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SELF-SIMILAR SOLUTIONS OF THE p-LAPLACE HEAT EQUATION: THE FAST DIFFUSION CASE

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We study the self-similar solutions of the equation $u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in $\mathbb{R}^N$, where $N \geq 1$, $p \in (1, 2)$. We provide a complete description of the signed solutions of the form $u(x,t) = (\pm t)^{-\alpha/\beta} w((\pm t)^{-1/\beta}|x|)$, regular or singular at $x = 0$, with α, β real, $\beta \neq 0$, and possibly not defined on all of $\mathbb{R}^N \times (0, \pm\infty)$.

1. Introduction and main results

In this article we study the existence of self-similar solutions of the degenerate parabolic equation involving the p-Laplace operator in $\mathbb{R}^N$, $N \geq 1$,

$$(E_u) \qquad u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

with $1 < p < 2$. In the sequel we set

$$\delta = \frac{p}{2-p},$$

so $\delta > 1$. Two critical values P_1, P_2 are involved in the problem

$$P_1 = \frac{2N}{N+1}, \qquad P_2 = \frac{2N}{N+2};$$

see [DiBenedetto and Herrero 1990], for example. They are connected with δ through the relations

$$p > P_1 \iff \delta > N, \qquad p > P_2 \iff \delta > \frac{N}{2}.$$

If $u(x,t)$ is a solution and $\alpha, \beta \in \mathbb{R}$, then $u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^\beta t)$ is a solution of (E_u) if and only if

$$\beta = p - (2-p)\alpha = (2-p)(\delta - \alpha);$$

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thus $\beta > 0$ if and only if $\alpha < \delta$. For given $\alpha \in \mathbb{R}$ such that $\alpha \neq \delta$, the natural way to construct particular solutions is to search for self-similar solutions, radially symmetric in x, of the form

$$u = u(x,t) = (\varepsilon\beta t)^{-\alpha/\beta} w(r), \quad r = (\varepsilon\beta t)^{-1/\beta}|x|, \tag{1–1}$$

where $\varepsilon = \pm 1$. By translation, for any real T, we obtain solutions defined for any $t > T$ when $\varepsilon\beta > 0$, or $t < T$ when $\varepsilon\beta < 0$. The hypersurfaces $\{r = \text{constant}\}$ are of focusing type if $\beta > 0$ and of spreading type if $\beta < 0$. We are led to the equation

$$(|w'|^{p-2}w')' + \frac{N-1}{r}|w'|^{p-2}w' + \varepsilon(rw' + \alpha w) = 0 \quad \text{in}(0,\infty). \tag{E_w}$$

If we look for solutions of (E_u) under the form

$$u = Ae^{-\varepsilon\mu t} w(r), \quad r = Me^{-\varepsilon\mu t/\delta}|x|, \quad \mu > 0,$$

then w solves (E_w) provided $M = \delta/\alpha$ and $A = (\delta^p/\alpha^{p-1}\mu)^{1/(2-p)}$, where $\alpha > 0$ is arbitrary. This is another motivation for studying equation (E_w) for any real α.

In the huge literature on self-similar solutions of parabolic equations, many results deal with positive solutions u defined and smooth on $\mathbb{R}^N \times (0,\infty)$. Equation (E_w) was studied in [Qi and Wang 1999] when $\alpha > 0$, $\varepsilon = 1$. In our work we provide an exhaustive description of the self-similar solutions of equation (E_u), possibly not defined on all of $(0,\infty)$, with constant or changing sign. In particular, for suitable values of α, we prove the existence of solutions w oscillating with respect to 0 as r tends to 0 or ∞, or constant-sign solutions oscillating with respect to some nonzero constant. Our main tool is the reduction of the problem to an autonomous system with two variables and two parameters, p and α. We are led to a dynamical system, which we study by phase-plane techniques. When $p = \frac{3}{2}$, this system is nearly quadratic, and many devices from the theory of algebraic dynamical systems can be used. In the general case such structures do not exist; then we use energy functions associated to the system. The behavior of the solutions presents great diversity, according to the possible values of p and α.

In the sequel we set

$$\eta = \frac{N-p}{p-1};$$

thus $\eta > 0$ if $N \geq 2$, and $\eta = -1$ if $N = 1$. Observe the relation connecting η, δ and N:

$$\frac{\delta - N}{p-1} = \delta - \eta = \frac{N-\eta}{2-p}. \tag{1–2}$$

Explicit solutions. Obviously if w is a solution of (E_w), so is $-w$. Many particular solutions are well-known.

The infinite point source solution U_∞. The simplest positive solutions of equation (E_w), which exist for any α such that $\varepsilon(\delta - N)(\delta - \alpha) > 0$, are given by

$$w(r) = \ell r^{-\delta}, \tag{1–3}$$

where

$$\ell = \Big(\varepsilon\delta^{p-1}\frac{\delta - N}{\delta - \alpha}\Big)^{1/(2-p)} > 0. \tag{1–4}$$

They correspond to a unique solution u of (E_u) called U_∞ in [Chasseigne and Vazquez 2002], singular at $x = 0$, for any $t \neq 0$:

$$U_\infty(x,t) = \Big(\frac{Ct}{|x|^p}\Big)^{1/(2-p)}, \quad C = (2-p)\delta^{p-1}(\delta - N).$$

The case $\alpha = N$. Here the equation (E_w) has a first integral

$$w + \varepsilon r^{-1}|w'|^{p-2}w' = Cr^{-N}. \tag{1–5}$$

All the solutions corresponding to $C = 0$ are given by

$$\begin{aligned} w &= w_{K,\varepsilon}(r) = \pm(\varepsilon\delta^{-1}r^{p'} + K)^{-\delta/p'},\\ u &= \pm u_{K,\varepsilon}(x,t) = (\varepsilon\beta_N t)^{-N/\beta_N}(\varepsilon\delta^{-1}(\varepsilon\beta_N t)^{-p'/\beta_N}|x|^{p'} + K)^{-(p-1)/(2-p)},\\ &\qquad K \in \mathbb{R}, \end{aligned} \tag{1–6}$$

with $\beta = \beta_N = (N+1)(p - P_1)$. For $p > P_1, \varepsilon = 1, K > 0$, the solutions are named after Barenblatt [1952]. For given $c > 0$, the function $u_{K,1}$, defined on $\mathbb{R}^N \times (0,\infty)$, is the unique solution of equation (E_u) with initial data $u(0) = c\delta_0$, δ_0 being the Dirac mass at 0 and K begin determined by $\int_{\mathbb{R}^N} u_K(x,t)\,dt = c$; see for example [Zhao 1993]. Moreover the functions $u_{K,1}$, with $K > 0$, are the only nonnegative solutions defined on $\mathbb{R}^N \times (0,\infty)$, such that $u(x,0) = 0$ for any $x \neq 0$; see [Kamin and Vázquez 1992]. In the case $K = 0$, we find again the function U_∞, and U_∞ is the limit of the functions $u_{K,1}$ as $K \to 0$, or equivalently $c \to \infty$.

The case $\alpha = \eta$. We exhibit a family of solutions of (E_w):

$$w(r) = Cr^{-\eta}, \quad u(t,x) = C|x|^{-\eta} = C|x|^{(p-N)/(p-1)}, \quad C \neq 0, \tag{1–7}$$

Solutions u, independent of t, are the fundamental p-harmonic solutions of the equation when $p > P_1$.

The case $\alpha = -p'$. Equation (E_w) admits solutions of the form

$$\begin{aligned} w(r) &= \pm K(N(Kp')^{p-2} + \varepsilon r^{p'}),\\ u(x,t) &= \pm K(N(Kp')^{p-2}t + \varepsilon|x|^{p'}), \quad K > 0, \end{aligned} \tag{1–8}$$

and the functions u are solutions of the form $\psi(t)+\Phi(|x|)$ with Φ nonconstant. They have constant sign when $\varepsilon=1$, and a changing sign when $\varepsilon=-1$.

The case $\alpha=0$. Here equation (E_w) can be explicitly solved: either $w'\equiv 0$ (hence $w\equiv a\in\mathbb{R}$, and u is a constant solution of (E_u)), or there exists $K\in\mathbb{R}$ such that

$$|w'| = r^{(1-N)/(p-1)}\times\begin{cases}\left(K+\dfrac{\varepsilon}{\delta-N}r^{N-\eta}\right)^{-1/(2-p)} & \text{if } \delta\neq N,\\ \left(\dfrac{2-p}{p-1}(K+\varepsilon\ln r)\right)^{-1/(2-p)} & \text{if } \delta=N,\end{cases} \tag{1–9}$$

which gives w by integration, up to a constant, and then $u(x,t)=w(|x|/(\varepsilon pt)^{1/p})$.

The case $N=1$ and $\alpha=(p-1)/(2-p)>0$. Here again we obtain explicit solutions:

$$w(r)=\pm\big(\varepsilon K(r-(K\alpha)^{p-1})\big)^{-\alpha},\quad u(x,t)=\pm\big(\varepsilon K(|x|-\varepsilon(K\alpha)^{p-1}t)\big)^{-\alpha},\quad K>0.$$

All the functions w above are defined on intervals of the form $(R,0)$, $R\geq 0$ if $\varepsilon=1$, and $(0,S)$, $S\leq\infty$ if $\varepsilon=-1$.

Note. When $\alpha=\delta$, equation (E_u) is invariant under the transformation $u_\lambda(x,t)=\lambda^\alpha u(\lambda x,t)$; searching solutions of the form $u(x,t)=|x|^{-\delta}\psi(t)$, we find again the function U_∞.

Different kinds of singularities. Consider equation (E_w). It is easy to get local existence and uniqueness near any point $r_1>0$; thus any solution w is defined on a maximal interval (R_w,S_w), with $0\leq R_w<S_w\leq\infty$; and in fact $S_w=\infty$ when $\varepsilon=1$, and $R_w=0$ when $\varepsilon=-1$ (see Theorem 2.2). Returning to solution the u of (E_u) associated to w by (1–1), it is defined on a subset of $\mathbb{R}^N\setminus\{0\}\times(0,\pm\infty)$:

$$D_w=\{(x,t): x\in\mathbb{R}^N,\ \varepsilon\beta t>0,\ (\varepsilon\beta t)^{1/\beta}R_w<|x|<(\varepsilon\beta t)^{1/\beta}S_w\}.$$

When w is defined on $(0,\infty)$, then u is defined on $\mathbb{R}^N\setminus\{0\}\times(0,\pm\infty)$.

Regular solutions. Among the solutions of (E_w) defined near 0, we also show the existence and uniqueness of solutions $w=w(\,.\,,a)\in C^2([0,S_w))$ such that, for some $a\in\mathbb{R}$,

$$w(0)=a,\qquad w'(0)=0. \tag{1–10}$$

These are called *regular solutions*. Obviously, they are defined on $[0,\infty)$ when $\varepsilon=1$. If w is regular, then $D_w=\mathbb{R}^N\times(0,\pm\infty)$, and $u(\,.\,,t)\in C^1(\mathbb{R}^N)$ for $t\neq 0$; we will say that u is *regular*. This does not imply the regularity up to $t=0$: indeed u presents a singularity at time $t=0$ if and only if $0<\alpha<\delta$. In the sequel we shall not mention the trivial solution $w\equiv 0$, corresponding to $a=0$.

Singular solutions. If $R_w = 0$ and w is not regular, u presents a singularity at $x = 0$ for $t \neq 0$, called a *standing singularity*. Following [Vazquez and Véron 1996; Chasseigne and Vazquez 2002], for such a solution, we say that $x = 0$ is a *weak singularity* if $x \mapsto w(|x|) \in L^1_{\text{loc}}(\mathbb{R}^N)$, or equivalently if $u(\,.\,,t) \in L^1_{\text{loc}}(\mathbb{R}^N)$ for $t \neq 0$; and a *strong singularity* if not. If u has a strong/weak singularity, and $\lim_{t\to 0} u(t,x) = 0$ for any $x \neq 0$, we call u a *strong/weak razor blade*. If $u(\,.\,,t) \in L^1(\mathbb{R}^N)$ for $t \neq 0$, then u is called *integrable*.

Solutions with a reduced domain. If $R_w > 0$ or $S_w < \infty$, we say that u and w have a *reduced domain*. Then D_w has a lateral boundary of the form $\Sigma_w = \{|x| = C(\varepsilon\beta t)^{1/\beta}\}$, of parabolic type if $\beta > 0$ and of hyperbolic type if $\beta < 0$, and u has an explosion near Σ_w. In Proposition 2.15 we calculate the blow-up rate, which is of the order of $d(x,t)^{-(p-1)/(2-p)}$, where $d(x,t)$ is the distance to Σ_w.

Main results. We give a summary of our main results, expressed in terms of the function u, avoiding for simplicity particular cases (such as $N = 1$, or $\alpha = \delta$, or $p = P_1$) and solutions with a reduced domain (although there exist many such). All cases omitted here and detailed statements in terms of w can be found inside each section. An important critical value of α is given by

$$\alpha^* = \delta + \frac{\delta(N-\delta)}{(p-1)(2\delta - N)}; \tag{1–11}$$

it appears when $\varepsilon = 1$, $p > P_2$, and then $\alpha^* > 0$, or $\varepsilon = -1$, $p < P_2$, and then $\alpha^* < 0$.

Note. To return from w to u, consider any solution w of (E_w) defined on $(0,\infty)$, such that for some $\lambda \geq 0$ and $\mu \in \mathbb{R}$, $\lim_{r\to 0} r^\lambda w = c \neq 0$ and $\lim_{r\to 0} r^\mu w = c' \neq 0$. Then:

(i) For fixed t, u has a singularity in $|x|^{-\lambda}$ near $x = 0$, and a behavior in $|x|^{-\mu}$ for large $|x|$. Thus $x = 0$ is a *weak singularity* if and only if $\lambda < N$, and u is integrable if and only if $\lambda < N < \mu$.

(ii) For fixed $x \neq 0$, the behavior of u near $t = 0$, depends on the sign of β:

$$\lim_{t\to 0} |x|^\mu |t|^{(\alpha-\mu)/\beta} u(x,t) = C \neq 0 \quad \text{if} \alpha < \delta,$$

$$\lim_{t\to 0} |x|^\lambda |t|^{(\alpha-\lambda)/\beta} u(x,t) = C \neq 0 \quad \text{if} \delta < \alpha.$$

Solutions defined for t > 0. Here we look for solutions u of (E_u) on $\mathbb{R}^N \setminus \{0\} \times (0,\infty)$ of the form (1–1). That means $\varepsilon\beta > 0$, or equivalently $\varepsilon = 1$ and $\alpha < \delta$ (see Section 3) or $\varepsilon = -1$, $\delta < \alpha$ (see Section 4). We begin with the case $\varepsilon = 1$, and examine the dependence on the sign of $p - P_1$. For proofs, see Theorems 3.2, 3.4 and 3.5.

Theorem 1.1. *Assume $\varepsilon = 1$, $-\infty < \alpha < \delta$, $p > P_1$, and $N \geq 2$. Then U_∞ is a solution on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$ and a strong razor blade. There exist also positive solutions having a strong singularity in $|x|^{-\delta}$ and satisfying $\lim_{t\to 0} |x|^\alpha u = L > 0$ (for $x \neq 0$). For $\alpha \leq N$, any function $u(. , t)$ has at most one zero at time t.*

(1) *For $\alpha < N$, all regular solutions on $\mathbb{R}^N \times (0, \infty)$ have constant sign, are not integrable, and they are solutions of (E_u) with initial data $L|x|^{-\alpha} \in L^1_{\text{loc}}(\mathbb{R}^N)$. There exist* ***positive integrable razor blades*** *having a singularity in $|x|^{-\eta}$. There exist also positive solutions having a weak regularity in $|x|^{-\eta}$ and satisfying $\lim_{t\to 0} |x|^\alpha u = L$; in particular if $\alpha = \eta$, then $u \equiv C|x|^{-\eta}$. There exist solutions with one zero and a weak or a strong singularity.*

(2) *For $\alpha = N$, all regular (Barenblatt) solutions have constant sign and are integrable. There exist solutions with one zero and a weak singularity.*

(3) *For $N < \alpha$, all regular solutions have at least one zero. If $\alpha < \alpha^*$, any solution has a* ***finite number of zeros****. If $N < \alpha^*$, there exists $\breve{\alpha} \in (\alpha^*, \delta)$ such that if $\breve{\alpha} < \alpha$, regular solutions are* ***oscillating around*** *0 for large $|x|$, and $r^\delta w$ is asymptotically periodic in $\ln r$; and there exists precisely a solution u such that $r^\delta w$ is* ***periodic in*** *$\ln r$.*

Theorem 1.2. *Assume $\varepsilon = 1$, $-\infty < \alpha < \delta$, and $p < P_1$. Then all regular solutions on $\mathbb{R}^N \times (0, \infty)$ have constant sign, are not integrable, and are solutions of (E_u) with initial data $L|x|^{-\alpha} \in L^1_{\text{loc}}(\mathbb{R}^N)$. There is no other solution on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$.*

If $\alpha > 0$, all the solutions w tend to 0 at ∞, whereas if $\alpha < 0$, some of the solutions are unbounded near ∞.

Next we come to the case $\varepsilon = -1$, which is treated in Theorems 4.1 and 4.2.

Theorem 1.3. *Assume $\varepsilon = -1$, $\delta < \alpha$, $p > P_1$, and $N \geq 2$. There is no regular solution on $\mathbb{R}^N \times (0, \infty)$. Besides the function U_∞, which is a strong razor blade, there exist* ***positive integrable razor blades*** *having a singularity in $|x|^{-\eta}$, and positive solutions having a strong singularity in $|x|^{-\alpha}$ and satisfying $\lim_{t\to 0} |x|^\alpha u = L$.*

Theorem 1.4. *Assume $\varepsilon = -1$, $\delta < \alpha$, $p < P_1$ ($N \geq 2$). There is no regular solution on $\mathbb{R}^N \times (0, \infty)$. There exists a positive solution on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$ with a singularity in $|x|^{-\alpha}$ (strong if and only if $N \leq \alpha$), and $\lim_{t\to 0} |x|^\alpha u = L$.*

Note. Weak singularities can occur even if $p > P_1$. For example, the solutions $u(t, x) = C|x|^{-\eta} = C|x|^{(p-N)/(p-1)}$ ($N \geq 2$) given in (1–7) have a weak singularity. There even exist positive solutions u with a standing singularity, and integrable; see Theorems 1.1 and 1.3. This is not contradictory with the regularizing effect $L^1_{\text{loc}}(\mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}^N)$, which concerns solutions in $(0, \infty) \times \mathbb{R}^N$. The functions constructed above are solutions in $(0, \infty) \times \mathbb{R}^N \setminus \{0\}$, and the singularity $x = 0$ is not removable.

Solutions defined for $t < 0$. Next we consider the solutions defined for $t < 0$, and more generally for $t < T$. They correspond to $\varepsilon = 1$, $\delta < \alpha$ (Section 5), or $\varepsilon = -1$, $\alpha < \delta$ (Section 6). A main question in that case is the *extinction problem*: do there exist regular solutions u vanishing identically on $\mathbb{R}$ at time T? Do there exist singular razor blades, vanishing on $\mathbb{R}^N \setminus \{0\}$ at time T? Are they integrable?

One of our most significant results is the existence of two critical values $\alpha_{\text{crit}} > 0$ (when $P_2 < p < P_1$) and $\alpha^{\text{crit}} < 0$ (when $1 < p < P_2$), for which the *regular solutions* $u_{\alpha_{\text{crit}}}$ are *positive, integrable*, and vanish identically at time 0. Another new phenomena is the existence of positive solutions such that $C_1 U_\infty \le u \le C_2 U_\infty$ for some $C_1, C_2 > 0$, with a periodicity property, see Theorems 1.6 and 1.8.

First assume $\varepsilon = 1$. From Theorems 5.1 when $p > P_1$ and 5.4, 5.6, 5.7 when $p < P_1$, we deduce:

Theorem 1.5. *Assume $\varepsilon = 1$, $\delta < \alpha$, $p > P_1$, with $N \ge 2$. Any solution u on $\mathbb{R}^N \setminus \{0\} \times (0, -\infty)$, in particular the regular ones, is **oscillating around** 0 for fixed $t < 0$ and large $|x|$, and $r^\delta w$ is asymptotically periodic in $\ln r$. There exists a solution such that $r^\delta w$ is **periodic** in $\ln r$. There exist weak integrable razor blades, with a singularity in $|x|^{-\eta}$.*

Theorem 1.6. *Assume $\varepsilon = 1$, $\delta < \alpha$, $p < P_1$. Then U_∞ is a solution on $\mathbb{R}^N \setminus \{0\} \times (0, -\infty)$, and a weak razor blade.*

(1) *If $p < P_2$, all regular solutions on $\mathbb{R}^N \times (0, -\infty)$ have **constant sign**, are not integrable, and vanish identically at $t = 0$, with $\|u(\,.\,, t)\|_{L^\infty(\mathbb{R}^N)} \le C|t|^{\alpha/|\beta|}$. All the solutions have a finite number of zeros.*

(2) *For $\alpha < \eta$, regular solutions have **constant sign**, with the same behavior (given by (1–6) if $\alpha = N$). There exists a positive solution u, which is not integrable, with a singularity in $|x|^{-\alpha}$ (a strong one if and only if $\alpha \ge N$), and $\lim_{t\to 0} |x|^\alpha u = L$. If $\alpha = \eta$, then $u(t, x) = C|x|^{-\eta}$ is a solution with a strong singularity.*

(3) *If $p > P_2$, there exists a **critical value** α_{crit} such that $\eta < \alpha_{\text{crit}} < \alpha^*$ and the **regular solutions** $u_{\alpha_{\text{crit}}}$ have **constant sign**, are **integrable**, and vanish identically at $t = 0$, with $\|u(\,.\,, t)\|_{L^\infty(\mathbb{R}^N)} \le C|t|^{\alpha/|\beta|}$.*

(4) *If $\alpha \in (\alpha_{\text{crit}}, \alpha^*)$, there exist **positive** solutions u such that $r^\delta w$ is **periodic** in $\ln r$; thus*

$$C_1 U_\infty \le u \le C_2 U_\infty \quad \textit{for some} C_1, C_2 > 0.$$

*There exist positive solutions u, with the same bounds, such that $r^\delta w$ is asymptotically periodic near 0 . There exist **positive integrable** solutions u such that $r^\delta w$ is asymptotically periodic near 0.*

(5) *If $\alpha_{\text{crit}} < \alpha$, all regular solutions are oscillating around* 0 *for fixed $t < 0$ and large $|x|$, and $r^\delta w$ is asymptotically periodic in* $\ln r$. *There exist solutions **oscillating around*** 0, *such that $r^\delta w$ is **periodic**. If $\alpha^* < \alpha$, there exist **positive integrable razor blades**, with a singularity in $|x|^{-\delta}$.*

Finally suppose $\varepsilon = -1$. From Theorems 6.1, 6.2 when $p > P_1$ and 6.4, 6.6, 6.8, 6.9 when $p < P_1$, we obtain:

Theorem 1.7. *Assume $\varepsilon = -1$, $\alpha < \delta$ and $p > P_1$, with $N \geq 2$. If $\alpha > 0$, there exist **positive** solutions u with a weak singularity in $|x|^{-\eta}$, integrable if and only if $\alpha > N$, and* $\lim_{t\to 0} |x|^\alpha u = L$. *If $\alpha < 0$, any solution has at least a zero. If $-p' < \alpha$, there is no regular solution on $\mathbb{R}^N \times (0, -\infty)$. If $\alpha = -p'$, all regular solutions, given by (1–8), have one zero.*

Theorem 1.8. *Assume $\varepsilon = -1$, $\alpha < \delta$ and $p < P_1$. Then U_∞ is a solution on $\mathbb{R}^N \setminus \{0\} \times (0, -\infty)$, and a weak razor blade.*

(1) *If $p > P_2$, all the solutions have a finite number of zeros. There exist **positive integrable razor blades**, with a singularity in $|x|^{-\delta}$.*

(2) *If $-p' < \alpha$, there is no regular solution on $\mathbb{R}^N \times (0, -\infty)$. There exist positive integrable razor blades as above. If $\alpha > 0$, there exist **positive** solutions u with a weak singularity in $|x|^{-\delta}$, integrable if and only if $\alpha > N$, and* $\lim_{t\to 0} |x|^\alpha u = L$. *If $-p' < \alpha < 0$, there exist solutions with one zero and the same behavior. If $\alpha = -p'$, all regular solutions, given by (1–8), have one zero.*

(3) *If $p < P_2$, there exists a critical value α^{crit} such that $\alpha^* < \alpha^{\text{crit}} < -p'$ for which the **regular solutions** $u_{\alpha^{\text{crit}}}$ have **constant sign**, are **integrable**, and vanish identically at $t = 0$, with $\|u(\,.\,,t)\|_{L^\infty(\mathbb{R}^N)} \leq C|t|^{\alpha/|\beta|}$.*

(4) *If $p < P_2$ and $\alpha \in (\alpha^*, \alpha^{\text{crit}})$, there exist **positive** solutions u such that $r^\delta w$ is **periodic** in* $\ln r$, *and thus*

$$C_1 U_\infty \leq u \leq C_2 U_\infty \quad \textit{for some} C_1, C_2 > 0.$$

*There exist positive solutions with a weak singularity in $|x|^{-\delta}$, with the same bounds, such that $r^\delta w$ is asymptotically periodic near ∞. The regular solutions have **constant sign**, are not integrable, vanish identically at $t = 0$, and $r^\delta w$ is asymptotically periodic near ∞.*

(5) *If $p < P_2$ and $\alpha < \alpha^{\text{crit}}$, there exist solutions **oscillating around*** 0, *such that $r^\delta w$ is **periodic**. There exist solutions **oscillating around*** 0, *integrable, such that $r^\delta w$ is asymptotically periodic. If $\alpha \leq \alpha^*$, all regular solutions have **constant sign**, are not integrable, and vanish identically at $t = 0$.*

Note. If $p < P_1$, recall that the Harnack inequality does not hold, as can be shown by the regular positive solutions constructed in Theorem 1.6, in particular those given by (1–6) when $\alpha = N$. The two kinds of regular, integrable, solutions constructed for the critical values $\alpha_{\text{crit}} > 0$ and $\alpha^{\text{crit}} < 0$ are of different types: the first, constructed for $p > P_2$, disappears in a spreading way, the second, for $p < P_2$, disappears in a focusing way.

The case $p > 2$ will be treated in a second article [Bidaut-Véron 2006b], where we complete the results of [Gil and Vázquez 1997].

2. General properties

Different formulations of the problem. In the remainder of the article we can assume that $\boldsymbol{\alpha \neq 0}$, since the solutions are given explicitly by (1–9) when $\alpha = 0$. Defining

$$J_N(r) = r^N(w + \varepsilon r^{-1}|w'|^{p-2}w'),\qquad J_\alpha(r) = r^{\alpha-N}J_N(r), \tag{2–1}$$

(E_w) can be written in an equivalent way under the form

$$J_N'(r) = r^{N-1}(N-\alpha)w,\qquad J_\alpha'(r) = -\varepsilon(N-\alpha)r^{\alpha-2}|w'|^{p-2}w'. \tag{2–2}$$

If $\alpha = N$, then J_N is constant, so we find again (1–5).

