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AN OPTIMAL SYSTOLIC INEQUALITY FOR CAT(0) METRICS IN GENUS TWO

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We prove an optimal systolic inequality for CAT(0) metrics on a genus 2 surface. We use a Voronoi cell technique, introduced by C. Bavard in the hyperbolic context. The equality is saturated by a flat singular metric in the conformal class defined by the smooth completion of the curve $y^2 = x^5 - x$. Thus, among all CAT(0) metrics, the one with the best systolic ratio is composed of six flat regular octagons centered at the Weierstrass points of the Bolza surface.

1. Hyperelliptic surfaces of nonpositive curvature

Over half a century ago, a student of C. Loewner's named P. Pu [1952] presented in this journal the first two optimal systolic inequalities, which came to be known as the Loewner inequality for the torus and Pu's inequality for the real projective plane. (See (5–2) on page 104 for the latter.)

The last couple of years have seen the discovery of a number of new systolic inequalities [Ammann 2004; Bangert and Katz 2003; 2004; Bangert et al. 2005; 2006a; 2006b; Ivanov and Katz 2004; Katz 2006; Katz and Lescop 2005; Katz and Sabourau 2006; Katz et al. 2006; Sabourau 2004], as well as near-optimal asymptotic bounds [Hamilton 2005; Katz 2003; Katz and Sabourau 2005; Katz et al. 2005; Rudyak and Sabourau ≥ 2006; Sabourau 2006; ≥ 2006]. A number of questions posed in [Croke and Katz 2003] have thus been answered. A general framework for systolic geometry in a topological context is proposed in [Katz and Rudyak 2005; 2006]. See [Katz ≥ 2006] for an overview of systolic problems. The homotopy 1-systole, denoted $\mathrm{sys}\pi_1(X)$, of a compact metric space X is the least length of a noncontractible loop of X.

Given a metric $\mathcal{G}$ on a surface, let $\mathrm{SR}(\mathcal{G})$ denote its systolic ratio

$$\mathrm{SR}(\mathcal{G}) = \frac{\mathrm{sys}\pi_1(\mathcal{G})^2}{\mathrm{area}(\mathcal{G})}.$$

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The *optimal systolic ratio* of a compact Riemann surface Σ is defined as $\mathrm{SR}(\Sigma) = \sup_{\mathcal{G}} \mathrm{SR}(\mathcal{G})$, where the supremum is over all metrics in the conformal type of Σ. Finally, given a smooth compact surface M, its optimal systolic ratio is defined by setting $\mathrm{SR}(M) = \sup_{\Sigma} \mathrm{SR}(\Sigma)$, where the supremum is over all conformal structures Σ on M. The latter ratio is known for the Klein bottle — see the bound (5–1) on page 104 — in addition to the torus and real projective plane already mentioned.

In the class of all metrics without any curvature restrictions, no singular flat metric on a surface of genus 2 can give the optimal systolic ratio in this genus [Sabourau 2004]. The best available upper bound for the systolic ratio of an arbitrary genus 2 surface is $\gamma_2 \simeq 1.1547$ [Katz and Sabourau 2006].

The precise value of SR for the genus 2 surface has so far eluded researchers [Calabi 1996; Bryant 1996]. We propose an answer in the framework of negatively curved, or more generally, CAT(0) metrics.

The term "CAT(0) space" evokes an extension of the notion of a manifold of nonpositive curvature to encompass singular spaces. We will use the term to refer to surfaces with metrics with only mild quotient singularities, defined below. Here the condition of nonpositive curvature translates into a lower bound of 2π for the total angle at the singularity. We need such an extension so as to encompass the metric that saturates our optimal inequality (1–1).

A mild quotient singularity is defined as follows. Consider a smooth metric on $\mathbb{R}^2$. Let $q \geq 1$ be an integer. Consider the q-fold cover X_q of $\mathbb{R}^2 \setminus \{0\}$ with the induced metric. We compactify X_q in the neighborhood of the origin to obtain a complete metric space $X_q^c = X_q \cup \{0\}$.

Definition 1.1. Suppose X_q^c admits an isometric action of $\mathbb{Z}_p$ fixing the origin. Then we can form the orbit space $Y_{p,q} = X_q^c/\mathbb{Z}_p$. The space $Y_{p,q}$ is then called mildly singular at the origin.

