Pacific Journal of Mathematics

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Volume 227 No. 1 September 2006

PACIFIC JOURNAL OF MATHEMATICS
Vol. 227, No. 1, 2006

AN EICHLER–ZAGIER MAP FOR JACOBI FORMS OF HALF-INTEGRAL WEIGHT

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We construct an Eichler–Zagier map for Jacobi cusp forms of half-integral weight. As an application, we show there exists no Hecke-equivariant map from index 1 to index p (p prime), when the weight is half-integral.

The aim of this paper is to generalize the Eichler–Zagier map for Jacobi forms of half-integral weight, which is formally defined as

$$\mathcal{Z}_m : \sum_{\substack{0>D,r\in\mathbb{Z}\\ D\equiv r^2\ (4m)}} c(D,r)\, e\left(\frac{r^2-D}{4m}\tau + rz\right) \mapsto \sum_{0>D\in\mathbb{Z}} \left(\sum_{\substack{r\ (2m)\\ r^2\equiv D\ (4m)}} c(D,r) \right) e(|D|\tau).$$

We prove that it is a Hecke-equivariant map from Jacobi cusp forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4M)$, index m and character χ (k and χ are even) into a certain subspace of cusp forms of weight k on $\Gamma_1(16m^2M)$. First we derive this assertion for $m=1$ by proving that $\mathcal{Z}_1$ maps respective Poincaré series. For the general index m, we apply certain operator I_m (see (2) for the definition) which changes the index m into index 1 and then apply $\mathcal{Z}_1$ to obtain the required mapping property.

In order to give a Maass relation for each prime p for Siegel modular forms of half-integral weight and degree two, Y. Tanigawa [1986] obtained a Hecke-equivariant map from the space of index 1 Jacobi forms of half-integral weight into certain modular forms of integral weight and he constructed the map V_{p^2} from the space of Jacobi forms of index 1 into index p^2. As a natural question, he asked the existence of a connection between Jacobi forms of index 1 and index p (p is a prime) in the case of half-integral weight. We show that there is no such Hecke-equivariant map as an application of the nature of the map $\mathcal{Z}_m$.

Notation and background. Throughout this paper, unless otherwise specified, the letters k, m, M, N will stand for natural numbers and τ for an element of $\mathcal{H}$, the complex upper half-plane.

MSC2000: primary 11F11, 11F50; secondary 11F37.
Keywords: Modular forms, Siegel modular forms, Jacobi forms.

For a complex number z, we write $\sqrt{z}$ for the square root with argument in $(-\pi/2, \pi/2]$, and we set $z^{k/2} = (\sqrt{z})^k$ for any $k \in \mathbb{Z}$.

For integers a, b, let $\left(\frac{a}{b}\right)$ denote the generalized quadratic residue symbol. Let $S_k(N, \psi)$ denote the space of cusp forms of weight k and level N with character ψ. We write the Fourier expansion of a modular form f as

$$f(\tau) = \sum_{n \geq 1} a_f(n) e^{2\pi i n \tau}.$$

For $z \in \mathbb{C}$ and $c, d \in \mathbb{Z}$, we put $e_d^c(z) = e^{2\pi i c z/d}$. We also write $e_1^c(z) = e^c(z)$, $e_c^1(z) = e_c(z)$, and $e_1^1(z) = e(z)$. The symbol $a \equiv \square\ (b)$ means that a is a square modulo b. For two forms f and g (either in the space of modular forms of integral weight or in the space of Jacobi forms of half-integral weight), $\langle f, g\rangle$ denotes the Petersson inner product of f and g. For a Dirichlet character ψ modulo $4m$, the twisting operator on modular forms of integral weight is given by

$$R_\psi = \frac{1}{W_\psi} \sum_{u \bmod 4m} \overline{\psi}(u) \begin{pmatrix} 4m & u \\ 0 & 4m \end{pmatrix}, \tag{1}$$

where $W_\psi = \sum_{u \bmod (4m)} \psi(u) e(u/4m)$. It follows that $\langle f \mid R_\psi, g\rangle = \langle f, g \mid R_{\overline{\psi}}\rangle$, where $f, g \in S_k(\Gamma_1(16mM))$ and

$$R_\psi : \sum_{n \geq 1} a_f(n) e(n\tau) \mapsto \sum_{n \geq 1} \psi(n) a_f(n) e(n\tau).$$

