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We construct new examples of complete locally conformally flat manifolds of negative curvature by means of warped product and multiply warped product structures. Special attention is paid to those spaces with one-dimensional base, thus generalizing the Robertson–Walker spacetimes, and to those with higher-dimensional base of constant curvature.

1. Introduction

Locally conformally flat structures on Riemannian manifolds are natural generalizations of isothermal coordinate systems, which are available on Riemann surfaces. However, not every higher-dimensional Riemannian manifold admits a locally conformally flat structure, and it is difficult to provide a classification of those that do; this is still an open problem. Some partial results are known. A compact simply connected locally conformally flat manifold must be a Euclidean sphere [Kuiper 1949; Schoen and Yau 1988]. Locally symmetric manifolds which are locally conformally flat are either of constant sectional curvature or locally isometric to a product of two spaces of constant opposite sectional curvature [Lafontaine 1988; Yau 1973]. Complete locally conformally flat manifolds with nonnegative Ricci curvature have been studied by several authors; Zhu [1994] showed that their universal cover is in the conformal class of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{R} \times \mathbb{S}^{n-1}$, where $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$ are spheres of constant sectional curvature. Such conformal equivalence can be specialized to isometric equivalence under some extra assumptions on the scalar curvature and the sign of the Ricci curvatures [Cheng 2001; Tani 1967] (see also [Carron and Herzlich 2004] and the references therein). In spite of the results on locally conformally flat manifolds of nonnegative curvature, to the best of our knowledge, there is a lack of information as concerns negative curvature. Henceforth, our purpose on this work is to construct new examples of complete locally conformally flat Riemannian manifolds with nonpositive curvature.

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Since their introduction by Bishop and O'Neill [1969], warped products have been a powerful tool for constructing manifolds of nonpositive curvature (see also [Bertola and Gouthier 2001]). Our aim, then, is to investigate the existence of locally conformally flat structures on manifolds equipped with a warped product structure, or more generally on multiply warped spaces, as being a natural generalization of warped products (see for example [Tojeiro 2004] and the references therein). Other generalizations of warped product structures, like twisted or multiply quasiwarped [Meumertzheim et al. 1999; Ponge and Reckziegel 1993; Tojeiro 2004] are not of interest for our purposes, since they reduce to warped and multiply warped spaces, respectively, if they are locally conformally flat [Brozos-Vázquez et al. 2005]. Another motivation for the consideration of locally conformally flat structures on manifolds equipped with a warped product metric comes from the fact that the Schouten tensor is Codazzi for any locally conformally flat manifold. Moreover, although the local structure of Codazzi tensors is not yet completely understood, the existence of such a tensor leads to warped product decompositions of the manifold in many cases [Bivens et al. 1981; Tojeiro 2004].

This paper is organized as follows. In Section 2 we recall basic facts on the curvature of warped and multiply warped spaces. Locally conformally flat multiply warped spaces are investigated in Section 3. Our approach relies on the fact that any multiply warped space is in the conformal class of a suitable product, a fact previously observed for warped product metrics [Lafontaine 1988], which has several implications on the geometry of the fibers and the base of the multiply warped space. A local description of locally conformally flat spaces with the underlying structure of a multiply warped product is then obtained from the fact that any warping function must define a global conformal transformation on the base which makes it of constant sectional curvature. Then the situation when the base has dimension 2 or higher reduces to the existence of nontrivial solutions of some Obata type equations on the base (sometimes called concircular transformations; see [Kühnel 1988; Tashiro 1965]) together with some compatibility conditions among the different warping functions. This analysis is carried out in Section 3A. Conditions become much weaker when the base is assumed to be one-dimensional, as shown in Section 3B, in accordance with Roberston–Walker type metrics, which are locally conformally flat independently of the warping function. Some global consequences are obtained in Section 4, where locally conformally flat warped product manifolds with complete base of constant curvature are classified, as well as multiply warped ones if the base is further assumed to be simply connected.

Applications of the results in Section 3 have already been found by R. Tojeiro in the study of conformal immersions into the Euclidean space [2006]. Moreover, multiply warped spaces with hyperbolic space as the base are of key interest, providing some new examples of complete locally conformally flat manifolds with nonpositive sectional curvature, and with nonpositive Ricci curvatures but no sign requirement on the sectional curvature.

2. Preliminaries

Let (B, g_B), (F_1, g_1), ..., (F_k, g_k) be Riemannian manifolds. The product manifold $M = B \times F_1 \times \cdots \times F_k$, equipped with the metric

$$g = g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k,$$

where $f_1, \ldots, f_k : B \to \mathbb{R}$ are positive functions, is called a *multiply warped product*. B is the *base*, $F_1, \ldots, F_k$ are the *fibers* and $f_1, \ldots, f_k$ are the *warping functions*. We will denote a multiply warped product manifold as above by $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$.

Remark 2.1. The general form of multiply warped products is slightly flexible, so we must adopt some criteria to identify multiply warped products with different form but which are essentially the same. They are:

C1. Warping functions are supposed to be nonconstant and any two warping functions which are multiples one to each other are written as the same function and the metric of the fiber is multiplied by the constant in order to do not modify the metric of the multiply warped product.

C2. Fibers with the same warping function are joined in one fiber.

Moreover, the possible order of the fibers is irrelevant for our purposes.

Next we fix some notation and criteria to be used in what follows. Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. The Riemann curvature tensor R is the $(1, 3)$-tensor field on M defined by $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, for all vector fields $X, Y, Z \in \mathfrak{X}(M)$. The Ricci tensor is the contraction of the curvature tensor given by $\rho(X, Y) = \text{trace}\{U \rightsquigarrow R(X, U)Y\}$ and the scalar curvature is obtained by contracting the Ricci tensor, $\tau = \text{trace}(\rho)$. For a vector field X on M the *divergence* of X is defined by $div\, X = \text{trace}\, \nabla X$. The *gradient* of a function $f : (M, g) \to \mathbb{R}$ is determined by $g(\nabla f, X) = X(f)$ and the *Laplacian* of f is defined by $\Delta f = div \nabla f$. Also, the linear map $h_f(X) = \nabla_X \nabla f$ is called the *Hessian tensor* of f on (M, g), and $H_f(X, Y) = g(h_f(X), Y)$ is called the *Hessian form* of f. Finally, note that $\Delta f = \text{trace}\, h_f$.