We shall often use the following logarithmic substitution; for given $d \in \mathbb{R}$, setting

$$w(r) = r^{-d}y_d(\tau),\qquad Y_d = -r^{(d+1)(p-1)}|w'|^{p-2}w',\qquad \tau = \ln r, \tag{2–3}$$

we obtain the equivalent system

$$y_d' = dy_d - |Y_d|^{(2-p)/(p-1)}Y_d,\qquad Y_d' = (p-1)(d-\eta)Y_d + \varepsilon e^{(p+(p-2)d)\tau}(\alpha y_d - |Y_d|^{(2-p)/(p-1)}Y_d). \tag{2–4}$$

And y_d, Y_d satisfy the equations

$$y_d'' + (\eta - 2d)y_d' - d(\eta - d)y_d + \frac{\varepsilon}{p-1}e^{((p-2)d+p)\tau}|dy_d - y_d'|^{2-p}(y_d' + (\alpha - d)y_d) = 0, \tag{2–5}$$

$$Y_d'' + (p-1)(\eta - 2d - p')Y_d' + \varepsilon e^{((p-2)d+p)\tau}|Y_d|^{\frac{2-p}{p-1}}\Big(\frac{Y_d'}{p-1} + (\alpha - d)Y_d\Big) - (p-1)^2(\eta - d)(p' + d)Y_d = 0. \tag{2–6}$$

Reduction to an autonomous system. The substitution (2–3) with $d = \delta$ is the most helpful: setting

$$y = y_d, \quad w(r) = r^{-\delta} y(\tau), \quad Y = -r^{(\delta+1)(p-1)}|w'|^{p-2}w', \quad \tau = \ln r, \tag{2–7}$$

we are led to the *autonomous* system that plays a key role in the sequel:

$$(S) \qquad \begin{aligned} y' &= \delta y - |Y|^{(2-p)/(p-1)}Y, \\ Y' &= (\delta - N)Y + \varepsilon(\alpha y - |Y|^{(2-p)/(p-1)}Y). \end{aligned}$$

Since $N - \delta p = \eta - 2\delta$ and $N - \delta = (p-1)(\eta - \delta)$, Equation (2–5) takes the form

$$(E_y) \quad (p-1)y'' + (N - \delta p)y' + \delta(\delta - N)y + \varepsilon|\delta y - y'|^{2-p}(y' + (\alpha - \delta)y) = 0,$$

while Equation (2–6) becomes

$$(E_Y) \quad Y'' + (N - 2\delta)Y' + \frac{\varepsilon}{p-1}|Y|^{(2-p)/(p-1)}Y' + \varepsilon(\alpha - \delta)|Y|^{(2-p)/(p-1)}Y + \delta(\delta - N)Y = 0.$$

When w has constant sign, we define two functions associated to (y, Y):

$$\begin{aligned} \zeta(\tau) &= \frac{|Y|^{(2-p)/(p-1)}Y}{y}(\tau) = -\frac{rw'(r)}{w(r)}, \\ \sigma(\tau) &= \frac{Y}{y}(\tau) = -\frac{|w'(r)|^{p-2}w'(r)}{rw(r)}. \end{aligned} \tag{2–8}$$

They play an essential role in the asymptotic behavior: ζ describes the behavior of w'/w and σ is the slope in the phase plane (y, Y). They satisfy the equations

$$\zeta' = \zeta(\zeta - \eta) + \frac{\varepsilon}{p-1}|\zeta y|^{2-p}(\alpha - \zeta) = \zeta\left(\zeta - \eta + \frac{\varepsilon(\alpha - \zeta)}{(p-1)\sigma}\right), \tag{2–9}$$

$$\sigma' = \varepsilon(\alpha - N) + (|\sigma y|^{\frac{2-p}{p-1}}\sigma - N)(\sigma - \varepsilon) = \varepsilon(\alpha - N) + (\zeta - N)(\sigma - \varepsilon). \tag{2–10}$$

Note. Since (S) is autonomous, for any solution w of (E_w) of the problem, all the functions $w_\xi(r) = \xi^\delta w(\xi r)$, $\xi > 0$, are also solutions. From uniqueness, all regular solutions are completely described from one of them: $w(r, a) = aw(a^{1/\delta}r, 1)$; thus they present the same behavior at infinity.

System (S) will be studied by using phase plane techniques, which was not done in [Qi and Wang 1999], and gives our main results. The set of trajectories of system (S) in the phase plane (y, Y) is symmetric with respect to $(0, 0)$. We define

$$\mathcal{M} = \{(y, Y) \in \mathbb{R}^2 : |Y|^{(2-p)/(p-1)}Y = \delta y\}, \tag{2–11}$$

which is the set of the extremal points of y. We denote the four quadrants by

$$\mathcal{Q}_1 = (0, \infty) \times (0, \infty), \quad \mathcal{Q}_2 = (-\infty, 0) \times (0, \infty), \quad \mathcal{Q}_3 = -\mathcal{Q}_1, \quad \mathcal{Q}_4 = -\mathcal{Q}_2.$$

Remarks 2.1. (i) The field at any point $(\xi, 0)$ with $\xi > 0$ satisfies $y' = -\xi^{1/(p-1)} < 0$, and so points toward $\mathcal{Q}_2$. The field at any point $(\varphi, 0)$ with $\varphi > 0$ satisfies $Y' = \varepsilon\alpha\varphi$, and so points toward $\mathcal{Q}_1$ if $\varepsilon\alpha > 0$ and toward $\mathcal{Q}_4$ if $\varepsilon\alpha < 0$.

(ii) The pair (y, Y) defined by (2–7) is related to J_N by the identity

$$J_N(r) = r^{N-\delta}(y(\tau) - \varepsilon Y(\tau)), \qquad \tau = \ln r, \tag{2–12}$$

and the formulae (2–2) can be recovered from the relations

$$\begin{aligned}(y-\varepsilon Y)' &= (\delta-\alpha)y + \varepsilon(N-\delta)Y = (\delta-\alpha)(y-\varepsilon Y) + \varepsilon(N-\alpha)Y \\ &= (\delta-N)(y-\varepsilon Y) + (N-\alpha)y.\end{aligned} \tag{2–13}$$

(iii) In the sequel the monotonicity of the functions y_d, Y_d, in particular y, Y, ζ and σ plays an important role. At any extremal point τ, these functions satisfy

$$y_d''(\tau) = y_d(\tau)\Big(d(\eta-d) - \frac{\varepsilon(\alpha-d)}{p-1}e^{((p-2)d+p)\tau}|dy_{d(\tau)}|^{2-p}\Big), \tag{2–14}$$

$$\begin{aligned}Y_d''(\tau) = Y_d(\tau)\big((p-1)^2(\eta-d)(p'+d) \\ -\varepsilon(\alpha-d)e^{((p-2)d+p)\tau}|Y_d(\tau)|^{(2-p)/(p-1)}\big),\end{aligned} \tag{2–15}$$

$$\begin{aligned}(p-1)y''(\tau) &= \delta^{2-p}y(\tau)\big(\delta^{p-1}(N-\delta) - \varepsilon(\alpha-\delta)|y(\tau)|^{2-p}\big) \\ &= -|Y(\tau)|^{(2-p)/(p-1)}Y'(\tau),\end{aligned} \tag{2–16}$$

$$Y''(\tau) = Y(\tau)\big(\delta(N-\delta) - \varepsilon(\alpha-\delta)|Y(\tau)|^{(2-p)/(p-1)}\big) = \varepsilon\alpha y'(\tau), \tag{2–17}$$

$$\begin{aligned}(p-1)\zeta''(\tau) &= \varepsilon(2-p)\big((\alpha-\zeta)|\zeta|^{2-p}|y|^{-p}yy'\big)(\tau) \\ &= \varepsilon(2-p)\big((\alpha-\zeta)(\delta-\zeta)|\zeta y|^{2-p}\big)(\tau),\end{aligned} \tag{2–18}$$

$$\begin{aligned}(p-1)\sigma''(\tau) &= (2-p)\big((\sigma-\varepsilon)|\sigma|^{(2-p)/(p-1)}Y|y|^{(4-3p)/(p-1)}y'\big)(\tau) \\ &= \zeta'(\tau)(\sigma(\tau)-\varepsilon).\end{aligned} \tag{2–19}$$

Energy functions for the system (*S*). There is a classical energy function associated to equation (E_w):

$$E(r) = \frac{1}{p'}|w'|^p + \varepsilon\frac{\alpha}{2}w^2, \tag{2–20}$$

which is nonincreasing when $\varepsilon = 1$, since $E'(r) = -(N-1)r^{-1}|w'|^p - \varepsilon r w'^2$. This is not sufficient in our study: we need energy functions adapted to y and Y.

Using the ideas of [Bidaut-Véron 1989], we construct two of them by using the Anderson and Leighton formula [1968].

We find a first function W given by

$$W(\tau)=\mathcal{W}(y(\tau),Y(\tau)),\qquad\text{where}\tag{2–21}$$

$$\mathcal{W}(y,Y)=\varepsilon\left(\frac{(2\delta-N)\delta^{p-1}}{p}|y|^p+\frac{|Y|^{p'}}{p'}-\delta yY\right)+\frac{\alpha-\delta}{2}y^2.$$

It satisfies

$$\begin{aligned}W'(\tau)&=\varepsilon(2\delta-N)(\delta y-|Y|^{(2-p)/(p-1)}Y)(|\delta y|^{p-2}\delta y-Y)-(\delta y-|Y|^{(2-p)/(p-1)}Y)^2\\&=(\delta y-|Y|^{(2-p)/(p-1)}Y)(|\delta y|^{p-2}\delta y-Y)\\&\qquad\times\left(\varepsilon(2\delta-N)-\frac{\delta y-|Y|^{(2-p)/(p-1)}Y}{|\delta y|^{p-2}\delta y-Y}\right).\end{aligned}$$

When $\varepsilon(2\delta-N)\le 0$, then W is nonincreasing. When $\varepsilon(2\delta-N)>0$, we consider the curve

$$\mathcal{L}=\left\{(y,Y)\in\mathbb{R}^2:H(y,Y)=\varepsilon(2\delta-N)\right\},$$

where

$$H(y,Y):=\frac{\delta y-|Y|^{(2-p)/(p-1)}Y}{|\delta y|^{p-2}\delta y-Y}$$

and by convention this quotient takes the value $|\delta y|^{2-p}/(p-1)$ if $|\delta y|^{p-2}\delta y=Y$. $\mathcal{L}$ is a closed curve surrounding $(0,0)$, symmetric with respect to $(0,0)$. Let $\mathcal{S}_{\mathcal{L}}$ be the domain with boundary $\mathcal{L}$ and containing $(0,0)$:

$$\mathcal{S}_{\mathcal{L}}=\left\{(y,Y)\in\mathbb{R}^2:H(y,Y)\le\varepsilon(2\delta-N)\right\}.\tag{2–22}$$

Then $W'(\tau)\ge 0$ if $(y(\tau),Y(\tau))\in\mathcal{S}_{\mathcal{L}}$ and $W'(\tau)\le 0$ if $(y(\tau),Y(\tau))\notin\mathcal{S}_{\mathcal{L}}$. Observe that $\mathcal{S}_{\mathcal{L}}$ is bounded: indeed, for any $(y,Y)\in\mathbb{R}^2$,

$$H(y,Y)\ge\tfrac{1}{2}\left((\delta y)^{2-p}+|Y|^{(2-p)/(p-1)}\right).\tag{2–23}$$

Also $\mathcal{S}_{\mathcal{L}}$ is connected; more precisely, for any $(y,Y)\in\mathcal{S}_{\mathcal{L}}$ and any $\theta\in[0,1]$, we have $(\theta y,\theta^{p-1}Y)\in\mathcal{S}_{\mathcal{L}}$.

A second function, denoted by V, is also given by Anderson formula, or by multiplication by Y' in (E_Y): let

$$V(\tau)=\mathcal{V}(Y(\tau),Y'(\tau)),\qquad\text{where}\tag{2–24}$$

$$\mathcal{V}(Y,Z)=\frac{\varepsilon}{2}\left(\delta(\delta-N)Y^2+Y'^2\right)+\frac{\alpha-\delta}{p'}|Y|^{p'};$$

then

$$V'(\tau)=\left(\varepsilon(2\delta-N)-\frac{1}{p-1}|Y|^{(2-p)/(p-1)}\right)Y'^2.$$

When $\varepsilon(2\delta - N)$ is not positive, V is nonincreasing. When it is positive, we have $V'(\tau) \geq 0$ whenever $|Y(\tau)| \leq D$, where

$$D = \big(\varepsilon(2\delta - N)(p-1)\big)^{(p-1)/(2-p)}. \tag{2–25}$$

The function W gives more information on the system, because $\mathcal{S}_{\mathcal{L}}$ is bounded, whereas the set of zeros of V' is unbounded.

Stationary points of system (S). If $\alpha = \delta = N$, system (S) has infinitely many stationary points, given by $\pm(k, (\delta k)^{p-1})$, $k \geq 0$. Otherwise, if $\varepsilon(\delta - N)(\delta - \alpha) \leq 0$, the system has a unique stationary point $(0, 0)$. If $\varepsilon(\delta - N)(\delta - \alpha) > 0$, it admits the three stationary points

$$(0,0), \quad M_\ell = (\ell, (\delta\ell)^{p-1}) \in \mathcal{Q}_1, \quad M'_\ell = -M_\ell \in \mathcal{Q}_3, \tag{2–26}$$

where ℓ is defined in (1–4). In that case, we find again that $w \equiv \ell r^{-\delta}$ is a particular solution of equation (E_w).

***Local behavior at* (0, 0).** The linearized problem at $(0, 0)$ is given by

$$y' = \delta y, \quad Y' = (\delta - N)Y + \varepsilon\alpha y,$$

and has eigenvalues $\mu_1 = \delta - N$ and $\mu_2 = \delta$. Thus $(0, 0)$ is a saddle point when $\delta < N$ and a source when $N < \delta$. One can choose a basis of eigenvectors $v_1 = (0, -1)$ and $v_2 = (N, \varepsilon\alpha)$.

Local behavior at M_ℓ. Setting

$$y = \ell + \bar{y}, \quad Y = (\delta\ell)^{p-1} + \bar{Y}, \tag{2–27}$$

system (S) is equivalent in $\mathcal{Q}_1$ to

$$\bar{y}' = \delta\bar{y} - \varepsilon\nu(\alpha)\bar{Y} - \Psi(\bar{Y}), \quad \bar{Y}' = \varepsilon\alpha\bar{y} + (\delta - N - \nu(\alpha))\bar{Y} - \varepsilon\Psi(\bar{Y}), \tag{2–28}$$

where

$$\begin{aligned} \nu(\alpha) &= \frac{\delta(N-\delta)}{(p-1)(\alpha-\delta)}, \\ \Psi(\vartheta) &= \big((\delta\ell)^{p-1} + \vartheta\big)^{1/(p-1)} - \delta\ell - \frac{(\delta\ell)^{2-p}}{p-1}\vartheta, \end{aligned} \tag{2–29}$$

with $\vartheta > -(\delta\ell)^{p-1}$. The linearized problem is given by

$$\bar{y}' = \delta\bar{y} - \varepsilon\nu(\alpha)\bar{Y}, \quad \bar{Y}' = \varepsilon\alpha\bar{y} + (\delta - N - \nu(\alpha))\bar{Y}.$$

Its eigenvalues $\lambda_1 \leq \lambda_2$ are the solutions of the equation

$$\lambda^2 - (2\delta - N - \nu(\alpha))\lambda + p'(N - \delta) = 0. \tag{2–30}$$

The discriminant Δ of this equation is

$$\Delta = (2\delta - N - \nu(\alpha))^2 - 4p'(N-\delta) = (N+\nu(\alpha))^2 - 4\nu(\alpha)\alpha. \tag{2–31}$$

The critical value α^* of α, given in (1–11), arises when $\varepsilon(\delta - N/2) > 0$:

$$\alpha = \alpha^* \iff \lambda_1 + \lambda_2 = 0.$$

When $\delta < N$ and $\varepsilon = 1$, then $\delta < \alpha$ and M_ℓ is a sink when $\delta \le N/2$ or $\delta > N/2$ and $\alpha < \alpha^*$, and a source when $\delta > N/2$ and $\alpha > \alpha^*$. When $\delta < N$, and $\varepsilon = -1$, then $\alpha < \delta$ and M_ℓ is a source when $\delta \ge N/2$ or $\delta < N/2$ and $\alpha > \alpha^*$, and a sink when $\delta < N/2$ and $\alpha < \alpha^*$. When $N < \delta$, then M_ℓ is always a saddle point, but, as we will see later, the value α^* also plays a role.

More specifically, the sign of α^* and its position with respect to N or η play a role. By computation,

$$\alpha^* = \frac{p'(\delta^2 - 3\delta + 2N)}{2(2\delta - N)} = \eta + \frac{(\delta - N)^2}{(p-1)(2\delta - N)} = N + \frac{(\delta - N)(\delta^2 - (N+3)\delta + N)}{(2\delta - N)(\delta - 1)}. \tag{2–32}$$

Thus, if $\varepsilon = 1$, then $\alpha^* > \eta > 0$ if $N \ge 2$; if $N = 1$, $\alpha^* > 0$ if $p > \frac{4}{3}$. If $\varepsilon = -1$, then $\alpha^* < -p' < 0$.

Otherwise, when $\Delta > 0$ a basis of eigenvectors $u_1 = (-\varepsilon\nu(\alpha), \lambda_1 - \delta)$, $u_2 = (\varepsilon\nu(\alpha), \delta - \lambda_2)$ can be chosen. If $\Delta \ge 0$, then δ is exterior to the roots if $\varepsilon\alpha > 0$, and $\lambda_1 < \delta < \lambda_2$ if $\varepsilon\alpha < 0$.

Existence of solutions of equation (E_w).

Theorem 2.2. (i) *Take $r_1 > 0$ ($r_1 \ge 0$ if $N = 1$) and let a, a' be reals. There exists a unique solution w of equation (E_w) in a neighborhood $\mathcal{V}$ of r_1, such that $w \in C^2(\mathcal{V})$ and $w(r_1) = a$, $w'(r_1) = a'$. It has a unique extension to a maximal interval of the form*

$$\begin{aligned} &(R_w, \infty) \ \text{with } 0 \le R_w && \text{if } \varepsilon = 1, \\ &(0, S_w) \ \text{with } S_w \le \infty && \text{if } \varepsilon = -1. \end{aligned}$$

If $0 < R_w$ or $S_w < \infty$, as the case may be, w is monotone near R_w or S_w with an infinite limit.

(ii) *For any $a \in \mathbb{R}$, there exists a unique regular solution w of (E_w) satisfying (1–10), and*

$$\lim_{r\to 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N. \tag{2–33}$$

(iii) *If* $N \geq 2$, *any solution defined near* 0 *and bounded is regular. If* $N = 1$, *it satisfies* $\lim_{r\to 0} w' = b \in \mathbb{R}$, *and* $\lim_{r\to 0} w = a \in \mathbb{R}$.

Proof. (i) Local existence and uniqueness near $r_1 > 0$ follow directly from Cauchy's theorem applied to equation (E_w) or to system (S), since the map $\xi \mapsto f_p(\xi) = |\xi|^{(2-p)/(p-1)}\xi$ is of class C^1. If $N = 1$, we can take $r_1 = 0$, obtain a local solution in a neighborhood of 0 in $\mathbb{R}$ and reduce it to $[0, \infty)$.

Any local solution around r_1 has a unique extension to a maximal interval (R_w, S_w). Suppose that $0 < R_w$ (or $S_w < \infty$) and that w is oscillating around 0 near R_w (or S_w). Making the substitution (2–3), with $d \neq 0$, if τ is a maximal point of $|y_d|$, we see that (2–14) holds. If we take d such that $\varepsilon(d-\alpha) > 0$, the sequence $(y_d(\tau))$ stays bounded, since the exponential has a positive limit; for that reason y_d stays bounded, w is bounded near R_w (or S_w) and then also J_N', J_N and w', which is contradictory. Thus w keeps a constant sign, for example $w > 0$, near R_w (or S_w). At each extremal point r such that $w(r) > 0$, we find $(|w'|^{p-2}w')'(r) = -\varepsilon\alpha w(r)$; thus r is unique since $\alpha \neq 0$. Thus w is strictly monotone near R_w (or S_w), and w and $|w'|$ tend to ∞.

First suppose $\varepsilon = 1$. We show that $S_w = \infty$. This is easy when $\alpha > 0$: since E is nondecreasing, w and w' are bounded for $r > r_1$. Assume $\alpha < 0$ and $S_w < \infty$. Then for example w is positive near S_w, nondecreasing, and $\lim_{r\to S_w} w = \infty$. Then J_α is nonincreasing and nonnegative near S_w; hence again w and w' are bounded, which is contradictory.

Next suppose $\varepsilon = -1$. If $R_w > 0$, for example, w is positive and nonincreasing and $\lim_{r\to R_w} w = \infty$. Then either $\alpha < N$ and J_N is nonnegative and nondecreasing near R_w, and thus bounded, or $\alpha \geq N$ and J_α is nonnegative and nondecreasing near R_w, and still bounded. In either case we reach a contradiction, then $R_w = 0$.

(ii) By symmetry we can suppose $a \geq 0$. Let $\rho > 0$. By (2–1) and (2–2), any regular solution w on $[0, \rho]$ satisfies

$$
\begin{aligned}
w(r) &= a - \varepsilon \int_0^r f_p(sT(w))\, ds, \\
T(w)(r) &= w(r) + (\alpha - N)\int_0^1 \theta^{N-1} w(r\theta)\, d\theta.
\end{aligned}
\tag{2–34}
$$

Conversely, any function $w \in C^0([0, \rho])$ that solves (2–34) satisfies $w \in C^1((0, \rho])$ and $|w'|^{p-2}w'(r) = rT(w)$; hence $|w'|^{p-2}w' \in C^1((0, \rho])$ and w satisfies (E_w) in $(0, \rho]$. And $\lim_{r\to 0} rT(w) = 0$, thus $w \in C^1([0, \rho])$ and $|w'|^{p-2}w' \in C^1([0, \rho])$. Then w satisfies (E_w) in $[0, \rho]$ and $w'(0) = 0$. From (E_w), we have

$$
\lim_{r\to 0} |w'|^{p-2}w'/rw = -\varepsilon\alpha/N,
$$

and therefore $w-a=O(r^{p'})$ near 0. We look for w of the form $a+r^{p'}\zeta(r)$, with

$$\zeta\in\mathcal{B}_{\rho,M}=\{\zeta\in C^0([0,\rho]):\|\zeta\|_{C^0([0,\rho])}=\max_{r\in[0,\rho]}|\zeta(r)|\le M\}.$$

We are led to the problem $\zeta=\Theta(\zeta)$, where

$$\begin{aligned}\Theta(\zeta)(r)&=-\varepsilon\int_0^1\theta^{1/(p-1)}f_p\big(T(a+(r\theta)^{p'}\zeta(r\theta))\big)\,d\theta\\&=-\varepsilon\int_0^1\theta^{1/(p-1)}f_p\Big(\frac{\alpha a}{N}+T((r\theta)^{p'}\zeta(r\theta))\Big)\,d\theta.\end{aligned}$$

Taking for example $M=(|\alpha|a)^{1/(p-1)}$, it follows that Θ is a strict contraction from $\mathcal{B}_{\rho,M}$ into itself for ρ small enough, hence existence and uniqueness hold in $[0,\rho]$.

(iii) If w is defined in $(0,\rho)$ and bounded, then J_N' is integrable. Set

$$l=\lim_{r\to 0}J_N(r).$$

Then $|w'|^{p-2}w'=\varepsilon l r^{1-N}(1+o(1)$. If $N\ge 2$, this implies $l=0$; thus from above, w is regular. If $N=1$, then $\lim_{r\to 0}w'=b\in\mathbb{R}$, and $\lim_{r\to 0}w=a\in\mathbb{R}$. □

Definition. Suppose $p>1$. Let $\mathcal{T}_r$ be the trajectory in the plane (y,Y) (see (2–7)) starting from $(0,0)$ at $-\infty$, with slope $\varepsilon\alpha/N$ and $y>0$ near time $-\infty$. Its opposite $-\mathcal{T}_r$ is also a trajectory with the same properties (except that $y<0$). Both are called *regular trajectories*. In this situation we say that y is regular. Observe that $\mathcal{T}_r$ starts in $\mathcal{Q}_1$ if $\varepsilon\alpha>0$, and in $\mathcal{Q}_4$ if $\varepsilon\alpha<0$.

Remark 2.3. Let w be any solution of (E_w) such that $w>0$ on some interval I.

(i) The function w has at most one extremal point on I, since $(|w'|^{p-2}w')'=-\varepsilon\alpha w$, and this point is a maximum if $\varepsilon\alpha>0$ and a minimum if $\varepsilon\alpha<0$.

(ii) From (2–33), if w is regular and $w>0$ in $(0,r_1)$, $r_1\le\infty$, then $w'<0$ in $(0,r_1)$ when $\varepsilon\alpha>0$; thus $\mathcal{T}_r$ is in $\mathcal{Q}_1$. And $w'>0$ in $(0,r_1)$ when $\varepsilon\alpha<0$; hence $\mathcal{T}_r$ is in $\mathcal{Q}_3$ in $(-\infty,\ln r_1)$.

Remark 2.4. In the case $\delta\ne N$, we can give a shorter proof of Theorem 2.2(ii). Indeed, $(0,0)$ is either a source or a saddle point. Thus there exists precisely one trajectory starting from $(0,0)$ at $-\infty$, with $y>0$, with slope $\varepsilon\alpha/N$. The corresponding solutions are regular: the slope σ defined in (2–8) satisfies $\lim_{\tau\to-\infty}\sigma=\varepsilon\alpha/N$. Thus $\lim_{r\to 0}|w'|^{p-2}w'/rw=-\varepsilon\alpha/N$, implying that $w^{(2-p)/(p-1)}$ has a limit $a>0$. Since $\lim_{r\to 0}w'=0$, this function w satisfies (1–10), and any a is obtained by scaling.

Notation. For any point $P_0=(y_0,Y_0)\in\mathbb{R}^2\setminus\{(0,0)\}$, the unique trajectory in the phase plane (y,Y) going through P_0 is denoted by $\mathcal{T}_{[P_0]}$. Notice that $\mathcal{T}_{[-P_0]}=-\mathcal{T}_{[P_0]}$, from the symmetry of system (S).

First sign properties.