For a natural number d, the operators $U(d)$ and $B(d)$ are defined on formal power series by

$$U(d) : \sum_{n \geq 1} a(n) e(n\tau) \mapsto \sum_{n \geq 1} a(nd) e(n\tau),$$

$$B(d) : \sum_{n \geq 1} a(n) e(n\tau) \mapsto \sum_{n \geq 1} a(n) e(nd\tau).$$

For $n \geq 1$, let P_n denote the n-th Poincaré series in $S_k(N, \psi)$ whose ℓ-th Fourier coefficient is given by

$$g_n(\ell) = \delta(\ell, n) + 2\pi i^{-k} (\ell/n)^{(k-1)/2} \sum_{c \geq 1,\ N \mid c} K_{N,\chi}(n, \ell; c) J_{k-1}\left(\frac{4\pi\sqrt{n\ell}}{c}\right),$$

where $\delta(\ell, n)$ is the Kronecker-delta function, $J_{k-1}(x)$ is the Bessel function of order $k-1$ and $K_{N,\chi}(n, \ell; c)$ is the Kloosterman sum defined by

$$K_{N,\chi}(n, \ell; c) = \frac{1}{c} \sum_{\substack{d(c)^* \\ dd^{-1} \equiv 1\ (c)}} \overline{\psi}(d) e_c(nd^{-1} + \ell d).$$

1. A certain space of cusp forms of integral weight

For $m, M \in \mathbb{N}$, let $\chi \bmod M$ be a Dirichlet character and $\chi_m(n) = \left(\frac{m}{n}\right)$ be the quadratic character modulo m or $4m$ according as $m \equiv 1$ or $m \equiv 3 \pmod 4$.

Let

$$S = \left\{\ell \in \mathbb{N} : 1 \le \ell \le 4m,\ \ell \equiv \square\ (4m)\right\},$$
$$S^* = \left\{\ell \in S : p^2 \mid 4mM \text{ implies } p \nmid \ell, \text{ with } p \text{ prime}\right\}.$$

If $\ell \in S$, define

$$S_k^{\square,\ell}(16mM, \chi\chi_m) := S_k(16mM, \chi\chi_m) \,\big|\, R_\ell,$$

where

$$R_\ell : \sum_{n\ge 1} a(n)e(n\tau) \mapsto \sum_{\substack{n\ge 1\\ -n\equiv \ell\ (4m)}} a(n)e(n\tau).$$

For $\ell \in S$, let $t = (\ell, 4m)$. A formal computation shows that

$$R_\ell = U(t)R(\ell)B(t),$$

with

$$R(\ell) = \frac{1}{\varphi(4m/t)} \sum_{\psi \bmod 4m/t} \overline{\psi}(-\ell/t)R_\psi,$$

where $\varphi(n)$ is the Euler totient function. Using the mapping properties of $U(t)$, R_ψ and $B(t)$ in the said order, we verify that $S_k^{\square,\ell}(16mM, \chi\chi_m)$ is a subspace of $S_k(\Gamma_1(16m^2M))$. Finally we define

$$S_k^{\square}(16mM, \chi\chi_m) = \sum_{\ell\in S} S_k^{\square,\ell}(16mM, \chi\chi_m).$$

2. Jacobi forms of half-integral weight

For $\alpha = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, let $\tilde\alpha = (\alpha, \phi(\tau))$, where $\phi(\tau)$ is a holomorphic function on $\mathcal{H}$ such that $\phi^2(\tau) = t(c\tau+d)$, with $t \in \{1, -1\}$. Then the set $G = \{\tilde\alpha : \alpha \in \mathrm{SL}_2(\mathbb{R})\}$ is a group with group law

$$\big(\alpha_1, \phi_1(\tau)\big)\,\big(\alpha_2, \phi_2(\tau)\big) = \big(\alpha_1\alpha_2, \phi_1(\alpha_2\tau)\phi_2(\tau)\big).$$

If $\alpha \in \Gamma_0(4)$, set

$$j(\alpha, \tau) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(c\tau+d)^{1/2}.$$

We set $\alpha^* = \big(\alpha, j(\alpha,\tau)\big)$; the association $\alpha \mapsto \alpha^*$ is an injective map from $\Gamma_0(4)$ into G. Let G^J be the set of all triplets $[\tilde\alpha, X, s]$, $\alpha \in \mathrm{SL}_2(\mathbb{R})$, $X \in \mathbb{R}^2$, $s \in \mathbb{C}$, $|s| = 1$. Then G^J is a group, with group law given by

$$[\tilde\alpha_1, X_1, s_1]\,[\tilde\alpha_2, X_2, s_2] = \left[\tilde\alpha_1\tilde\alpha_2,\ X_1\alpha_2 + X_2,\ s_1s_2\cdot\left(\det\begin{pmatrix} X_1\alpha_2\\ X_2\end{pmatrix}\right)\right].$$

The stroke operator $\big|_{k+1/2,m}$ is defined on functions $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ by

$$\phi\big|_{k+1/2,m}[\tilde{\alpha}, X, s] = \\ s^m \phi(\tau)^{-2k-1} e^m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + 2\lambda^2\tau + 2\lambda z + \lambda\mu\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right),$$

where $[\tilde{\alpha}, X, s] \in G^J$.