In order to study the properties of multiply warped products, we need some properties of their curvature tensor, obtained essentially in the same way as for warped products [Bishop and O'Neill 1969]. Therefore proofs are omitted. The nonzero components of the curvature tensor are

$$
\begin{aligned}
&R_{XY}Z = R^B_{XY}Z, && R_{V_i X}Y = \frac{1}{f_i}H_{f_i}(X,Y)V_i,\\
(1)\quad &R_{XU_i}V_i = \frac{\langle U_i, V_i\rangle}{f_i}\nabla_X\nabla f_i, && R_{U_jU_i}V_i = \frac{\langle U_i, V_i\rangle}{f_i f_j}\langle \nabla f_i, \nabla f_j\rangle U_j \ \text{ if } i\neq j,\\
&R_{U_iV_i}W_i = R^{F_i}_{U_iV_i}W_i - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2}\big(\langle U_i, W_i\rangle V_i - \langle V_i, W_i\rangle U_i\big),
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{L}(B)$ and $U_i, V_i, W_i \in \mathfrak{L}(F_i)$, where R^B and R^{F_i} denote the curvature tensor of (B, g_B) and (F_i, g_i), respectively. Here $H_{f_i}(X, Y)$ and ∇f_i denote the Hessian tensor and the gradient of the warping function f_i with respect to the Riemannian structure of (B, g_B). A straightforward calculation from (1) shows that the sectional curvature of M satisfies

$$
\begin{aligned}
&K_{XY} = K^B_{XY}, && K_{XU_i} = -\frac{H_{f_i}(X,X)}{f_i\|X\|^2},\\
(2)\quad &K_{U_iV_i} = \frac{1}{f_i^2}K^{F_i}_{U_iV_i} - \frac{\|\nabla f_i\|^2}{f_i^2}, && K_{U_iV_j} = -\frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} \ \text{ if } i \neq j,
\end{aligned}
$$

where K^B and K^{F_i} denote the sectional curvatures on the base B and the fiber F_i. Here the sectional curvature of a plane π is taken with the sign convention $K(\pi) = R(X, Y, X, Y)$, for any orthonormal base $\{X, Y\}$ of π.

3. Locally conformally flat multiply warped spaces

Recall that a Riemannian manifold (M, g) is *locally conformally flat* if every point in M admits a coordinate neighborhood $\mathfrak{U}$ which is conformal to Euclidean space $\mathbb{R}^n$; equivalently, if there is a diffeomorphism $\Phi : V \subset \mathbb{R}^n \to \mathfrak{U}$ such that $\Phi^* g = \Psi^2 g_{\mathbb{R}^s}$ for some positive function Ψ. Any surface is locally conformally flat, but not every higher-dimensional Riemannian manifold admits a locally conformally flat structure. Necessary and sufficient conditions for the existence of such a structure are the nullity of the Weyl tensor $W = R - C \odot g$ when $\dim M \geq 4$, and, in dimension three, the condition that the Schouten tensor

$$C = \frac{1}{n-2}\left(\rho - \frac{\tau}{2(n-1)}g\right)$$

be a Codazzi tensor. Here $\odot$ represents the Kulkarni–Nomizu product (see [Lafontaine 1988], for example). A nonflat locally decomposable Riemannian manifold is locally conformally flat if and only if it is locally equivalent to the product $N(c)\times\mathbb{R}$ of an interval and a space of constant sectional curvature, or to the product

$N_1(c) \times N_2(-c)$ of two spaces of opposite constant sectional curvature [Lafontaine 1988; Yau 1973].

Lemma 3.1. *Let $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a locally conformally flat multiply warped space.*

(i) *(B, g_B) is locally conformally flat.*

(ii) *(F_i, g_i) is a space of constant sectional curvature for all $i = 1, \ldots, k$, provided that* $\dim F_i \geq 2$.

Proof. For any $i = 1, \ldots, k$, write the multiply warped metric as

$$g = f_i^2 \left(\frac{1}{f_i^2} g_B \oplus \frac{f_1^2}{f_i^2} g_1 \oplus \cdots \oplus g_i \oplus \cdots \oplus \frac{f_k^2}{f_i^2} g_k \right).$$

Since f_i maps B to $\mathbb{R}^+$, this expression shows that g is in the conformal class of a suitable product metric tensor. Hence, the multiply warped metric is locally conformally flat if and only if so is the product metric of (F_i, g_i) and the multiply warped $\tilde{B} \times_{f_1/f_i} F_1 \times \cdots \times \widehat{F_i} \times \cdots \times_{f_k/f_i} F_k$ with base $\tilde{B} \equiv (B, f_i^{-2} g_B)$. This shows that either $\dim F_i = 1$ or otherwise it is of constant sectional curvature, and moreover that $\tilde{B} \times_{f_1/f_i} F_1 \times \cdots \times \widehat{F_i} \times \cdots \times_{f_k/f_i} F_k$ is of constant sectional curvature. Now the result is obtained by iterating this process. $\square$

Remark 3.2. Note from the previous proof that if $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is locally conformally flat, then so is $B \times_{f_1} F_1 \times \cdots \times_{f_{k-1}} F_{k-1}$.

3A. ***Locally conformally flat multiply warped spaces with base of dimension at least 2.*** Although the fibers of any locally conformally flat multiply warped space are of constant curvature, this necessary condition does not suffice for local conformal flatness since it strongly depends on the warping functions. In this section we obtain a local description of such warping functions. As a consequence, we will show the existence of some limitations on the number of fibers of a locally conformally flat multiply warped space and also on their geometries. Assuming that the base (B, g_B) is of constant sectional curvature, the necessary and sufficient conditions for local conformal flatness are as follows.

Theorem 3.3. *Let $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a multiply warped space with s-dimensional base B of constant sectional curvature, where $s \geq 2$. Then M is locally conformally flat if and only if the warping functions satisfy*

$$H_{f_i} = \frac{\Delta f_i}{s} g_B, \tag{3}$$

$$\frac{\Delta f_i}{f_i} + \frac{\Delta f_j}{f_j} = s \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} - s K^B \text{ if } i \neq j, \tag{4}$$

$$K^{F_i} = \|\nabla f_i\|^2 - \frac{2}{s} f_i \Delta f_i - f_i^2 K^B \text{ if } \dim F_i \geq 2, \tag{5}$$

where $i, j = 1, \dots, k$ and K^B and K^{F_i} denote the sectional curvatures of the base (B, g_B) and the fibers (F_i, g_i).

