^\theta = \int \sum_{i,j} L_{ij}^k v_{ij} v^{1-\delta}\eta^\theta - \frac{1}{n}\int \sum_{i,j} L_{ij}^k (\Delta v) v^{1-\delta}\delta_{ij}\eta^\theta.$$

The last term vanishes since L^k is trace-free. Integrating by parts and using (2-8),

$$\begin{aligned}
&\int \sum_{i,j} L_{ij}^k E_{ij} v^{-\delta}\eta^\theta \\
&= -\int \sum_{i,j} (\partial_i L_{ij}^k) v_j v^{1-\delta}\eta^\theta - (1-\delta)\int \sum_{i,j} L_{ij}^k v_i v_j v^{-\delta}\eta^\theta - \int \sum_{i,j} L_{ij}^k v_i \eta_j v^{1-\delta}\eta^{\theta-1} \\
&= -\frac{n-k}{n}\int \sum_i (\partial_i \sigma_k) v_i v^{1-\delta}\eta^\theta - (1+n-\delta)\int \sum_{i,j} L_{ij}^k v_i v_j v^{-\delta}\eta^\theta \\
&\qquad - \int \sum_{i,j} L_{ij}^k v_i \eta_j v^{1-\delta}\eta^{\theta-1}.
\end{aligned}$$

Group in $E_1(\eta)$ all the terms containing derivatives of η. Now compute, using (2-1), (2-2), and (2-3):

$$
\begin{aligned}
\text{(3-3)}\quad \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta}\eta^\theta &= \frac{n-k}{n}\int \sigma_k |\nabla v|^2 v^{-\delta}\eta^\theta - \int \sum_{i,j} T^k_{ij} v_i v_j v^{-\delta}\eta^\theta \\
&= -\frac{k}{n}\int \sigma_k |\nabla v|^2 v^{-\delta}\eta^\theta + \int \sum_{i,j,l} T^{k-1}_{il} A_{lj} v_i v_j v^{-\delta}\eta^\theta \\
&= -\frac{k}{n}\int \sigma_k |\nabla v|^2 v^{-\delta}\eta^\theta + \int \sum_{i,j,l} T^{k-1}_{il} v_{lj} v_i v_j v^{1-\delta}\eta^\theta \\
&\qquad -\frac{1}{2}\int \sum_{i,j} T^{k-1}_{ij} v_i v_j v^{-\delta}\eta^\theta .
\end{aligned}
$$

The middle term can be handled similarly to [González 2005b, Section 4]:

$$
\begin{aligned}
\text{(3-4)}\quad \int \sum_{i,j,l} T^{k-1}_{il} v_{lj} v_i v_j v^{1-\delta}\eta^\theta &= \frac{1}{2}\int \sum_{i,l} \partial_l(|\nabla v|^2)\, T^{k-1}_{il} v_i v^{1-\delta}\eta^\theta \\
= -\frac{\delta-1}{2}\int \sum_{i,l} T^{k-1}_{il} v_i v_l |\nabla v|^2 v^{-\delta}\eta^\theta &- \frac{1}{2}\int \sum_{i,l} \partial_l(T^{k-1}_{il} v_i)\, |\nabla v|^2 v^{1-\delta}\eta^\theta \\
&-\frac{1}{2}\int \sum_{i,l} T^{k-1}_{il} v_i \eta_l |\nabla v|^2 v^{1-\delta}\eta^{\theta-1}.
\end{aligned}
$$

To eliminate the term $\partial_l(T^{k-1}_{il} v_i)$ from (3-4), just use the equality (2-5) and then substitute (3-4) into (3-3):

$$
\begin{aligned}
\text{(3-5)}\quad & \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta}\eta^\theta \\
&= -k\,\frac{n+2}{2n}\int \sigma_k |\nabla v|^2 v^{-\delta}\eta^\theta \;-\; \frac{2+n-\delta}{2}\int \sum_{i,j} T^{k-1}_{ij} v_i v_j |\nabla v|^2 v^{-\delta}\eta^\theta \\
&\qquad + \frac{n-k+1}{4}\int \sigma_{k-1} |\nabla v|^4 v^{-\delta}\eta^\theta \;+\; E_1(\eta) \\
&= -k\,\frac{n+2}{2n}\int \sigma_k |\nabla v|^2 v^{-\delta}\eta^\theta \;+\; \mathcal{B}_{k-1} \;+\; E_1(\eta),
\end{aligned}
$$

where we have defined, for k fixed and $s = 1, \dots, k-1$,

$$
\mathcal{B}_{k-s} = -\frac{s+1+n-\delta}{s+1}\int \sum_{i,j} T^{k-s}_{ij} v_i v_j |\nabla v|^{2s} v^{-\delta}\eta^\theta \;+\; \frac{n-k+s}{2(s+1)}\int \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta}\eta^\theta .
$$