Proposition 2.5. *Let $w \not\equiv 0$ be any solution of (E_w).*

(i) *If $\varepsilon = 1$ and $\alpha \le \max(N, \eta)$, then w has at most one zero, and no zero if w is regular.*

(ii) *If $\varepsilon = 1$ and $N < \min(\delta, \alpha)$ and w is regular, then w has at least one zero.*

(iii) *If $\varepsilon = -1$ and $\alpha \ge \min(0, \eta)$, then w has at most one zero. If $\alpha > 0$ and w is regular, then it has no zero.*

(iv) *If $\varepsilon = -1$ and $-p' \le \alpha < \min(0, \eta)$, then w' has at most one zero; consequently w has at most two zeros, and at most one if w is regular.*

Proof. (i) Let $\varepsilon = 1$. Take two consecutive zeros $\rho_0 < \rho_1$ of w, with $w > 0$ on (ρ_0, ρ_1), so $w'(\rho_1) < 0 < w'(\rho_0)$. If $\alpha \le N$, we find, using the function J_N of (2–1),

$$J_N(\rho_1) - J_N(\rho_0) = -\rho_1^{N-1}|w'(\rho_1)|^{p-2} - \rho_0^{N-1}w'(\rho_0)^{p-1} = (N-\alpha)\int_{\rho_0}^{\rho_1} s^{N-1} w\,ds,$$

which is contradictory; thus w has at most one zero. If w is regular with $w(0) > 0$ and ρ_1 is a first zero, then

$$J_N(\rho_1) = -\rho_1^{N-1}|w'(\rho_1)|^{p-1} = (N-\alpha)\int_0^{\rho_1} s^{N-1} w\,ds \ge 0,$$

again a contradiction. Next suppose $0 < \alpha \le \eta$ and use the substitution (2–3), with $d = \alpha$. Then y_α has at most one zero: indeed, if y_α has a maximal point τ where it is positive, and is not constant, then from (2–14),

$$y''_\alpha(\tau) = \alpha(\eta - \alpha) y_d(\tau); \tag{2–35}$$

hence $y''_\alpha(\tau) < 0$, which is impossible. In the same way the regular solution satisfies $\lim_{\tau \to -\infty} y_\alpha = 0$ since $\alpha > 0$, and y_α has no maximal point; thus y_α is positive and increasing.

(ii) Let $\varepsilon = 1$ and $w > 0$ on $[0, \infty)$. If $N < \alpha$, then $J_N(r) = (N-\alpha)\int_0^r s^{N-1} w\,ds < 0$. The function $r \mapsto \delta r^{p'} - w^{(p-2)/(p-1)}$ is nonincreasing; hence $w = O(r^{-\delta})$ at ∞, so y is bounded at ∞. For any $r \ge 1$, one gets $J_N(r) \le J_N(1) < 0$, hence $y(\tau) + |J_N(1)|e^{(\delta-N)\tau} \le Y(\tau)$ for any $\tau \ge 0$, from (2–12). Then $\lim_{\tau\to\infty} Y = \infty$, implying by (S) that $\lim_{\tau\to\infty} y' = -\infty$, which is impossible.

(iii) Let $\varepsilon = -1$ and $\alpha \ge \min(\eta, 0)$. We use again the substitution (2–3) for some $d \ne 0$. If y_d is not constant and has a maximal point where it is positive, then (2–14) holds. Taking $d \in (0, \min(\alpha, \eta))$ if $N \ge 2$ and $\alpha > 0$ and $d = -1$ if $N = 1$ and $\eta = -1 \le \alpha$, we reach a contradiction. Now suppose w is regular and $\alpha > 0$. Then $w' > 0$ near 0, from Theorem 2.2, and as long as w stays positive, any extremal point r is a strict minimum; thus in fact $w' > 0$ on $[0, S_w)$.

(iv) Let $\varepsilon = -1$ and $-p' \le \alpha < \min(0, \eta)$. Suppose that w' has two consecutive zeros $\rho_1 < \rho_2$, and use (2–3) again with $d = \alpha$. If the function Y_α is not constant and has a maximal point τ where it is positive, we get from (2–15)

$$Y''_\alpha(\tau) = (p-1)^2(\eta - \alpha)(p' + \alpha)Y_\alpha(\tau); \tag{2–36}$$

thus $Y''_\alpha(\tau) < 0$, and Y_α has at most one zero. Next consider regular solutions: they satisfy $Y_\alpha = e^{(\alpha(p-1)+p)\tau}(|\alpha|a/N)(1+o(1))$ near $-\infty$, by Theorem 2.2 and (2–3). Thus $\lim_{\tau\to-\infty} Y_\alpha = 0$; as above Y_α cannot have any extremal point, so Y_α is positive and increasing. Then $w' < 0$ from (2–3), and w has at most one zero. □

Remark 2.6. From (2–35) and (2–36) we see that if $0 < \alpha \le \eta$ then y_α has only minimal points on any set where it is positive, and the same conclusion holds for Y_α when $-p' \le \alpha \le \min(\eta, 0)$).

Proposition 2.7. *Let y be any solution of (E_y), linked with w by (2–7), and having constant sign in a semi-interval around the point $\ln R_w$ or $\ln S_w$.*

(i) *If y is not strictly monotone near that same point, then $R_w = 0$ or $S_w = \infty$. If y is not constant, then either $\varepsilon = 1$ and $\delta < N < \alpha$ or $\varepsilon = -1$ and $\alpha < \delta < N$. In any case, y oscillates around ℓ.*

(ii) *If y is strictly monotone near $\ln R_w$ or $\ln S_w$, then also Y, ζ, σ are monotone near the same point.*

Proof. Let $s = R_w$ or S_w, and suppose that y has constant sign near s. Then so does Y, by Remark 2.3.

(i) At each point τ where $y'(\tau) = 0$, we have $y''(\tau) \ne 0$, and (2–16) holds with $y > 0$. Suppose that y is not strictly monotone near s. There exists a strictly monotone sequence (τ_n) converging to s and such that $y'(\tau_n) = 0$, $y''(\tau_{2n}) > 0$, $y''(\tau_{2n+1}) < 0$. Then either $\varepsilon = 1$ and $\delta < \min(\alpha, N)$, or $\varepsilon = -1$ and $\alpha < \delta < N$; and $y(\tau_{2n}) < \ell < y(\tau_{2n+1})$. This cannot happen if s is finite, because y tends to ∞. It is also impossible when $\varepsilon = 1$ and $\alpha \le N$; indeed, there exist at least two points $\theta_1 < \theta_2$ such that $y(\theta_1) = y(\theta_2) = \ell$ and $y \ge \ell$ on (θ_1, θ_2), with $y'(\theta_1) > 0 > y'(\theta_2)$. Then from (S), $Y(\theta_1) < (\delta\ell)^{p-1} < Y(\theta_2)$. And from (2–13), $(e^{(N-\delta)\tau}(y - Y))' = (N-\alpha)e^{(N-\delta)\tau}y$; and the constant $(\ell, (\delta\ell)^{p-1})$ is also a solution of (S), hence

$$(e^{(N-\delta)\tau}(y - \ell - Y + (\delta\ell)^{p-1}))' = (N-\alpha)e^{(N-\delta)\tau}(y - \ell) \ge 0$$

on (θ_1, θ_2). A contradiction follows by integration on this interval.

(ii) Suppose y strictly monotone near s. At any extremal point τ of Y, we find $Y''(\tau) = \varepsilon\alpha y'(\tau)$ from (2–17); hence $y'(\tau) \ne 0$, and $Y''(\tau)$ has constant sign; thus τ is unique, and Y is strictly monotone near s.

Next consider the function ζ satisfying (2–9). If there exists τ_0 such that $\zeta(\tau_0)=\alpha$, then $\zeta'(\tau_0)=\alpha(\alpha-\eta)$. If $\alpha\neq\eta$, then τ_0 is unique, so $\alpha-\zeta$ has a constant sign near s. Then also $\zeta''(\tau)$ has constant sign at any extremal point τ of ζ, from (2–18). Then ζ is strictly monotone near s. If $\alpha=\eta$, then $\zeta\equiv\alpha$.

Finally consider σ, which satisfies (2–10). At each point τ such that $\sigma'(\tau)=0$, (2–19) holds and Y has a constant sign. If there exists τ_0 such that $\sigma(\tau_0)=\varepsilon$, then $\sigma'(\tau_0)=\varepsilon(\alpha-N)$. If $\alpha\neq N$, then τ_0 is unique, and $\sigma-\varepsilon$ has constant sign near s. Thus $\sigma''(\tau)$ has constant sign at any extremal point τ of σ, by (2–19), since Y has constant sign near s. If $\alpha=N$, then $\sigma\equiv\varepsilon$. □

Behavior of w near 0 or ∞. Here we suppose w is defined near 0 or ∞, which means the function y of (2–7) is defined near $\pm\infty$. We study the behavior of y and then return to w. First we suppose y monotone, so we can assume $y>0$ near $\pm\infty$. We do not look for a priori estimates, which could be obtained by successive approximations as in [Bidaut-Véron 2006a]. Our method is based on monotonicity and L'Hospital's rule, which is much more rapid and efficient.

Proposition 2.8. *Let (y,Y) be any solution of (S) such that y is strictly monotone and $y>0$ near $s=\pm\infty$. Then ζ has a finite limit λ near s, which is equal to $0,\alpha,\eta,\delta$. More precisely, we are in one of the following cases*:

(i) *(y,Y) converges to a stationary point different from $(0,0)$. Then $\lambda=\delta$, and $\varepsilon(\delta-N)(\delta-\alpha)>0$ or $\alpha=\delta=N$.*

(ii) *(y,Y) converges to $(0,0)$. Then*
- *either $\lambda=0$, $s=-\infty$, and y is regular, or $N=1$;*
- *or $\lambda=\eta$; then either $(s=\infty,\delta<N)$ or $(s=\infty,\delta=N,\varepsilon(\alpha-N)<0))$ or $(s=-\infty,N<\delta)$ or $(s=-\infty,\delta=N,\varepsilon(\alpha-N)>0))$.*

(iii) *$\lim_{\tau\to s}y=\infty$ and $\lambda=\alpha$. Then either $(s=\infty,\alpha<\delta)$ or $(s=\infty,\alpha=\delta,\varepsilon(\delta-N)<0)$ or $(s=-\infty,\delta<\alpha)$ or $(s=-\infty,\alpha=\delta,\varepsilon(\delta-N)>0)$.*

Proof. From Proposition 2.7, the functions y,Y,σ,ζ are monotone; hence ζ has a limit $\lambda\in[-\infty,\infty]$ and σ has a limit $\mu\in[-\infty,\infty]$, and (y,Y) converges to a stationary point, or $\lim y=\infty$. Then $\lim|Y|=\infty$, since $\alpha\neq 0$ from system (S). To apply the L'Hospital's rule, we consider the two quotients

$$\frac{Y'}{y'}=\frac{(\delta-N)\sigma+\varepsilon(\alpha-\zeta)}{\delta-\zeta} \tag{2–37}$$

and

$$\frac{(|Y|^{(2-p)/(p-1)}Y)'}{y'}=\frac{\zeta(\delta-N+\varepsilon(\alpha-\zeta)/\sigma)}{(p-1)(\delta-\zeta)}=\frac{\zeta(\delta-N)+\varepsilon(\alpha-\zeta)|\zeta y|^{2-p}}{(p-1)(\delta-\zeta)}. \tag{2–38}$$

(i) *First case:* $\varepsilon(\delta - N)(\delta - \alpha) > 0$ and (y, Y) converges to the point M_ℓ defined by (2–26). Then obviously $\lambda = \delta$; or $\alpha = \delta = N$ and $\lim_{\tau \to s} y = k > 0$; then $\lim_{\tau \to s} Y = (\delta k)^{p-1}$, so $\lambda = \delta$.

(ii) *Second case:* (y, Y) converges to $(0, 0)$. Then λ is finite; indeed, if $\lambda = \pm\infty$, the quotient (2–38) converges to $(N-\delta)/(p-1)$, because $|\zeta y| = |Y|^{1/(p-1)} = o(1)$; thus $\zeta = |Y|^{(2-p)/(p-1)} Y/y$ has the same limit, from L'Hospital's rule, which is contradictory.

We next consider N in relation to δ. If $N < \delta$, then $(0, 0)$ is a source, thus $s = -\infty$. Using the eigenvectors, either $\mu = \varepsilon\alpha/N$, then $|\zeta|^{p-1} = |\mu| y^{2-p}(1 + o(1))$, thus $\lambda = 0$ and w is regular, from Remark 2.4. Or $\mu = \pm\infty$; then $\lambda = \lambda(\delta - N)/(p-1)(\delta - \zeta)$ from (2–38), thus $\lambda = 0$ or $\lambda = \eta$. If $\lambda = 0$, then ζ'/ζ converges to $-\eta$ from (2–9), and $s = -\infty$, thus necessarily $\eta < 0$, which means $N = 1$.

If $\delta < N$ (so $N \geq 2$), then $(0, 0)$ is a saddle point. Thus either $s = -\infty$ and $\mu = \varepsilon\alpha/N$, $\lambda = 0$ and w is regular. Or $s = \infty$, $\mu = \pm\infty$, and as above, $\lambda = 0$ or $\lambda = \eta$. Now if $\lambda = 0$ the quotient (2–37) converges to $\mp\infty$, which is contradictory. Thus $\lambda = \eta$.

If $\delta = N$ (so $N \geq 2$), either $\lambda = 0$, so $y' > 0$, $s = -\infty$, and $\mu = \varepsilon\alpha/N$ by (2–38); or else $\lambda > 0$, in which case $\lambda = N = \eta$ from (2–38). Moreover if $s = \infty$, then $\varepsilon(\alpha - N) < 0$; if $s = -\infty$, then $\varepsilon(\alpha - N) > 0$. Indeed $(\varepsilon y - Y)' = \varepsilon(N - \alpha)y$ and $y - \varepsilon Y$ converges to 0; thus if $s = \infty$ and $\varepsilon(N - \alpha) \geq 0$, or $s = -\infty$ and $\varepsilon(N - \alpha) \leq 0$, then $\mu \leq \varepsilon$, but $\mu = \infty$, we reach again a contradiction.

(iii) *Third case:* y tends to ∞. If $s = \infty$, then $y' > 0$, thus $\zeta < \delta$; if $s = -\infty$, then $\zeta > \delta$. If $\lambda = \pm\infty$, then the quotient (2–38) converges to $\varepsilon\infty$; thus $\lambda = \varepsilon\infty$ and $s = -\varepsilon\infty$. In any case, $\zeta' < 0$, so $|\mu| \leq 1/(p-1)$ by (2–9), and $\mu = \varepsilon$ by (2–37); thus $Y' = -\varepsilon|Y|^{(2-p)/(p-1)} Y(1 + o(1))$, and we reach a contradiction by integration. Thus λ is finite; moreover $\lambda \neq 0$ for otherwise we would have $\mu = 0$, seeing that $\sigma = |\zeta y|^{p-2}\zeta$; but $\mu = \alpha/\delta$ by (2–37).

If $\alpha \neq \delta$, then $\lambda = \alpha$ or δ, by (2–38). In turn $\sigma = |\lambda y|^{p-2}\lambda(1 + o(1))$, thus $\mu = 0$. From (2–37), necessarily $\lambda = \alpha$. And if $s = \infty$, then $y' > 0$, thus $\zeta < \delta$, thus $\alpha < \delta$. If $s = -\infty$, then similarly $\alpha > \delta$.

If $\alpha = \delta$, then $\lambda = \alpha = \delta \neq N$, and $\varepsilon(\delta - N)(\delta - \zeta) < 0$ from (2–38); thus if $s = \infty$, then $\varepsilon(\delta - N) < 0$ since $\zeta < \delta$; if $s = -\infty$, then $\varepsilon(\delta - N) > 0$. □

Next we improve Proposition 2.8 by giving a precise behavior of w in any case:

Proposition 2.9. *We keep the assumptions of Proposition 2.8.*

(i) *If* $\lambda = \alpha \neq \delta$, *then* $\lim r^\alpha w = L > 0$ (*near* 0, *or* ∞).

(ii) *If* $\lambda = \eta > 0$, $\eta \neq N$, *then* $\lim r^\eta w = c > 0$.

(iii) *If* $\lambda = \alpha = \delta \neq N$, *then*

$$\lim r^{\delta}(\ln r)^{-1/(2-p)} w = \kappa := \left((2-p)\delta^{p-1}|N-\delta|\right)^{1/(2-p)}. \tag{2–39}$$

(iv) *If* $\lambda = \eta = N = \delta \neq \alpha$, *then*

$$\lim r^{N}(\ln r)^{(N+1)/2} w = \rho := \frac{1}{N}\left(\frac{N(N-1)}{2|\alpha - N|}\right)^{(N+1)/2}. \tag{2–40}$$

(v) *If* $N = 1$, $\lambda = \eta = -1$ *or* $\lambda = 0$ (*near* 0) *then*

$$\lim_{r\to 0} w = a \in \mathbb{R}, \quad \lim_{r\to 0} w' = b; \tag{2–41}$$

and $b \neq 0$; *moreover*, $a = 0$ (*hence* $b > 0$) *if and only if* $\lambda = -1$.

Proof. (i) Let $\lambda = \alpha \neq \delta$. From (2–8) we have $rw'(r) = -\alpha w(r)(1 + O(1))$. Next we apply Proposition 2.8, and are led to two cases:

If $s = \infty$ *and* $\alpha < \delta$, then for any $\gamma > 0$ we have $w = O(r^{-\alpha+\gamma})$ and $1/w = O(r^{\alpha+\gamma})$ near ∞ and $w' = O(r^{-\alpha-1+\gamma})$. Then $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1+\gamma})$, so J'_α is integrable, hence J_α has a limit L, and $\lim r^\alpha w = L$, seeing that $J_\alpha(r) = r^\alpha w(1 + o(1))$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p+\gamma})$, which contradicts the estimate of $1/w = O(r^{\alpha+\gamma})$ for γ small enough. Thus $L > 0$.

Otherwise, we have $s = -\infty$ and $\delta < \alpha$; hence $\lim_{\tau\to s} y = \infty$, $w = O(r^{-\alpha-\gamma})$, $1/w = O(r^{\alpha-\nu})$, $w' = O(r^{-\alpha-1-\gamma})$ near 0, and $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1-\gamma})$. Thus J'_α is still integrable; hence $\lim r^\alpha w = L \geq 0$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p-\gamma})$, which contradicts the estimate of $1/w$. Therefore we again obtain $L > 0$.

(ii) Let $\lambda = \eta > 0$, $\eta \neq N$. From Proposition 2.8, either $s = \infty$, $\delta < N$ or $s = -\infty$, $N < \delta$. As above we get $w = O(r^{-\eta\pm\gamma})$ and $1/w = O(r^{\eta\pm\gamma})$ near ∞ or 0. Here we make the substitution (2–3) with $d = \eta$. We find $y_\eta = O(e^{\pm\gamma\tau})$, $1/y_\eta = O(e^{\pm\gamma\tau})$, $y'_\eta = O(e^{\pm\gamma\tau})$, thus $Y_\eta = O(e^{\pm\gamma\tau})$, and from (2–4), $Y'_\eta = O(e^{\pm\gamma\tau})$. Substituting in (2–4), we deduce $Y'_\eta = O(e^{(2-p)((\delta-\eta)\pm\gamma)\tau})$. When $s = \infty$, then $\delta < \eta$, when $s = -\infty$, then $\delta > \eta$ from (1–2). In any case, Y'_η is integrable, hence Y_η has a limit k, and $Y_\eta - k = O(e^{(2-p)((\delta-\eta)\pm\gamma)\tau})$. Now $(e^{-\eta\tau}y_\eta)' = -e^{-\eta\tau}Y_\eta^{1/(p-1)}$, thus y_η has a limit $c = k^{1/(p-1)}/\eta$; in other words, $\lim r^\eta w = c$. If $c = 0$, then $Y_\eta = O(e^{(2-p)((\delta-\eta)\pm\gamma)\tau})$, $y_\eta = O(e^{((2-p)((\delta-\eta)\pm\gamma)/(p-1))\tau})$, which contradicts $1/y_\eta = O(e^{\gamma\tau})$ for γ small.

(iii) Now suppose $\lambda = \alpha = \delta \neq N$. Then either $s = \infty$ and $\varepsilon(\delta - N) < 0$ or $s = -\infty$ and $\varepsilon(\delta - N) > 0$; moreover, $\lim_{\tau\to s} y = \infty$. Then $Y = (\delta y)^{p-1}(1 + o(1))$, and $\mu = 0$; hence $y - \varepsilon Y = y(1 + o(1))$, and from (2–13),

$$(y - \varepsilon Y)' = \varepsilon(N - \delta)Y = \varepsilon(N - \delta)\delta^{p-1}(y - \varepsilon Y)^{p-1}(1 + o(1)).$$

Then $y = (|N - \delta|\delta^{p-1}(2-p)|\tau|)^{1/(2-p)}(1 + o(1))$, which is equivalent to (2–39) by (2–7).

(iv) Let $\lambda = \eta = N = \delta \neq \alpha$. Then either $s = \infty$ and $\varepsilon(\alpha - N) < 0$ or $s = -\infty$ and $\varepsilon(\alpha - N) > 0$; moreover, $\lim_{\tau\to s} y = 0$. Then $Y = (Ny)^{p-1}(1+o(1))$ and $\mu = \infty$, so $Y - \varepsilon y = Y(1+o(1))$, and from (2–13) we have

$$(Y - \varepsilon y)' = \varepsilon(\alpha - N)y = \varepsilon(\alpha - N)N^{-1}(Y - \varepsilon y)^{1/(p-1)}(1+o(1)).$$

Hence $y = c|\tau|^{-(N+1)/2}(1+o(1))$ with $c = N^{-1}\big(N(N-1)/2|\alpha - N|\big)^{(N+1)/2}$, and (2–40) follows from (2–7).

(v) Let $\lambda = 0$. Then also $rw' = o(w)$; thus by integration we get $w + |w'| = O(r^{-k})$ for any $k > 0$. Then J_1' is integrable, so J_1 has a limit at 0, and $\lim_{r\to 0} rw = 0$. Therefore $\lim_{r\to 0} w' = b \in \mathbb{R}$ and $\lim_{r\to 0} w = a \geq 0$. Then $b \neq 0$, since regular solutions satisfy (2–33), and $a \neq 0$, since $a = 0$ would imply $w = -br(1+o(1))$, $\zeta = -1$. If $\lambda = \eta = -1$, then from (2–8), w is nondecreasing, so it has a limit $a \geq 0$ at 0, leading to $w' = -a\lambda r^{-1}(1+o(1))$, and by integration $a = 0$. And $((w')^{p-1})' = \varepsilon(1-\alpha)w(1+o(1))$, so w' has a limit $b \neq 0$. □

Next we consider the cases where y is not monotone and possibly changes sign.

Proposition 2.10. *Assume* $\varepsilon = 1$.

(i) *Suppose that* $N \leq \delta < \alpha$, *or* $N < \delta \leq \alpha$. *Then any solution* y *has a infinite number of zeros near* ∞.

(ii) *Suppose that* y *has a infinite number of zeros near* $\pm\infty$. *Then either*

$$N < \alpha < \delta \text{ and } |y| < \ell \text{ and } |Y| < (\delta\ell)^{p-1} \text{ near } \pm\infty,$$

or $N < \delta = \alpha$, *or* $\max(\delta, N, \eta) < \alpha$. *If moreover* $\delta < N < \alpha$, *then* $|y|$ *exceeds* ℓ *at its extremal points and* $|Y|$ *exceeds* $(\delta\ell)^{p-1}$ *at its extremal points.*

Proof. (i) Suppose the conclusion does not hold. Then for example $y > 0$ for large τ; and y is monotone, from Proposition 2.7(i). Applying Proposition 2.8 with $s = \infty$, we reach a contradiction.

(ii) Suppose that y is oscillating around 0 near $\pm\infty$. Then from (2–16), at the extremal points,

$$|y(\tau)|^{2-p}(\delta - \alpha) < (\delta - N)\delta^{p-1}, \tag{2–42}$$

and the inequality is strict, because in case of equality, y is constant by uniqueness. Similarly Y is oscillating around 0, and at the extremal points one finds, from (2–17),

$$|Y(\tau)|^{(2-p)/(p-1)}(\delta - \alpha) < (\delta - N)\delta. \tag{2–43}$$

Then $\max(N, \eta) < \alpha$, thanks to Proposition 2.5; and the conclusions follow from (2–42) and (2–43). □

We can complete these results according to the sign of $\delta - N/2$:

Proposition 2.11. *Suppose that $\varepsilon(\delta - N/2) \le 0$. Then any solution y has a finite number of zeros near $\ln R_w$ or $\ln S_w$. If y is defined near $\pm\infty$ and nonmonotone, then (y, Y) converges to $\pm M_\ell$. There is no cycle or homoclinic orbit in $\mathbb{R}^2$.*

Proof. (i) Suppose that y has an infinity of zeros. Then $R_w = 0$ or $S_w = \infty$, and there exists a strictly monotone sequence (r_n) of consecutive zeros of w, converging to 0 or ∞. Since $\varepsilon(\delta - N/2) \le 0$, the energy function V defined in (2–24) is nonincreasing. We claim that V is bounded. This is not easy to prove; we define for the purpose a function

$$U(r) = r^N\left(\tfrac{1}{2}w^2 + \varepsilon r^{-1}|w'|^{p-2}w'w\right) = e^{(N-2\delta)\tau}y\left(\tfrac{1}{2}y - \varepsilon Y\right).$$

Then

$$U'(r) = r^{N-1}\left((\tfrac{1}{2}N - \alpha)w^2 + \varepsilon|w'|^p\right) = e^{(N-1-2\delta)\tau}\left((\tfrac{1}{2}N - \alpha)y^2 + \varepsilon|Y|^{p'}\right).$$

If $\varepsilon = 1$, then $\delta \le N/2 < N < \alpha$. If $\varepsilon = -1$, then $\alpha < 0$, by Proposition 2.10. Then $U(r_n) = 0$ and $\varepsilon U'(r_n) > 0$. Therefore there exists another sequence (s_n) such that $s_n \in (r_n, r_{n+1})$, $U(s_n) = 0$, and $\varepsilon U'(s_n) \le 0$. At the point $\tau_n = e^{s_n}$ we find $2^{1-p'}y^{2p'} = 2|Y|^{p'} \le \varepsilon(2\alpha - N)y^2$, so $(y(\tau_n), Y(\tau_n))$ is bounded. Hence $(V(\tau_n))$ is bounded, so V is bounded near $\pm\infty$. Therefore V has a finite limit χ, and Y and Y' are bounded because $\varepsilon(\alpha - \delta) > 0$; in turn, (y, Y) is bounded. Otherwise $(0, 0)$ and $\pm M_\ell$ are not in the limit set at $\pm\infty$, since $(0, 0)$ is a saddle point, and $\pm M_\ell$ is a source or a sink. Then the trajectory has a limit cycle, and there exists a periodic solution (y, Y). The corresponding function V is periodic and monotone, hence constant; then $V' \equiv 0$ implies that Y is constant and hence also y, by (S). But this is a contradiction.

(ii) Suppose that y is positive near $\pm\infty$, and nonmonotone. If $\varepsilon = 1$, then $\delta \le N/2 < N < \alpha$; if $\varepsilon = -1$, then $\alpha < \delta < N$, by Proposition 2.7, and y oscillates around ℓ. There exists a sequence of minimal points (τ_n), where $y(\tau_n) < \ell$, and $|Y(\tau_n)| = \delta y(\tau_n)$; thus again $(y(\tau_n), Y(\tau_n))$ is bounded, and as above (y, Y) is bounded. The trajectory has no limit cycle, and hence converges to M_ℓ. Finally, if there is an homoclinic orbit, then $\mathcal{T}_r$ is homoclinic. Then $\lim_{\tau\to-\infty} V = \lim_{\tau\to\infty} V = 0$; hence $V \equiv 0$, and as above (y, Y) is constant, so $(y, Y) \equiv (0, 0)$, again a contradiction. □

Proposition 2.12. *If y is not monotone near $\varepsilon\infty$ (positive or changing sign), then y and Y are bounded.*

Proof. From Proposition 2.11, it follows that $\varepsilon(\delta - N/2) > 0$. When $\varepsilon = 1$, and y is changing sign and $N < \alpha < \delta$, then $|y|$ is bounded by ℓ from above. Apart from this case, if y is changing sign, then $\varepsilon(\alpha - \delta) > 0$, from Proposition 2.11. If y stays positive, either $\varepsilon = 1$, $\delta < \min(\alpha, N)$, or $\varepsilon = -1$, $\alpha < \delta < N$, by Proposition 2.7.