The total angle at the singularity is then $2\pi q/p$, and the CAT(0) condition is $q/p \geq 1$.

Remark 1.2. Alternatively, a point is singular of total angle $2\pi(1+\beta)$ if the metric is of the form $e^{h(z)}|z|^{2\beta}|dz|^2$ in its neighborhood, where $|dz|^2 = dx^2 + dy^2$. See [Troyanov 1990, p. 915].

Theorem 1.3. *Every* CAT(0) *metric* $\mathcal{G}$ *on a surface* Σ_2 *of genus* 2 *satisfies the optimal inequality*

$$\mathrm{SR}(\Sigma_2, \mathcal{G}) \leq \frac{1}{3} \cot \frac{\pi}{8} = \frac{1}{3}\big(\sqrt{2}+1\big) = 0.8047\ldots \tag{1–1}$$

The inequality is saturated by a singular flat metric, with 16 *conical singularities, in the conformal class of the Bolza surface.*

The Bolza surface is described in Section 2. The optimal metric is described in more detail in Section 3. Theorem 1.3 is proved in Section 4 based on the octahedral triangulation of S^2.

Remark 1.4. A similar optimal inequality can be proved for hyperelliptic surfaces of genus 5 based on the icosahedral triangulation [Bavard 1986].

2. Distinguishing 16 points on the Bolza surface

The Bolza surface $\mathcal{B}$ is the smooth completion of the affine algebraic curve

$$y^2 = x^5 - x. \tag{2–1}$$

It is the unique Riemann surface of genus 2 with a group of holomorphic automorphisms of order 48. (A way of passing from an affine hyperelliptic surface to its smooth completion is described in [Miranda 1995, p. 60–61].)

Definition 2.1. A conformal involution J of a compact Riemann surface Σ of genus g is called *hyperelliptic* if J has precisely $2g+2$ fixed points. The fixed points of J are called the Weierstrass points of Σ.

The quotient Riemann surface Σ/J is then necessarily the Riemann sphere, denoted henceforth S^2. Let $Q : \Sigma \to S^2$ be the conformal ramified double cover, with $2g+2$ branch points. Thus, J acts on Σ by sheet interchange. Recall that every surface of genus 2 is hyperelliptic, that is, admits a hyperelliptic involution [Farkas and Kra 1992, Proposition III.7.2].

We make note of 16 special points on $\mathcal{B}$. We call a point *special* if it is a fixed point of an order 3 automorphism of $\mathcal{B}$.

Consider the regular octahedral triangulation of $S^2 = \mathbb{C} \cup \infty$. Its set of vertices is conformal to the set of roots of the polynomial $x^5 - x$ of formula (2–1), together with the unique point at infinity. Thus the six points in question can be thought of as the ramification points of the ramified conformal double cover $Q : \mathcal{B} \to S^2$, while the 16 special points of $\mathcal{B}$ project to the eight vertices of the cubic subdivision dual to the octahedral triangulation.

In other words, the x-coordinates of the ramification points are

$$\{0, \infty, 1, -1, i, -i\},$$

which stereographically correspond to the vertices of a regular inscribed octahedron. The conformal type therefore admits the symmetries of the cube. If one includes both the hyperelliptic involution and the real (antiholomorphic) involution of $\mathcal{B}$ corresponding to the complex conjugation $(x, y) \to (\bar{x}, \bar{y})$ of $\mathbb{C}^2$, one obtains the full symmetry group $\mathrm{Aut}(\mathcal{B})$, of order

$$|\mathrm{Aut}(\mathcal{B})| = 96; \tag{2–2}$$

see [Kuusalo and Näätänen 1995, p. 404] for more details.

Lemma 2.2. *The hyperbolic metric of* $\mathcal{B}$ *admits* 12 *systolic loops. The* 12 *loops are in one-to-one correspondence with the edges of the octahedral decomposition of* S^2*. The correspondence is given by taking the inverse image under* Q *of an edge. The* 12 *systolic loops cut the surface into* 16 *hyperbolic triangles. The centers of the triangles are the* 16 *special points.*

See [Schmutz 1993, §5] for further details. The Bolza surface is extremal for two distinct problems:

- systole of hyperbolic surfaces [Bavard 1992; Schmutz 1993, Theorem 5.2];
- conformal systole of Riemann surfaces [Buser and Sarnak 1994].