The computations in (3-4) can be redone for T^{k-s}, and thus

$$\mathcal{B}_{k-s} = \tilde{d}_{k-s} \int \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \eta^{\theta} + \tilde{c}_{k-s-1} \mathcal{B}_{k-s-1} + E_1(\eta), \tag{3-6}$$

with

$$\tilde{d}_{k-s} = -\frac{s+n+1-\delta}{s+1}\left(1+\frac{k-s}{2(s+1)}\right)+\frac{n-k+s}{2(s+1)}, \quad \tilde{c}_{k-s} = \frac{(s+n+1-\delta)(s+2)}{2(s+1)^2}.$$

The last step is

$$\mathcal{B}_1 = \tilde{d}_1 \int \sigma_1 |\nabla v|^{2k} v^{-\delta} \eta + \tilde{c}_1 \tilde{d}_0 \int |\nabla v|^{2(k+1)} v^{-\delta} \eta.$$

Substitute (3-6) into (3-5), inductively. This proves (3-1) for some constants c_{k-s} and d_{k-s} obtained from $\tilde{c}_{k-s}$ and $\tilde{d}_{k-s}$. Note that $c_{k-s} > 0$ if $\delta < n+1$. We also want $d_{k-s} > 0$ for $s = 1, \dots, k$, and this is achieved when δ is close enough to $n+1$ because $n > 2k$. □

Lemma 3.2. *With the same hypothesis as in the previous lemma,*

$$\int_U v^{\alpha/k-\gamma} \eta^{\theta} \lesssim \left(-1+\gamma-\frac{n}{2}\right) \int_U \sigma_k |\nabla v|^2 v^{-\gamma} \eta^{\theta} + E_2(\eta), \quad \textit{where} \tag{3-7}$$

$$E_2(\eta) \lesssim \left| \int_U \sum_i v_i \eta_i v^{1-\gamma} \eta^{\theta-1} \right|.$$

Proof. Since $\sigma_k(v) = v^{\alpha}$ and $\sigma_k \le C(n,k)\sigma_1^k$ (Lemma 2.2), we get $\sigma_1(v) \gtrsim v^{\alpha/k}$. It is easy to see that

$$\int \sigma_1 v^{-\gamma} \eta^{\theta} = \left(-1+\gamma-\frac{n}{2}\right) \int |\nabla v|^2 v^{-\gamma} \eta^{\theta} + E_2(\eta),$$

and the lemma is proved. □

4. Main estimates

Here we obtain the needed L^p estimate, as a consequence of (3-1). The terms on the left-hand side of (3-1) will be "good" terms, and we will give an estimate of the error terms.

Proposition 4.1. *Take $n > 2k$, $\alpha \in (0,k)$, and let v be a solution of (1-3). We have*

$$\int_{\rho<|x|<M\rho} v^{\alpha(k+1)/k-\delta} \lesssim \frac{1}{\rho^{2(k+1)}} \int_{A_\rho \cup A_{M\rho}} v^{2(k+1)-\delta} + \frac{1}{\rho^2} \int_{A_\rho \cup A_{M\rho}} v^{2+\alpha-\delta}, \tag{4-1}$$

for δ smaller than but close enough to $n+1$, and for $A_\rho = \left\{\frac{1}{2}\rho < |x| < \rho\right\}$ and $A_{M\rho} = \left\{M\rho < |x| < 2M\rho\right\}$; the constants depend on M but not on ρ.

Proof. If we take $\alpha-\delta=-\gamma$, then $-1-\frac{1}{2}n+\gamma>0$, and the preceding lemma allows us, in (3-1), to replace

$$\int |\nabla v|^2 v^{\alpha-\delta}\eta^\theta \qquad \text{by} \qquad \int v^{\alpha(k+1)/k-\delta}\eta^\theta + E_2(\eta).$$

Let η be a smooth cutoff function such that

$$\eta=\begin{cases}1 & \text{if } \rho<|x|<M\rho,\\ 0 & \text{if } 0<|x|<\frac{1}{2}\rho \text{ and } 2M\rho<|x|,\end{cases}$$

$|\nabla\eta|\lesssim 1/\rho$, and $|D^2\eta|\lesssim 1/\rho^2$. The error $E_1(\eta)$ in (3-2) is of one of these two types:

$$E_{11}(\eta)\lesssim \left|\int_{A_\rho\cup A_{M\rho}}\sum_{i,j} L^k_{ij} v_i\eta_j v^{1-\delta}\eta^{\theta-1}\right|, \quad \text{or}$$

$$E_{12}(\eta)\lesssim \sum_{s=1}^{k}\left|\int_{A_\rho\cup A_{M\rho}}\sum_{i,j} T^{k-s}_{ij} v_j\eta_j |\nabla v|^{2s} v^{1-\delta}\eta^{\theta-1}\right|.$$

These will be handled as in the proof of [González 2005a, Theorem 1.1], but here we present a clearer proof for this particular cutoff.