In any case $\varepsilon(\alpha-\delta)>0$. Here we use the energy function W defined by (2–21). We can write $\mathcal{W}(y,Y)$ in the form

$$\mathcal{W}(y,Y)=\varepsilon(F(y,Y)+G(y)), \tag{2–44}$$

with

$$F(y,Y)=\frac{|Y|^{p'}}{p'}-\delta yY+\frac{|\delta y|^p}{p},\quad G(y)=\frac{(\delta-N)\delta^{p-1}}{p}|y|^p+\frac{\varepsilon(\alpha-\delta)}{2}y^2. \tag{2–45}$$

Observe that $F(y,Y)\geq 0$, so $\varepsilon\mathcal{W}(y,Y)\geq G(y)>0$ for large $|y|$. Then $W'(\tau)\leq 0$ whenever $(y(\tau),Y(\tau))\notin\mathcal{S}_{\mathcal{L}}$, where $\mathcal{S}_{\mathcal{L}}$ is given in (2–22). Let τ_0 be arbitrary in the interval of definition of y. Since $\mathcal{S}_{\mathcal{L}}$ is bounded, there exists $k>0$ large enough that $\varepsilon W(\tau)\leq k$ for any τ such that $\varepsilon(\tau-\tau_0)\geq 0$ and $(y(\tau),Y(\tau))\in\mathcal{S}_{\mathcal{L}}$, and we can choose $k>W(\tau_0)$. Then $\varepsilon W(\tau)\leq k$ for $\varepsilon(\tau-\tau_0)\geq 0$; hence y and Y are bounded near $\varepsilon\infty$. □

Further sign properties. We can improve Proposition 2.5 using Propositions 2.8 and 2.9:

Proposition 2.13. *Assume $\varepsilon=1$, $-\infty<\alpha\leq\delta$ and $\alpha<N$. Then all regular solutions have constant sign, y is strictly monotone and $\lim_{\tau\to\infty}\zeta=\alpha$. Any solution has at most one zero, and $\lim_{\tau\to\infty}\zeta=\alpha$.*

Proof. Regular solutions have constant sign by Proposition 2.5. Moreover J_N is increasing from 0; thus it is positive for $r>0$, which means $Y<y$. And y is monotone near ∞, by Proposition 2.7. From Proposition 2.8, we have three possibilities: either $\alpha<N<\delta$ and $\lim_{\tau\to\infty}\zeta=\delta$, in which case $\lim_{\tau\to\infty}Y/y=(\delta-\alpha)/(\delta-N)>1$, which is impossible; or $\delta\leq N$ and $\lim_{\tau\to\infty}\zeta=\eta\geq N$, in which case $\lim_{\tau\to\infty}Y/y=\infty$, which is also contradictory, or finally $\lim_{\tau\to\infty}\zeta=\alpha$. Moreover y is increasing on $\mathbb{R}$ from 0 to ∞; indeed, if y has a local maximum for some τ, we get $\alpha<N<\delta$ and $y(\tau)\leq\ell$ from (2–16), and moreover $\ell<\delta^{(p-1)/(2-p)}$; but $\delta y(\tau)=Y(\tau)^{1/(p-1)}<y(\tau)^{1/(p-1)}$, which is contradictory.

For the second statement, we see from Proposition 2.5 that any solution $w\not\equiv 0$ has at most one zero. If $w(r_1)=0$ and, say, $w>0$ on (r_1,∞), we get $w'(r_1)>0$; thus $J_N(r)\geq J_N(r_1)>0$ for $r\geq r_1$, and we conclude as above. □

Proposition 2.14. *Assume $\varepsilon=-1$.*

(i) *If $\alpha<0$ and $N\leq\delta$, all regular solutions have at least one zero.*

(ii) *If $0<\alpha$, all regular solutions have constant sign and satisfy $S_w<\infty$.*

(iii) *If $-p'<\alpha<\min(0,\eta)$, all regular solutions have precisely one zero and $S_w<\infty$.*

Proof. (i) Let $\alpha < 0$ and $N \le \delta$. Since $\varepsilon\alpha > 0$, the trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. Suppose that y stays positive. Then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, from Remark 2.3. If $N \le \delta$, then y is monotone, since it can only have minimal points, from (2–16); and $(0,0)$ is the only stationary point. Then $\lim_{\tau\to\infty} y = \infty$, and $\lim_{\tau\to\infty} \zeta = \alpha < 0$ from Proposition 2.8; thus (y, Y) is in $\mathcal{Q}_4$ for large τ, which is impossible.

(ii) Let $0 < \alpha$. Then $\varepsilon\alpha < 0$, so that $\mathcal{T}_r$ starts in $\mathcal{Q}_4$. Moreover $y > 0$ on $\mathbb{R}$, by Proposition 2.5. And $\mathcal{T}_r$ stays in $\mathcal{Q}_4$, by Remark 2.1(i) on page 211. Thus $y' = \delta y + |Y|^{1/(p-1)} > 0$. If $S_w = \infty$, we see from Proposition 2.8 that $\lim_{\tau\to\infty} \zeta = \alpha > 0$; hence (y, Y) ends up in $\mathcal{Q}_1$, which is false. Then $S_w < \infty$.

(iii) Let $-p' < \alpha < \min(0, \eta)$. Then $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. By Proposition 2.5, Y_α stays positive, $\mathcal{T}_r$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, and Y_α is increasing:

$$Y'_\alpha = -(p-1)(\eta-\alpha)Y_\alpha + e^{(p-(2-p)\alpha)\tau}\big(Y_\alpha^{1/(p-1)} - \alpha y_\alpha\big) > 0.$$

Suppose that $S_w = \infty$. Then $\lim_{\tau\to\infty} Y_\alpha(\tau) \ge C > 0$, so $r^{\alpha+1} w'(r) \le -C^{1/(p-1)}$ for large r, and, by integration, $r^\alpha w(r) \le -C^{1/(p-1)}/2$. In particular, we obtain from (2–3) that $\lim_{\tau\to\infty} y = -\infty$. From Propositions 2.7, 2.8, and 2.9, it follows that $\lim_{r\to\infty} r^\alpha w = L < 0$; thus $\lim_{\tau\to\infty} Y_\alpha(\tau) = (\alpha L)^{p-1}$. And there exists a unique τ_0 such that $y_\alpha(\tau_0) = 0$, by Remark 2.1(i). But

$$\begin{aligned} Y''_\alpha(\tau) &- (p-1)^2(\eta-\alpha)(\alpha+p')Y_\alpha \\ &= \frac{Y'_\alpha}{Y_\alpha}\Big(\frac{1}{p-1}e^{(p-(2-p)\alpha)\tau}Y_\alpha^{1/(p-1)} - (p-1)(\eta-2\alpha-p')Y_\alpha\Big) \\ &\ge \frac{Y'_\alpha}{Y_\alpha}\Big(\frac{\alpha}{p-1}e^{(p-(2-p)\alpha)\tau}y_\alpha + (\eta-\alpha)(2-p) + (p-1)(\alpha+p')Y_\alpha\Big). \end{aligned} \tag{2–46}$$

Thus $Y''_\alpha(\tau) > 0$ for any $\tau \ge \tau_0$, an impossibility. Then $S_w < \infty$, $\lim_{\tau\to\ln S_w} Y/y = -1$, and y has a zero. □

Behavior of w near $R_w > 0$ or $S_w < \infty$.

Proposition 2.15. *Let w be any solution of (E_w) with a reduced domain (so either $\varepsilon = 1$ and $R_w > 0$ or $\varepsilon = -1$ and $S_w < \infty$). Let $s = R_w$ or S_w. Then*

$$\lim_{r\to s} |r-s|^{(p-1)/(2-p)} s^{1/(2-p)} w = \pm\Big(\frac{p-1}{2-p}\Big)^{\frac{p-1}{2-p}} \quad \text{and} \quad \lim_{\tau\to\ln s} \sigma = \varepsilon. \tag{2–47}$$

Proof. From Proposition 2.5, we can suppose that εw is decreasing near s and $\lim_{r\to s} w = \infty$; thus $y > 0$, $\varepsilon Y > 0$ near $\ln s$, and $\lim_{\tau\to\ln s} y = \infty$. Also, σ is monotone near $\ln s$, by Proposition 2.7; thus it has a limit μ such that $\varepsilon\mu \in [0, \infty]$. Suppose that $\mu = 0$. Then $Y = o(y) = o(y - \varepsilon Y)$; from (2–13) we get

$$(y-\varepsilon Y)' = (\delta-\alpha)(y-\varepsilon Y) + \varepsilon(N-\alpha)Y = (\delta-\alpha+o(1))(y-\varepsilon Y),$$

so y cannot blow up in finite time. In the same way, if $\mu = \infty$, then $y = o(\varepsilon Y) = o(\varepsilon Y - y)$, and

$$(y - \varepsilon Y)' = (\delta - N)(y - \varepsilon Y) + (N - \alpha)y = (\delta - N + o(1))(y - \varepsilon Y),$$

again leading to a contradiction; thus $\varepsilon\mu \in (0, \infty)$. Therefore $\lim_{\tau\to\ln R_w} \zeta = \varepsilon\infty$ and $\mu = \varepsilon$ from (2–37); then $w'w^{-1/(p-1)} + (\varepsilon + o(1))r^{1/(p-1)} = 0$, and (2–47) holds. □

More information on stationary points.

The Hopf bifurcation point. When $\varepsilon(\delta - N/2) > 0$, a Hopf bifurcation appears at the critical value $\alpha = \alpha^*$ given by (1–11). Then some cycles do appear near α^*, by the Poincaré–Andronov–Hopf theorem; see [Hale and Koçak 1991, p. 344]. We get more precise results by using the Lyapunov test for a weak sink or source; it requires an expansion up to the order 3 near M_ℓ, in a suitable basis of eigenvectors, where the linearized problem has a rotation matrix.

Theorem 2.16. *Let* $\varepsilon(\delta - N/2) > 0$.

(i) *Suppose* $\varepsilon = 1$. *If* $\alpha = \alpha^*$, *then* M_ℓ *is a weak source. If* $\alpha < \alpha^*$ *with* $\alpha^* - \alpha$ *small enough, there exists a unique limit cycle in* $\mathfrak{Q}_1$ *attracting at* $-\infty$.

(ii) *Suppose* $\varepsilon = -1$. *If* $\alpha = \alpha^*$, *then* M_ℓ *is a weak sink. If* $\alpha > \alpha^*$ *with* $\alpha - \alpha^*$ *small enough, there exists a unique limit cycle in* $\mathfrak{Q}_1$, *attracting at* ∞.

Proof. The eigenvalues are given by $\lambda_1 = -ib$, $\lambda_2 = ib$, with $b = \sqrt{p'(N-\delta)}$. From (2–29) we get

$$\nu(\alpha^*) = 2\delta - N = \frac{\delta(N-\delta)}{(p-1)(\alpha^* - \delta)} = \frac{\varepsilon(\delta\ell)^{2-p}}{(p-1)}.$$

First we make the substitution (2–27) as above, which leads to (2–28). The function Ψ defined in (2–29) has an expansion near $t = 0$ of the form

$$\Psi(\vartheta) = B_2\vartheta^2 + B_3\vartheta^3 + \cdots,$$

where

$$B_2 = \frac{(2-p)(\delta\ell)^{3-2p}}{2(p-1)^2}, \quad B_3 = \frac{(2-p)(3-2p)(\delta\ell)^{4-3p}}{6(p-1)^6} = \frac{2(3-2p)B_2^2}{3(2-p)\nu(\alpha^*)}.$$

Next we make the substitution

$$\tau = -\theta/b, \quad \bar{y}(\tau) = \varepsilon\nu(\alpha)x_1(\theta), \quad \bar{Y}(\tau) = \delta x_1(\theta) + bx_2(\theta),$$

and obtain

$$x_1'(\theta) = x_2 + \frac{\varepsilon}{b\nu(\alpha)}\Psi(\delta x_1 + bx_2), \quad x_2'(\theta) = -x_1 - \frac{\varepsilon(N-\delta)}{b^2\nu(\alpha)}\Psi(\delta x_1 + bx_2).$$

We write the expansion of order 3 in the form

$$x_1' = x_2 + \varepsilon(a_{2,0}x_1^2 + a_{1,1}x_1x_2 + a_{0,2}x_2^2 + a_{3,0}x_1^3 + a_{2,1}x_1^2x_2 + a_{1,2}x_1x_2^2 + a_{0,3}x_2^3 + \cdots),$$
$$x_1' = -x_1 + \varepsilon(b_{2,0}x_1^2 + b_{1,1}x_1x_2 + b_{0,2}x_2^2 + b_{3,0}x_1^3 + b_{2,1}x_1^2x_2 + b_{1,2}x_1x_2^2 + b_{0,3}x_2^3 + \cdots),$$

and we compute the Lyapunov coefficient

$$L_C = \varepsilon(3a_{3,0} + a_{1,2} + b_{2,1} + 3b_{0,3}) \\ -a_{2,0}a_{1,1} + b_{1,1}b_{0,2} - 2a_{0,2}b_{0,2} - a_{0,2}a_{1,1} + 2a_{2,0}b_{2,0} + b_{1,1}b_{2,0}.$$

After simplification, we obtain

$$\frac{(2-p)b\nu(\alpha)^2}{2B_2^2(\delta^2+b^2)} L_C = (N-2\delta)(1-\varepsilon(3-2p)) \\ = \begin{cases} 2(N-2\delta)(p-1) < 0 & \text{if } \varepsilon = 1, \\ 2(N-2\delta)(2-p) > 0 & \text{if } \varepsilon = -1, \end{cases}.$$

The nature of M_ℓ follows from [Hubbard and West 1995, p. 292], taking into account that θ has opposite sign from τ. If $\varepsilon = 1$, M_ℓ is a week source, and there exists a small limit cycle attracting at $-\infty$ for all α near α^* such that M_ℓ is a sink; this means that $\alpha < \alpha^*$. If $\varepsilon = -1$, M_ℓ is a weak sink and there exists a small limit cycle attracting at ∞ for all α near α^* such that M_ℓ is a source; this means $\alpha^* < \alpha$. □

Node points or spiral points. When the system (S) has three stationary points, and M_ℓ is a source or a sink (so $\delta < N$), it is interesting to know if M_ℓ is a node point. When α^* exists, it is a spiral point, by (2–30).

If $\varepsilon = 1$, we see from (2–31) that M_ℓ is a node point when $\delta \le N/2 - \sqrt{p'(N-\delta)}$ or $\delta > N/2 - \sqrt{p'(N-\delta)}$ and $\alpha \le \alpha_1$, or $\delta > N/2 + \sqrt{p'(N-\delta)}$ and $\alpha_2 \le \alpha$, where

(2–48)
$$\alpha_1 = \delta + \frac{\delta(N-\delta)}{(p-1)(2\delta - N + 2\sqrt{p'(N-\delta)})}, \\ \alpha_2 = \delta + \frac{\delta(N-\delta)}{(p-1)(2\delta - N - 2\sqrt{p'(N-\delta)})}.$$

If $\varepsilon = -1$, M_ℓ is a node when $\delta \ge N/2 + \sqrt{p'(N-\delta)}$, or $\delta < N/2 + \sqrt{p'(N-\delta)}$ and $\alpha_2 \le \alpha$, or $\delta < N/2 - \sqrt{p'(N-\delta)}$ and $\alpha \le \alpha_1$. In any case $\alpha_1 < \alpha_2$.

Remarks 2.17. (i) Let $\varepsilon = 1$. One can verify that $N \le \alpha_1$ and that $N = \alpha_1$ if and only if $N = \delta/(p-1) = p'/(2-p)$. Also $\alpha_1 < \eta$ if and only if $\delta^2 + (7-N)\delta + N > 0$, which is true for $N \le 14$, but not always.

(ii) Let $\varepsilon = -1$. It is easy to see that $\alpha_2 \le 0$ and that $\alpha_2 = 0$ if and only if $N(2-p) = \delta$, or equivalently $N = p/(2-p)^2$. Also $\alpha_2 > -p'$ if and only if $\delta^2 + 7\delta - 8N < 0$, which is true for $\delta < N/2 < 9$, but not always.

Nonexistence of cycles. If the system (S) admits a cycle $\mathcal{O}$ in $\mathbb{R}^2$, then $\mathcal{O}$ surrounds at least one stationary point. If it surrounds $(0, 0)$, the corresponding solutions y are not of constant sign. If it only surrounds M_ℓ, then it stays in $\mathcal{Q}_1$, so y stays positive. Indeed $\alpha \neq 0$ from (1–9), and $\mathcal{O}$ cannot intersect $\{(\varphi, 0), \varphi > 0\}$ at two points, and similarly $\{(0, \xi), \xi > 0\}$, by Remark 2.1(i) on page 211.

For suitable values of α, δ, N, we can show that cycles cannot exist, by using Bendixson's criterion or the Poincaré map. Writing (S) under the form

$$y' = f_1(y, Y), \quad Y' = f_2(y, Y), \tag{2–49}$$

we obtain

$$\frac{\partial f_1}{\partial y}(y, Y) + \frac{\partial f_2}{\partial Y}(y, Y) = 2\delta - N - \varepsilon |Y|^{(2-p)/(p-1)}. \tag{2–50}$$

For example, as a direct consequence of Bendixson's criterion, if $\varepsilon(\delta - N/2) < 0$, we find again the nonexistence of any cycle in $\mathbb{R}^2$, which was obtained in Proposition 2.11. Now we consider cycles in $\mathcal{Q}_1$.

First we extend to system (S) a general property of quadratic systems, proved in [Chicone and Tian 1982], stating that there cannot exist a closed orbit surrounding a node point. Note that the restriction of our system to $\mathcal{Q}_1$ is quadratic if $p = \frac{3}{2}$.

Theorem 2.18. *Let $\delta < N$ and $\varepsilon(\delta - \alpha) < 0$. When M_ℓ is a node point, there is no cycle or homoclinic orbit in $\mathcal{Q}_1$.*

Proof. We use the linearization (2–27), (2–28), (2–29). Consider the line L with equation $A\bar{y} + \bar{Y} = 0$, where A is a real parameter. The points of L are in $\mathcal{Q}_1$ whenever $-(\delta\ell)^{p-1} < \bar{Y}$ and $-\ell < \bar{y}$. As in [Chicone and Tian 1982], we study the orientation of the vector field along L: we find

$$A\bar{y}' + \bar{Y}' = (\varepsilon\nu(\alpha)A^2 + (N + \nu(\alpha))A + \varepsilon\alpha)\bar{y} - (A + \varepsilon)\Psi(\bar{Y}).$$

Using (2–31), apart from the case $\varepsilon = 1$, $\alpha = N = \alpha_1$, we can find an A such that $\varepsilon\nu(\alpha)A^2 + (N + \nu(\alpha))A + \varepsilon\alpha = 0$, and $A + \varepsilon \neq 0$. Moreover $\Psi(\bar{Y}) \ge 0$ on $L \cap \mathcal{Q}_1$; indeed, $(p-1)\Psi'(t) = ((\delta\ell)^{p-1} + t)^{(2-p)/(p-1)} - t^{(2-p)/(p-1)}$, so Ψ has a minimum on $(-(\delta\ell)^{p-1}, \infty)$ at 0, and hence is nonnegative on this interval. Then the orientation of the vector field does not change along $L \cap \mathcal{Q}_1$; in particular no cycle can exist in $\mathcal{Q}_1$; and similarly no homoclinic trajectory can exist. In the case $\varepsilon = 1$, $\alpha = N = \alpha_1$, $Y \equiv y \in [0, \ell)$ defines the trajectory $\mathcal{T}_r$, corresponding to the solutions given by (1–6) with $K > 0$, and again no cycle can exist in $\mathcal{Q}_1$: it would intersect $\mathcal{T}_r$. □

Next we prove the nonexistence of cycles on one side of the Hopf bifurcation point:

Theorem 2.19. *Assume $\delta < N$ and $\varepsilon(\delta-\alpha) < 0 < \varepsilon(\delta - N/2)$. If $\varepsilon(\alpha-\alpha^*) \geq 0$, there exists no cycle or homoclinic orbit in $\mathcal{Q}_1$.*

Proof. M_ℓ is a source or weak source if $\varepsilon = 1$, and a sink or weak sink if $\varepsilon = -1$. Suppose there exists a cycle in $\mathcal{Q}_1$. Then any trajectory starting from M_ℓ at $-\varepsilon\infty$ has a limit cycle in $\mathcal{Q}_1$, which is attracting at $\varepsilon\infty$. Such a cycle is not unstable (if $\varepsilon = 1$) or not stable (if $\varepsilon = -1$); in other words the Floquet integral on the period $[0, \mathcal{P}]$ is nonpositive if $\varepsilon = 1$ and nonnegative if $\varepsilon = -1$. From (2–50) we then get

$$\text{(2–51)}\quad \varepsilon\int_0^{\mathcal{P}}\left(\frac{\partial f_1}{\partial y}(y,Y)+\frac{\partial f_2}{\partial Y}(y,Y)\right)d\tau = \int_0^{\mathcal{P}}\left(|2\delta-N|-\frac{1}{p-1}Y^{(2-p)/(p-1)}\right)d\tau \leq 0.$$

Now, from (2–28),

$$0=\delta\int_0^{\mathcal{P}}\bar{y}\,d\tau-\nu(\alpha)\int_0^{\mathcal{P}}\bar{Y}\,d\tau-\int_0^{\mathcal{P}}\Psi(\bar{Y})\,d\tau,$$
$$0=\alpha\int_0^{\mathcal{P}}\bar{y}\,d\tau+(\delta-N-\nu(\alpha))\int_0^{\mathcal{P}}\bar{Y}\,d\tau-\int_0^{\mathcal{P}}\Psi(\bar{Y})\,d\tau.$$

Moreover, since Ψ is nonnegative,

$$\int_0^{\mathcal{P}}\Psi(\bar{Y})\,d\tau=-p'\int_0^{\mathcal{P}}\bar{y}\,d\tau=-\frac{p'(N-\delta)}{\alpha-\delta}\int_0^{\mathcal{P}}\bar{Y}\,d\tau>0;$$

and since $y'=\delta y-Y^{1/(p-1)}$,

$$\int_0^{\mathcal{P}}Y^{1/(p-1)}dt=\delta\int_0^{\mathcal{P}}y\,dt<\delta\ell\mathcal{P}.$$

From this, (2–51), and Jensen's inequality, it follows that

$$(p-1)|2\delta-N|\leq\int_0^{\mathcal{P}}Y^{(2-p)/(p-1)}\,d\tau \leq \mathcal{P}^{p-1}\left(\int_0^{\mathcal{P}}Y^{1/(p-1)}d\tau\right)^{2-p}<(\delta\ell)^{2-p}=\frac{\varepsilon\delta(N-\delta)}{\alpha-\delta}.$$

Hence $\varepsilon(\alpha-\alpha^*)<0$, a contradiction. Next, suppose that there is an homoclinic orbit. From [Hubbard and West 1995, Theorem 9.3, p. 303] we see that the saddle connection is repelling if $\varepsilon = 1$ and attracting if $\varepsilon = -1$, because the sum of the eigenvalues μ_1, μ_2 of the linearized problem at $(0, 0)$ is $2\delta - N$. That means that the solutions just inside it spiral toward the loop near $-\varepsilon\infty$. Because M_ℓ is a source or weak source or sink or weak sink, such solutions have a limit cycle that is attracting at $\varepsilon\infty$. As before, we reach a contradiction. □

Finally we get the nonexistence of cycles in nonobvious cases, where we have shown that any solution has at most one or two zeros.

Theorem 2.20. *Assume $\delta < N$ and $\varepsilon(\delta-\alpha) < 0 < \varepsilon(\delta - N/2)$. If $\varepsilon = 1$ and $\alpha \le \eta$, or $\varepsilon = -1$ and $-p' \le \alpha < 0$, there exists no cycle and no homoclinic orbit in $\mathcal{Q}_1$.*

Proof. (i) Suppose there exists a cycle. There are two possibilities:

Suppose $\varepsilon = 1$ and $\alpha \le \eta$. M_ℓ is a sink since $\alpha < \alpha^*$, so any trajectory converging to M_ℓ at ∞ has a limit cycle $\mathcal{O}$ in $\mathcal{Q}_1$, attracting at $-\infty$. Let (y, Y) describe the orbit $\mathcal{O}$, of period $\mathcal{P}$. Then $\mathcal{O}$ is not stable, so the Floquet integral is nonnegative, and from (2–51),

$$\int_0^{\mathcal{P}} \left(2\delta - N - \frac{1}{p-1} Y^{(2-p)/(p-1)}\right) d\tau \ge 0.$$

Otherwise y is bounded from above and below; thus the function y_α, defined by (2–3) with $d = \alpha$, satisfies $\lim_{\tau\to-\infty} y_\alpha = 0$ and $\lim_{\tau\to\infty} y_\alpha = \infty$; moreover y_α has only minimal points, from (2–35), since $\alpha \le \eta$; thus $y'_\alpha > 0$ on $\mathbb{R}$. From (2–5) and (2–4) with $d = \alpha$,

$$\frac{y''_\alpha}{y'_\alpha} + \eta - 2\alpha + \frac{1}{p-1} Y^{(2-p)/(p-1)} = \frac{\alpha(\eta-\alpha) y_\alpha}{y'_\alpha} = \frac{\alpha(\eta-\alpha) y_\alpha}{\alpha y_\alpha - Y_\alpha^{1/(p-1)}} > \eta - \alpha.$$

Upon integration over $[0, \mathcal{P}]$, this implies $\eta - 2\alpha + 2\delta - N > \eta - \alpha$, which is impossible, since $\delta - N + \delta - \alpha < 0$.

Alternatively, *suppose $\varepsilon = -1$ and $-p' \le \alpha < 0$.* M_ℓ is a source since $\alpha^* < \alpha$, and any trajectory converging to it at $-\infty$ has a limit cycle $\mathcal{O}'$ attracting at ∞. Let (y, Y) describe the orbit $\mathcal{O}'$, of period $\mathcal{P}$. Then $\mathcal{O}'$ is not unstable, so the Floquet integral is nonpositive, hence

$$\int_0^{\mathcal{P}} (2\delta - N + \frac{1}{p-1} Y^{(2-p)/(p-1)})\, d\tau \le 0.$$

Moreover Y is bounded from above and below; thus Y_α, defined by (2–3) with $d = \alpha$, satisfies $\lim_{\tau\to-\infty} Y_\alpha = \infty$, $\lim_{\tau\to\infty} Y_\alpha = 0$. And Y_α has only minimal points, by (2–36), since $-p' \le \alpha < 0$; thus $Y'_\alpha < 0$ on $\mathbb{R}$. From (2–6) and (2–4) we get

$$\begin{aligned}\frac{Y''_\alpha}{Y'_\alpha} + (p-1)(\eta - 2\alpha - p') - \frac{1}{p-1} Y^{(2-p)/(p-1)} &= \frac{(p-1)^2(\eta-\alpha)(p'+\alpha) Y_\alpha}{Y'_\alpha} \\ &< -(p-1)(p'+\alpha).\end{aligned}$$

Upon integration over $[0, \mathcal{P}]$, this implies $(p-1)(\eta - 2\alpha - p') + 2\delta - N < -(p-1) \times (p'+\alpha)$, which means $p\delta + (p-1)|\alpha| < 0$; but this is false.