The square of the conformal systole of a Riemann surface is also known as its Seshadri constant [Kong 2003]. The Bolza surface is also conjectured to be extremal for the first eigenvalue of the Laplacian. Such extremality has been verified numerically [Jakobson et al. 2005]. The evidence above suggests that the systolically extremal surface may lie in the conformal class of $\mathcal{B}$, as well. Meanwhile, we have the following result, proved in Section 5.

Theorem 2.3. *The Bolza surface* $\mathcal{B}$ *satisfies* $\mathrm{SR}(\mathcal{B}) \leq \frac{\pi}{3}$.

Note that Theorems 2.3 and 1.3 imply that $\mathrm{SR}(\mathcal{B}) \in [0.8, 1.05]$.

3. A flat singular metric in genus two

The optimal systolic ratio of a genus 2 surface $(\Sigma_2, \mathcal{G})$ is unknown, but it satisfies the Loewner inequality [Katz and Sabourau 2006]. Here we discuss a *lower* bound for the optimal systolic ratio in genus 2, briefly described in [Croke and Katz 2003].

The example of M. Berger (see [Gromov 1983, Example 5.6.B′]) in genus 2 is a singular flat metric with conical singularities. It has systolic ratio $\mathrm{SR} = 0.6666$. This ratio was improved by F. Jenni [Jenni 1984], who identified the hyperbolic genus 2 surface with the optimal systolic ratio among all hyperbolic genus 2 surfaces (see also C. Bavard [Bavard 1992] and P. Schmutz [Schmutz 1993, Theorem 5.2]). The surface in question is a (2,3,8) triangle surface. Its conformal class is that of the Bolza surface (Section 2). It admits a regular hyperbolic octagon as a fundamental domain, and has 12 systolic loops of length $2x$, where $x = \cosh^{-1}(1+\sqrt{2})$. It has

$$\mathrm{sys}\pi_1 = 2\log\bigl(1+\sqrt{2}+\sqrt{2+2\sqrt{2}}\,\bigr),$$

area 4π, and systolic ratio $\mathrm{SR} \simeq 0.7437$. This ratio can be improved to 0.8047, as we shall see. The history for genus 2 so far can be summarized as follows:

$$\mathrm{SR}(\mathcal{G}) = \frac{\mathrm{sys}\pi_1(\mathcal{G})^2}{\mathrm{area}(\mathcal{G})} = \begin{cases} 0.6666 & \text{(Berger)} \\ 0.7437 & \text{(Jenni)} \\ 0.8047 & \text{(our metric } \mathcal{G}_{\mathbb{O}} \text{ on Bolza surface)} \end{cases}$$

Proposition 3.1. *The conformal class of the Bolza surface $\mathcal{B}$ admits a metric, denoted $\mathcal{G}_{\mathbb{O}}$, with the following properties*:

(1) *the metric is singular flat, with conical singularities precisely at the* 16 *special points of Section 2*;

(2) *each singularity is of total angle $\frac{9}{4}\pi$, so that the metric $\mathcal{G}_{\mathbb{O}}$ is* CAT(0);

(3) *the metric is glued from six flat regular octagons, centered on the Weierstrass points, while the* 1*-skeleton projects under $Q : \mathcal{B} \to S^2$ to that of the dual cube in S^2*;

(4) *the systolic ratio equals* $\mathrm{SR}(\mathcal{G}_{\mathbb{O}}) = \frac{1}{3}\left(\sqrt{2}+1\right) > \frac{4}{5}$.

Proof. The octahedral triangulation of the sphere, discussed in Section 2, lifts to a triangulation of $\mathcal{B}$ consisting of 16 triangles, which we think of as being "equilateral". Here eight equilateral triangles are connected cyclically around each of the six Weierstrass vertices of the triangulation of $\mathcal{B}$.

We further subdivide each equilateral triangle into three isosceles triangles, with a common vertex at the center of the equilateral triangle. We equip each of the 48 isosceles triangles with a flat metric with obtuse angle $\frac{3}{4}\pi$.