To understand E_{11}, substitute $L^k=(1-k/n)\,\sigma_k I-T^k$, so that

$$E_{11}(\eta)\lesssim \int_{A_\rho\cup A_{M\rho}}\sigma_k v_i\eta_i v^{1-\delta}\eta^{\theta-1}+\int_{A_\rho\cup A_{M\rho}}T^k_{ij}v_i\eta_j v^{1-\delta}\eta^{\theta-1}. \tag{4-2}$$

We cannot use the standard trick — to estimate the norm $\|T^k\|\lesssim\sigma_k$ as in Lemma 2.7 — because we cannot conclude that T^k is positive definite from the information on $\sigma_1,\dots,\sigma_k$, and we need to write everything in terms of smaller T^{k-s}'s. An inductive process is needed.

Substitute $T^k_{ij}=\sigma_k\delta_{ij}-A_{il}T^{k-1}_{lj}$ and $A_{il}=vv_{il}-\frac{1}{2}|\nabla v|^2\delta_{il}$ in (4-2). Together with Lemma 2.7, we have

$$E_{11}(\eta)\lesssim \int \sigma_k|\nabla v|\,|\nabla\eta|\,v^{1-\delta}\eta^{\theta-1}+\int\sigma_{k-1}|\nabla v|^3|\nabla\eta|\,v^{1-\delta}\eta^{\theta-1} +\left|\int T^{k-1}_{lj}v_{il}v_i\eta_j v^{2-\delta}\eta^{\theta-1}\right|. \tag{4-3}$$

For the last term, proceed as in (3-4):

$$\begin{aligned}\int T^{k-1}_{lj}v_{il}v_i\eta_j v^{2-\delta}\eta^{\theta-1} &= \frac{1}{2}\int \partial_l\big(|\nabla v|^2\big)\,T^{k-1}_{lj}\eta_j v^{2-\delta}\eta^{\theta-1}\\ &= -\frac{1}{2}\int(\partial_l T^{k-1}_{lj})\,|\nabla v|^2\eta_j v^{2-\delta}\eta^{\theta-1}-\frac{1}{2}\int T^{k-1}_{lj}|\nabla v|^2\eta_{lj}v^{2-\delta}\eta^{\theta-2}\\ &\quad -\frac{2-\delta}{2}\int T^{k-1}_{lj}\eta_l v_j|\nabla v|^2 v^{1-\delta}\eta^{\theta-1}.\end{aligned} \tag{4-4}$$

Note that (2-5) helps to compute $\partial_l T_{lj}^{k-1}$, and thus, from (4-4) and Lemma 2.7,

$$\left|\int T_{lj}^{k-1} v_{il} v_i \eta_j v^{2-\delta}\eta^{\theta-1}\right| \lesssim \int \sigma_{k-1}|D^2\eta||\nabla v|^2 v^{2-\delta}\eta^{\theta-2} + \int \sigma_{k-1}|\nabla v|^3|\nabla\eta| v^{1-\delta}\eta^{\theta-1}. \tag{4-5}$$

Young's inequality for a small ε, together with (4-3) and (4-5), gives

$$E_{11}(\eta) \lesssim \varepsilon\int \sigma_k|\nabla v|^2\eta^\theta v^{-\delta} + \frac{C_\varepsilon}{\rho^2}\int_{A_\rho\cup A_{M\rho}} \sigma_k v^{2-\delta}\eta^{\theta-2} + \varepsilon\int \sigma_{k-1}|\nabla v|^4\eta^\theta v^{-\delta} + \frac{C_\varepsilon}{\rho^4}\int_{A_\rho\cup A_{M\rho}} \sigma_{k-1} v^{4-\delta}\eta^{\theta-4}. \tag{4-6}$$

To finish the estimate, we just need (4-7) from the next lemma, applied iteratively:

$$E_{11}(\eta) \lesssim \varepsilon\sum_{s=0}^{k}\int \sigma_{k-s}|\nabla v|^{2(s+1)}\eta^\theta v^{-\delta} + \frac{C_\varepsilon}{\rho^{2(k+1)}}\int_{A_\rho\cup A_{M\rho}} v^{2(k+1)-\delta}.$$

The estimate for $E_{12}(\eta)$ follows in a similar manner. For the error in $E_2(\eta)$, defined in 3-7, we use Young's inequality with $p = q = 2$:

$$E_2(\eta) \lesssim \int |\nabla v||\nabla\eta| v^{1-\gamma}\eta^{\theta-1} \lesssim \varepsilon\int |\nabla v|^2 v^{\alpha-\delta}\eta^\theta + \frac{C_\varepsilon}{\rho^2}\int_{A_\rho\cup A_{M\rho}} v^{2+\alpha-\delta}.$$

Putting it all together in (3-1), and taking into account that $\sum_{i,j} L_{ij}^k E_{ij} \geq 0$,

$$\int_{\rho<|x|<M\rho} v^{\alpha(k+1)/k-\delta} \leq \int v^{\alpha(k+1)/k-\delta}\eta^\theta \lesssim \frac{1}{\rho^{2(k+1)}}\int_{A_\rho\cup A_{M\rho}} v^{2(k+1)-\delta} + \frac{1}{\rho^2}\int_{A_\rho\cup A_{M\rho}} v^{2+\alpha-\delta}. \qquad \square$$