(ii) Suppose there exists an homoclinic orbit. Since $\delta < N$, the origin is a saddle point, so $\mathcal{T}_r$ is the only trajectory starting from $(0, 0)$ in $\mathcal{Q}_1$, and there exists a unique trajectory $\mathcal{T}_s$ converging to $(0, 0)$, lying in $\mathcal{Q}_1$ for large τ, having infinite slope at $(0, 0)$, and satisfying $\lim_{r\to 0} r^\eta w = c > 0$.

If $\varepsilon = 1$, then $\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} e^{-\alpha\tau} y_\alpha = a > 0$, so $\lim_{\tau\to-\infty} y_\alpha = 0$; also y_α has only minimal points, so it is increasing and positive; and $\mathcal{T}_s$ satisfies $\lim_{\tau\to\infty} e^{(\eta-\alpha)\tau} y_\alpha = c > 0$. If $\alpha < \eta$, then $\lim_{\tau\to\infty} y_\alpha = 0$, thus $\mathcal{T}_r \neq \mathcal{T}_s$. If $\alpha = \eta$, $\mathcal{T}_s$ is given explicitly by (1–7), that means y_α is constant, thus again $\mathcal{T}_r \neq \mathcal{T}_s$.

If $\varepsilon = -1$, then $\mathcal{T}_s$ satisfies $\lim_{\tau\to-\infty} e^{(\eta-\alpha)(p-1)\tau} Y_\alpha > 0$, because $\lim_{\tau\to-\infty} \zeta = \eta$; so $\lim_{\tau\to\infty} Y_\alpha = 0$. Moreover Y_α has only minimal points, and hence is increasing and positive; otherwise $\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} e^{-(\alpha(p-1)+p)\tau} Y_\alpha = -a\alpha/N > 0$, by (2–33). If $\alpha > -p'$, we get $\lim_{\tau\to\infty} Y_\alpha = 0$, which implies $\mathcal{T}_r \neq \mathcal{T}_s$. If $\alpha = -p'$, then $\mathcal{T}_r$ is given explicitly by (1–8); in other words Y_α is constant, and again $\mathcal{T}_r \neq \mathcal{T}_s$. □

Boundedness of cycles. When there do exist cycles, except for a few cases, we cannot prove their uniqueness, but we can show:

Theorem 2.21. *When nonempty, the set $\mathcal{C}$ of cycles of system (S) is bounded in $\mathbb{R}^2$.*

Proof. Suppose there exists a cycle $\mathbb{O}$ in $\mathbb{R}^2$. By Propositions 2.5, 2.7, 2.10, 2.11 and Theorem 2.20, this can happen only in four cases: $\varepsilon = 1$, $N < \alpha < \delta$; $\varepsilon = 1$, $N < \delta = \alpha$; $\varepsilon = 1$, $\max(\delta, N, \eta) < \alpha$, $N/2 < \delta$; $\varepsilon = -1$, $\delta < N/2$, $\alpha < -p'$. In the first case, $\mathcal{C}$ is bounded and lies in $(-\ell, \ell) \times (-(\delta\ell)^{p-1}, (\delta\ell)^{p-1})$, by Proposition 2.10. In the other cases we use the energy function W. Let (y, Y) describe the trajectory $\mathbb{O}$. Then W is periodic, and its maximum and minimum points are precisely the points of the curve $\mathcal{L}$. Indeed if $W'(\tau_1) = 0$ and the point $(y(\tau_1), Y(\tau_1)$ is not on $\mathcal{L}$, it lies on the curve $\mathcal{M}$ defined in (2–11); hence $y'(\tau_1) = 0$ and $y''(\tau_1) \neq 0$, since $\mathbb{O}$ is not just a stationary point. Therefore $(\delta y - |Y|^{(2-p)/(p-1)} Y)(|\delta y|)^{p-2}\delta y - Y) > 0$ near τ_1; then W' has constant sign, and τ_1 is not a maximum or a minimum. In this way we obtain estimates for W independently of the trajectory:

$$\max_{\tau\in\mathbb{R}} |W(\tau)| = M = \max_{(y,Y)\in\mathcal{L}} |\mathcal{W}(y, Y)|.$$

At the maximal points τ of y, one has $|Y(\tau)|^{(2-p)/(p-1)} Y(\tau) = \delta y(\tau)$, so

$$W(\tau) = \frac{\varepsilon(\delta - N)\delta^{p-1}}{p} |y(\tau)|^p + \frac{\alpha - \delta}{2} y^2(\tau).$$

By the Hölder inequality, y is bounded by a constant independent of the trajectory, and

$$\frac{|Y|^{p'}}{p'} \leq \delta y Y + \frac{|2\delta - N|\delta^{p-1}}{p} |y|^p + \frac{|\alpha - \delta|}{2} y^2 + M.$$

Thus Y is also uniformly bounded, and $\mathcal{C}$ is bounded. □

3. The case $\varepsilon = 1$, $\alpha < \delta$ or $\alpha = \delta < N$

Lemma 3.1. *Assume* $\varepsilon = 1$ *and* $-\infty < \max(\alpha, N) < \delta (\alpha \neq 0)$. *In the phase plane* (y, Y), *there exist*

(i) *a trajectory* $\mathcal{T}_1$ *converging to* M_ℓ *at* ∞, *such that* y *is increasing as long as it is positive*;

(ii) *a trajectory* $\mathcal{T}_2$ *in* $\mathcal{Q}_1 \cup \mathcal{Q}_4$ *converging to* M_ℓ *at* $-\infty$, *and unbounded at* ∞, *with* $\lim_{\tau\to\infty} \zeta = \alpha$;

(iii) *a trajectory* $\mathcal{T}_3$ *converging to* M_ℓ *at* $-\infty$, *such that* y *has at least one zero*;

(iv) *a trajectory* $\mathcal{T}_4$ *in* $\mathcal{Q}_1$, *converging to* M_ℓ *at* ∞, *with* $\lim_{\tau\to\ln R_w} Y/y = 1$;

(v) *trajectories* $\mathcal{T}_5$ *in* $\mathcal{Q}_1 \cup \mathcal{Q}_4$ *unbounded at* $\pm\infty$, *with*

$$\lim_{\tau\to\infty} \zeta = \alpha \quad \text{and} \quad \lim_{\tau\to\ln R_w} Y/y = 1.$$

Proof. Here the system (S) has three stationary points, defined by (2–26). The point $(0, 0)$ is a source, and the point M_ℓ is a saddle point. The eigenvalues satisfy $\lambda_1 < 0 < \lambda_2 < \delta$. The eigenvectors $u_1 = (-\nu(\alpha), \lambda_1 - \delta)$ and $u_2 = (\nu(\alpha), \delta - \lambda_2)$ form a positively oriented basis, and u_1 points toward $\mathcal{Q}_3$, while u_2 points toward $\mathcal{Q}_1$. There exist four particular trajectories converging to M_ℓ at $\pm\infty$, namely:

- $\mathcal{T}_1$ converging to M_ℓ at ∞, with tangent vector u_1; then $y < \ell$ and $Y < (\delta\ell)^{p-1}$ and $y' > 0$ near ∞; as above, y cannot have a local minimum, so $y' > 0$ whenever $y > 0$.
- $\mathcal{T}_2$ converging to M_ℓ at $-\infty$, with tangent vector u_2; then $y' > 0$ near $-\infty$. If y has a local maximum at some τ, then $y''(\tau) \leq 0$, so that $y(\tau) \leq \ell$ from (2–16), which is impossible. Then y is increasing on $\mathbb{R}$ and $\lim_{\tau\to\infty} y = \infty$, and $\lim_{\tau\to\infty} \zeta = \alpha$ from Proposition 2.8. In particular $\mathcal{T}_2$ stays in $\mathcal{Q}_1$ if $\alpha > 0$, and enters $\mathcal{Q}_4$ if $\alpha < 0$.
- $\mathcal{T}_3$ converging to M_ℓ at $-\infty$, with tangent vector $-u_2$; then $y' < 0$ near $-\infty$. If y has a local minimum at some τ, then $y(\tau) \geq \ell$, which is still impossible. Thus y is decreasing at long as the trajectory stays in $\mathcal{Q}_1$. It cannot stay in it, because it cannot converge to $(0, 0)$. It cannot enter $\mathcal{Q}_4$ by Remark 2.1(i) on page 211. Then it enters $\mathcal{Q}_2$ and y has at least one zero.
- $\mathcal{T}_4$ converging to M_ℓ at ∞, with tangent vector $-u_1$; then $y' < 0$ near ∞. As above, y cannot have a local maximum, it is decreasing and $\lim_{\tau\to\ln R_w} y = \infty$. From Proposition 2.8, y cannot be defined near $-\infty$, hence $R_w > 0$ and $\lim_{\tau\to\ln R_w} Y/y = 1$.

For any trajectory $\mathcal{T}$ in the domain delimited by $\mathcal{T}_2$ and $\mathcal{T}_4$, the function y is positive, and $\mathcal{T}$ cannot converge to M_ℓ at ∞, and y is monotone for large τ from

Proposition 2.7, because $\alpha < \delta$; thus $\lim_{\tau\to\infty} \zeta = \alpha$ from Proposition 2.8, and y is not defined near $-\infty$, and $\mathcal{T}$ is of type (5). □

We now study the various global behaviors, according to the values of α. The results are expressed in terms of w.

$\alpha \leq N < \delta$

Theorem 3.2. *Assume the* $\varepsilon = 1$ *and* $-\infty < \alpha \leq N < \delta$, *with* $\alpha \neq 0$. *All regular solutions* w *of* (E_w) *have constant sign, and* $\lim_{r\to\infty} r^\alpha |w| = L > 0$ *if* $\alpha < N$, $\lim_{r\to\infty} r^\delta |w| = \ell$ *if* $\alpha = N$. *And* $w(r) = \ell r^{-\delta}$ *is also a solution. There exist solutions satisfying any one of these characterizations*:

(1) (*only if* $\alpha < N$) w *is positive*, $\lim_{r\to 0} r^\eta w = c > 0$, *if* $N \geq 2$ (*and* (2–41) *holds with* $a > 0 > b$ *if* $N = 1$), *and* $\lim_{r\to\infty} r^\delta w = \ell$;

(2) w *is positive*, $\lim_{r\to 0} r^\delta w = \ell$, $\lim_{r\to\infty} r^\alpha w = L > 0$;

(3) w *has precisely one zero*, $\lim_{r\to 0} r^\delta w = \ell$, $\lim_{r\to\infty} r^\alpha w(r) = L < 0$;

(4) w *is positive*, $R_w > 0$, $\lim_{r\to\infty} r^\delta w = \ell$;

(5) w *is positive*, $R_w > 0$, $\lim_{r\to\infty} r^\alpha w = L > 0$;

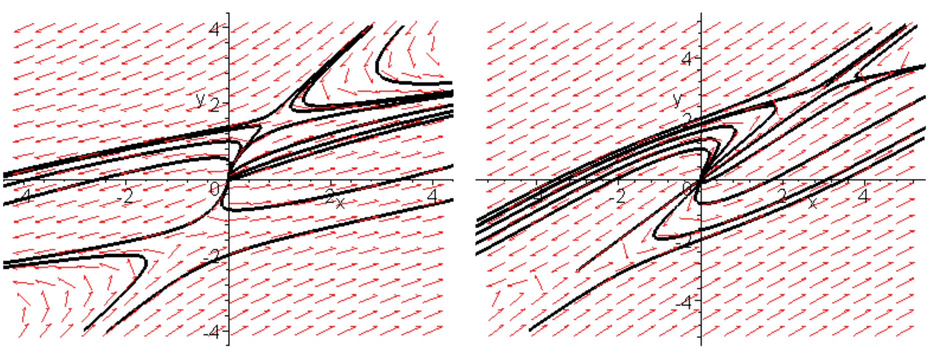
(6) w *has one zero*, $R_w > 0$, *and* $\lim_{r\to\infty} r^\alpha w = L \neq 0$;

(7) (*only if* $\alpha < N$) w *is positive*, $\lim_{r\to 0} r^\eta w = c > 0$ *if* $N \geq 2$ (*and* (2–41) *holds with* $a > 0 > b$ *if* $N = 1$), *and* $\lim_{r\to\infty} r^\alpha w = L > 0$;

(8) w *has one zero*, *with* $\lim_{r\to 0} r^\eta w = c > 0$ *if* $N \geq 2$ (*and* (2–41) *holds with* $a > 0 > b$ *if* $N = 1$), *and* $\lim_{r\to\infty} r^\alpha w = -L < 0$;

(9) $N = 1$, $w > 0$ *and* (2–41) *holds with* $a \geq 0$, $b > 0$ *and* $\lim_{r\to\infty} r^\alpha w = L$.

Up to symmetry, all the solutions of (E_w) *are as above.*



$\alpha = 1 < N$ $\alpha = 2 = N$

Figure 1. Theorem 3.2: $N = 2 < \delta = 3$.

Proof. (i) We first assume that $\alpha \neq N$, and refer to Figure 1, left. The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$ for $\alpha > 0$, in $\mathcal{Q}_4$ for $\alpha < 0$, and y stays positive. Then $\lim_{\tau\to\infty} y = \infty$, and $\lim_{\tau\to\infty} \zeta = \alpha$, and $\lim_{r\to\infty} r^\alpha w = L > 0$, by Propositions 2.10 and 2.13, since $\alpha < N$. Moreover y is increasing: indeed if it has a local maximum, at this point $y \le \ell$, and then y has no local minimum, since at such a point $y \ge \ell$, so that y cannot tend to ∞. Then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and Y is increasing from 0 to ∞. Indeed each extremal point τ of Y is a local minimum, from (2–17). If $\alpha < 0$, in the same way, then Y is decreasing from 0 to $-\infty$, and $\mathcal{T}_r$ stays in $\mathcal{Q}_4$.

First we follow the trajectory $\mathcal{T}_1$: it does not intersect $\mathcal{T}_r$, and cannot enter $\mathcal{Q}_2$ by Remark 2.1(i). Thus y stays positive and increasing. It cannot enter $\mathcal{Q}_4$, seeing that it does not meet $\mathcal{T}_r$ if $\alpha > 0$, or (by the same remark) if $\alpha < 0$. Thus $\mathcal{T}_1$ stays in $\mathcal{Q}_1$, and (y, Y) converges necessarily to $(0, 0)$. If $N \ge 2$, then $\lim_{\tau\to-\infty} \zeta = \eta$, $\lim_{r\to 0} r^\eta w = c > 0$ from Proposition 2.8 and 2.9. If $N = 1$, since $\mathcal{T}_1$ stays in $\mathcal{Q}_1$, then necessarily $\lim_{\tau\to-\infty} \zeta = 0$, thus (2–41) holds with $a > 0 > b$.

Next we follow $\mathcal{T}_3$: here y has a zero, which is unique by Proposition 2.5, since $\alpha < N$. Then $y < 0$, and $\lim_{\tau\to\infty} y = -\infty$, $\lim_{r\to\infty} r^\alpha w = -L < 0$ by Propositions 2.8 and 2.9. $\mathcal{T}_3$ stays in $\mathcal{Q}_2$ if $\alpha < 0$, or goes from $\mathcal{Q}_2$ into $\mathcal{Q}_3$ if $\alpha > 0$.

Trajectories $\mathcal{T}_2$, $\mathcal{T}_4$, $\mathcal{T}_5$ of Lemma 3.1 yield solutions w of type (2), (4), (5).

For any trajectories $\mathcal{T}_6$ in the domain delimited by $\mathcal{T}_3$, $\mathcal{T}_4$, y has one zero, and $\lim_{r\to\infty} r^\alpha w = L \neq 0$; and w is of type (6).

The solutions of type (7) correspond to the trajectories $\mathcal{T}$ in the domain delimited by $\mathcal{T}_r$, $\mathcal{T}_1$, $\mathcal{T}_2$. Indeed $\lim_{\tau\to\infty} y = \infty$, and $\lim_{r\to\infty} r^\alpha w = L > 0$. And $\lim_{\tau\to-\infty} y = 0$. If $N \ge 2$, then $\lim_{\tau\to-\infty} \zeta = \eta$, $\lim_{r\to 0} r^\eta w = c > 0$, from Proposition 2.8 and 2.9. If $N = 1$, $\mathcal{T}$ cannot meet $\mathcal{T}_r$, thus necessarily $\lim_{\tau\to-\infty} \zeta = 0$, and (2–41) holds with $a > 0 > b$.

Up to a change of w into $-w$, the solutions of type (8) and (9) correspond to the trajectories in the domain delimited by $-\mathcal{T}_r$, $\mathcal{T}_1$, $\mathcal{T}_3$. Indeed they satisfy $\lim_{\tau\to\infty} y = -\infty$, and $\lim_{r\to\infty} r^\alpha w = L < 0$; and $\lim_{\tau\to-\infty} y = 0$. If $N \ge 2$, then $\lim_{r\to 0} r^\eta w = c > 0$ and w has a zero. If $N = 1$, either (2–41) holds with $a = 0 > b$ and w stays negative, or $a < 0$, $b < 0$ and w has a zero. Such solutions exist from Theorem 2.2. By symmetry, all the solutions are described.

(ii) Now assume $\alpha = N$ (Figure 1, right). Then M_ℓ belongs to the line $y = Y$, and

$$u_1 = (-\delta/(p-1), -\delta/(p-1))$$

has the same direction. Moreover J_N is constant, which means $y - Y = Ce^{(\delta-N)\tau}$, with $C \in \mathbb{R}$. The solutions corresponding to $C = 0$ satisfy $y \equiv Y$, thus $\mathcal{T}_1 = \mathcal{T}_r = \{(\xi, \xi) : \xi \in [0, \ell)\}$, corresponding to the regular Barenblatt solutions. And $\mathcal{T}_4 = \{(\xi, \xi) : \xi > \ell\}$ yields the solutions defined by (1–6) for $K < 0$. All other solutions exist as before, apart from type (7). □

Note. The trajectory $\mathcal{T}_1$ is the only one joining the stationary points (0, 0) and M_ℓ. Hence, for $\alpha < N$, solutions w of type (1) are unique, up to the scaling mentioned in the note on page 210. Solutions of types (2), (4), and (5) are also unique.

$N < \alpha < \delta$

Here we prove that some periodic trajectories can exist, according to the value of α with respect to α^*. By (2–32), $N < \alpha^*$ whenever $\delta^2 - (N+3)\delta + N > 0$, and in particular for any $p \le \frac{3}{2}$. Our main tool is the Poincaré–Bendixson theorem, using the level curves of the energy function $\mathcal{W}$:

Lemma 3.3. *Assume $\varepsilon = 1$ and $N < \alpha < \delta$. Consider, for $k \in \mathbb{R}$, the level curves*

$$\mathcal{C}_k = \{(y, Y) \in \mathbb{R}^2 : \mathcal{W}(y, Y) = k\}$$

of the function $\mathcal{W}$ defined in (2–21). They are symmetric with respect to (0, 0). *Let*

$$k_\ell = \mathcal{W}(\ell, (\delta\ell)^{p-1}) = \tfrac{1}{2}(\delta - N)\delta^{p-2}\ell^p.$$

If $k > k_\ell$, then $\mathcal{C}_k$ has two unbounded connected components. If $0 < k < k_\ell$, $\mathcal{C}_k$ has three connected components, of which one is bounded. If $k = k_\ell$, $\mathcal{C}_{k_\ell}$ is connected with a double point at M_ℓ. If $k = 0$, one of the three connected components of $\mathcal{C}_0$ is $\{(0, 0)\}$. *If $k < 0$, $\mathcal{C}_k$ has two unbounded connected components.*

Proof. The energy k_ℓ of the statement is positive. Also $(y, Y) \in C_k$ if and only if $F(y) = k - G(y)$, where F, G are defined in (2–45). By symmetry we can reduce the study of C_k to the set $y > 0$. Let $\varphi(s) = |s|^{p'}/p' - s + 1/p$ for any $s \in \mathbb{R}$, and set $\theta = Y/(\delta y)^{p-1}$. Then (2–44) reduces to

$$\varphi(\theta) = (k - G(y))/(\delta y)^p.$$

The function φ is decreasing on $(-\infty, 1)$ from ∞ to 0, and increasing on $(1, \infty)$ from 0 to ∞. Let ψ_1 be the inverse of the restriction of φ to $(-\infty, 1]$ and ψ_2 the inverse of the restriction of φ to $[1, \infty)$, both defined on $[0, \infty)$. For any $y > 0$,

$$y \in \mathcal{C}_k \iff Y = \Phi_1(y) < (\delta y)^{p-1} \text{ or } Y = \Phi_2(y) \ge (\delta y)^{p-1},$$

where

$$\Phi_i(y) = (\delta y)^{p-1}\psi_i\Big(\frac{k - G(y)}{(\delta y)^p}\Big) \quad \text{for } i = 1, 2, \tag{3–1}$$

Φ_1 lies below $\mathcal{M}$ whereas Φ_2 lies above $\mathcal{M}$, and $\Phi_1, \Phi_2 \in C^1((0, \infty))$. The function G has a maximal point at $y = \ell$, and $G(\ell) = k_\ell$. Using symmetry we see that either $k > k_\ell$ and y ranges over $\mathbb{R}$, in which case $\mathcal{C}_k$ has two unbounded connected components; or $0 < k < k_\ell$ and $\mathcal{C}_k$ has three connected components, one of which, $\mathcal{C}_k^b$, is bounded; or $k = k_\ell$ and $\mathcal{C}_{k_\ell}$ is connected with a double point at M_ℓ; or yet $k = 0$ and one of the three connected components of $\mathcal{C}_0$ is $\{(0, 0)\}$; or $k < 0$ and $\mathcal{C}_k$

has two unbounded connected components. The unbounded components satisfy $\lim_{|y|\to\infty} Y/y^{2/p'} = \pm(p'(\delta\alpha)/2)^{1/p'}$, by (3–1). The zeros of Φ_i' are contained in

$$\mathcal{N} = \{(y, Y) \in \mathbb{R}^2 : y > 0,\ \delta Y = -(\delta-\alpha)y + (2\delta - N)(\delta y)^{p-1}\},$$

and $\mathcal{N}$ lies above $\mathcal{M}$ as long as $y < \ell$.

We now describe $\mathscr{C}_k^b$ when $0 < k \le k_\ell$. The function Φ_1 is increasing on a segment $[0, \bar{y}]$ such that $\bar{y} < \ell$, and $\Phi_1(0) = -(kp')^{1/p'}$ and $(\bar{y}, \Phi_1(\bar{y})) \in \mathcal{M}$, with an infinite slope at this point; Φ_2 is increasing on some interval $[0, \tilde{y})$ such that $(\tilde{y}, \Phi_2(\tilde{y})) \in \mathcal{N}$ and then decreasing on $(\tilde{y}, \bar{y}]$, and $\Phi_2(0) = (kp')^{1/p'}$ and $\Phi_2(\bar{y}) = \Phi_1(\bar{y})$. By symmetry with respect to $(0, 0)$, the curve $\mathscr{C}_k^b$ is completely described.

Next consider $\mathscr{C}_{k_\ell}$ for $y > 0$: the function Φ_2 is increasing on $[0, \infty)$ from $(p'k_\ell)^{1/p'}$ to ∞, and $\Phi_2(\ell) = (\delta\ell)^{p-1}$; the function Φ_1 is increasing on some interval $[0, \hat{y})$ such that $(\hat{y}, \Phi_1(\hat{y})) \in \mathcal{N}$, so $\hat{y} > \ell$; and $(\hat{y}, \Phi_1(\hat{y}))$ is below $\mathcal{M}$, and $\Phi_1(\ell) = (\delta\ell)^{p-1}$, and Φ_1 is decreasing on $(\hat{y}, \infty)$, with $\lim_{y\to\infty} \Phi_1 = -\infty$. Setting $\mathscr{C}_{k_\ell,1} = \{(y, \Phi_1(y)) : y > \ell\}$ and $\mathscr{C}_{k_\ell,2} = \{(y, \Phi_2(y)) : y > \ell\}$, one has $\mathscr{C}_{k_\ell} = \mathscr{C}_{k_\ell}^b \cup \pm\mathscr{C}_{k_\ell,1} \cup \mathscr{C}_{k_\ell,2}$. □

Theorem 3.4. *Assume $\varepsilon = 1$ and $N < \alpha < \delta$. Then $w(r) = \ell r^{-\delta}$ is a solution of (E_w).*

(i) *If $\alpha \le \alpha^*$, any solution of (E_w) has at most a finite number of zeros.*

(ii) *There exist $\check{\alpha}$ such that $\max(N, \alpha^*) < \check{\alpha} < \delta$, such that if $\alpha > \check{\alpha}$, in the phase plane (y, Y), there exists a cycle surrounding $(0, 0)$.*

(iii) *Let α be such that there exists no such cycle. Then all regular solutions have a finite positive number of zeros and $\lim_{r\to\infty} r^\alpha w = L_r \ne 0$ or $\lim_{r\to\infty} r^\delta w = \pm\ell$. There exist solutions of types (2)–(6) of Theorem 3.2, and solutions such that*

(1′) *(only if $L_r \ne 0$) $\lim_{r\to 0} r^\delta w = \ell$, and $\lim_{r\to 0} r^\eta w = c \ne 0$ (or (2–41) holds if $N = 1$);*

(7′) *$\lim_{r\to 0} r^\eta w = c \ne 0$ (or (2–41) holds if $N = 1$) and $\lim_{r\to\infty} r^\alpha w = L \ne 0$.*

(iv) *For any α such that there exists such a cycle, there exist solutions w which oscillate near 0 and ∞, and $r^\delta w$ is periodic in $\ln r$. All regular solutions w oscillate near ∞, and $r^\delta w$ is asymptotically periodic in $\ln r$. There exist solutions of types (2), (4), (5), and solutions*

(1″) *with precisely one zero, $R_w > 0$, and $\lim_{r\to\infty} r^\delta w = \ell$;*

(3″) *such that $\lim_{r\to 0} r^\delta w = \ell$, and oscillating near ∞;*

(9) *such that $\lim_{r\to 0} r^\eta w = c \ne 0$ (or (2–41) holds if $N = 1$) and oscillating near ∞;*

(10) *with precisely one zero, $R_w > 0$, and $\lim_{r\to\infty} r^\alpha w = L \ne 0$;*

(11) *with $R_w > 0$ and oscillating near ∞.*

Proof. There always exist solutions of type (2), (4), and (5), by Lemma 3.1.