Each of the six Weierstrass vertices of the original triangulation is a smooth point, since the total angle is eight times $\pi/4$. Each equilateral triangle possesses a singularity at the center with total angle $\frac{9}{4}\pi > 2\pi$. Alternatively, we can apply the Gauss–Bonnet formula $\sum_\sigma \alpha(\sigma) = 2g-2$ in genus 2, with 16 isometric singularities. Here the sum is over all singularities σ of a singular flat metric on a surface of genus g, where the cone angle at singularity σ is $2\pi(1+\alpha(\sigma))$. Since the metric $\mathcal{G}_{\mathbb{O}}$ is smooth at a Weierstrass point of $\mathcal{B}$, the metric has only 16 singularities, precisely at the special points of Section 2, proving items 1 and 2 of the proposition.

Let x denote the side length of the equilateral triangle. The barycentric subdivision of each equilateral triangle consists of six copies of a flat right angle triangle, denoted $\mathcal{R}$, with side $x/2$ and adjacent angle $\pi/8$. We thus obtain a decomposition of the metric $\mathcal{G}_{\mathbb{O}}$ into 96 copies of the triangle $\mathcal{R}$, which can be thought of as a fundamental domain for the action of $\mathrm{Aut}(\mathcal{B})$; see (2–2).

We have $\mathrm{sys}\pi_1(\mathcal{G}_{\mathbb{O}}) = 2x$ by Lemma 3.2, proving item 4 of the proposition. To prove item 3, note that the union of the 16 triangles $\mathcal{R}$ with a common Weierstrass

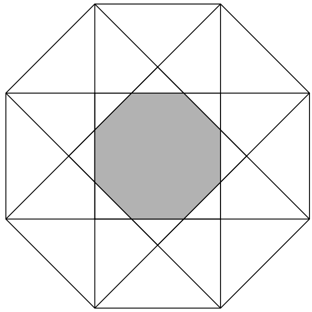

Figure 1. Flat regular octagon obtained as the union of 16 right triangles $\mathcal{R}$ with side $x/2$ and adjacent angle $\pi/8$. The shaded interior octagon represents the region with four geodesic loops through every point.

vertex is a flat regular octagon. The latter is represented in Figure 1, together with the systolic loops passing through it. □

Lemma 3.2. *The systole of the singular flat* CAT(0) *metric on the Bolza surface equals twice the distance between a pair of adjacent Weierstrass points.*

Proof. Consider the smooth closed geodesic $\gamma \subset \mathcal{B}$ that is the inverse image under the map $Q : \mathcal{B} \to S^2$ of an edge of the octahedron; see Lemma 2.2. Let x be the distance between a pair of opposite sides of the regular flat octagon in $\mathcal{B}$, or equivalently, the distance between a pair of adjacent Weierstrass points. Thus, $\operatorname{length}(\gamma) = 2x$. Consider a loop $\alpha \subset \mathcal{B}$ whose length satisfies

$$\operatorname{length}(\alpha) < 2x. \tag{3–1}$$

We will prove that there are two possibilities for α: it is either contractible, or freely homotopic to one of the 12 geodesics γ of the type described above. On the other hand, γ is necessarily length minimizing in its free homotopy class, by the CAT(0) property of the metric [Bridson and Haefliger 1999, Theorem 6.8]. This will rule out the second possibility, and prove the lemma.

Denote by $\mathcal{B}^{(1)}$ the graph on $\mathcal{B}$ given by the inverse image under Q of the 1-skeleton of the cubic subdivision of S^2. The graph $\mathcal{B}^{(1)}$ partitions the surface into six regular octagons, denoted Ω_k:

$$\mathcal{B} = \bigcup_{k=1}^{6} \Omega_k. \tag{3–2}$$

We will deform α to a loop $\beta \subset \mathcal{B}^{(1)}$ as follows. The partition (3–2) induces a partition of the loop α into arcs α_i, each lying in its respective octagon Ω_{k_i}. We deform each α_i, without increasing length, to the line segment $[p_i, q_i] \subset \Omega_{k_i}$. The boundary points p_i, q_i of α_i split the boundary closed curve $\partial\Omega_{k_i} \subset \mathcal{B}^{(1)}$ into a pair of paths. Let $\beta_i \subset \partial\Omega_{k_i}$ be the shorter of the two paths. Denote by y the distance between adjacent vertices of the octagon. Then clearly

$$\text{length}(\beta_i) \le \frac{4y}{x}\,\text{length}(\alpha_i). \tag{3–3}$$

We first deform the loop α into the graph $\mathcal{B}^{(1)}$. The deformation fixes the intersection points $\alpha \cap \mathcal{B}^{(1)}$. Inside Ω_{k_i} we deform the arc α_i to the path β_i. The length of the resulting loop is at most