Lemma 4.2. *For all $\varepsilon > 0$ and $s = 0, \dots, k-1$, and for θ a big positive integer,*

$$\frac{1}{\rho^{2(s+1)}}\int \sigma_{k-s} v^{2(s+1)-\delta}\eta^{\theta-2(s+1)} \leq \varepsilon\int \sigma_{k-s-1}|\nabla v|^{2(s+2)}\eta^\theta v^{-\delta} + \frac{C_\varepsilon}{\rho^{2(s+2)}}\int_{\{|\nabla\eta|\neq 0\}} \sigma_{k-s-1}\,\eta^{\theta-2(s+2)} v^{2(s+2)-\delta}. \tag{4-7}$$

Proof. First use the "divergence" formula (2-6) for σ_{k-s} with integration by parts:

$$(k-s)\int \sigma_{k-s} v^{2(s+1)-\delta}\eta^{\theta-2(s+1)} = \frac{n-k+s+1}{2}\int \sigma_{k-s-1}|\nabla v|^2\eta^{\theta-2(s+1)}v^{2(s+1)-\delta} - (n+2(s+1)-\delta+1)\int T^{k-s-1}_{ij}v_iv_j\eta^{\theta-2(s+1)}v^{2(s+1)-\delta} - \int T^{k-s-1}_{ij}v_i\eta_j\eta^{\theta-2(s+1)-1}v^{2(s+1)-\delta+1}. \tag{4-8}$$

Use Lemma 2.7 again to bound the norm of the Newton tensor in (4-8):

$$\int \sigma_{k-s} v^{2(s+1)-\delta}\eta^{\theta-2(s+1)} \lesssim \int \sigma_{k-s-1}|\nabla v|^2\eta^{\theta-2(s+1)}v^{2(s+1)-\delta} + \frac{1}{\rho}\int \sigma_{k-s-1}|\nabla v|\eta^{\theta-2(s+1)-1}v^{2(s+1)-\delta+1}. \tag{4-9}$$

Young's inequality with ε and $p=s+2,\ q=(s+2)/(s+1)$ now reads

$$\int \sigma_{k-s-1}|\nabla v|^2\eta^{\theta-2(s+1)}v^{2(s+1)-\delta} \lesssim \varepsilon\rho^{2(s+1)}\int \sigma_{k-s-1}|\nabla v|^{2(s+1)}\eta^{\theta}v^{-\delta} + \frac{C_\varepsilon}{\rho^2}\int \sigma_{k-s-1}\eta^{\theta-2(s+2)}v^{2(s+2)-\delta}. \tag{4-10}$$

For the second part in (4-9), take $p=2(s+2)$ and $q=\dfrac{2(s+2)}{2(s+2)-1}$:

$$\frac{1}{\rho}\int \sigma_{k-s-1}|\nabla v|\eta^{\theta-2(s+2)-1}v^{2(s+1)-\delta+1} \lesssim \varepsilon\rho^{2(s+1)}\int \sigma_{k-s-1}|\nabla v|^{2(s+2)}\eta^{\theta}v^{-\delta} + \frac{C_\varepsilon}{\rho^2}\int \sigma_{k-s-1}\eta^{\theta-2(s+2)}v^{2(s+2)-\delta}. \tag{4-11}$$

The lemma is proved by substituting (4-10) and (4-11) into (4-9). □

Proposition 4.3. *For $n\geq 2(k+1)$, $\alpha\in(0,k)$, and v a solution of (1-3), we have*

$$\int_{\rho<|x|<M\rho} v^{\alpha(k+1)/k-\delta} \leq C\rho^{n-(\delta-\alpha(k+1)/k)/(1-\alpha/2k)}, \tag{4-12}$$

where C depends on M and δ, but not on ρ.

Proof. Use Hölder's inequality with

$$p=\frac{\delta-\alpha(k+1)/k}{\delta-2(k+1)} \qquad \text{and} \qquad q=\frac{p}{p-1}$$

to get

$$\frac{1}{\rho^{2(k+1)}}\int_{A_\rho\cup A_{M\rho}} v^{2(k+1)-\delta} \leq \varepsilon\int_{A_\rho\cup A_{M\rho}} v^{\alpha(k+1)/k-\delta} + C_\varepsilon\rho^{n-2(k+1)q}, \tag{4-13}$$

for some small ε, to be chosen later. Also, a Hölder estimate with

$$\tilde{p} = \frac{\delta - \alpha(k+1)/k}{\delta - 2 - \alpha} \qquad \text{and} \qquad \tilde{q} = \frac{\tilde{p}}{\tilde{p} - 1}$$

gives

$$\frac{1}{\rho^2}\int_{A_\rho \cup A_{M\rho}} v^{2+\alpha-\delta} \le \varepsilon \int_{A_\rho \cup A_{M\rho}} v^{\alpha(k+1)/k-\delta} + C_\varepsilon \rho^{n-2\tilde{q}}. \tag{4-14}$$