The Jacobi group for $\Gamma_0(4N)$ is a subgroup $\Gamma_0^J(4N)^*$ of G^J, given by

$$\Gamma_0^J(4N)^* = \left\{[\alpha^*, X] : \alpha \in \Gamma_0(4N),\, X \in \mathbb{Z}^2\right\}.$$

A *Jacobi form* $\phi(\tau, z)$ of weight $k + \frac{1}{2}$ and index m for the group $\Gamma_0(4M)$, with character χ, is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following conditions:

(i) $\phi\big|_{k+1/2,m}[\gamma^*, X](\tau, z) = \chi(d)\phi(\tau, z)$, where χ is a Dirichlet character mod $4M$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4M)$.

(ii) For every $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the image $\phi\big|_{k+1/2,m}[\tilde{\alpha}, (0,0)](\tau, z)$ has a Fourier development of the form

$$\sum_{\substack{n,r\in\mathbb{Q} \\ r^2\le 4nm}} c_\alpha(n, r)\, e\,(n\tau + rz)\,,$$

where the sum ranges over rational numbers n, r with bounded denominators subject to the condition $r^2 \le 4nm$.

Further, if $r^2 < 4nm$ whenever $c_\alpha(n, r) \neq 0$, then ϕ is called a *Jacobi cusp form*. We denote by $J_{k+1/2,m}(4M, \chi)$ the space of Jacobi forms of weight $k + \frac{1}{2}$, index m for $\Gamma_0(4M)$ with character χ, and by $J^{\mathrm{cusp}}_{k+1/2,m}(4M, \chi)$ the subspace of $J_{k+1/2,m}(4M, \chi)$ consisting of Jacobi cusp forms. A Jacobi form ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2\le 4nm}} c(n, r)\, e\,(n\tau + rz)\,.$$

Since $c(n, r) = c(n', r')$ if $r'^2 - 4n'm = r^2 - 4nm$ and $r' \equiv r \pmod{2m}$, we write the Fourier expansion of ϕ as

$$\phi(\tau, z) = \sum_{\substack{0\ge D,\, r\in\mathbb{Z} \\ D\equiv r^2\ (4m)}} c_\phi(D, r)\, e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

Let $D < 0$ be a discriminant and r an integer modulo $2m$ with $D \equiv r^2\ (4m)$. Then the (D, r)-th Poincaré series, denoted by $P_{(D,r)}$, is defined by

$$P_{(D,r)}(\tau, z) = \sum_{\gamma \in \Gamma_0(4M)^J_\infty \backslash \Gamma_0(4M)^J} \overline{\chi}(\gamma) e(n\tau + rz) \big|_{k+1/2,m} \gamma.$$

We state the following proposition without proof.

Proposition 2.1. *The Poincaré series $P_{(D,r)}$ lies in $J^{\text{cusp}}_{k+1/2,m}(4M, \chi)$ and satisfies*

$$\langle \phi, P_{(D,r)} \rangle = \alpha_{k,m} |D|^{-k+1}\, c_\phi(D, r),$$

for each $\phi \in J^{\text{cusp}}_{k+1/2,m}(4M, \chi)$, where $\alpha_{k,m} = \Gamma(k-1) m^{k-3/2}/(2\pi^{k-1})$. It has a Fourier development of the form

$$P_{(D,r)}(\tau, z) = \sum_{\substack{0 > D',\, r' \in \mathbb{Z} \\ D' \equiv r'^2\ (4m)}} \big(g_{D,r}(D', r') + \chi(-1) g_{D,r}(D', -r')\big)\, e\left(\frac{r'^2 - D'}{4m}\tau + r'z\right),$$

where $D = r^2 - 4mn$, $D' = r'^2 - 4mn'$, and $g_{D,r}(D', r')$ is given by

$$\delta_m(D, r, D', r') + i^{-k-3/2}\pi \sqrt{\frac{2}{m}} \left(\frac{D'}{D}\right)^{k/2} \sum_{\substack{c \geq 1 \\ 4M | c}} H_{m,c,\chi}(D, r, D', r')\, J_k\left(\frac{\pi\sqrt{DD'}}{mc}\right),$$

with

$$\delta_m(D, r, D', r') = \begin{cases} 1 & \text{if } D' = D \text{ and } r' \equiv r \ (\text{mod } 2m), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$H_{m,c,\chi}(D, r, D', r') = c^{-3/2} e^{-rr'/(2mc)} \times \sum_{\substack{d,\, \lambda(c) \\ dd^{-1} \equiv 1\ (c)}} \overline{\chi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{1/2} e_c\left(d^{-1}(m\lambda^2 + r\lambda + n) + dn' - \lambda r'\right).$$