$$\frac{4y}{x}\,\text{length}(\alpha) < 8y$$

by (3–1) and (3–3). Therefore, its homotopy class in $\mathcal{B}^{(1)}$ can be represented by an imbedded loop $\beta \subset \mathcal{B}^{(1)}$ of length at most $8y$. Thus, β contains fewer than eight edges of $\mathcal{B}^{(1)}$. Since the number of edges must be even, its image under Q must retract to a circuit with at most six edges in the 1-skeleton of the cubical subdivision of S^2. If the number is four, then the circuit lies in the boundary of a square face of the cube in S^2. But the boundary of a face does not lift to $\mathcal{B}$, since it surrounds a single ramification point, namely the center of the square face.

Hence there must be six edges in the circuit. There are two types of circuits with six edges in the 1-skeleton of the cubical subdivision of S^2:

(a) the boundary of the union of a pair of adjacent squares;

(b) a path consisting of the edges meeting a suitable great circle.

However, a path of type (b) surrounds an odd number, namely 3, of ramification points, and hence does not lift to the genus 2 surface. Meanwhile, a path of type (a) surrounds two ramification points, and hence does lift to the surface. Such a path is freely homotopic in $\mathcal{B}$ to one of the 12 geodesics of type γ (Lemma 2.2), completing the proof. □

4. Voronoi cells and Euler characteristic

The following proposition provides a preliminary lower bound on the area of hyperelliptic surfaces with nonpositive curvature.

Proposition 4.1. *Every J-invariant* CAT(0) *metric $\mathcal{G}$ on a closed hyperelliptic surface Σ_g of genus g satisfies the bound*

$$\text{SR}(\Sigma_g, \mathcal{G}) \le 8((g+1)\pi)^{-1}.$$

Proof. To prove this scale-invariant inequality, we normalize the metric on $\Sigma = \Sigma_g$ to unit systole, that is, $\text{sys}\pi_1(\Sigma, \mathcal{G}) = 1$. The preimage by $Q : \Sigma \to S^2$ of an arc of S^2 joining two distinct branch points forms a noncontractible loop on Σ. Therefore the distance between two Weierstrass points is at least $\frac{1}{2}\operatorname{sys}\pi_1(\Sigma,\mathcal{G}) = \frac{1}{2}$. Thus we obtain $2g+2$ disjoint disks of radius $\frac{1}{4}$, centered at the Weierstrass points. Since the metric is CAT(0), the area of each disk is at least $\frac{\pi}{16}$. Thus,

$$\operatorname{area}(\Sigma,\mathcal{G}) \geq \frac{g+1}{8}\pi. \qquad \square$$

An *optimal* lower bound requires a more precise estimate on the area of the Voronoi cells. The idea is to replace area of balls by area of polygons, where control over the number of sides is provided by the Euler characteristic [Bavard 1992].

Denote by $u : \tilde{\Sigma} \to \Sigma$ its universal cover. Let $\{x_i \mid i \in \mathbb{N}\}$ be an enumeration of the lifts of Weierstrass points on $\tilde{\Sigma}$. The Voronoi cell $V_i \subset \tilde{\Sigma}$ centered at x_i is defined as the set of points closer to x_i than to any other lift of a Weierstrass point. In formulas,

$$V_i = \left\{x \in \tilde{\Sigma} \;\middle|\; d(x,x_i) \leq d(x,x_j) \text{ for every } j \neq i\right\}.$$

The Voronoi cells on $\tilde{\Sigma}$ are polygons whose edges are arcs of the equidistant curves between a pair of lifts of Weierstrass points. Note that these edges are not necessarily geodesics. The Voronoi cells on $\tilde{\Sigma}$ are topological disks, while their projections $u(V_i) \subset \Sigma$ may have more complicated topology. Thus, the surface Σ decomposes into $2g+2$ images of Voronoi cells, centered at the $2g+2$ Weierstrass points. By the number of sides of $u(V)$ we will mean the number of sides of the polygon V.