Proof. Condition (3) is equivalent to the constancy of the sectional curvature of the base of a locally conformally flat multiply warped space. Since (B, g_B) is locally conformally flat and $(B, f_i^{-2} g_B)$ is a space of constant sectional curvature by Lemma 3.1, we see that (B, g_B) is of constant sectional curvature if and only if the conformal deformation $g_B \mapsto f_i^{-2} g_B$ preserves the (unique) eigenspaces of the Ricci tensor, and this occurs if and only if f is a solution of the Möbius equation; this proves (3) (see [Kühnel 1988; Osgood and Stowe 1992]).

Next, consider the Weyl curvature tensor given by

$$W(X, Y, Z, T) = R(X, Y, Z, T) + \frac{\tau}{(n-1)(n-2)}\big(\langle X, Z\rangle\langle Y, T\rangle - \langle Y, Z\rangle\langle X, T\rangle\big)$$
$$-\frac{1}{n-2}\big(\rho(X, Z)\langle Y, T\rangle - \rho(Y, Z)\langle X, T\rangle + \langle X, Z\rangle\rho(Y, T) - \langle Y, Z\rangle\rho(X, T)\big).$$

Also note from (1) that the nonzero components of the Ricci tensor of a multiply warped space $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ are given by

$$\text{(6)}\qquad \begin{aligned} \rho(X, Y) &= \rho^B(X, Y) - \sum_i d_i \frac{H_{f_i}(X, Y)}{f_i}, \\ \rho(U_a, V_a) &= \rho^{F_a}(U_a, V_a) \\ &\qquad -\langle U_a, V_a\rangle\left(\frac{\Delta f_a}{f_a} + (d_a - 1)\frac{\langle\nabla f_a, \nabla f_a\rangle}{f_a^2} + \sum_{i\neq a} d_i \frac{\langle\nabla f_a, \nabla f_i\rangle}{f_a f_i}\right) \end{aligned}$$

for all $X, Y \in \mathfrak{L}(B)$ and $U_a, V_a \in \mathfrak{L}(F_a)$, where $d_i = \dim F_i$ and ρ^B and ρ^{F_i} denote the Ricci tensor of the base (B, g_B) and the fibers (F_i, g_i). The scalar curvature of M satisfies

$$\text{(7)}\qquad \begin{aligned} \tau = \tau^B &+ \sum_i \frac{1}{f_i^2}\tau^{F_i} \\ &- 2\sum_i d_i \frac{\Delta f_i}{f_i} - \sum_i d_i(d_i - 1)\frac{\langle\nabla f_i, \nabla f_i\rangle}{f_i^2} - \sum_i \sum_{j\neq i} d_i d_j \frac{\langle\nabla f_i, \nabla f_j\rangle}{f_i f_j}, \end{aligned}$$

where τ^B and τ^{F_i} denote the scalar curvatures of the base and the fibers.

Now, in order to show the necessity of (4) and (5), note that if M is locally conformally flat, then it follows from Remark 3.2 that the warped product space $B \times_{f_a} F_a$ is also locally conformally flat, for all $a = 1, \dots, k$, and thus its Weyl tensor vanishes. A straightforward calculation from (6) and (7) using that $H_{f_a} = (\Delta f_a / s)\, g_B$ shows that

$$W(X, Y, X, Y) = \frac{d_a(d_a - 1)}{(s+d_a-1)(s+d_a-2)}\left(K^B + \frac{2}{s}\frac{\Delta f_a}{f_a} + \frac{K^{F_a}}{f_a^2} - \frac{\langle\nabla f_a, \nabla f_a\rangle}{f_a^2}\right)$$

for all orthogonal unit vectors $X, Y \in \mathfrak{L}(B)$, whence (5). We proceed in an analogous way to show the necessity of (4), just considering the multiply warped space $B \times_{f_a} F_a \times_{f_b} F_b$, which is also locally conformally flat for all $a \neq b \in \{1, \ldots, k\}$. After some calculations from (6) and (7) and using the already proved Equation (5), we have

$$W(X,Y,X,Y) = \frac{2d_a d_b}{(s+d_a+d_b-1)(s+d_a+d_b-2)}\left(K^B + \frac{1}{s}\frac{\Delta f_a}{f_a} + \frac{1}{s}\frac{\Delta f_b}{f_b} - \frac{\langle \nabla f_a, \nabla f_b\rangle}{f_a f_b}\right)$$

for all orthogonal unit vectors $X, Y \in \mathfrak{L}(B)$, which proves (4).

Next we show that conditions (3)–(5) are indeed sufficient for M to be locally conformally flat. Note first that the a-priori nonzero components of the Weyl tensor in a local orthonormal frame $\{X, Y, \ldots, U_1, V_1, \ldots, U_a, V_a, \ldots\}$ with $X, Y, \ldots$ in $\mathfrak{L}(B)$ and $U_a, V_a, \ldots$ in $\mathfrak{L}(F_a)$ are those given by $W(X,Y,X,Y)$, $W(X,U_a,X,U_a)$, $W(U_a,U_b,U_a,U_b)$ and $W(U_a,V_a,U_a,V_a)$. Now, a long but straightforward calculation from (6) and (7), using the equalities $H_{f_i} = (\Delta f_i/s)\, g_B$, shows that

$$\begin{aligned} W(X,Y,X,Y) = &\sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i}\right)\\ &+ \sum_i \sum_{j\neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j}\right), \end{aligned}$$

for all $X, Y \in \mathfrak{L}(B)$. Also, for $X \in \mathfrak{L}(B)$ and $U_a \in \mathfrak{L}(F_a)$, one has

$$\begin{aligned} W(X,U_a,X,U_a) = &\sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i}\right)\\ &+ \sum_i \sum_{j\neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j}\right)\\ &+ \sum_{i\neq a} \frac{d_i}{n-2}\left(\frac{\langle \nabla f_a, \nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{s f_a} - \frac{\Delta f_i}{s f_i} - K^B\right)\\ &+ \frac{d_a-1}{n-2}\left(\frac{\langle \nabla f_a, \nabla f_a\rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{s f_a} - K^B\right). \end{aligned}$$