3. The Eichler–Zagier map

First we consider the space $J^{\text{cusp}}_{k+1/2,1}(4M, \chi)$. Put $D = D_0\ell^2$, $r = r_0\ell$ in Proposition 2.1. In the Fourier coefficient of $P_{(D_0\ell^2, r_0\ell)}$, the Kloosterman-type sum is periodic as a function of ℓ of period $2c$. Hence, for any h (mod $2c$), its Fourier transform (after replacing ℓ by ℓd and λ by λd) becomes

$$\frac{1}{2c^{5/2}} \sum_{\substack{\ell(2c),\, d(c)^* \\ \lambda(c)}} \overline{\chi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{1/2} \times e_{2c}\big(d(2\lambda^2 + 2r_0\ell\lambda + 2n_0\ell^2 + 2n - 2r\lambda - r_0\ell r - h\ell)\big).$$

Since $4|c$, the sum over λ is nonzero only if $r_0\ell \equiv r \pmod 2$. Hence, the sum over λ becomes

$$\sum_{\lambda(c)} e_c(d\lambda^2)\, e_c\Big(-d\Big(\frac{r_0\ell - r}{2}\Big)^2\Big).$$

Again, the fact that $4\,|\,c$ and $\gcd(c, d) = 1$ gives the identity

$$\frac{1}{\sqrt{2ic}} \sum_{\lambda(c)} e_c(d\lambda^2) = \Big(\frac{c}{d}\Big)\Big(\frac{-4}{d}\Big)^{-1/2}.$$

Thus, the Fourier transform simplifies to

$$\frac{\sqrt{i}}{\sqrt{2}c^2} \sum_{\ell(2c),d(c)^*} \overline{\chi}(d) e_{4c}\big(d(D_0\ell^2 + D - 2h\ell)\big)$$
$$= \frac{\sqrt{i}}{4\sqrt{2}c^2} \sum_{\ell(2c),d(4c)^*} \overline{\chi}(d)\, e_{4c}\big(d(D_0\ell^2 + D - 2h\ell)\big),$$

which is the Fourier transform of the corresponding Kloosterman sum of integral weight.

More precisely:

Theorem 3.1. *The Eichler–Zagier map $\mathcal{Z}_1$ maps $J^{\text{cusp}}_{k+1/2,1}(4M,\chi)$ into $S_k^{\square}(16M,\chi)$.*

Proof. We shall prove that the (D, r)-th Fourier coefficient of $P_{(D_0\ell^2,r_0\ell)}$ is equal (up to constant) $|D|$-th Fourier coefficient of $P_{|D_0|\ell^2}$. It is easy to see that

$$\delta_1(D_0\ell^2, r_0\ell, D, r) = \delta_{|D_0|\ell^2,|D|}.$$

We consider both the Kloosterman sums as periodic functions of period $2c$. The arguments put forth above shows that for each $c \geq 1$, with $4M|c$, the Fourier transform of $H_{1,c,\chi}(D_0\ell^2, r_0\ell, D, r)$ is equal to (up to the required constants) the Fourier transform of the Kloosterman sum (corresponding to integral weight) $K_{16M,\chi}(|D_0|\ell^2, |D|; 4c)$. This proves the theorem. □

The index-changing operator I_m. If $\phi \in J^{\text{cusp}}_{k+1/2,m}(4M, \chi)$, define I_m by

(2) $$\phi \,\big|\, I_m(\tau, z) = \sum_{\lambda \pmod m} e(\lambda^2\tau + 2\lambda z)\phi(m\tau, z + \lambda\tau).$$

Proposition 3.2. *I_m maps $J^{\text{cusp}}_{k+1/2,m}(4M, \chi)$ into $J^{\text{cusp}}_{k+1/2,1}(4mM, \chi\chi_m)$. The Fourier development of $\phi \,\big|\, I_m$ is of the form*

$$\phi \,\big|\, I_m(\tau, z) = \sum_{\substack{0<D,\, r\in\mathbb{Z} \\ D\equiv r^2 \pmod 4}} \Bigg(\sum_{\substack{s \pmod{2m} \\ s\equiv r \pmod 2}} c_\phi(D, s) \Bigg) e\Big(\frac{r^2 - D}{4}\tau + rz\Big).$$