Lemma 4.2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two* CAT(0) *metrics lying in the same conformal class. Then, the averaged metric $\mathcal{G} = \frac{1}{2}(\mathcal{G}_1+\mathcal{G}_2)$ is* CAT(0), *as well.*

Proof. Choose a point $x \in \Sigma$ and a metric $\mathcal{G}_0$ in its conformal class, which is flat in a neighborhood of x. Every metric $\mathcal{G} = H\,\mathcal{G}_0$ conformal to it satisfies

$$K_{\mathcal{G}} H = K_{\mathcal{G}_0} - \tfrac{1}{2}\Delta \log H$$

(see [Gallot et al. 1990, p. 265]), where $K_{\mathcal{G}}$ and $K_{\mathcal{G}_0}$ are the Gaussian curvatures of $\mathcal{G}$ and $\mathcal{G}_0$, and Δ is the Laplacian of $\mathcal{G}_0$ with $\Delta f = \operatorname{div}\nabla f$. Thus, the metrics $\mathcal{G}_i$ can be written as $\mathcal{G}_i = e^{h_i}\,\mathcal{G}_0$, where h_i is subharmonic in the neighborhood of x, that is, $\Delta h_i \geq 0$. A simple computation shows that

$$\Delta \log H \geq \frac{e^{h_1}\Delta h_1 + e^{h_2}\Delta h_2}{2H} \geq 0,$$

where $H = \frac{1}{2}(e^{h_1}+e^{h_2})$, proving the lemma if both points are regular. For singular points with positive angle excess, the CAT(0) property for the averaged metric is immediate from Remark 1.2. $\square$

Proof of Theorem 1.3. Since averaging by J can only improve the systolic ratio, we may assume without loss of generality that our metric is already J-invariant.

There exists an extension of the notions of tangent plane and exponential map to surfaces with singularities. Namely, let $A \in \Sigma$. There exists a CAT(0) piecewise flat plane T_A with conical singularities and a covering $\exp_A : T_A \to \Sigma$ with the following properties:

(1) $\exp_A$ sends the origin O of T_A to A;

(2) $\exp_A$ takes the conical singularities of T_A to the singularities of Σ;

(3) $\exp_A$ sends every pair of geodesic arcs issuing from the origin $O \in T_A$ to a pair of geodesic arcs of the same lengths and forming the same angle at their basepoint A.

By the Rauch Comparison Theorem, the exponential map $\exp_A$ does not decrease distances.

Now assume $A \in \Sigma$ is a Weierstrass point, and let $B \in \Sigma$ be another Weierstrass point. Fix a lift B_0 of B to the tangent plane T_A, along a minimizing arc. Consider the equidistant line

$$L_{O,B_0} \subset T_A$$

between the origin $O \in T_A$ and the point B_0. Consider a point $X_0 \in L_{O,B_0}$. Let $X = \exp_A(X_0)$. Since the exponential map does not decrease distances, we have

$$\mathrm{dist}_\Sigma(A, X) = \mathrm{dist}_{T_A}(O, X_0) = \mathrm{dist}_{T_A}(B_0, X_0) \leq \mathrm{dist}_\Sigma(B, X).$$

Now consider the polygon in the tangent plane T_A, obtained as the intersection of the half-spaces containing the origin, defined by the lines L_{O,B_0}, as B runs over all Weierstrass points. It follows from the preceding equality that the exponential image of this polygon is contained in the Voronoi cell of A. Since the exponential map does not decrease distances, the area of the polygon is a lower bound for the area of the Voronoi cell. If k is the number of sides of V, then V is partitioned into k triangles with angle θ_i at O, whose area is bounded below by

$$\left(\frac{\mathrm{sys}\pi_1}{4}\right)^2 \tan\frac{\theta_i}{2} \tag{4–1}$$

since $\mathrm{dist}(A, B) \geq \frac{1}{2}\,\mathrm{sys}\pi_1$.

Consider the graph on S^2 defined by the projections of the Voronoi cells to the sphere. Thus we have $f = 6$ faces. Applying the formula $v - e + f = 2$ and the well-known fact that $3v \leq 2e$, we obtain

$$e \leq 3f - 6 = 12.$$

Hence the spherical graph has at most 12 edges. Note that the maximum is attained by the 1-skeleton of the cubical subdivision.