Next, given $U_a \in \mathfrak{L}(F_a)$ and $U_b \in \mathfrak{L}(F_b)$, where $a \neq b$, we get

$$
\begin{aligned}
W(U_a,U_b,U_a,U_b) = {} & \sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i,\nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i}\right) \\
& + \sum_i \sum_{j\neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i,\nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j}\right) \\
& + \sum_{i\neq a} \frac{d_i}{n-2}\left(\frac{\langle\nabla f_a,\nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{s f_a} - \frac{\Delta f_i}{s f_i} - K^B\right) \\
& + \sum_{i\neq b} \frac{d_i}{n-2}\left(\frac{\langle\nabla f_b,\nabla f_i\rangle}{f_b f_i} - \frac{\Delta f_b}{s f_b} - \frac{\Delta f_i}{s f_i} - K^B\right) \\
& + \frac{d_a-1}{n-2}\left(\frac{\langle\nabla f_a,\nabla f_a\rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{s f_a} - K^B\right) \\
& + \frac{d_b-1}{n-2}\left(\frac{\langle\nabla f_b,\nabla f_b\rangle}{f_b^2} - \frac{K^{F_b}}{f_b^2} - \frac{2\Delta f_b}{s f_b} - K^B\right) \\
& + \left(K^B - \frac{\langle\nabla f_a,\nabla f_b\rangle}{f_a f_b} + \frac{\Delta f_a}{s f_a} + \frac{\Delta f_b}{s f_b}\right),
\end{aligned}
$$

$$
\begin{aligned}
W(U_a,V_a,U_a,V_a) = {} & \sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i,\nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i}\right) \\
& + \sum_i \sum_{j\neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i,\nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j}\right) \\
& + \sum_{i\neq a} \frac{2d_i}{n-2}\left(\frac{\langle\nabla f_a,\nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{s f_a} - \frac{\Delta f_i}{s f_i} - K^B\right) \\
& + \frac{2(d_a-1)}{n-2}\left(\frac{\langle\nabla f_a,\nabla f_a\rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{s f_a} - K^B\right) \\
& + \left(K^B - \frac{\langle\nabla f_a,\nabla f_a\rangle}{f_a^2} + \frac{K^{F_a}}{f_a^2} + \frac{2\Delta f_a}{s f_a}\right),
\end{aligned}
$$

for all $U_a, V_a \in \mathfrak{L}(F_a)$.

It follows from these expressions that the compatibility conditions (4) and (5) suffice to show the local conformal flatness of the multiply warped space M. $\square$

Although Equations (3)–(5) characterize the warping functions of a locally conformally flat multiply warped space with base of constant curvature, they are difficult to deal with. However, they become simpler if the base is assumed to be locally Euclidean:

Theorem 3.4. *Let $M = \mathfrak{U}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a multiply warped space, where $\mathfrak{U}^s \subset \mathbb{R}^s$ with $s \geq 2$. Then M is locally conformally flat if and only if the warping functions satisfy*

$$f_i(\vec{\boldsymbol{x}}) = a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i \tag{8}$$

for all $\vec{\boldsymbol{x}} \in \mathfrak{U}^s$, where $a_i > 0$, $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{b}}_i \in \mathbb{R}^s$. Moreover the warping functions are compatible in the sense that

$$\langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{b}}_j \rangle = 2(a_i c_j + a_j c_i), \qquad i \neq j \tag{9}$$

and the sectional curvature of each fiber of $\dim F_i \geq 2$ *is given by*

$$K^{F_i} = \|\vec{\boldsymbol{b}}_i\|^2 - 4a_i c_i, \quad i, j = 1, \ldots, k. \tag{10}$$

Proof. It follows from [Osgood and Stowe 1992] that the solutions of the Möbius equation in Euclidean space are given by $f_i(\vec{\boldsymbol{x}}) = a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i$ for some $a_i, c_i \in \mathbb{R}$ and $\vec{\boldsymbol{b}}_i \in \mathbb{R}^s$. The result follows by observing the equivalence between (4) and (5) in Theorem 3.3 and (9) and (10) in Theorem 3.4. □

Remark 3.5. We explain how the previous theorem can be extended for not necessarily flat locally conformally flat bases to get a local description of locally conformally flat multiply warped spaces. Since (B, g_B) is locally conformally flat, there exist local coordinates such that $g_B = \boldsymbol{\Psi}^2 g_{\mathfrak{U}^s}$. In such coordinates, the multiply warped metric satisfies

$$g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k = \boldsymbol{\Psi}^2 \left(g_{\mathfrak{U}^s} \oplus \left(\frac{f_1}{\boldsymbol{\Psi}} \right)^2 g_1 \oplus \cdots \oplus \left(\frac{f_k}{\boldsymbol{\Psi}} \right)^2 g_k \right).$$

Therefore the multiply warped product $g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k$ is locally conformally flat if and only if $g_{\mathfrak{U}^s} \oplus (f_1/\boldsymbol{\Psi})^2 g_1 \oplus \cdots \oplus (f_k/\boldsymbol{\Psi})^2 g_k$ is. Hence the warping functions are determined locally by Theorem 3.3 up to a conformal factor $\boldsymbol{\Psi}$, since the warping functions, in local coordinates where $g_B = \boldsymbol{\Psi}^2 g_{\mathfrak{U}^s}$, are given by $f_i(\boldsymbol{x}) = \big(a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i\big) \boldsymbol{\Psi}$ for all $i = 1, \ldots, k$.

Remark 3.6. Locally conformally flat multiply warped spaces can now be easily constructed as follows. Since any warping function of a locally conformally flat multiply warped space $M = \mathfrak{U}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is completely determined by scalars $a_i, c_i \in \mathbb{R}$ and vectors $\vec{\boldsymbol{b}}_i = (b_{i1}, \ldots, b_{is}) \in \mathbb{R}^s$, consider the vectors $\vec{\xi}_i = (b_{i1}, \ldots, b_{is}, a_i, c_i)$ in $\mathbb{R}^{s+2}$. Next, define a Lorentzian inner product in $\mathbb{R}^{s+2}$ by

$$\left(\begin{array}{ccc|cc} 1 & & & & \\ & \ddots^{s} & & & \\ & & 1 & & \\ \hline & & & 0 & -2 \\ & & & -2 & 0 \end{array} \right)$$

and note that equations (9) and (10) of Theorem 3.4 are interpreted in terms of the orthogonality $\vec{\xi}_i \perp \vec{\xi}_j$ (for all $i \neq j$) and $K^{F_i} = \|\vec{\xi}_i\|^2$ (whenever $\dim F_i \geq 2$), respectively. Thus Remark 3.5 has the following consequences:

(i) *A locally conformally flat space $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ has, at most, $s+2$ different fibers, where $s = \dim B$.*

(i) *For the sectional curvatures of the fibers (F_i, g_i) of a locally conformally flat multiply warped space, we have, whenever $\dim F_i \geq 2$:*

(ii.1) *At most $\dim B + 1$ fibers have positive curvature.*

(ii.2) *At most one fiber has nonpositive curvature.*

(iii) *For any locally conformally flat manifold (B^s, g_B), there exists $s+2$ locally defined warping functions $f_i : \mathfrak{U} \subset B \to \mathbb{R}^+$ and (F_i, g_i) spaces of constant curvature such that $M = \mathfrak{U} \times_{f_1} F_1 \times \cdots \times_{f_{s+2}} F_{s+2}$ is locally conformally flat.*

3B. ***Multiply warped spaces with one-dimensional base.*** Recall that a warped product $I \times_f F$ with one-dimensional base is locally conformally flat if and only if the fiber is a space of constant sectional curvature. Local conformal flatness is independent of the warping function f [Lafontaine 1988], in opposition to the case of higher-dimensional base just considered. In what remains of this section we look at the local structure of a locally conformally flat multiply warped space with one-dimensional base.

The characterization in the next theorem is essentially independent of the last warping function, as in the case of metrics of Robertson–Walker type.

Theorem 3.7. *Let $M = I \times_{f_1} F_1 \cdots \times_{f_k} F_k$ be a multiply warped space with one-dimensional base I. Then M is locally conformally flat if and only if, up to a reparametrization of I, one of the following conditions holds:*

(i) *$M = I \times_f F$ is a warped product with fiber F of constant sectional curvature (if $\dim F \geq 2$) and any (positive) warping function f.*

(ii) *$M = I \times_{f_1} F_1 \times_{f_2} F_2$ is a multiply warped product with two fibers of constant sectional curvature (if $\dim F_i \geq 2$) and warping functions*

$$f_1 = (\xi \circ f)\frac{1}{f'}, \qquad f_2 = \frac{1}{f'}$$

where f is a strictly increasing function and ξ is a warping function making $I \times_\xi F_1$ of constant sectional curvature and $(\xi \circ f) > 0$.

(iii) *$M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ is a multiply warped product with three fibers of constant sectional curvature (if $\dim F_i \geq 2$) and warping functions*

$$f_1 = (\xi_1 \circ f)\frac{1}{f'}, \qquad f_2 = (\xi_2 \circ f)\frac{1}{f'}, \qquad f_3 = \frac{1}{f'}$$

where f is a strictly increasing function and ξ_i are warping functions making $I \times_{\xi_1} F_1 \times_{\xi_2} F_2$ of constant sectional curvature such that $\xi_i \circ f > 0$ for $i = 1, 2$.

Proof. This is a local consideration. Proceeding as in Lemma 3.1 it follows that M is locally conformally flat if and only if

$$\frac{1}{f_k^2} g_I \oplus \frac{f_1^2}{f_k^2} g_1 \oplus \cdots \oplus \frac{f_{k-1}^2}{f_k^2} g_{k-1}$$

is of constant sectional curvature opposite to K^{F_k} (if $\dim F_k \geq 2$), and hence $k \leq 3$ (see Remark 3.8). Now, since f_k is strictly positive, it defines a reparametrization on I by $\tau = \int 1/f_k$ to obtain a multiply warped metric $d\tau^2 \oplus \xi_1(\tau)^2 g_1 \oplus \cdots \oplus \xi_{k-1}(\tau)^2 g_{k-1}$ of constant sectional curvature, where the warping functions ξ_i are given in Remark 3.8. Hence $f_i(t) = \xi_i\left(\int 1/f_k\right) f_k(t)$ for $i = 1, \ldots, k-1$, and there are no constraints on the last warping function f_k. □

Remark 3.8. Observe from (2) that, if a multiply warped space M with one-dimensional base is of constant sectional curvature κ, then the warping functions satisfy $f_i'' + \kappa f_i = 0$ and $f_i'^2 + \kappa f_i^2 = K^{F_i}$, which is just an adjustment of the sectional curvatures of the fibers since $f_i'^2 + \kappa f_i^2$ is necessarily constant. Moreover, the necessary compatibility conditions among the different warping functions are given by $f_i' f_j' + \kappa f_i f_j = 0$, $(i \neq j)$, from where it follows that no more than two fibers are admissible. As a consequence, one obtains the following (see also [Mignemi and Schmidt 1998]):

(i) If $K^M = 0$, then $M = I \times_{\xi_1} F_1$ or $M = I \times_{\xi_1} F_1 \times_{\xi_2} F_2$, with warping functions given by $\xi_i(t) = a_i t + b_i$ and $K^{F_i} = a_i^2$ whenever $\dim F_i \geq 2$ for $i = 1, 2$. If the two fibers are different we have $a_1 a_2 = 0$.

(ii) If $K^M = c^2$, then $M = I \times_{\xi_1} F_1$ or $M = I \times_{\xi_1} F_1 \times_{\xi_2} F_2$, with warping functions given by $\xi_i(t) = a_i \sin ct + b_i \cos ct$ and $K^{F_i} = c^2(a_i^2 + b_i^2)$, whenever $\dim F_i \geq 2$ for $i = 1, 2$. If the two fibers are different we have $a_1 a_2 + b_1 b_2 = 0$.

(iii) If $K^M = -c^2$, then $M = I \times_{\xi_1} F_1$ or $M = I \times_{\xi_1} F_1 \times_{\xi_2} F_2$, with warping functions given by $\xi_i(t) = a_i \sinh ct + b_i \cosh ct$ and $K^{F_i} = c^2(a_i^2 - b_i^2)$, provided that $\dim F_i \geq 2$ for $i = 1, 2$. If the two fibers are different we have $a_1 a_2 - b_1 b_2 = 0$.

Remark 3.9. A generalization of the notion of warped product structures $B \times_f F$ to warped bundles has been developed in [Bishop and O'Neill 1969], where it is shown that those results which are local on B remain valid in the warped bundle framework. Therefore, previous results in this section can be generalized to warped bundles.