When $\alpha \in (0, k)$ and δ is close enough to $n+1$, then p, $\tilde{p} > 1$. Look at the powers of ρ in (4-13) and (4-14):

$$n - 2(k+1)q = n - 2\tilde{q} = n - \frac{\delta - \alpha(k+1)/k}{1 - \alpha/2k}.$$

Choosing ε small enough, we conclude from (4-1) that

$$\int_{\rho<|x|<M\rho} v^{\alpha(k+1)/k-\delta} \le C\rho^{n-(\delta-\alpha(k+1)/k)/(1-\alpha/2k)}. \qquad \square$$

5. Proof of Theorem 1.1

The next proposition is similar to the study of the critical problem in [González 2005a]. In particular, a volume finiteness condition gives regularity near the singularity.

Proposition 5.1. *Take $\alpha \in (0, k)$ and $n > 2k$, and let v be a solution of (1-3) on $B_\rho(x_0) \subset B$, with $v > 0$ and $v \in \Gamma_k^+$. If*

$$\int_{B_\rho(x_0)} v^{(\alpha-2k)n/(2k)} \le a$$

for some small enough a (not depending on ρ), then

$$\sup_{B_{\rho/2}(x_0)} |v^{-1}| \le \frac{C}{\rho^{n/p}} \|v^{-1}\|_{L^p(B_\rho(x_0))} \tag{5-1}$$

for all $p > (n-2k)k/(k+1)$. In particular, if

$$\int_{\varepsilon<|x|<1} v^{(\alpha-2k)n/(2k)} < C < \infty \tag{5-2}$$

for some constant C independent of ε, the function v is bounded near the origin.

Proof. The argument is similar to [González 2005a, Theorem 1.2] for the critical problem. Condition (5-2) is analogous to its volume smallness condition. $\square$

Proof of Theorem 1.1. Fix x_0 small enough and take $2R = |x_0|$. First, note that Hölder estimates with

$$r = \frac{\delta - (k+1)/k}{(2k-\alpha)n/(2k)} > 1 \qquad \text{and} \qquad 1 = \frac{1}{r} + \frac{1}{s}$$

give, independently of x_0,

$$\int_{B_R(x_0)} v^{(\alpha-2k)n/(2k)} \le \left(\int_{R\le|x|\le 3R} v^{(k+1)/k-\delta}\right)^{1/r} \varepsilon^{n/s} \lesssim R^{\left(n-\frac{\delta-\alpha(k+1)/k}{1-\alpha/(2k)}\right)\frac{1}{r}\frac{n}{s}} \lesssim R^0 < \infty. \tag{5-3}$$

We cannot apply Proposition 5.1 directly to v. However, we could have started with the function $\tilde v(y) = A^{2k/(2k-\alpha)} v(y/A)$ that still satisfies the same equation $\sigma_k(\tilde v) = \tilde v^\alpha$, for some A big enough and of the form

$$A = (\text{constant}) \int_{R\le|x|\le 3R} v^{(\alpha-2k)n/(2k)}.$$

Since we are interested only in the local behavior near zero, we can assume that (5-1) gives an estimate for v,

$$\sup_{B_{R/2}(x_0)} |v^{-1}| \le \frac{C}{R^{n/p}} \|v^{-1}\|_{L^p(B_R(x_0))}$$

for all $p > (n-2k)k/(k+1)$, and with C depending on

$$\int_{R\le|x|\le 3R} v^{(\alpha-2k)n/(2k)}.$$

This estimate is uniformly bounded by a constant, independently of R, because of (5-3). It is also true that

$$\sup_{B_{R/2}(x_0)} |v^{-1}| \le \frac{C}{|x_0|^{n/p}} \|v^{-1}\|_{L^p(\{R\le|x|\le 3R\})} \tag{5-4}$$

for all $p > (n-2k)k/(k+1)$. Set $p = \delta - \alpha(k+1)/k$; this choice is valid when $\alpha \in (0,k)$ and $n > 2k$. Use (4-12) again:

$$\int_{R\le|x|\le 3R} v^{-p} \le C|x_0|^{n-\frac{p}{1-\alpha/(2k)}},$$

and thus, from (5-4), we arrive at

$$v^{-1}(x_0) \le \frac{C}{|x_0|^{2k/(2k-\alpha)}},$$

as desired. □

Corollary 5.2 (Harnack). *Under these hypotheses, there exists $M_0 > 0$ such that, for all $\rho > 0$ and $M \le M_0$,*

$$\sup_{\rho\le|x|\le\rho M} v^{-1} \le C \inf_{\rho\le|x|\le\rho M} v^{-1}, \tag{5-5}$$

where C is independent of v, ρ, and M.