The area of a flat isosceles triangle with third angle θ and with unit altitude from the third vertex is $\tan(\theta/2)$. This formula provides a lower bound for the area of the Voronoi cells as in (4–1). The proof is completed by Jensen's inequality applied to the convex function $\tan(x/2)$ when $0 < x < \pi$. In the boundary case of equality, we have $e = 12$, all angles θ_i as in (4–1) must be equal, curvature must be zero because of equality in the Rauch Comparison Theorem, and we easily deduce that each Voronoi cell is a regular octagon. To minimize the area of the octagon, we must choose θ as small as possible. The CAT(0) hypothesis at the center of the octagon imposes a lower bound $\theta \geq \frac{\pi}{4}$. Hence the optimal systolic ratio is achieved for the regular flat octagon with a smooth point at the center. □

5. Arbitrary metrics on the Bolza surface

The conformal class of the Bolza surface $\mathcal{B}$ of Section 2 is likely to contain a systolically optimal surface in genus 2, as discussed in Section 1.

Theorem 5.1. *Every metric $\mathcal{G}$ in the conformal class of the Bolza surface satisfies the bound*

$$\mathrm{SR}(\mathcal{G}) \leq \frac{\pi}{3} = 1.0471\ldots.$$

Remark 5.2. In particular, every metric $\mathcal{G}$ in the conformal class of the Bolza surface satisfies Bavard's inequality

$$\mathrm{SR}(\mathcal{G}) \leq \frac{\pi}{2^{3/2}} \simeq 1.1107 \tag{5–1}$$

for the Klein bottle [Bavard 1986]. This suggests a possible monotonicity of $\chi(\Sigma)$ as a function of $\mathrm{SR}(\Sigma)$.

Lemma 5.3. *Let $\mathcal{G}$ be an* $\mathrm{Aut}(\mathcal{B})$*-invariant metric on $\mathcal{B}$. Let*

$$\delta(J) = \min_{x \in \mathcal{B}^{(1)}} \mathrm{dist}(x, J(x))$$

be the displacement on the 1*-skeleton $\mathcal{B}^{(1)}$ of the Voronoi subdivision of $\mathcal{B}$. Then*

$$\mathrm{area}(\mathcal{B}, \mathcal{G}) \geq 6\left(\tfrac{2}{\pi}\right)\delta(J)^2.$$

Proof. Consider the Voronoi subdivision with respect to the set of six Weierstrass points on $\mathcal{B}$. Since each Voronoi cell $\Omega \subset \mathcal{B}$ is J-invariant, we can identify all pairs of opposite points of the boundary $\partial\Omega$, to obtain a projective plane

$$\mathbb{RP}^2 = \Omega/\sim.$$

We now apply Pu's inequality [Pu 1952] to each of the six Voronoi cells, to obtain

$$\mathrm{area}(\mathbb{RP}^2) \geq \frac{2}{\pi}\,\mathrm{sys}\pi_1(\mathbb{RP}^2)^2. \tag{5–2}$$

The lemma now follows from the bound $\operatorname{sys}\pi_1(\mathbb{RP}^2) \geq \delta(J)$. □

Lemma 5.4. *Every* $\operatorname{Aut}(\mathcal{B})$*-invariant metric* $\mathcal{G}$ *on* $\mathcal{B}$ *satisfies the bound*

$$2\delta(J) \geq \operatorname{sys}\pi_1(\mathcal{B}, \mathcal{G}).$$

Proof. Let p, $J(p) \in \partial\Omega$ satisfy $\operatorname{dist}(p, J(p)) = \delta(J)$. Let $\alpha \subset \Omega$ be a minimizing path joining p to $J(p)$. Let $\Omega' \subset \mathcal{B}$ be the adjacent Voronoi cell containing this pair of boundary points, and $r : \mathcal{B} \to \mathcal{B}$ the anticonformal involution that switches Ω and Ω', and fixes their common boundary. The loop $\alpha \cup r(\alpha)$ belongs to the free homotopy class of the noncontractible loop $\gamma \subset \mathcal{B}$ obtained as the inverse image under $Q : \mathcal{B} \to S^2$ of the edge of the octahedral decomposition of S^2 joining the images of the centers of Ω and Ω' (Lemma 2.2). Since the length of $\alpha \cup r(\alpha)$ is $2\delta(J)$, the lemma follows. □

Proof of Theorem 5.1. We may assume that the metric on $\mathcal{B}$ is $\operatorname{Aut}(\mathcal{B})$-invariant, since averaging the metric by a finite group of holomorphic and antiholomorphic diffeomorphisms can only improve the systolic ratio. We combine the inequalities of Lemmas 5.4 and 5.3 to prove the theorem. □

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