4. Some global considerations

The existence of nontrivial globally defined solutions of (3) on complete manifolds has significant geometrical consequences [Kühnel 1988]. They leads to:

Theorem 4.1. *Let $M = B \times_f F$ be a locally conformally flat warped product space with complete base (B, g_B) of constant curvature. Then one of the following occurs*:

(i) *B is isometric to the Euclidean space $\mathbb{R}^s$ and the warping function is given by $f(\vec{\boldsymbol{x}}) = a\|\vec{\boldsymbol{x}}\|^2 + \langle\vec{\boldsymbol{b}}, \vec{\boldsymbol{x}}\rangle + c$. Moreover $4ac - \|\vec{\boldsymbol{b}}\|^2 > 0$, $a > 0$ and the fiber F is either one-dimensional or $K^F = \|\vec{\boldsymbol{b}}\|^2 - 4ac < 0$.*

(ii) *B is isometric to a Euclidean sphere $\mathbb{S}^s$ and the warping function is given by*

$$f = -\frac{s-1}{\tau}\psi + \kappa,$$

where τ denotes the scalar curvature of $\mathbb{S}^s$, ψ is the restriction to the sphere of a function Ψ on $\mathbb{R}^{s+1}$ defined by $\Psi(\vec{\boldsymbol{x}}) = \langle\vec{\boldsymbol{a}}, \vec{\boldsymbol{x}}\rangle$ for any $\vec{\boldsymbol{a}} \in \mathbb{R}^{s+1}$, and κ is a constant greater than $(s-1)\|\vec{\boldsymbol{a}}\|/\tau$. Moreover F is either one-dimensional or of constant negative curvature

$$K^F = \frac{(s-1)^2}{\tau^2}\|\vec{\boldsymbol{a}}\|^2 - \kappa^2.$$

(iii) *B is isometric to a warped product $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N$, where N is a complete flat manifold and the warping function is given by $f(t) = \frac{\alpha}{\beta}e^{\beta t+\gamma} + c$ for some constants $\beta, c > 0$. Moreover F is either one-dimensional or $K^F = c^2\beta^2$.*

(iv) *B is isometric to the hyperbolic space $\mathbb{H}^s$ and the warping function is given by*

$$f(\vec{\boldsymbol{x}}) = \frac{a\|\vec{\boldsymbol{x}}\|^2 + \langle\vec{\boldsymbol{b}}, \vec{\boldsymbol{x}}\rangle + c}{x_s},$$

for some $\vec{\boldsymbol{b}} \in \mathbb{R}^s$, where $a > 0$ and either $4ac - (b_1^2 + b_2^2 + \cdots + b_s^2) > 0$ or $4ac - (b_1^2 + b_2^2 + \cdots + b_{s-1}^2) \geq 0$ and $b_s \geq 0$. Moreover the fiber F is either one-dimensional or $K^F = \|\vec{\boldsymbol{b}}\|^2 - 4ac$.

Proof. Since any warping function f defines a global conformal transformation that makes $(B, f^{-2}g_B)$ have constant curvature, it follows from [Kühnel 1988] that B is either a complete and simply connected space form or a warped product $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N$, where N is complete Ricci flat, and thus flat since B is necessarily locally conformally flat. Now the result will follow after a case by case consideration of the possible warping functions and the curvature of the induced metric $f^{-2}g_B$.

Next, observe that a solution of the Möbius equation in $\mathbb{R}^s$, $f(\vec{\boldsymbol{x}}) = a\|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}, \vec{\boldsymbol{x}}\rangle + c$, is everywhere positive if and only if $4ac - \|\vec{\boldsymbol{b}}\|^2 > 0$, $a > 0$, and (i) is obtained since the conformal metric $f^{-2}g_{\mathbb{R}^s}$ has constant curvature $4ac - \|\vec{\boldsymbol{b}}\|^2 > 0$.

If $B \equiv \mathbb{S}^s$, it follows from [Brozos-Vázquez et al. 2005; Xu 1993] that any warping function is given by

$$f = -\frac{s-1}{\tau}\psi + \kappa,$$

where τ is as in the theorem's statement, ψ is a first eigenfunction of the Laplacian and κ is a constant making f positive. Hence ψ is the restriction to the sphere of a function Ψ on $\mathbb{R}^{s+1}$ defined by

$$\Psi(\vec{\boldsymbol{x}}) = \langle \vec{\boldsymbol{a}}, \vec{\boldsymbol{x}}\rangle$$

for $0 \neq \vec{\boldsymbol{a}} \in \mathbb{R}^{s+1}$, [Berger et al. 1971] and the sectional curvature of $(\mathbb{S}^s, f^{-2}g_{\mathbb{S}})$ is the constant $\kappa^2 - \big((s-1)^2/\tau^2\big)\|\vec{\boldsymbol{a}}\|^2 > 0$, proving (ii).

In case (iii) the warping function f gives rise to a warped product decomposition of B as $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N$, where the warping function is of the form $f(t) = (\alpha/\beta)\, e^{\beta t+\gamma} + c$ for some positive constant c [Kühnel 1988]. This defines a global conformal transformation such that $(B, f^{-2}g_B)$ has constant curvature $-c^2\beta^2$; hence the result.

Finally, assume B to be hyperbolic space. We work in the half-space model, with domain $\{x_s < 0\}$ and metric obtained by a conformal deformation of the Euclidean metric: $(\mathbb{H}^s, x_s^{-2}g_{\mathbb{R}^s})$. The general form of the warping functions then arises from Remark 3.5. Next note that $a\|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}, \vec{\boldsymbol{x}}\rangle + c$ is positive in hyperbolic space if and only if $a > 0$ and the minimum of the paraboloid is positive $(4ac - \|\vec{\boldsymbol{b}}\|^2 > 0)$ or occurs on the lower half-space (so $-b_s/(2a) \le 0$) and the intersection of the paraboloid and the hyperplane $x_s = 0$ is positive, which gives

$$4ac - (b_1^2 + b_2^2 + \cdots + b_{s-1}^2) \ge 0.$$

Further note that the induced metric $f^{-2}g_B$ is of constant curvature $4ac - \|\vec{\boldsymbol{b}}\|^2 > 0$ but it has no preferred sign in opposition to case (i). □