Proof. Once we get a supremum estimate (5-4) for a ball, standard elliptic theory yields the infimum estimate. If we write $v^{-2}=u^{2/(n-2)}$, then u is a superharmonic function. To finish, use a covering argument for the annulus $\{\rho \le |x| \le \rho M\}$. □

Corollary 5.3. *If v is a solution of (1-3), then either v^{-1} is bounded near the origin, or $v^{-1}(x) \to \infty$ as $x \to 0$.*

Proof. The argument follows the steps of [Gidas and Spruck 1981, Corollary 3.3], by using the second part of Proposition 5.1. □

6. Proof of Theorem 1.2

We have proved the estimate

$$v^{-1}(x) \le \frac{C}{|x|^{2k/(2k-\alpha)}} \tag{6-1}$$

Now we would like to get the opposite inequality. Suppose that

$$\liminf_{x\to 0} |x|^{2k/(2k-\alpha)} v^{-1}(x) = 0.$$

From the Harnack estimate (5-5) follows that

$$\lim_{x\to 0} |x|^{2k/(2k-\alpha)} v^{-1}(x) = 0. \tag{6-2}$$

We want to see that in this case the function v^{-1} is bounded near the origin and thus that the theorem follows. It suffices to establish (5-2).

Let's review two results from [González 2005a]:

Proposition 6.1. *Let v be a solution with $v^{-1} \in \mathscr{C}^3(U)$, $v > 0$, $v \in \Gamma_k^+$, and $n > 2k$. For all $\varphi \in \mathscr{C}_0^\infty(U)$ and θ a big positive integer,*

$$\int_U \sigma_k \varphi^\theta v^{-\gamma} \ge \sum_{s=1}^{k} c_{k-s}(\gamma) \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^\theta v^{-\gamma} + E(\varphi), \tag{6-3}$$

where

$$E(\varphi) \lesssim \sum_{s=1}^{k} \left| \int_U \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^{\theta-1} v^{1-\gamma} \right|, \tag{6-4}$$

and where the coefficients $c_{k-s}(\gamma)$ are positive for all γ with

$$\gamma > n - \frac{n-2k}{k+1}. \tag{6-5}$$

Proposition 6.2. *For all $\varepsilon > 0$, the error term (6-4) can be estimated by*

$$E(\varphi) \le \varepsilon \sum_{s=1}^{k} \int \sigma_{k-s} |\nabla v|^{2s} \varphi^\theta v^{-\gamma} + C_\varepsilon \sum_{U_k} \int U_k(\varphi) \varphi^{\theta-\alpha_k} v^{2k-\gamma},$$

where the $U_k(\varphi)$'s are groups of derivatives of φ of order $2k$, and $\alpha_k \in \mathbb{R}$ are constants depending on each of the U_k's. These concepts are defined inductively in the following manner:

- *For a fixed $s = 1, \ldots, k$, the starting point is*

$$U_s(\varphi)\varphi^{\alpha_s} = |\nabla\varphi|^{2s}\varphi^{-2s}.$$

- *For each integer $l = 0, 1, \ldots$ and $m = s + l$, and once given $U_m\varphi^{\alpha_m}$, the following step is of one of these three shapes:*

$$U_{m+1}\varphi^{-\alpha_{m+1}} = \begin{cases} U_m^{(m+1)/m}\varphi^{-\alpha_m(m+1)/m}, \\ \left|\nabla U_m\right|^{\frac{2(m+1)}{2(m+1)-1}}\varphi^{-\alpha_m\frac{2(m+1)}{2(m+1)-1}}, \\ \left(|\nabla\varphi|^2 U_m\right)\varphi^{-\alpha_m-2}. \end{cases} \tag{6-6}$$

- *The ending point is when $m = s + l$ reaches k.*

We will use (6-3) for a suitable cutoff function. Take $\varphi = \eta r$ with $\eta \in \mathscr{C}_0^\infty(B\backslash\{0\})$, such that

$$\eta = \begin{cases} 1 & \text{if } \varepsilon < |x| < R, \\ 0 & \text{if } |x| < \frac{1}{2}\varepsilon \text{ and } |x| > 2R, \end{cases}$$

and so that the derivatives have a good bound on $\frac{1}{2}\varepsilon < |x| < \varepsilon$ and $R < |x| < 2R$. The value of γ will be chosen later. Rewrite (6-3) as

$$\int \sigma_k v^{-\gamma}\varphi^\theta \gtrsim \sum_{s=1}^k \int \sigma_{k-s}|\nabla v|^{2s} v^{-\gamma}\varphi^\theta - \int T_{ij}^{k-1} v_i\varphi_j\varphi^{\theta-1}v^{1-\gamma} + \tilde{E}(\varphi), \tag{6-7}$$

with

$$\tilde{E}(\varphi) \lesssim \sum_{s=2}^k \left| \int T_{ij}^{k-s} v_i\varphi_j|\nabla v|^{2(s-1)}\varphi^{\theta-1}v^{1-\gamma} \right|,$$

since we will look more carefully at the term in T^{k-1}. Integration by parts gives

$$\begin{aligned} -\int \sum_{i,j} T_{ij}^{k-1} v_i\varphi_j\varphi^{\theta-1}v^{1-\gamma} &= -\frac{1}{2-\gamma}\int\sum_{i,j} T_{ij}^{k-1}\partial_i(v^{2-\gamma})\varphi_j\varphi^{\theta-1} \\ &= \frac{1}{2-\gamma}\int\sum_{i,j} T_{ij}^{k-1}\varphi_{ij}\varphi^{\theta-1}v^{2-\gamma} + \frac{1}{2-\gamma}\int\sum_{i,j}\partial_i(T_{ij}^{k-1})\varphi_j\varphi^{\theta-1}v^{2-\gamma} \\ &\qquad + \frac{\theta-1}{2-\gamma}\int\sum_{i,j} T_{ij}^{k-1}\varphi_i\varphi_j\varphi^{\theta-2}v^{2-\gamma}. \end{aligned}$$