(i) Assume $\alpha \le \alpha^*$ (see Figure 2, left). Consider any trajectory $\mathcal{T}$. Suppose y has infinitely many zeros near $\pm\infty$. From Proposition 2.10, $\mathcal{T}$ is contained in the set

$$\mathcal{D} = \{(y, Y) \in \mathbb{R}^2 : |y| < \ell, |Y| < (\delta\ell)^{p-1}\}$$

near $\pm\infty$. Then $\mathcal{T}$ is bounded near $\pm\infty$, hence the limit set at $\pm\infty$ is contained in $\mathcal{D}$. But $M_\ell \notin \mathcal{D}$, and $(0, 0)$ is a source and a node point, so it cannot be in the limit set Γ at ∞. From the Poincaré–Bendixson theorem, Γ is a closed orbit, so that there exists a cycle. Moreover, from (2–25), (2–49) and (2–50),

$$\frac{\partial f_1}{\partial y}(y, Y) + \frac{\partial f_2}{\partial Y}(y, Y) = \frac{1}{p-1}(D^{(2-p)/(p-1)} - |Y|^{(2-p)/(p-1)});$$

thus, by Bendixson's criterion, the set $\{|Y| < D\}$ contains no cycle. Now note that

(3–2) $$\alpha \le \alpha^* \iff (\delta\ell)^{p-1} \le D.$$

Then there is no cycle in $\mathcal{D}$, and we reach a contradiction.

(ii) Now assume $\alpha > \max(N, \alpha^*)$. The curve $\mathcal{L}$ intersects $\mathcal{M}$ at $(\delta^{-1}D^{1/(p-1)}, D)$. Then

$$\mathcal{S}_{\mathcal{L}} \cap \mathcal{M} = \{(\delta^{-1}(\theta D)^{1/(p-1)}, \theta D) : \theta \in [0, 1]\};$$

and $D < (\delta\ell)^{p-1}$ by (3–2), so $\mathcal{S}_{\mathcal{L}}$ does not contain M_ℓ. We can find $k_1 > 0$ small enough that $\mathcal{C}^b_{k_1}$ is interior to $\mathcal{S}_{\mathcal{L}}$. Next we search for $k \in (0, k_\ell)$ such that $\mathcal{L}$ is in the domain delimited by C^b_k. By symmetry we only consider the points of $\mathcal{L}$ such that $y \ge 0$. In any case for any point of $\mathcal{L}$ we have $|\delta y|^p + |Y|^{p'} \le M = (2(2\delta - N))^\delta$, by (2–23) and by convexity. By a straightforward computation this implies that

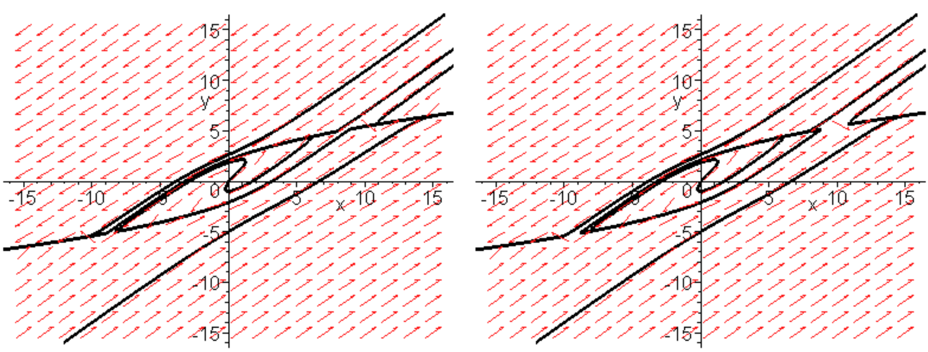


Figure 2. Theorem 3.4: $\varepsilon = 1$, $N = 2 < \alpha < \delta = 3$.

$\mathcal{W}(y, Y) \le KM$, where $K = \max(2/p',\ (3\delta - N)/\delta p)$. Let $\check{\alpha} = \check{\alpha}(\delta, N)$ be given by $KM = k_\ell$. This means that

$$\delta - \check{\alpha} = \Big(\frac{\delta - N}{2K\delta^{2-p}}\Big)^{1/\delta} \frac{\delta^{p-1}(\delta - N)}{2(2\delta - N)}.$$

If $\alpha > \check{\alpha}$, there exists $k_2 < k_\ell$ such that $\mathcal{L}$ is contained in the set

$$\{(y, Y) \in \mathbb{R}^2 : \mathcal{W}(y, Y) < k_2\},$$

which has three connected components; because $\mathcal{S}_{\mathcal{L}}$ is connected, it is contained in the interior to $\mathcal{C}_{k_2^b}$. Then the domain delimited by $\mathcal{C}^b_{k_1}$ and $\mathcal{C}^b_{k_2}$ is bounded and forward invariant. It does not contain any stationary point, and so it contains a cycle, by the Poincaré–Bendixson theorem (see Figure 2, right).

(iii) Let α be such that there exists no cycle. Since $N < \alpha$, all regular solutions y have at least one zero. They have a finite number of zeros. For if not, (y, Y) is bounded near ∞, so it has a limit cycle. Then either $\lim_{\tau\to\infty} y = \pm\infty$ and $\lim_{\tau\to\infty} \zeta = \alpha > 0$, so that the trajectory $\mathcal{T}_r$ ends up in $\mathcal{Q}_1 \cup \mathcal{Q}_3$ and $\lim r^\alpha w = L_r \neq 0$, or else $\lim_{\tau\to\infty} y = \pm\ell$ and $\lim_{r\to\infty} r^\delta w = \pm\ell$.

$\mathcal{T}_3$ cannot meet $\mathcal{T}_r$ or $-\mathcal{T}_r$, thus y has a unique zero, and $\lim_{\tau\to\infty} y = -\infty$, and $\lim_{\tau\to\infty}\zeta = \alpha$. The same happens for the trajectories $\mathcal{T}_6$ in the domain delimited by $\mathcal{T}_3$, $\mathcal{T}_4$. Thus there exist solutions of types (3) and (6).

Suppose $L_r \neq 0$ and consider $\mathcal{T}_1$: the trajectories $\mathcal{T}_r$, $-\mathcal{T}_r$, $\mathcal{T}_1$ have a last intersection point at time τ_0 with the half-axis $\{y = 0,\ Y < 0\}$ at some points P_r, P_r', P_1, and $P_1 \in [P_r, P_r']$. The domain delimited by $\mathcal{T}_r$, $-\mathcal{T}_r$ and $[P_r, P_r']$ is bounded and backward invariant, by Remark 2.1(i) on page 211. Then $\mathcal{T}_1$ stays in it for $\tau < \tau_0$, it has a finite number of zeros, and converges to $(0, 0)$ near $-\infty$; thus w is of type $(1')$. If $N \ge 2$, then $\lim_{\tau\to\infty}\zeta = \eta$, so that y has at least one zero.

Since $(0, 0)$ is a source, there exist other solutions converging to $(0, 0)$ near $-\infty$, they have a finite number of zeros, and $\lim_{\tau\to\infty}\zeta = \alpha$, and w is of type $(7')$.

(iv) Let α such that there exists a cycle, thus $\mathcal{T}_r$ has a limit cycle $\mathbb{O}$.

Consider again $\mathcal{T}_1$. Since $M_\ell \notin \mathcal{S}_{\mathcal{L}}$, the function W is decreasing near ∞, so that $W(\tau) > k_\ell$; thus $\mathcal{T}_1$ is exterior to $\mathcal{C}^b_{k_\ell}$ for large τ, in the domain exterior to $\mathcal{C}^b_{k_\ell}$ delimited by $\mathcal{C}_{k_\ell,1}$ and $-\mathcal{C}_{k_\ell,2}$; and it cannot cut $\mathcal{C}_{k_\ell}$. Moreover y is decreasing at long as $y > 0$, then $\mathcal{T}_1$ enters $\mathcal{Q}_4$ as τ decreases. It cannot stay in it, because it would converge to $(0, 0)$, which is impossible. Then y has at least one zero, and $\mathcal{T}_1$ enters $\mathcal{Q}_3$. It stays in it, since it cannot cross $-\mathcal{C}_{k_\ell,2}$. Thus y has a unique zero, and $\lim_{\tau\to-\infty} y = -\infty$, and $R_w > 0$ from Proposition 2.8, because $\mathcal{T}_1$ cannot converge to $(0, 0)$ at $-\infty$, and w is of type $(1'')$.

Next consider $\mathcal{T}_3$. Here W is decreasing near $-\infty$, hence $W(\tau) < k_\ell$; thus $\mathcal{T}_3$ is in the interior of $\mathcal{C}^b_{k_\ell}$ near $-\infty$. Now the domain delimited by $\mathcal{C}^b_{k_1}$ and $\mathcal{C}^b_{k_\ell}$ is forward invariant, thus $\mathcal{T}_3$ stays in it; then it is bounded, and has a limit cycle at ∞, and w is of type ($3''$).

The solutions of type (9) correspond to trajectories $\mathcal{T}$ in the domain delimited by $\mathbb{O}$, and distinct from $\mathcal{T}_r$. Indeed $\mathcal{T}$ is bounded, in particular the limit-set at $-\infty$ is $(0, 0)$, or a closed orbit. But $\mathcal{T}$ cannot intersect $\mathcal{T}_r$. Then $\mathcal{T}$ converges to $(0, 0)$ near $-\infty$.

The solutions of type (10) correspond to a trajectory $\mathcal{T}$ in the domain delimited by $\mathcal{T}_1 \cup \mathcal{T}_2$ (or its opposite): indeed y has constant sign near ∞ and near $\ln R_w$, and $\lim_{r\to\infty} r^\alpha w = L \neq 0$, and $R_w > 0$, from Proposition 2.8. Then $\mathcal{T}$ starts in $\mathcal{Q}_3$, and ends up in $\mathcal{Q}_1$; and y has at most one zero, because at such a point $y' = -|Y|^{1/(p-1)}Y > 0$, thus it has precisely one zero.

Solutions of type (11) correspond to a trajectory $\mathcal{T}$ in the domain delimited by $\mathcal{T}_1, \mathcal{T}_4, -\mathcal{T}_1, -\mathcal{T}_4$. Then y cannot have constant sign near ∞: indeed this implies $\lim \zeta = \alpha > 0$; this is impossible since the line $Y = y$ is an asymptotic direction for $\mathcal{T}_1, \mathcal{T}_4$. Thus $\mathcal{T}$ is bounded near ∞, and it has a limit cycle at ∞. Near $-\infty$, y a constant sign, because $\mathcal{T}$ cannot meet $\mathcal{T}_3$; and $R_w > 0$ from Proposition 2.8, and $\mathcal{T}$ has the same asymptotic direction $Y = y$ as $\mathcal{T}_1, \mathcal{T}_4$. □

Note. From numerical studies, we conjecture that $\check{\alpha}$ is unique, and the number of zeros of w increases with α in the range $(N, \check{\alpha})$; and moreover there exists $\alpha_1 = N < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} < \ldots$, such that regular solutions have n zeros for any $\alpha \in (\alpha_n, \alpha_{n+1})$, with $\lim_{r\to\infty} r^\alpha w = L_r \neq 0$, and $n+1$ zeros for $\alpha = \alpha_{n+1}$, with $\lim_{r\to\infty} r^\delta w = \pm\ell$.

$\alpha \leq \delta \leq N, \alpha \neq N$

Here $(0, 0)$ is the only stationary point, and $N \geq 2$.

Theorem 3.5. *Assume $\varepsilon = 1$ and $-\infty < \alpha \leq \delta \leq N, \alpha \neq 0, N$. Then all regular solutions of (E_w) have constant sign, and the positive ones satisfy $\lim_{r\to\infty} r^\alpha w(r) = L > 0$ if $\alpha \neq \delta$, or (2–39) holds if $\alpha = \delta$. All the other solutions have a reduced domain ($R_w > 0$). Among them, there exist solutions satisfying any one of these characterizations*:

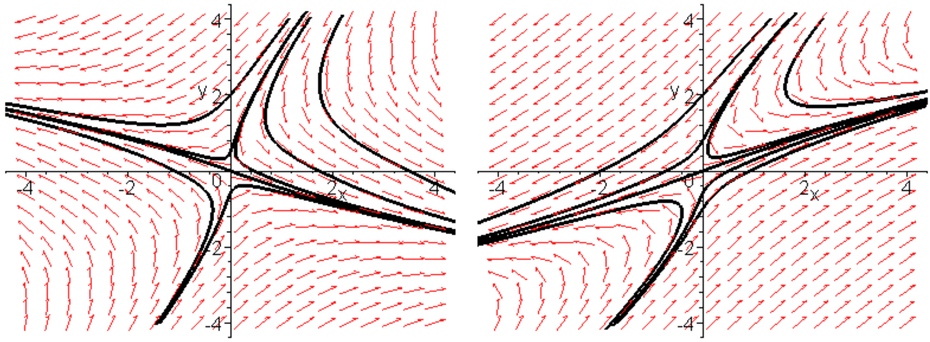
(1) *w is positive, $\lim_{r\to\infty} r^\eta w = c \neq 0$ if $\delta < N$, or $\lim_{r\to\infty} r^N (\ln r)^{(N+1)/2} w = \varrho$ defined in (2–40) if $\delta = N$;*

(2) *w is positive, $\lim_{r\to\infty} r^\alpha w = L > 0$ if $\alpha \neq \delta$, or (2–39) holds if $\alpha = \delta$;*

(3) *w has one zero, and $\lim_{r\to\infty} r^\alpha w = L \neq 0$ if $\alpha \neq \delta$, or (2–39) holds if $\alpha = \delta$.*

Up to symmetry, all the solutions are as above.

Proof. Any solution has at most one zero, by Proposition 2.5. The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_4$ if $\alpha < 0$ (Figure 3, left) and in $\mathcal{Q}_1$ if $\alpha > 0$ (Figure 3, right). Moreover y



$\alpha = -2$ $\alpha = 2$

Figure 3. Theorem 3.5: $\varepsilon = 1$, $\alpha < \delta = 3 < N = 4$.

stays positive, and $\lim_{\tau\to\infty} y = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha$, by Proposition 2.13. Then $\lim_{r\to\infty} r^\alpha w(r) = L > 0$ if $\alpha < \delta$, or (2–39) holds if $\alpha = \delta$, from Proposition 2.9. Moreover y is increasing: indeed it has no local maximum from (2–16). Thus $\mathcal{T}_r$ does not meet $\mathcal{M}$, and so stays below $\mathcal{M}$. If $\alpha > 0$, then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and Y is increasing from 0 to ∞; indeed each extremal point τ of Y is a local minimum, by (2–17). Likewise, if $\alpha < 0$ the function Y is decreasing from 0 to $-\infty$, and $\mathcal{T}_r$ stays in $\mathcal{Q}_4$. The only solutions y defined on $(0, \infty)$ are the regular ones, by Proposition 2.8.

For any point $P = (\varphi, (\delta\varphi)^{p-1}) \in \mathbb{R}^2$ with $\varphi > 0$, in other words on the curve $\mathcal{M}$, the trajectory $\mathcal{T}_{[P]}$ intersects $\mathcal{M}$ transversally: the vector field is $(0, -(N-\alpha)\varphi)$. Moreover the solution going through this point at time τ_0 satisfies $y''(\tau_0) > 0$ from (E_y), then τ_0 is a point of local minimum. From (2–16), τ_0 is unique, so that it is a minimum. Then $y > 0$, $\lim_{\tau\to\infty} \zeta = \alpha$, $\lim_{\tau\to\ln R_w} Y/y = 1$, and $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1$ if $\alpha > 0$, or goes from $\mathcal{Q}_1$ into $\mathcal{Q}_3$ if $\alpha < 0$. The corresponding w is of type (2).

For any point $P = (0, \xi)$, $\xi > 0$, the trajectory $\mathcal{T}_{[P]}$ goes through P from $\mathcal{Q}_1$ into $\mathcal{Q}_2$, by Remark 2.1(i). Then y has only one zero, and as above, it is decreasing on $\mathbb{R}$ and $\lim_{\tau\to\infty} y = -\infty$, and $\lim_{\tau\to\infty} \zeta = \alpha$, $\lim_{\tau\to\ln R_w} Y/y = 1$. Thus $\mathcal{T}_{[P]}$ starts in $\mathcal{Q}_1$, then stays in $\mathcal{Q}_2$ if $\alpha < 0$, and enters $\mathcal{Q}_3$ and stays in it if $\alpha > 0$. The corresponding w is of type (3).

It remains to prove the existence of a solution of type (1). If $\delta < N$, then $(0, 0)$ is a saddle point. There exists a trajectory $\mathcal{T}_1$ converging to $(0, 0)$ at ∞, with $y > 0$, and $\lim_{\tau\to\infty} \zeta = \eta > 0$, thus in $\mathcal{Q}_1$ near ∞, with $y' < 0$. As above, y has no local maximum, it is increasing, so that $y > 0$. If $\delta = N$, we consider the sets

$$\mathcal{A} = \{P \in (0, \infty) \times \mathbb{R} : \mathcal{T}_{[P]} \cap \mathcal{M} \neq \varnothing\},$$

$$\mathcal{B} = \{P \in (0, \infty) \times \mathbb{R} : \mathcal{T}_{[P]} \cap \{(0, \xi) : \xi > 0\} \neq \varnothing\}.$$

They are nonempty, and open, because the intersections are transverse. Since $\mathcal{T}_r$ is below $\mathcal{M}$, the sets $\mathcal{A}$ and $\mathcal{B}$ are contained in the domain $\mathcal{R}$ of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ above $\mathcal{T}_r$, and $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}$. As a result there exists at least a trajectory $\mathcal{T}_1$ above $\mathcal{T}_r$, which does not intersects $\mathcal{M}$ and the set $\{(0, \xi) : \xi > 0\}$. The corresponding y is monotone. Suppose that y is increasing, then $\lim_{\tau\to-\infty} y = 0$; it is impossible since $\mathcal{T}_1 \neq \mathcal{T}_r$. Then y is decreasing, and $\lim_{\tau\to\infty} y = 0$. In any case w is of type (1), by Propositions 2.8 and 2.9. All the solutions are described, because any solution has at most one zero, and at most one extremum point. And $\mathcal{T}_1$ is unique when $\delta < N$. □

4. The case $\varepsilon = -1$, $\delta < \alpha$

$N < \delta < \alpha$

Theorem 4.1. *Assume $\varepsilon = -1$ and $N < \delta < \alpha$. Then all regular solutions of (E_w) have constant sign and satisfy $S_w < \infty$. And $w \equiv \ell r^{-\delta}$ is a solution. There exist solutions satisfying any one of these characterizations*:

(1) *w is positive*, $\lim_{r\to0} r^\eta w = c \neq 0$ *if* $N \geq 2$ (*and* $\lim_{r\to0} w = a > 0$, $\lim_{r\to0} w' = b(a) < 0$ *if* $N = 1$) *and* $\lim_{r\to\infty} r^\delta w = \ell$;

(2) *w is positive*, $\lim_{r\to0} r^\delta w = \ell$ *and* $S_w < \infty$;

(3) *w has one zero*, $\lim_{r\to0} r^\delta w = \ell$ *and* $S_w < \infty$;

(4) *w is positive*, $\lim_{r\to0} r^\alpha w = L \neq 0$ *and* $\lim_{r\to\infty} r^\delta w = \ell$;

(5) *w is positive*, $\lim_{r\to0} r^\alpha w = L \neq 0$ *and* $S_w < \infty$;

(6) *w has one zero*, $\lim_{r\to0} r^\alpha w = L \neq 0$ *and* $S_w < \infty$;

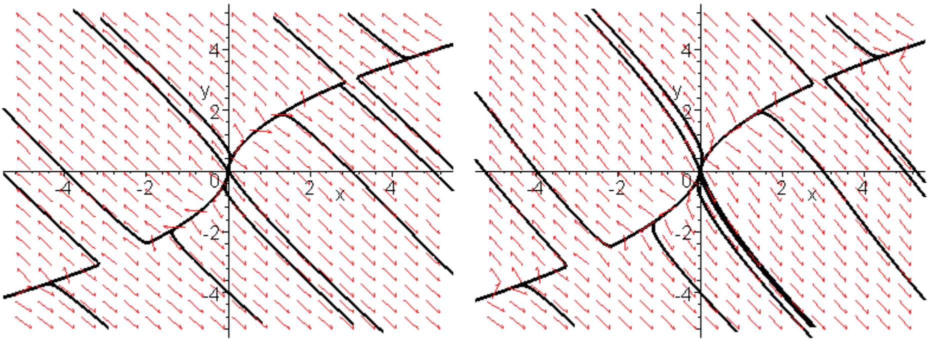
(7) *w is positive*, $\lim_{r\to0} r^\eta w = c \neq 0$ *if* $N \geq 2$ (*and* $\lim_{r\to0} w = a > 0$ *for any* $a > 0$ *and* $\lim_{r\to0} w' = b < 0$, $b \neq b(a)$ *if* $N = 1$), *and* $S_w < \infty$;

(8) *w has one zero and the same behavior*;

(9) (*only if* $N = 1$) *w is positive*, $\lim_{r\to0} w = a > 0$, *and* $\lim_{r\to0} w' = b > 0$, *and* $S_w < \infty$.

Up to symmetry, all solutions are as above.

Proof. Here we still have three stationary points, $(0, 0)$ is a source and M_ℓ a saddle point (see Figure 4). By Propositions 2.5 and 2.14, all regular solutions have constant sign and satisfy $S_w < \infty$. Also, $\mathcal{T}_r$ stays in $\mathcal{Q}_4$ by Remark 2.3, and $\lim_{\tau\to\ln S_w} Y/y = -\infty$ by Proposition 2.15. Since $\alpha > 0$, any solution y has at most one zero, by Proposition 2.5, and y is monotone near $\ln S_w$ (finite or not) and near $-\infty$, by Proposition 2.7. In the linearization near M_ℓ the eigenvectors $u_1 = (\nu(\alpha), \lambda_1 - \delta)$ and $u_2 = (-\nu(\alpha), \delta - \lambda_2)$ form a positively oriented basis, where



$N = 2,\ \alpha = 4$ $\qquad$ $N = 1,\ \alpha = 5$

Figure 4. Theorem 4.1: $\varepsilon = -1,\ N < \delta = 3 < \alpha$.

now $\nu(\alpha) < 0$ and $\lambda_1 < \delta < \lambda_2$; thus u_1 points toward $\mathcal{Q}_3$ and u_2 points toward $\mathcal{Q}_4$. There exist four particular trajectories converging to M_ℓ near $\pm\infty$, namely:

- $\mathcal{T}_1$ converging to M_ℓ at ∞, with tangent vector u_1. Here y is increasing near ∞, and as long as $y > 0$; indeed, if there exists a minimal point τ, (E_y) shows that $y(\tau) > \ell$. And $\mathcal{T}_1$ stays in $\mathcal{Q}_1$ on $\mathbb{R}$, by Remark 2.1(i) on page 211. Therefore $\mathcal{T}_1$ converges to $(0, 0)$ at $-\infty$, and w is of type (1), where $b(a)$ is a function of a, by the note on page 210.

- $\mathcal{T}_2$ converging to M_ℓ at $-\infty$, with tangent vector u_2. Here again $y' > 0$ as long as $y > 0$. Also $Y' < 0$ near $-\infty$, and Y is decreasing as long as $Y > 0$: if there exists a minimal point of Y in $\mathcal{Q}_1$, (E_Y) shows that $Y(\tau) > (\delta\ell)^{p-1}$. But (y, Y) cannot stay in $\mathcal{Q}_1$, as this would imply $\lim_{\tau\to\infty} y = \infty$, which is impossible by Proposition 2.8. Thus $\mathcal{T}_2$ enters $\mathcal{Q}_4$ at some point $(\xi_2, 0)$ with $\xi_2 > 0$ and stays in it since $y' > 0$. Thus $S_w < \infty$ and $\lim_{\tau\to\infty} Y/y = -1$, and w is of type (2).

- $\mathcal{T}_3$ converging to M_ℓ at $-\infty$, with tangent vector $-u_2$. Here again $y' < 0$ as long as $y > 0$. And $Y' > 0$ as long as $Y > 0$; thus $Y' > 0$ on $\mathbb{R}$. Then again (y, Y) cannot stay in $\mathcal{Q}_1$, so y has a unique zero, and $\mathcal{T}_3$ enters $\mathcal{Q}_2$ at some point $(0, \xi_3)$ with $\xi_3 > 0$ and stays in it. Hence $S_w < \infty$ and $\lim_{\tau\to\infty} Y/y = -1$, and w is of type (3).

- $\mathcal{T}_4$ converging to M_ℓ at ∞, with tangent vector $-u_1$. In the same way, y is decreasing near ∞, and y is everywhere decreasing: if there exists a maximal point τ, then $y(\tau) < \ell$ by (E_y). Then Y stays positive, thus $\mathcal{T}_4$ stays in $\mathcal{Q}_1$. By Proposition 2.8, $\lim_{\tau\to-\infty} y = \infty$ and $\lim_{\tau\to-\infty} \zeta = \alpha$, so w is of type (4).

Next we describe all the other trajectories $\mathcal{T}_{[P]}$ with one point P in the domain $\mathcal{R}$ above $\mathcal{T}_r \cup (-\mathcal{T}_r)$.

If $P = (\varphi, 0)$ with $\varphi > \xi_2$, then $\mathcal{T}_{[P]}$ stays in $\mathfrak{Q}_4$ after P, because it cannot meet $\mathcal{T}_2$; before P it stays in $\mathfrak{Q}_1$, by Remark 2.1(i). Thus again $S_w = \infty$, and $\lim_{\tau\to-\infty} \zeta = \alpha > 0$, and y has a unique minimal point, and w is of type (5). For any P is in the domain delimited by $\mathcal{T}_2$, $\mathcal{T}_4$, the trajectory $\mathcal{T}_{[P]}$ is of the same type.

If $P = (0, \xi)$ with $\xi > \xi_3$, then $\mathcal{T}_{[P]}$ stays in $\mathfrak{Q}_2$ after P, in $\mathfrak{Q}_1$ before P, since it cannot meet $\mathcal{T}_2$, $\mathcal{T}_4$. Then $\lim_{\tau\to-\infty} \zeta = \alpha > 0$, and $S_w = \infty$, and w is of type (6). If P is in the domain delimited by $\mathcal{T}_3$, $\mathcal{T}_4$, then $\mathcal{T}_{[P]}$ is of the same type.

If $P = (\varphi, 0)$ with $\varphi \in (0, \xi_2)$, then $\mathcal{T}_{[P]}$ stays in $\mathfrak{Q}_4$ after P, in $\mathfrak{Q}_1$ before P; it cannot meet $\mathcal{T}_r$, thus $S_w < \infty$; and $\mathcal{T}_{[P]}$ converges to $(0, 0)$ in $\mathfrak{Q}_1$ at $-\infty$; thus w is of type (7), by Theorem 2.2. If P is in the domain delimited by $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_r$, then $\mathcal{T}_{[P]}$ is of the same type.

If $P = (0, \xi)$ for some $\xi \in (0, \xi_3)$, then $\mathcal{T}_{[P]}$ stays in $\mathfrak{Q}_2$ after P, in $\mathfrak{Q}_1$ before P; and $\mathcal{T}$ cannot meet $-\mathcal{T}_r$, so that $S_w < \infty$. Then $\mathcal{T}_{[P]}$ converges to $(0, 0)$ in $\mathfrak{Q}_1$ at $-\infty$, and w is of type (8).