Theorem 4.2. *Let $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$, $k \ge 2$, be a locally conformally flat multiply warped product space with complete and simply connected base (B, g_B) of constant curvature. Then B is isometric to the hyperbolic space $\mathbb{H}^s$ and for each $k \le s+2$ there exists locally conformally flat multiply warped spaces $M = \mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$.*

Proof. First of all, note that since B is assumed to be simply connected, the possibly warping functions reduce to cases (i), (ii) and (iv) in Theorem 4.1. Next, in order to show that a locally conformally flat multiply warped space whose base is the

Euclidean space or the sphere reduces to a warped product, an analysis of the curvature of the induced metric $(B, f^{-2}g_B)$ is needed. Assuming that the space $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is locally conformally flat, so is $M_{ij} = B \times_{f_i} F_i \times_{f_j} F_j$, whose metric tensor can be expressed as

$$g_{M_{ij}} \equiv f_j^2 \left(\frac{1}{f_j^2} g_B \oplus \frac{1}{f_j^2} f_i^2 g_i \oplus g_j \right).$$

This shows that $M_{\hat{\jmath} i} = B \times F_i$, equipped with the metric $(1/f_j^2)\, g_B \oplus (f_i^2/f_j^2)\, g_i$, has constant sectional curvature $K^{M_{\hat{\jmath} i}}$. Since $M_{\hat{\jmath} i}$ can be viewed as a warped product, it follows from (2) that

$$K^{M_{\hat{\jmath} i}}(X \wedge U) = -\frac{f_j^3}{f_i} \widehat{H}_{f_i/f_j}(X, X)$$

for all unit vectors $X \in \mathfrak{L}(B)$, $U \in \mathfrak{L}(F_i)$, where $\widehat{H}_{f_i/f_j}$ denotes the Hessian of f_i/f_j with respect to the conformal metric $f_j^{-2} g_B$. Now, since

$$\widehat{H}_{f_i/f_j} = \frac{1}{f_j} \left(H_{f_i} - \frac{f_i}{f_j} H_{f_j} - \frac{1}{f_j} g_B(\nabla f_j, \nabla f_i) g_B + \frac{f_i}{f_j^2} g_B(\nabla f_j, \nabla f_j) g_B \right)$$

(see [García-Río and Kupeli 1999]), one gets

$$-K^{M_{\hat{\jmath} i}} \frac{f_i}{f_j} g_B = f_j H_{f_i} - f_i H_{f_j} - g_B(\nabla f_j, \nabla f_i) g_B + \frac{f_i}{f_j} g_B(\nabla f_j, \nabla f_j) g_B. \tag{11}$$

Proceeding similarly, and expressing the metric tensor of $M_{ij} = B \times_{f_i} F_i \times_{f_j} F_j$ as

$$g_{M_{ij}} \equiv f_i^2 \left(\frac{1}{f_i^2} g_B \oplus \frac{1}{f_i^2} f_j^2 g_j \oplus g_i \right),$$

one also gets

$$-K^{M_{\hat{\imath} j}} \frac{f_j}{f_i} g_B = f_i H_{f_j} - f_j H_{f_i} - g_B(\nabla f_i, \nabla f_j) g_B + \frac{f_j}{f_i} g_B(\nabla f_i, \nabla f_i) g_B. \tag{12}$$

Now it follows from (11) and (12) that

$$-K^{M_{\hat{\jmath} i}} f_i^2 - K^{M_{\hat{\imath} j}} f_j^2 = \| f_j \nabla f_i - f_i \nabla f_j \|^2. \tag{13}$$

As an immediate application of this equality we have:

Proposition 4.3. *If $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a locally conformally flat multiply warped space, then the (constant) sectional curvature of $(B, f_i^{-2} g_B)$ cannot be nonnegative for two different warping functions.*

Proof. If $\| f_j \nabla f_i - f_i \nabla f_j \|^2 = 0$, then $\nabla \ln(f_i/f_j) = 0$, implying f_i is a multiple of f_j in opposition to Remark 2.1. This shows there exist no nontrivial locally conformally flat multiply warped metrics having Euclidean space or the sphere for base. □

Finally, in order to show the existence of complete locally conformally flat multiply warped products with base $\mathbb{H}^s$ and the maximum number of fibers, just consider the set of functions

$$\begin{aligned}
\bar{f}_1(\vec{x}) &= \tfrac{1}{4}(s+4)\|\vec{x}\|^2 + x_1 + \cdots + x_{s-1} + (s+2)x_s + s + 1,\\
\bar{f}_2(\vec{x}) &= \tfrac{1}{4}(s+4)\|\vec{x}\|^2 + x_1 + \cdots + x_{s-1} + s x_s + s - 1,\\
\bar{f}_3(\vec{x}) &= \|\vec{x}\|^2 + 3x_s + 2,\\
\bar{f}_4(\vec{x}) &= \|\vec{x}\|^2 + x_{s-1} + 2x_s + 2,\\
\bar{f}_5(\vec{x}) &= \|\vec{x}\|^2 + x_{s-2} + 2x_s + 2,\\
&\;\vdots\\
\bar{f}_{s+2}(\vec{x}) &= \|\vec{x}\|^2 + x_1 + 2x_s + 2.
\end{aligned}$$

These functions are positive in hyperbolic space and satisfy the compatibility conditions in Theorem 3.4. Hence, proceeding as in Remark 3.5, one sees that $f_i(\vec{x}) = \bar{f}_i(\vec{x})/x_s$ are positive warping functions on $\mathbb{H}^s$ that define a locally conformally flat multiply warped space for either one- or higher-dimensional fibers of suitable constant curvature as in Remark 3.6. This completes the proof of the theorem. □

Remark 4.4. If $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is locally conformally flat with compact base B, then $k = 1$. Indeed, let f_i, f_j be two distinct warping functions. Proceeding as in Lemma 3.1, we conclude that $(B, f_i^{-2} g_B)$ and $(B, f_i^{-2} g_B)$ are of constant sectional curvature. Since f_i/f_j is not constant it follows that $(B, f_i^{-2} g_B)$ and $(B, f_i^{-2} g_B)$ are conformal metrics of constant curvature, and thus Euclidean spheres [Kühnel 1988], from which the result follows.

Examples of complete locally conformally flat manifolds of nonpositive curvature. Proceeding as in [Bishop and O'Neill 1969], note that a multiply warped manifold $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is complete if and only if the base and all the fibers are so. In such a case, the sectional curvature is nonpositive if and only if the following conditions are satisfied:

(a) the sectional curvatures of the base and the fibers are nonpositive: $K^B \leq 0$ and $K^{F_i} \leq 0$.