Substituting (2-5) into this, we get

$$-(n+2-\gamma)\int\sum_{i,j}T_{ij}^{k-1}v_i\varphi_j\varphi^{\theta-1}v^{1-\gamma}=\int\sum_{i,j}T_{ij}^{k-1}\varphi_{ij}\varphi^{\theta-1}v^{2-\gamma}$$
$$-(n-k+1)\int\sum_{i}\sigma_{k-1}v_i\varphi_i\varphi^{\theta-1}v^{1-\gamma}+(\theta-1)\int\sum_{i,j}T_{ij}^{k-1}\varphi_i\varphi_j\varphi^{\theta-2}v^{2-\gamma}.$$

Now substitute this into (6-7):

$$\text{(6-8)}\quad \int\sigma_k\varphi^\theta v^{-\gamma}\gtrsim\sum_{s=1}^{k}\int\sigma_{k-s}|\nabla v|^{2s}\varphi^\theta v^{-\gamma}+\frac{1}{n+2-\gamma}\int\sum_{i,j}T_{ij}^{k-1}\varphi_{ij}\varphi^{\theta-2}v^{2-\gamma}$$
$$+\frac{\theta-1}{n+2-\gamma}\int\sum_{i,j}T_{ij}^{k-1}\varphi_i\varphi_j\varphi^{\theta-2}v^{2-\gamma}$$
$$-\frac{n-k+1}{n+2-\gamma}\int\sum_{i}\sigma_{k-1}v_i\varphi_i\varphi^{\theta-1}v^{1-\gamma}+\tilde{E}(\varphi).$$

Group all the error terms into

$$E(\varphi)\lesssim\sum_{s=1}^{k}\int\sigma_{k-s}|\nabla\varphi|\,|\nabla v|\,\varphi^{\theta-1}v^{1-\gamma}.$$

Compute

$$\varphi_i=\frac{x_i}{r}\eta+E_1(\varphi),$$
$$\varphi_{ij}=r^{-1}\Big(-\frac{x_ix_j}{r^2}+\delta_{ij}\Big)\eta+E_1(\varphi),$$
$$\sum_{i,j}T_{ij}^{k-1}\varphi_{ij}=r^{-1}\Big(-\sum_{i,j}T_{ij}^{k-1}\frac{x_ix_j}{r^2}+(n-k+1)\sigma_{k-1}\Big)\eta+E_1(\varphi).$$

Since T^{k-1} is positive definite and trace $T^{k-1}=(n-k+1)\sigma_{k-1}$, as long as we keep $1<\theta$ we have

$$\sum_{i,j}T_{ij}^{k-1}\big(\varphi_{ij}+(\theta-1)\varphi_i\varphi_jr^{-\theta}\big)\ge C(\theta)\,\sigma_{k-1}r^{-1}\eta^2+E_1(\varphi),$$

for some $C(\theta)>0$. If we keep $\gamma<n+2$, we can conclude from (6-8) that

$$\text{(6-9)}\quad E(\varphi)+E_1(\varphi)+\int\sigma_k\varphi^\theta v^{-\gamma}$$
$$\gtrsim\sum_{s=1}^{k}\int\sigma_{k-s}|\nabla v|^{2s}\varphi^\theta v^{-\gamma}+\int\sigma_{k-1}r^{-2}\varphi^\theta v^{2-\gamma}.$$

We have not been very precise with the errors $E_1(\varphi)$; however, they are of a similar type to $E(\varphi)$ and can be treated in the same manner. Note that, in the positive cone,

$$\sigma_{k-1} \gtrsim \sigma_k^{(k-1)/k} = v^{\alpha(k-1)/k},$$

so, with (6-9) we have actually proved

$$E(\varphi) \gtrsim \int \left(v^{\alpha(k-1)/k+2-\gamma} r^{-2} - v^{\alpha-\gamma}\right)\varphi^\theta + \sum_{s=1}^{k} \int \sigma_{k-s} |\nabla v|^{2s} \varphi^\theta v^{-\gamma}. \tag{6-10}$$