If P lies in the domain delimited by $\mathcal{T}_1$, $\mathcal{T}_3$ and $-\mathcal{T}_r$, either y has one zero, and $\mathcal{T}_{[P]}$ is of the same type; or $y < 0$ on $\mathbb{R}$, and $y' = \delta y - Y^{1/(p-1)} < 0$. Hence $S_w < \infty$ and $\mathcal{T}_{[P]}$ converges to $(0, 0)$ in $\mathfrak{Q}_2$ at $-\infty$. It implies $N = 1$ (see Figure 4, right), and $-w$ is of type (9), by Propositions 2.8 and 2.9; and such a solution does exist, by Theorem 2.2. Up to symmetry, all the solutions have been obtained. Here again, up to a scaling, the solutions w of types (1)–(4) are unique. □

$\delta \le \min(\alpha, N)$ (*apart from* $\alpha = \delta = N$)

Theorem 4.2. *Suppose $\varepsilon = -1$ and $\delta \le \min(\alpha, N)$ (apart from $\alpha = \delta = N$). Then all regular solutions of (E_w) have constant sign and a reduced domain ($S_w < \infty$). There exist solutions satisfying any one of these characterizations*:

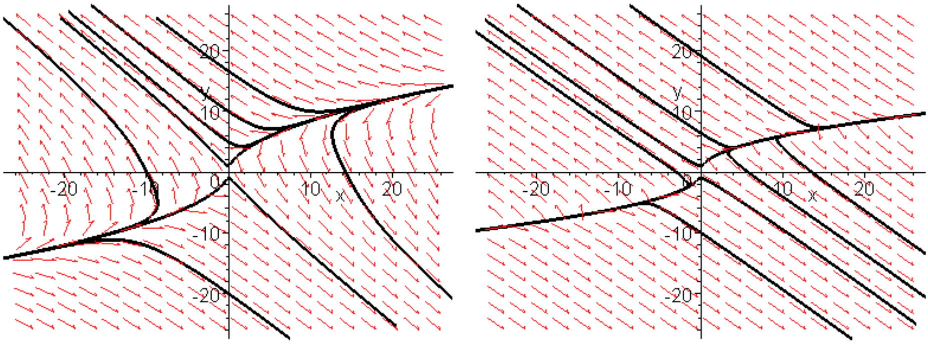
(1) *w is positive, $\lim_{r\to 0} r^\alpha w = L \neq 0$ and $\lim_{r\to\infty} r^\eta w = c \neq 0$ if $\delta < N$, or (2–40) holds if $\delta = N < \alpha$;*

(2) *w is positive, $\lim_{r\to 0} r^\alpha w = L \neq 0$ if $\delta < \alpha$, or (2–39) holds if $\alpha = \delta < N$, and $S_w < \infty$;*

(3) *w has one zero and the same behavior.*

Up to symmetry, all solutions are as above.

Proof. Here $(0, 0)$ is the only one stationary point, and $N \ge 2$ (Figure 5). By Propositions 2.5 and 2.14, all regular solutions have constant sign, and $S_w < \infty$. Moreover $w' > 0$ near 0, by Theorem 2.2; and w can only have minimal points, by Remark 2.3, so $w' > 0$ on $(0, S_w)$. In other words, $\mathcal{T}_r$ stays in $\mathfrak{Q}_4$, and $\lim_{\tau\to\ln S_w} Y/y = -1$. By Propositions 2.5 and 2.7, any solution y has at most one zero and is monotone at the extremities. By Proposition 2.8, apart from $\mathcal{T}_r$, any



$N < \alpha = 8$ $\delta < \alpha = 3.1 < N$

Figure 5. Theorem 4.2: $\varepsilon = -1$, $\delta = 3 < N = 4$.

trajectory $\mathcal{T}$ satisfies $\lim_{\tau\to-\infty} |y| = \infty$, and so $\lim_{\tau\to-\infty} \zeta = \alpha > 0$; hence $\mathcal{T}$ starts from $\mathcal{Q}_1$ or $\mathcal{Q}_3$ at $-\infty$.

For any $P = (\varphi, 0)$ with $\varphi > 0$, the trajectory $\mathcal{T}_{[P]}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_4$ at P, by Remark 2.1(i) on page 211; it stays in $\mathcal{Q}_4$ after P, since it cannot meet $\mathcal{T}_r$; and it stays in $\mathcal{Q}_1$ before P: it cannot start from $\mathcal{Q}_3$, because it does not meet $-\mathcal{T}_r$. Thus y remains positive and w is of type (2).

For any $P = (0, \xi)$ with $\xi > 0$, $\mathcal{T}_{[P]}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_2$ by the same remark; thus $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_2$ after P, since it cannot meet $-\mathcal{T}_r$, and in $\mathcal{Q}_1$ before P, and w is of type (3).

It remains to prove the existence of solutions of type (1). If $\delta < N$, the origin is a saddle point, so there exists a trajectory $\mathcal{T}_1$ converging to $(0, 0)$ at ∞; and $\lim_{\tau\to\infty} \zeta = \eta > 0$, by Proposition 2.8. Thus $\mathcal{T}_1$ lies in $\mathcal{Q}_1$ for large τ, and stays there, because $\mathcal{Q}_1$ is backward invariant. The conclusion follows. If $\delta = N$, we consider the sets

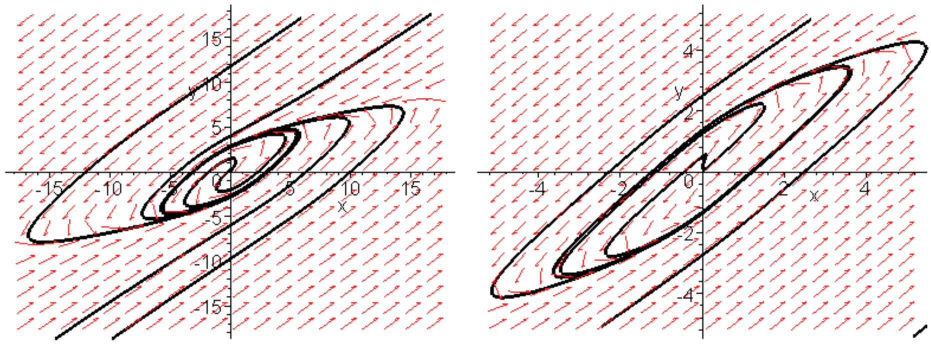
$$\mathcal{A} = \{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \varnothing\},$$
$$\mathcal{B} = \{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(0, \xi) : \xi > 0\} \neq \varnothing\}.$$

They are nonempty and open, since the vector field is transverse at $(\varphi, 0)$ and $(0, \xi)$; thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{Q}_1$. Hence there exists a trajectory $\mathcal{T}_1$ staying in $\mathcal{Q}_1$; therefore $S_w = \infty$ and $\mathcal{T}_1$ converges to $(0, 0)$ at ∞, and w is of type (1), by Proposition 2.9. All solutions have been described, up to symmetry. □

5. The case $\varepsilon = 1$, $\delta \leq \alpha$

$N \leq \delta \leq \alpha$

Theorem 5.1. *Assume $\varepsilon = 1$, $N \leq \delta \leq \alpha$ and $\alpha \neq N$.*



$N = 2 < \delta$ $\qquad$ $N = 3 = \delta$

Figure 6. Theorem 5.1: $\varepsilon = 1, \; < \delta = 3 < \alpha = 3.5$.

(i) *There exists a cycle surrounding* $(0, 0)$, *and thus also solutions* w *of* (E_w) *with changing sign and such that* $r^\delta w$ *is periodic in* $\ln r$. *All other solutions* w, *in particular the regular ones, are oscillating near* ∞, *and* $r^\delta w$ *is asymptotically periodic in* $\ln r$. *There exist solutions* w *such that* $\lim_{r\to 0} r^\eta w = c \neq 0$ *if* $2 \le N < \delta$ *and* (2–40) *holds if* $N = \delta$, *or* (2–41) *holds if* $N = 1$.

(ii) *There exist solutions such that* $R_w > 0$, *or* $\lim_{r\to 0} r^\alpha w = L \neq 0$ *if* $\alpha \neq \delta$, *or* (2–39) *holds if* $\alpha = \delta$.

Proof. (i) Here $(0, 0)$ is the only stationary point. From Proposition 2.8, any trajectory is bounded and y is oscillating around 0 near ∞.

First assume $N < \delta < \alpha$ (Figure 6, left). Then $(0, 0)$ is a source and all trajectories have a limit cycle at ∞ or are periodic. In particular there exists at least one cycle, with orbit $\mathcal{O}_p$. The trajectory $\mathcal{T}_r$ has a limit cycle $\mathcal{O} \subseteq O_p$. There exist also trajectories $\mathcal{T}_s$ starting from $(0, 0)$ with an infinite slope, such that $\lim_{r\to 0} r^\eta w = c \neq 0$ if $N \ge 2$ or (2–41) if $N = 1$, and all the $\mathcal{T}_s$ have the same limit cycle $\mathcal{O}$.

Next assume $N = \delta < \alpha$ (Figure 6, right). Then $\mathcal{T}_r$ cannot converge to $(0, 0)$, since it would intersect itself. Thus again the limit set at ∞ is a closed orbit $\mathcal{O}$. No trajectory can converge to $(0, 0)$ at ∞, as it would spiral around this point and hence intersect $\mathcal{T}_r$. Consider any trajectory $\mathcal{T} \neq \mathcal{T}_r$ in the connected component of $\mathcal{O}$ containing $(0, 0)$. $\mathcal{T}$ is bounded, so its limit set at $-\infty$ is $(0, 0)$ or a closed orbit. The second case is impossible, since $\mathcal{T}$ does not meet $\mathcal{T}_r$. Thus $\mathcal{T}$ is of the form $\mathcal{T}_s$, and the corresponding w satisfies (2–40).

(ii) By Theorem 2.21, all cycles are contained in a ball B of $\mathbb{R}^2$. Take any point P_0 exterior to B. Then $\mathcal{T}_{[P_0]}$ has a limit cycle at ∞ contained in B. If it has a limit cycle at $-\infty$, it is contained in B, so $\mathcal{T}_{[P_0]}$ is contained in B, which is impossible. Thus y has constant sign near $\ln R_w$. By Proposition 2.8, either $R_w > 0$ or y is defined near $-\infty$. $\square$

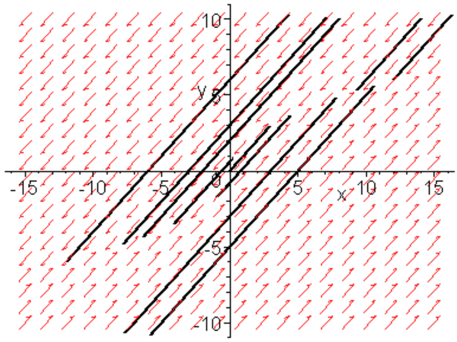


Figure 7. Theorem 5.2: $\varepsilon = 1,\ \alpha = \delta = N = 3$.

Theorem 5.2. *Assume* $\varepsilon = 1$ *and* $\alpha = \delta = N$. *All regular solutions of* (E_w) *have constant sign, and are given by* (1–6). *For any* $k \in \mathbb{R}$, $w(r) = kr^{-N}$ *is a solution. There exist solutions satisfying any one of these characterizations*:

(1) w *is positive*, $\lim_{r\to 0} r^N w = c_1 > 0$, $\lim_{r\to 0} r^N w = c_2 > 0$ $(c_2 \neq c_1)$;

(2) w *has one zero*, $\lim_{r\to 0} r^N w = c_1 > 0$ *and* $\lim_{r\to\infty} r^N w = c_2 < 0$;

(3) w *is positive*, $R_w > 0$, *and* $\lim_{r\to 0} r^N w = c \neq 0$;

(4) w *has one zero and the same behavior.*

Up to symmetry, all solutions are as above.

Proof. Since $\alpha = N$, equation (E_w) admits the first integral (1–5), so $J_N \equiv C$ for $C \in \mathbb{R}$. We gave in (1–6) the regular (Barenblatt) solutions for the case $C = 0$. Since $\delta = N$, (1–5) is equivalent to the equation $Y \equiv y - C$, by (2–12) (refer to Figure 7). For any $k \in \mathbb{R}$, $(y, Y) \equiv (k, |Nk|^{p-2}Nk)$ is a solution of the system (S) located on the curve $\mathcal{M}$, so that $w(r) = kr^{-N}$ is a solution. Any solution has at most one zero, by Proposition 2.5. By Propositions 2.8 and 2.10, any trajectory converges to a point $(k, |Nk|^{p-2}Nk)$ of $\mathcal{M}$ at ∞. Let $\bar{C} < 0$ be such that the line $Y = y - \bar{C}$ is tangent to $\mathcal{M}$. For any $C \in (\bar{C}, 0)$, the line $Y = y - C$ cuts $\mathcal{M}$ at three points $k_1 < 0 < k_2 < k_3$. And $y' > 0$ if the trajectory is below $\mathcal{M}$ and $y' < 0$ if it is above $\mathcal{M}$. We find two solutions y defined on $\mathbb{R}$: one is positive and $\lim_{\tau\to-\infty} y = k_2$, $\lim_{\tau\to-\infty} y = k_3$, and the other has one zero. All other solutions satisfy $R_w > 0$, $\lim_{\tau\to\ln R_w} Y/y = 1$; some of them are positive, the others have one zero. □

$\delta < \min(\alpha, N)$

Here the system has three stationary points: $(0, 0)$ is a saddle point, while M_ℓ, M'_ℓ are sinks if $\delta \leq N/2$, or $N/2 < \delta$ and $\alpha < \alpha^*$, and sources when $N/2 < \delta$ and $\alpha > \alpha^*$, and node points whenever $\alpha \leq \alpha_1$ or $\alpha_2 \leq \alpha$, where α_1, α_2 are defined in (2–48) (recall that α_1 can be greater or less than η). This case is one of the most delicate, since two types of periodic trajectories can appear, either surrounding $(0, 0)$, corresponding to changing sign solutions, or located in $\mathcal{Q}_1$ or $\mathcal{Q}_3$, corresponding to solutions of constant sign. Notice that $\delta < N$ implies $\delta < N < \eta$ by (1–2), and $N/2 < \delta$ implies $\eta < \alpha^*$ by (2–32).

Remark 5.3. (i) $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. Since $(0, 0)$ is a saddle point, Propositions 2.8 and 2.9 show there is a unique trajectory $\mathcal{T}_s$ converging to $(0, 0)$, residing in $\mathcal{Q}_1$ for large τ, having an infinite slope at $(0, 0)$, and satisfying $\lim_{r\to 0} r^\eta w = c > 0$. Moreover if $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, then $\mathcal{T}_s$ stays in it, and is bounded and contained in the domain delimited by $\mathcal{Q}_1 \cap \mathcal{T}_r$, by Remark 2.1(i). Thus if $\mathcal{T}_r$ is homoclinic, it stays in $\mathcal{Q}_1$.

(ii) Any trajectory $\mathcal{T}$ is bounded near ∞, by Propositions 2.8 and 2.12. From the strong form of the Poincaré–Bendixson theorem [Hubbard and West 1995, p. 239], any trajectory $\mathcal{T}$ bounded at $\pm\infty$ either converges to $(0, 0)$ or $\pm M_\ell$, or its limit set $\Gamma_\pm$ at $\pm\infty$ is a cycle, or it is homoclinic hence $\mathcal{T} = \mathcal{T}_r$ and $\Gamma_\pm = \overline{\mathcal{T}_r}$ (indeed, for any $P \in \Gamma_\pm$, $\mathcal{T}_{[P]}$ converges at ∞ and $-\infty$ to $(0, 0)$ or $\pm M_\ell$; if one of them is $\pm M_\ell$, then $\pm M_\ell \in \overline{\mathcal{T}_{[P]}} \subset \Gamma_\pm$, and M_ℓ is a source or a sink, so $\mathcal{T}$ converges to $\pm M_\ell$; otherwise $\mathcal{T}$ is homoclinic and $\mathcal{T}_{[P]} = \mathcal{T}_r$).

(iii) If there exists a limit cycle around $(0, 0)$, then by (2–42) this cycle also surrounds the points $\pm M_\ell$.

We begin with the case $\alpha \le \eta$, where there exists no cycle in $\mathcal{Q}_1$ and no homoclinic orbit, by Theorem 2.20.

Theorem 5.4. *Assume that* $\varepsilon = 1$ *and* $\delta < \min(\alpha, N)$, *and* $\alpha \le \eta$. *Then all regular solutions of* (E_w) *have constant sign, and* $\lim_{r\to\infty} r^\delta |w(r)| = \ell$. *And* $w(r) = \ell r^{-\delta}$ *is a solution.*

If $\alpha < \eta$, *there exist solutions satisfying any one of these characterizations*:

(1) w *is positive*, $\lim_{r\to 0} r^\alpha w = L$ *and* $\lim_{r\to\infty} r^\delta w = \ell$;

(2) w *is positive*, $R_w > 0$ *and* $\lim_{r\to\infty} r^\eta w = c > 0$;

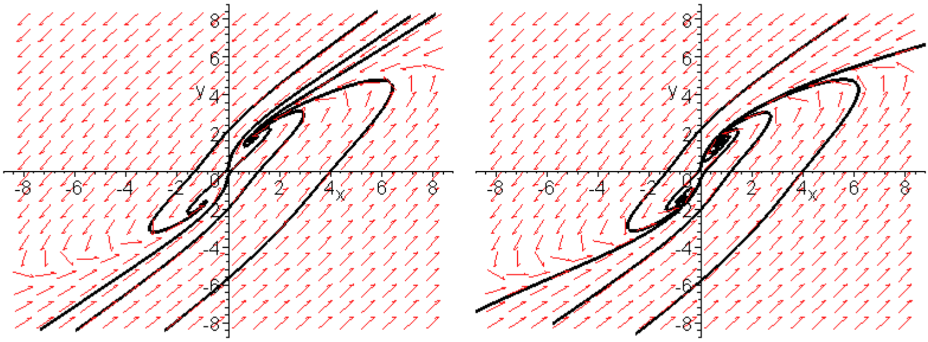
(3) w *is positive*, $R_w > 0$ *and* $\lim_{r\to\infty} r^\delta w = \ell$;

(4) w *has one zero*, $R_w > 0$ *and* $\lim_{r\to\infty} r^\delta w = \ell$;

If $\alpha = \eta$, *then* $w = Cr^{-\eta}$ *is a solution and there exist solutions of type* (4), *but not of type* (2) *or* (3).

Proof. By Proposition 2.5 and Remark 2.3, $\mathcal{T}_r$ stays in $\mathcal{Q}_1$ and converges to M_ℓ at ∞; indeed there is no cycle in $\mathcal{Q}_1$, by Propositions 2.8, 2.12 and 2.20.

(i) Assume $\alpha < \eta$ (Figure 8, left). Consider any trajectory in $\mathcal{Q}_1$. Then $Y_\alpha > 0$. If there exists τ such that $Y'_\alpha(\tau) = 0$, at this point $Y''_\alpha(\tau) \ge 0$ by (2–36), and τ



$\alpha = 4.7 < \eta$ $\alpha = 5 = \eta$

Figure 8. Theorem 5.4: $\varepsilon = 1$, $\delta = 3 < N = 4 < \eta = 5$.

is a local minimum. $\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} Y_\alpha = 0$, and so $Y'_\alpha > 0$ on $\mathbb{R}$. This is equivalent to $\alpha y > Y^{1/(p-1)} + (p-1)(\eta-\alpha)Y$. Therefore $\mathcal{T}_r$ stays strictly below the curve

$$\mathcal{M}_\alpha = \left\{(y, Y) \in \mathcal{Q}_1 : \alpha y = Y^{1/(p-1)} + (p-1)(\eta-\alpha)Y\right\}.$$

First consider $\mathcal{T}_s$. Since $\alpha < \eta$, this trajectory satisfies $\lim_{\tau\to\infty} Y_\alpha = 0$. Then $Y'_\alpha < 0$ on $(\ln R_w, \infty)$, so $\mathcal{T}_s$ stays strictly above $\mathcal{M}_\alpha$. Hence it stays above $\mathcal{M}$: indeed, if it meets $\mathcal{M}$ at a first point $(y_1, (\delta y_1)^{p-1})$, the function y has a maximum at this point. Thus by (2–16), we have $\ell < y_1$ and

$$(\alpha-\delta)y_1^{2-p} = \delta^{p-1}(p-1)(\eta-\alpha) < \delta^{p-1}(p-1)(\eta-\delta),$$

contradicting (1–2) and (1–4). This shows that $y' < 0$. Suppose that y is defined on $\mathbb{R}$; then $\lim_{\tau\to-\infty} y = \infty$ and $\lim_{\tau\to-\infty} \zeta = \alpha$. If $\zeta' > 0$ on $\mathbb{R}$, then $\zeta(\mathbb{R}) = (\alpha, \eta)$, which contradicts (2–9). Then ζ has at least one extremal point τ, and $\zeta(\tau)$ is exterior to (α, η), by (2–9); if it is a minimum, $\zeta(\tau) > \alpha$ by (2–18), since $y' < 0$, and if it is a maximum, $\zeta(\tau) < \alpha$. Thus we reach again a contradiction. Therefore $R_w > 0$ and $\lim_{\tau\to\ln R_w} Y/y = 1$, and the corresponding w is of type (2).

For any $P = (\varphi, 0)$, $\varphi > 0$, the trajectory $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1$ after P. The solution (y, Y) originating at P at time 0 satisfies $Y_\alpha(0) = 0$; hence $Y'_\alpha(\tau) > 0$ for any $\tau \geq 0$. Thus $\mathcal{T}_{[P]}$ stays below $\mathcal{M}_\alpha$. Moreover it enters $\mathcal{Q}_4$ as τ decreases. But $y' > 0$ in $\mathcal{Q}_4$, by (S); thus $\mathcal{T}_{[P]}$ does not stay in $\mathcal{Q}_4$, by Proposition 2.8; it goes into $\mathcal{Q}_3$ and must stay there because it cannot meet $-\mathcal{T}_s$. This shows that $R_w > 0$ and y has precisely one zero, and w is of type (4).

Next consider any trajectory $\mathcal{T}_{[P_1]}$ going through some point $P_1 = (y_1, Y_1)$ in $\mathcal{Q}_1$, lying below $\mathcal{T}_s$ and such that $\alpha y_1 < Y_1^{1/(p-1)}$. Such a trajectory exists because $y = Y$ is an asymptotic direction of $\mathcal{T}_s$. Let (y, Y) be the solution issuing from P_1 at time 0. Suppose y is defined on $\mathbb{R}$; then $\lim_{\tau\to-\infty} y = \infty$ and $\lim_{\tau\to-\infty} \zeta = \alpha$.

Also, $\zeta(0) > \alpha$. Then $\zeta > \delta$ on $(-\infty, 0)$: otherwise there would exist $\tau < 0$ such that $\zeta(\tau) = \alpha$ and $\zeta'(\tau) \geq 0$, contradicting (2–9). Thus $y' < 0$ on $(-\infty, \tau_1)$. Either $\zeta' > 0$ on $(-\infty, 0)$, in which case $\zeta > \eta > 0$ by (2–9), which is impossible; or ζ has at least an extremal point τ. If it is a minimum, then $\zeta(\tau) > \alpha$ from (2–18); if it is a maximum, then $\zeta(\tau) < \alpha$; and again we reach a contradiction. Therefore $R_w > 0$, and the trajectory stays in $\mathcal{Q}_1$ and converges to M_ℓ, because there is no cycle in $\mathcal{Q}_1$, by Theorem 2.20. Hence w is of type (3).

Let $\mathcal{O}$ be the domain of $\mathcal{Q}_1$ bounded above by $\mathcal{T}_s$. It is forward invariant. Any trajectory going through any point of $\mathcal{O}$ converges to M_ℓ at ∞. Either it meets the axis $Y = 0$ at some point $(\xi, 0)$ with $\xi > 0$, or it stays in $\mathcal{O}$, satisfies $R_w > 0$ and $\lim_{\tau \to \ln R_w} T/y = 1$, and meets $\mathcal{M}_\alpha$, since M_ℓ lies strictly below $\mathcal{M}_\alpha$. Let

$$\mathcal{A} = \{P \in \mathcal{O} : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \varnothing\},$$
$$\mathcal{B} = \{P \in \mathcal{O} : \mathcal{T}_{[P]} \cap \mathcal{M}_\alpha \neq \varnothing\}.$$

These sets are nonempty and open: indeed, one can check that the intersection with $\mathcal{M}_\alpha$ is transverse, because $\alpha \neq \eta$. Thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{O}$, so there exists a trajectory $\mathcal{T}_1$ with w of type (1).

(ii) Assume $\alpha = \eta$ (Figure 8, right). There is no positive solution with $R_w > 0$, thus no solution of type (2) or (3). Indeed all the trajectories stay below $\mathcal{T}_s$, and $\mathcal{T}_s$ is defined by the equation $\zeta \equiv \eta$, meaning that $w \equiv Cr^{-\eta}$, or equivalently $Y_\eta \equiv C$; thus $Y'_\eta \equiv 0$ and $\mathcal{T}_s = \mathcal{M}_\eta$. Consider any trajectory $\mathcal{T}_{[P]}$ going through some point $P = (\varphi, 0)$ with $\varphi > 0$, and the solution (y, Y) issuing from P at time 0. Then $Y_\eta(0) = 0$ and $Y_\eta < 0$, so $Y'_\eta = \eta y - |Y|^{(2-p)/(p-1)} Y > 0$ on $(-\infty, 0)$, seeing that $\mathcal{T}_{[P]}$ does not meet $-\mathcal{T}_s$. Suppose $R_w = 0$. Then $\mathcal{T}_{[P]}$ starts from $\mathcal{Q}_3$, with $\lim_{\tau \to -\infty} \zeta = \alpha = \eta$. Then $\lim_{\tau \to -\infty} y_\eta = L < 0$; thus $\lim_{\tau \to -\infty} Y_\eta = -(\alpha |L|)^{(2-p)/(p-1)}$. A straightforward computation gives

$$Y''_\eta = Y'_\eta \Big(N - \frac{1}{p-1} |Y|^{(2-p)/(p-1)} \Big).$$

This leads to $Y''_\eta < 0$ near $-\infty$, which is impossible; thus $R_w < \infty$ and w is of type (4). □

Remarks. (i) For $\alpha \leq \eta$, both trajectories $\mathcal{T}_r$ and $\mathcal{T}_s$ stay in $\mathcal{Q}_1$.

(ii) When $\alpha \leq N$, one can verify that the regular positive solution y is increasing and $y \leq \ell$ on $\mathbb{R}$, so $r^\delta w(r) \leq \ell$ for any $r \geq 0$.

(iii) When $\alpha = N$, we have $\mathcal{T}_r = \{(\xi, \xi) : \xi \in [0, \ell)\}$, and the corresponding solutions w are given by (1–6) with $K > 0$. And $\mathcal{T}_3 = \{(\xi, \xi) : \xi > \ell)\}$ is a trajectory corresponding to particular solutions w of type (3), given by (1–6) with $K < 0$.