(b) The warping functions are convex, i.e., H_{f_i} is positive semidefinite.

(c) $\langle \nabla f_i, \nabla f_j \rangle \geq 0$ for all $i \neq j$.

Condition (a) can be omitted whenever the base and the corresponding fiber are one-dimensional.

A complete locally conformally flat multiply warped space with simply connected base of constant curvature is of nonpositive sectional curvature if and only if one of the following conditions holds:

(i) $B \equiv \mathbb{R}^s$, and then $\mathbb{R}^s \times_f F$ is of nonpositive sectional curvature for any warping function f as in Theorem 4.1.

(ii) $B \equiv \mathbb{H}^s$, and then $\mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is of nonpositive sectional curvature if and only if the warping functions

$$f_i(\vec{\boldsymbol{x}}) = \frac{a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i}{x_s}$$

satisfy

$$f_i \geq 2b_{is} \quad \text{if } \dim F_i \geq 2 \qquad \text{and} \qquad 1 \geq \frac{b_{is}}{f_i} + \frac{b_{js}}{f_j} \quad \text{for all } i \neq j.$$

The simplest examples illustrating this situation are as follows.

(a) Let M be the product manifold $M = \mathbb{H}^2 \times F_1^d \times F_2$ equipped with the multiply warped metric tensor defined by the warping functions

$$f_1(\vec{\boldsymbol{x}}) = \kappa \frac{\frac{1}{2}\|\vec{\boldsymbol{x}}\|^2 + x_2 + 1}{x_2}, \qquad f_2(\vec{\boldsymbol{x}}) = \frac{\frac{1}{4}\|\vec{\boldsymbol{x}}\|^2 + x_2 + \frac{1}{2}}{x_2},$$

where F_2 is one-dimensional and F_1 is either one-dimensional or of negative sectional curvature $K^{F_1} = -\kappa^2$.

(b) Let M be the product manifold $M = \mathbb{H}^2 \times {F_1}^d \times F_2$ equipped with the multiply warped metric tensor defined by the warping functions

$$f_1(\vec{\boldsymbol{x}}) = \frac{\frac{1}{4}\|\vec{\boldsymbol{x}}\|^2 + x_1 + 1}{x_2}, \qquad f_2(\vec{\boldsymbol{x}}) = \frac{\frac{1}{2}\|\vec{\boldsymbol{x}}\|^2 + 2x_1 + x_2 + 2}{x_2}.$$

where F_2 is one-dimensional and F_1 is either one-dimensional or flat.

Further, if $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a locally conformally flat multiply warped space with base of constant sectional curvature, then it follows from (3)–(5) that M has at most $(k+1)$-different Ricci curvatures given by

$$\begin{aligned} \lambda_B &= (s-1)K^B - \frac{1}{s}\sum_i d_i \frac{\Delta f_i}{f_i}, \\ \lambda_{F_a} &= (s-1)K^B - \frac{1}{s}\sum_i d_i \frac{\Delta f_i}{f_i} - (n-2)\left(K^B + \frac{1}{s}\frac{\Delta f_a}{f_a}\right). \end{aligned} \tag{14}$$

A straightforward calculation shows that examples (a) have exactly three different Ricci curvatures, but only two occur in case (b).

Remark 4.5. Observe that the base and the fibers of a multiply warped product play completely different roles. For instance, if M is a warped product with compact base and nonpositive sectional curvature, then it follows from (2) that the warping function satisfies $H_f \geq 0$, and thus f is constant, which shows that M must be a direct product. In opposition, one can easily construct examples of locally conformally flat multiply warped spaces of nonpositive sectional curvature with compact fibers. In addition to examples (b) above, those metrics in Theorem 4.1(iii) can also be viewed as multiply warped metrics with one-dimensional base. A straightforward calculation shows that $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N \times_{\frac{\alpha}{\beta} e^{\beta t+\gamma}+c} F$ is of nonpositive sectional curvature if and only if F is one-dimensional. Further note that both N and F can be chosen to be compact. Further, $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N \times_{(\alpha/\beta)\, e^{\beta t+\gamma}+c} F$ has three distinct Ricci curvatures and therefore is not isometric to example (b), where only two distinct Ricci curvatures occur.

Remark 4.6. As an immediate application of (14), a locally conformally flat multiply warped space $M = \mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ ($s \geq 2$) has nonpositive Ricci curvature if and only if the warping functions

$$f_i(\vec{\boldsymbol{x}}) = \frac{a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i}{x_s}$$

satisfy

$$\sum_i d_i \frac{b_{is}}{f_i} \leq n-1 \quad \text{for all } i = 1, \ldots, k,$$

$$(n-2)\frac{b_{is}}{f_i} + \sum_j d_j \frac{b_{js}}{f_j} \leq n-1 \quad \text{for all } i \neq j \in \{1, \ldots, k\}.$$

The simplest examples of complete locally conformally flat manifolds with nonpositive Ricci curvature consist of

$$\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2 \times_{f_3} \mathbb{S}^2 \times_{f_4} \mathbb{H}^2$$

with warping functions

$$f_1(\vec{\boldsymbol{x}}) = \frac{\frac{3}{2}\|\vec{\boldsymbol{x}}\|^2 + x_1 + 4x_2 + 3}{x_2}, f_2(\vec{\boldsymbol{x}}) = \frac{\|\vec{\boldsymbol{x}}\|^2 + 3x_2 + 2}{x_2},$$
$$f_3(\vec{\boldsymbol{x}}) = \frac{\frac{1}{2}\|\vec{\boldsymbol{x}}\|^2 + x_1 + 2x_2 + 2}{x_2}, f_4(\vec{\boldsymbol{x}}) = \frac{\|\vec{\boldsymbol{x}}\|^2 + x_1 + 2x_2 + 1}{x_2}.$$

The same conclusions hold for the multiply warped spaces $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2 \times_{f_3} \mathbb{S}^2$, $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2$ and $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2$. Also note from (13) that if $\mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a locally conformally flat space of nonpositive sectional curvature, there is at most one fiber F_a with $\dim F_a \geq 2$, which must necessarily be of nonpositive sectional curvature. This shows that none of the examples above has nonpositive sectional curvature.

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