To handle $E(\varphi)$, we need to control the error terms that appear in Proposition 6.2. Using Lemma 6.3 below,

$$\begin{aligned}\int U_k(\varphi)\varphi^{\theta-\alpha_k} v^{2k-\gamma} \\ \lesssim \int r^{-2k}\varphi^\theta v^{2k-\gamma} + \frac{1}{\varepsilon^{2k}} \int_{\varepsilon/2<|x|<\varepsilon} r^\theta v^{2k-\gamma} + \frac{1}{R^{2k}} \int_{R<|x|<2R} r^\theta v^{2k-\gamma}\end{aligned} \tag{6-11}$$

Looking one-by-one at the terms above, notice that

$$\frac{1}{\varepsilon^{2k}} \int_{\varepsilon/2<|x|<\varepsilon} r^\theta v^{2k-\gamma} \to 0 \qquad \text{as } \varepsilon \to 0,$$

by using the previous estimate (6-1) and the definition of η, and as soon as

$$\gamma > n - \alpha\left(\frac{n-2k}{2k}\right). \tag{6-12}$$

A similar argument gives

$$\frac{1}{R^{2k}} \int_{R<|x|<2R} r^\theta v^{2k-\gamma} \leq C.$$

The other integral in (6-11) is bounded by

$$\int r^{-2k}\varphi^\theta v^{2k-\gamma} \lesssim \int \left(v^{\alpha(k-1)/k+2-\gamma} r^{-2}\right)\left(v^{-\alpha(k-1)/k-2+2k} r^{2-2k}\right)\varphi^\theta.$$

Our assumption (6-2) yields

$$v^{-\alpha(k-1)/k-2+2k} r^{2-2k} = o(1),$$

and thus from (6-10) we obtain

$$C \gtrsim \int \left(v^{\alpha(k-1)/k+2-\gamma} r^{-2} - v^{\alpha-\gamma}\right)\varphi^\theta.$$

Again, because of (6-2), we have

$$r^2 v^{\alpha/2-k} = o(1).$$

Theorem 1.1 gives

$$v^{\alpha-\gamma} \lesssim \left(v^{\alpha(k-1)/k+2-\gamma} r^{-2}\right)\left(r^2 v^{(\alpha-2k)/2}\right).$$

Comparing the orders, we quickly obtain

$$\int v^{\alpha(k-1)/k} r^{-2} v^{2-\gamma} \varphi^\theta < \infty. \tag{6-13}$$

This is precisely the term (5-2) that we need to estimate, because

$$\begin{aligned}\int_{\varepsilon\le|x|\le R} v^{(\alpha-2k)n/(2k)} &= \int v^{\alpha(k-1)/k+2-\gamma}\, v^{-\alpha(k-1)/k-n+\alpha n/(2k)-2+\gamma}\, \eta^\theta \\ &\lesssim \int v^{\alpha(k-1)/k+2-\gamma}\, r^{\left(-\alpha\frac{k-1}{k}-n+\alpha\frac{n}{2k}-2+\gamma\right)\left(\frac{2k}{2k-\alpha}\right)}\, \eta^\theta \\ &= \int v^{\alpha(k-1)/k+2-\gamma}\, r^{-2} \varphi^\theta,\end{aligned} \tag{6-14}$$

after using Theorem 1.1 and choosing θ and γ so that

$$\left(-\alpha\frac{k-1}{k} - n + \alpha\frac{n}{2k} - 2 + \gamma\right)\left(\frac{2k}{2k-\alpha}\right) = -2+\theta, \tag{6-15}$$

that is, picking

$$\gamma = n - \alpha\left(\frac{n-2k}{2k}\right) + \theta\left(1-\frac{\alpha}{2k}\right).$$

This is an admissible value for γ because, when $\alpha < 2k/(k+1)$, it can be chosen to satisfy (6-5), (6-12), $\gamma < n+2$, and $\theta > 1$.

Lemma 6.3. *For the cutoff $\varphi = r\eta$ constructed in the previous proof,*

$$U_k(\varphi)\varphi^{\theta-\alpha_k} \lesssim r^{-2k}\varphi^\theta + \varepsilon^{-2k} r^\theta \chi_{\{\varepsilon/2\le|x|<\varepsilon\}} + R^{-2k} r^\theta \chi_{\{R\le|x|<2R\}}$$

Proof. The definition of the U_k was given in Proposition 6.2. We are just interested in the orders of r and ε. For fixed $s = 1, \dots, k$, the initial step is

$$\begin{aligned}U_s(\varphi)\varphi^{\theta-2s} &= |\nabla\varphi|^{2s}\varphi^{\theta-2s} \\ &\lesssim |\nabla r|^{2s}\varphi^{\theta-2s}\eta^{2s} + |\nabla\eta|^{2s} r^{2s}\varphi^{\theta-2s} \lesssim r^{-2s}\varphi^\theta + \varepsilon^{-2s} r^\theta \eta^{2s}.\end{aligned}$$

Next, assume that the result is true for $m = s + l$:

$$U_m(\varphi)\varphi^{\theta-\alpha_m} \lesssim r^{-2m}\varphi^\theta + \varepsilon^{-2m} r^\theta \eta^{2m}.$$

The proof for $m+1$ follows easily from (6-6). □

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