Proof. It is easy to see that

$$\phi \,\big|\, I_m(\tau, z) = m^{-k/2-1/4} \sum_{\lambda \pmod m} \phi_{1/\sqrt{m}} \big|_{k,1} \big[\, \widetilde{\Delta}_m,\ (\lambda, 0)\big]\, (\tau, z),$$

where $\phi_{1/\sqrt{m}}(\tau, z) = \phi(\tau, z/\sqrt{m})$ and Δ_m is the diagonal matrix $\operatorname{diag}\big(\sqrt{m}, 1/\sqrt{m}\big)$. The proposition now follows directly from the preceding expression. □

Using the equality $\mathfrak{X}_m = I_m \mathfrak{X}_1$, together with Theorem 3.1 and Proposition 3.2, we have:

Theorem 3.3. *The map $\mathfrak{X}_m$ takes $J^{\text{cusp}}_{k+1/2,m}(4M, \chi)$ into $S_k^{\square}(16mM, \chi\chi_m)$.*

4. Half-integral weight Jacobi forms of index 1 and index p

In the case of integral weight Jacobi forms, the well-known map V_p is a Hecke-equivariant map from $J_{k,1}$ into $J_{k,p}$ (p is a prime). If we replace k by $k+\frac{1}{2}$, then we have a Hecke-equivariant map V_{p^2} from $J_{k+1/2,1}(4M)$ into $J_{k+1/2,p^2}(4M)$, which was given by Tanigawa [1986]. Therefore, existence of a Hecke-equivariant map from index 1 into p in the case of half-integral weight Jacobi forms seems to be a natural question.

As an application of the map $\mathfrak{X}_m$, we show that there does not exist a Hecke-equivariant map from $J^{\text{cusp}}_{k+1/2,1}(4)$ into $J^{\text{cusp}}_{k+1/2,p}(4)$.

Let

$$N = \begin{cases} p & \text{if } p \equiv 1 \pmod 4, \\ p^2 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Let $\psi \pmod N$ be a primitive Dirichlet character such that $\psi^2 = \chi_p$. Let R_ψ be the twisting operator defined as in (1). Then, R_ψ maps $S_k(16N^2, \chi_p)$ into $S_k(16N^2)$ and commutes with Hecke operators T_n, $(n, p) = 1$. Further, if $f \in S_k(16N^2, \chi_p)$, we have

$$\big(f \,\big|\, R_\psi\big) \,\big|\, W_p = f \,\big|\, R_\psi,$$

where W_p is the W-operator on $S_k(16N^2)$ for the prime p.

***Case 1:* $p \equiv 3 \pmod 4$.** Let $f \in S_k(4p, \chi_p)$ be a normalized Hecke eigenform. Since $f \,\big|\, R_\psi \in S_k(4p^4)$ and it is an eigenform for all the Hecke operators and the W operators, it is a newform in $S_k^{\text{new}}(4p^4)$. Hence, by the theory of newforms, it is not equivalent to a level-1 Hecke eigenform.

***Case 2:* $p \equiv 1 \pmod 4$.** Let $f \in S_k^{\text{new}}(4p, \chi_p)$ be a normalized Hecke eigenform. Then, $f \,\big|\, R_\psi \in S_k^{\text{new}}(4p^2)$. Since $f \,\big|\, R_{\overline{\psi}} \in S_k^{\text{new}}(4p^2)$, and $\psi^3 = \overline{\psi}$ (as ψ^2 is quadratic), we get $f \,\big|\, R_\psi$ and $f \,\big|\, R_\psi \,\big|\, R_{\chi_p}$ are newforms in $S_k^{\text{new}}(4p^2)$. Thus, the form f is not equivalent to a level-1 Hecke eigenform. Now, we let $f \in S_k(p, \chi_p)$.

Arguments as above again show that f is not equivalent to a level-1 Hecke eigenform.

Thus, we conclude that a normalized Hecke eigenform in $S_k(4p, \chi_p)$ is not equivalent to a normalized Hecke eigenform in $S_k(4)$. In view of the mapping property proved in Theorem 3.3, we have proved:

Theorem 4.1. *There is no Hecke-equivariant map from the space* $J^{\text{cusp}}_{k+1/2,1}(4)$ *into the space* $J^{\text{cusp}}_{k+1/2,p}(4)$.

In this connection the following question seems natural.

What contribution do half-integral weight Jacobi forms of square-free index make to the construction of a "Maass space" (*if one exists*) *for degree*-2 *Siegel modular forms of half-integral weight*?

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Received November 1, 2004.

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