Next we come to the most interesting case, where $\eta < \alpha$.

Lemma 5.5. *Assume $\varepsilon = 1$, $\delta < \min(\alpha, N)$ and $\eta < \alpha$. If $N/2 < \delta$ and $\alpha < \alpha^*$ and $\mathcal{T}_s$ stays in $\mathcal{Q}_1$, $\mathcal{T}_s$ has a limit cycle at $-\infty$ in $\mathcal{Q}_1$ or is homoclinic. If $\delta \le N/2$, then $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$.*

Proof. In any case M_ℓ is a sink, so $\mathcal{T}_s$ cannot converge to M_ℓ at $-\infty$. Suppose $\mathcal{T}_s$ has no limit cycle in $\mathcal{Q}_1$, and is not homoclinic and stays in $\mathcal{Q}_1$. (This happens when $\delta \le N/2$, by Proposition 2.11.) Then either $\lim_{\tau\to-\infty} y = \infty$ and $\lim_{r\to 0} r^\alpha w = \Lambda \neq 0$, or $R_w > 0$. In either case, for any $d \in (\eta, \alpha)$, the function $y_d(\tau) = r^d w = r^{d-\delta} y$ satisfies $\lim_{\tau\to\ln R_w} y_d = \infty = \lim_{\tau\to\infty} y_d$. Then it has a minimum point, contradicting (2–5). □

Theorem 5.6. *Assume $\varepsilon = 1$ and $N/2 < \delta < \min(\alpha, N)$. Then $w(r) = \ell r^{-\delta}$ is still a solution.*

(i) *There exists a (maximal) critical value α_{crit} of α, such that*

$$\max(\eta, \alpha_1) < \alpha_{\text{crit}} < \alpha^*,$$

and the regular trajectory is homoclinic: all regular solutions of (E_w) have constant sign and satisfy $\lim_{r\to\infty} r^\eta w = c \neq 0$.

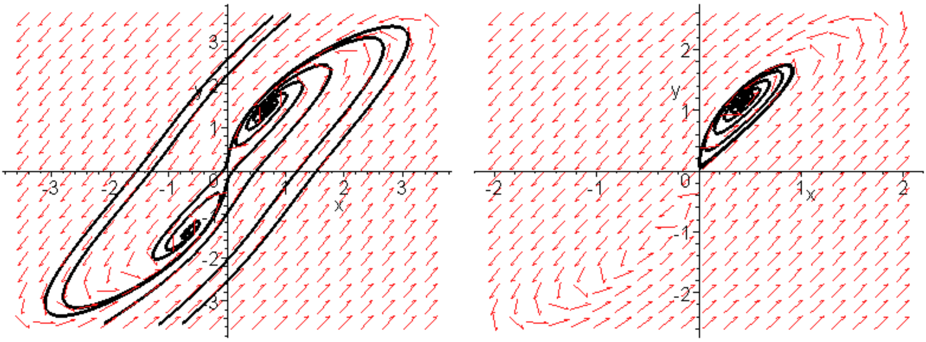
(ii) *For any $\alpha \in (\alpha_{\text{crit}}, \alpha^*)$, there does exist a cycle in $\mathcal{Q}_1$, in other words there exist positive solutions w such that $r^\delta w$ is periodic in $\ln r$. There exist positive solutions such that $r^\delta w$ is asymptotically periodic in $\ln r$ near 0 and $\lim_{r\to\infty} r^\delta w = \delta$. There exist positive solutions such that $r^\delta w$ is asymptotically periodic in $\ln r$ near 0 and $\lim_{r\to\infty} r^\eta w = c \neq 0$.*

(iii) *For any $\alpha \ge \alpha^*$ there does not exist such a cycle, but there exist positive solutions such that $\lim_{r\to 0} r^\delta w = \ell$ and $\lim_{r\to\infty} r^\eta w = c > 0$.*

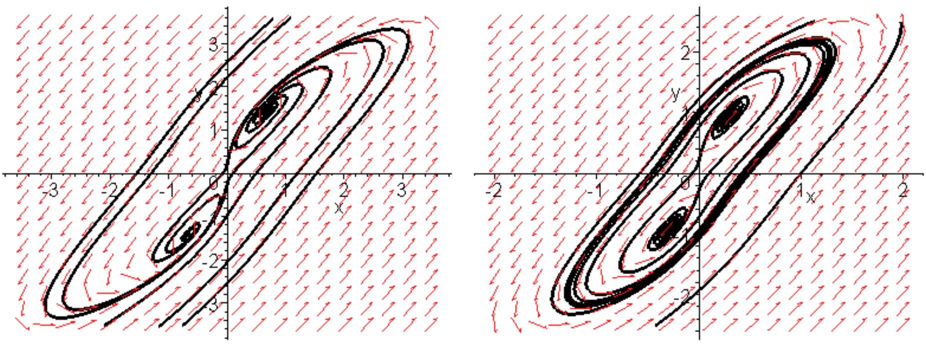
(iv) *For any $\alpha > \alpha_{\text{crit}}$, there exists also a cycle, surrounding $(0,0)$ and $\pm M_\ell$, thus $r^\delta w$ is changing sign and periodic in $\ln r$. All regular solutions change signs and are oscillating at ∞, and $r^\delta w$ is asymptotically periodic in $\ln r$. There exist solutions such that $R_w > 0$, or $\lim_{r\to 0} r^\alpha w = L \neq 0$, and oscillating at ∞, and $r^\delta w$ is asymptotically periodic in $\ln r$.*

Proof. (i) For any $\alpha \in (\alpha_1, \alpha_2)$ such that $\eta \le \alpha$, we have by Remark 5.3 three possibilities for the regular trajectory $\mathcal{T}_r$:

- $\mathcal{T}_r$ converges to M_ℓ and spirals around it, or else it has a limit cycle in $\mathcal{Q}_1$ around M_ℓ. Then $\mathcal{T}_r$ meets the set $\mathcal{E} = \{(\ell, Y) : Y > (\delta\ell)^{p-1}\}$ at a first point $(\ell, Y_r(\alpha))$. Note that ℓ and $\mathcal{E}$ depend continuously on α. Then $\mathcal{T}_s$ meets $\mathcal{E}$ at some last point $(\ell, Y_s(\alpha))$ such that $Y_s(\alpha) - Y_r(\alpha) > 0$. See Figure 9, top left.



$N < \eta = 5 < \alpha = 5.1$ $\quad$ $N < \eta = 5 < \alpha = 5.62$



$N < \alpha = 5.9 < \alpha^* = 6$ $\quad$ $N < \alpha^* = 6 < \alpha = 6.2$

Figure 9. Theorem 5.6: $\varepsilon = -1$, $N/2 < \delta = 3 < N = 4$.

- $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$; then $\mathcal{T}_s$ is bounded at $-\infty$, and so converges to M_ℓ at $-\infty$ and spirals around this point, or it has a limit cycle around M_ℓ. Then $\mathcal{T}_s$ meets $\mathcal{E}$ at a last point $(\ell, Y_s(\alpha))$ and $\mathcal{T}_r$ meets $\mathcal{E}$ at a first point $(\ell, Y_r(\alpha))$ such that $Y_s(\alpha) - Y_r(\alpha) < 0$. See Figure 9, bottom row.
- $\mathcal{T}_r$ is homoclinic, which is equivalent to $Y_s(\alpha) - Y_r(\alpha) = 0$. See Figure 9, top right.

Now the function $\alpha \mapsto g(\alpha) = Y_s(\alpha) - Y_r(\alpha)$ is continuous. If $\alpha_1 < \eta$, then $g(\eta)$ is defined and $g(\eta) > 0$, by Theorem 5.4. If $\eta \leq \alpha_1$, we observe that for $\alpha = \alpha_1$, the trajectory $\mathcal{T}_s$ leaves $\mathcal{Q}_1$, by Theorem 2.18, because α_1 is a sink, and does so transversally by Remark 2.1(i). The same holds for $\alpha = \alpha_1 + \gamma$ for γ small enough, by continuity, so $\mathcal{T}_r$ stays in $\mathcal{Q}_1$ and $g(\alpha_1 + \gamma) > 0$. If $\alpha \geq \alpha^*$ (Figure 9, bottom right), then M_ℓ is a source or a weak source, by Theorem 2.16; thus $\mathcal{T}_r$ cannot converge to M_ℓ. By Theorem 2.19, there exists no cycle in $\mathcal{Q}_1$ and no homoclinic orbit. By Remark 5.3(i), $\mathcal{T}_r$ cannot stay in $\mathcal{Q}_1$, so $g(\alpha) < 0$ for $\alpha^* \leq \alpha < \alpha_2$. As a consequence, there exists at least one $\alpha_{\text{crit}} \in (\max(\eta, \alpha_1), \alpha^*)$ such that $g(\alpha_{\text{crit}}) = 0$. If it is not unique, we can choose the largest one.

(ii) Suppose $\alpha < \alpha^*$. The existence and uniqueness of the desired cycle $\mathbb{O}$ in $\mathfrak{Q}_1$ follows by Theorem 2.16 when α is close to α^* (Figure 9, bottom left). In fact, existence holds for any $\alpha \in (\alpha_{\text{crit}}, \alpha^*)$; indeed $g(\alpha) < 0$ on this interval, and $\mathcal{T}_s$ cannot converge to M_ℓ at $-\infty$, so it has a limit cycle around M_ℓ at $-\infty$. Since M_ℓ is a sink, there exist also trajectories converging to M_ℓ at ∞, with a limit cycle at $-\infty$ contained in $\mathbb{O}$. Now $\mathcal{T}_r$ does not stay in $\mathfrak{Q}_1$ and is bounded at ∞, so it has a limit cycle at ∞ containing the three stationary points.

(iii) Suppose $\alpha \geq \alpha^*$. Then $\mathcal{T}_s$ stays in $\mathfrak{Q}_1$, is bounded on $\mathbb{R}$, and converges at $-\infty$ to M_ℓ. At the same time, $\mathcal{T}_r$ does not stay in $\mathfrak{Q}_1$ for the same reason as above; thus it has a limit cycle at ∞, containing the three stationary points (see Figure 9, bottom right).

(iv) For any $\alpha > \alpha_{\text{crit}}$, apart from $\mathcal{T}_s$ and the cycles, all the trajectories have a limit cycle at ∞ containing the three stationary points. By Theorem 2.21, all the cycles are contained in a ball B of $\mathbb{R}^2$. Take any point P exterior to B. By Remark 5.3(ii), $\mathcal{T}_{[P]}$ has a limit cycle at ∞ contained in B and cannot have a limit cycle at $-\infty$. Thus y has constant sign near $\ln R_w$. By Proposition 2.8, either $R_w > 0$ or y is defined near $-\infty$ and $\lim_{\tau\to-\infty} \zeta = L$, $\lim_{r\to 0} r^\alpha w = L$. □

Note. It is an open question whether α_{crit} is unique. It can be shown that if there exist two critical values $\alpha^1_{\text{crit}} > \alpha^2_{\text{crit}}$, the first orbit is contained in the second.

When $\delta \leq N/2$, or equivalently $p \leq P_2$, there are no cycles in $\mathbb{R}^2$ and we get:

Theorem 5.7. *Assume $\varepsilon = 1$, $\delta \leq N/2$, and $\delta < \alpha$. All regular solutions of (E_w) have constant sign, and $\lim_{r\to\infty} r^\delta |w| = \ell$. All solutions have a finite number of zeros. The function $w(r) = \ell r^{-\delta}$ is a solution. If $\alpha \leq \eta$, Theorem 5.4 applies. If $\eta < \alpha$, all other solutions have at least one zero. There exist solutions satisfying $\lim_{r\to\infty} r^\eta w = c \neq 0$ and having m zeros, for some $m > 0$. All other solutions satisfy $\lim_{r\to\infty} r^\delta w = \pm\ell$, and have m or $m+1$ zeros. There exist solutions with $m+1$ zeros.*

Proof. (i) By Proposition 2.11, all solutions have a finite number of zeros. Since $\delta \leq N/2$, the function W defined in (2–21) is nonincreasing. The regular solutions (y, Y) satisfy $\lim_{\tau\to-\infty} W(\tau) = 0$, so $W(\tau) \leq 0$ on $\mathbb{R}$. If $y(\tau_0) = 0$ for some real τ_0, then $W(\tau_0) = |Y(\tau_0)|^{p'} > 0$, and we reach a contradiction. From Propositions 2.8 and 2.11 we obtain $\lim_{\tau\to\infty} y = \pm\ell$, so $\lim_{r\to\infty} r^\delta w = \pm\ell$.

(ii) Assume $\eta < \alpha$. By Lemma 5.5, $\mathcal{T}_s$ does not stay in $\mathfrak{Q}_1$. By Propositions 2.8 and 2.15, $\mathcal{T}_s$ cannot stay in $\mathfrak{Q}_4$, so it intersects the line $y = 0$ at points $(0, \xi_1), \ldots, (0, \xi_m)$. By Remark 5.3, any trajectory other than $\mathcal{T}_s$ converges to $\pm M_\ell$. Given any $P = (0, \xi)$, with $\xi > |\xi_i|$ for $1 \le i \le m$, the trajectory $\mathcal{T}_{[P]}$ cannot intersect $\mathcal{T}_s$ or $-\mathcal{T}_s$, so y has $m+1$ zeros. Any other solution has m or $m+1$ zeros, because the trajectory does not meet $\mathcal{T}_r$ or $-\mathcal{T}_r$ or $\mathcal{T}_{[P]}$. Finally, $R_w > 0$ or $\lim_{r\to 0} r^{\alpha} w = L \neq 0$. □

Note. Theorems 5.4, 5.6 and 5.7 recover, in particular, the results in [Qi and Wang 1999, Theorem 2].

6. The case $\varepsilon = -1$, $\alpha \le \delta$

max(α, N) ≤ δ

Here $(0, 0)$ is the only stationary point, and it is a source when $\delta \neq N$. We first suppose $0 < \alpha$.

Theorem 6.1. *Suppose* $\varepsilon = -1$, $\max(\alpha, N) \le \delta$ *and* $0 < \alpha$.

(i) *Suppose $\alpha \neq N$ or $\alpha \neq \delta$. Then all regular solutions of* (E_w) *have constant sign and a reduced domain* ($S_w < \infty$). *There exist solutions satisfying any one of these characterizations*:

(1) *w is positive*, $\lim_{r\to 0} r^{\eta} w = c \neq 0$ *if* $N \ge 2$ ($\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b < 0$ *if* $N = 1$), *and* $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$ *if* $\alpha \neq \delta$, *or* (2–39) *holds if* $\alpha = \delta$;

(2) *w is positive*, $\lim_{r\to 0} r^{\eta} w = c \neq 0$ *if* $N \ge 2$ ($\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b \neq 0$, *or* $a = 0 < b$ *if* $N = 1$), *and* $S_w < \infty$;

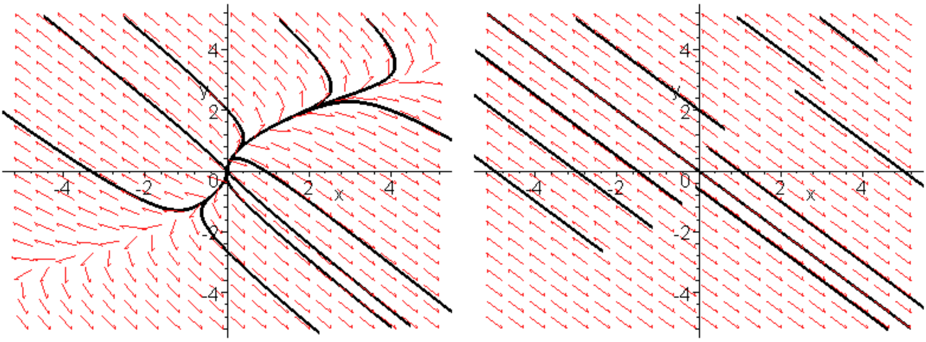
(3) *w has one zero*, $\lim_{r\to 0} r^{\eta} w = c \neq 0$ *if* $N \ge 2$ ($\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b < 0$ *if* $N = 1$), *and* $S_w < \infty$.

(ii) *Suppose $\alpha = \delta = N$. Then the regular solutions, given by* (1–6), *have constant sign, with $S_w < \infty$. For any $k \in \mathbb{R}$, $w(r) = kr^{-N}$ is a solution. Moreover there exist positive solutions such that* $\lim_{r\to 0} r^{N} w = c > 0$ *and* $S_w < \infty$, *and solutions with one zero, such that* $\lim_{r\to 0} r^{N} w = c > 0$ *and* $S_w < \infty$.

Up to symmetry, all solutions are as above.

Proof. (i) Here $\alpha \neq N$ or $\alpha \neq \delta$ (Figure 10, left). Since $\alpha > 0$, Propositions 2.5, 2.7 and 2.14 imply that $y > 0$ and $S_w < \infty$ for $\mathcal{T}_r$; and any solution y has at most one zero, and y, Y are monotone near $-\infty$ and near $\ln S_w$. By Proposition 2.8, any trajectory $\mathcal{T}$ converges to $(0, 0)$ at $-\infty$; and apart from $\mathcal{T}_r$, such a trajectory is tangent to the axis $y = 0$. Now suppose $y > 0$ near $-\infty$. If $N \ge 2$, then $\mathcal{T}$ starts in $\mathcal{Q}_1$, since $\lim_{\tau\to-\infty} \zeta = \eta > 0$; if $N = 1$, then $\lim_{r\to 0} w = a \ge 0$ and $\lim_{r\to 0} w' = b$, and $\mathcal{T}$ starts in $\mathcal{Q}_1$ if $b < 0$ and in $\mathcal{Q}_4$ if $b > 0$ (in particular when $a = 0$).

For any $P = (\varphi, 0)$ with $\varphi > 0$, the trajectory $\mathcal{T}_{[P]}$ satisfies $y > 0$ on $\mathbb{R}$, and by Remark 2.1(i), it stays in $\mathcal{Q}_4$ after P, because it cannot meet $\mathcal{T}_r$ (hence $S_w < \infty$); also it stays in $\mathcal{Q}_1$ before P, so w is of type (2). In the same way for any $P = (0, \xi)$



$N = 2 < \alpha = 2.5 < \delta = 3$ $\qquad\qquad$ $\alpha = \delta = N = 3$

Figure 10. Theorem 6.1: $\varepsilon = -1$.

with $\xi > 0$, the trajectory $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_2$ after P, since it cannot meet $-\mathcal{T}_r$ (hence $S_w < \infty$), and it stays in $\mathcal{Q}_1$ before P, so w is of type (3).

Next consider the sets

$$\mathcal{A} = \{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \varnothing\},$$
$$\mathcal{B} = \{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(0, \xi) : \xi > 0\} \neq \varnothing\}.$$

From the previous discussion we know they are nonempty and open, so $\mathcal{A} \cup \mathcal{B} \neq \mathcal{Q}_1$. There exists a trajectory $\mathcal{T}_1$ starting at $(0, 0)$ and staying in $\mathcal{Q}_1$. By Proposition 2.8, necessarily $\lim_{\tau\to\infty} y = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha > 0$, so w is of type (1) by Proposition 2.9.

Finally we describe all other trajectories $\mathcal{T}_{[P]}$ with one point P in the domain $\mathcal{R}$ above $\mathcal{T}_r \cup (-\mathcal{T}_r)$. If P is in the domain delimited by $\mathcal{T}_r, \mathcal{T}_1$, then w is still of the type (2). If P is in the domain delimited by $-\mathcal{T}_r, \mathcal{T}_1$, then either y has a zero and w is of type (3), or $N = 1$, $y < 0$ and $-w$ is of type (2). Up to a symmetry, all the solutions have been obtained.

(ii) Here $\alpha = \delta = N$ (Figure 10, right). Since $\alpha = N$ equation (1–5) holds, and the regular solutions relative to $C = 0$ are given by (1–6). Since $\delta = N$, (1–5) is equivalent to $y + Y \equiv C$, from (2–12). For any $k \in \mathbb{R}$, $(y, Y) \equiv P_k = (k, |Nk|^{p-2}Nk)$ is a solution of system (S), located on the curve $\mathcal{M}$, thus $w(r) = kr^{-N}$ is a solution of (E_w). Any solution has at most one zero, by Proposition 2.5. From Propositions 2.8, and 2.10, any other trajectory converges to a point $P_k \in \mathcal{M}$ at ∞, and $S_w < \infty$. There exists trajectories such that y has constant sign, and other ones such that y has one zero. All solutions have been obtained. □

Next we suppose $\alpha < 0$, and distinguish the cases $N \geq 2$ and $N = 1$.

Theorem 6.2. *Suppose $\varepsilon = -1$ and $\alpha < 0 < 2 \leq N \leq \delta$. Then any solution of (E_w) has a finite number of zeros. Regular solutions have at least one zero, and* *precisely one if* $-p' \leq \alpha$. *Any solution has at least one zero, and any nonregular solution satisfies* $\lim_{r\to 0} r^\eta w = c \neq 0$.

If $-p' < \alpha$, *all regular solutions have a reduced domain* ($S_w < \infty$), *and they fall into the following types, all of which occur*:

(1) *solutions with two zeros and* $S_w < \infty$;

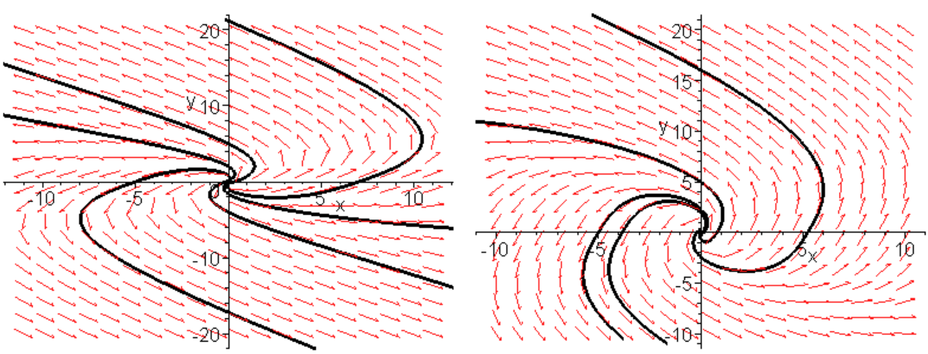
(2) *solutions with one zero and* $\lim_{r\to\infty} r^\alpha w = L \neq 0$;

(3) *solutions with one zero and* $S_w < \infty$.

If $\alpha = -p'$, *all regular solutions satisfy* $\lim_{r\to\infty} r^\alpha w = L \neq 0$. *The other solutions are of type* (1).

Proof. By Proposition 2.8, any trajectory converges necessarily to $(0,0)$ at $-\infty$, and apart from $\mathcal{T}_r$, it is tangent to the axis $y = 0$. Any solution y has a finite number of zeros, and y is monotone near $-\infty$, and near S_w (finite or not), by Propositions 2.7 and 2.11, since $\delta > N/2$. Either $S_w < \infty$, so $\lim_{\tau\to\ln S_w} Y/y = -1$, or $S_w = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha < 0$. In any case (y, Y) is in $\mathcal{Q}_2$ or $\mathcal{Q}_4$ for large τ. By Proposition 2.14, $\mathcal{T}_r$ has at least one zero, and starts in $\mathcal{Q}_1$. Since $N \geq 2$, any trajectory $\mathcal{T} \neq \pm\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} \zeta = \eta > 0$. Thus it starts in $\mathcal{Q}_1$ (or $\mathcal{Q}_3$), and has at least one zero. Any trajectory $\mathcal{T}$ starting in $\mathcal{Q}_1$ enters $\mathcal{Q}_2$, by Remark 2.1(i). And $y' = \delta y - Y^{1/(p-1)}$, so y decreases as long as $\mathcal{T}$ stays in $\mathcal{Q}_2$. Then either $\mathcal{T}$ enters $\mathcal{Q}_3$, hence also $\mathcal{Q}_4$, and y has at least two zeros; or it stays in $\mathcal{Q}_2$, and either $S_w < \infty$ and $\lim_{\tau\to\ln S_w} Y/y = -1$, or $S_w = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha$.

(i) Suppose $-p' < \alpha$ (Figure 11, left). Then $\mathcal{T}_r$ has precisely one zero, by Proposition 2.14, thus it stays in $\mathcal{Q}_2$, and $S_w < \infty$, $\lim_{\tau\to\ln S_w} Y/y = -1$. Any other solution has at most two zeros, because the trajectory does not meet $\pm\mathcal{T}_r$. Recall that the function Y_α defined by (2–3) with $d = \alpha$ has only minimal points on the



$-p' = -3 < \alpha = -2.5 < 0 < N$ $\qquad$ $-p' = -3 = \alpha < 0 < N$

Figure 11. Theorem 6.2: $\varepsilon = -1$, $N = 2 < \delta = 3$.

sets where it is positive, by Remark 2.6. By Proposition 2.14, $\mathcal{T}_r$ satisfies

$$Y'_\alpha = -(p-1)(\eta-\alpha)Y_\alpha + e^{(p-(2-p)\alpha)\tau}(Y_\alpha^{1/(p-1)} - \alpha y_\alpha) > 0,$$

which is equivalent to

$$Y^{1/(p-1)} - (p-1)(\eta-\alpha)Y > \alpha y. \tag{6–1}$$

$\mathcal{T}_r$ stays strictly to the right of the curve

$$\mathcal{N}_\alpha = \{(y, Y) \in \mathbb{R} \times (0, \infty) : \alpha y = Y^{1/(p-1)} - (p-1)(\eta-\alpha)Y\}, \tag{6–2}$$

which intersects the axis $y = 0$ at the points $(0, 0)$ and $(0, (p-1)(\eta-\alpha))$.

For $\bar{P} = (\varphi, 0)$ with $\varphi < 0$, the trajectory $\mathcal{T}_{[\bar{P}]}$ enters $\mathcal{Q}_3$ after $\bar{P}$, by Remark 2.1(i); the solution passing through $\bar{P}$ at $\tau = 0$ satisfies $Y_\alpha(0) = 0$ (so Y_α stays positive for $\tau < 0$) and $Y'_\alpha(\tau) < 0$, since Y_α has no maximal point. Thus $\mathcal{T}_{[\bar{P}]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ before P, to the left of $\mathcal{N}_\alpha$, and starts and $(0, 0)$ in $\mathcal{Q}_1$ and ends up in $\mathcal{Q}_4$. Hence y has two zeros. If $S_w = \infty$ then $\lim_{\tau\to\infty} |y| = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha < 0$; this is impossible, because $\mathcal{T}_{[\bar{P}]}$ does not meet $-\mathcal{T}_r$. Thus $S_w < \infty$, and w is of type (1).

Next consider $\mathcal{T}_{[P]}$, for $P = (\varphi, \xi) \in \mathcal{N}_\alpha$, with $\varphi \le 0$. The solution going through P at $\tau = 0$ satisfies $Y'_\alpha(0) = 0$, $Y_\alpha(0) > 0$, and 0 is a minimal point; hence $Y''_\alpha(0) > 0$. Indeed, if $Y''_\alpha(0) = 0$, then Y_α is constant on $\mathbb{R}$ by uniqueness; by (2–6), in turn, we have $Y_\alpha \equiv 0$ (since $\alpha \neq -p'$); but this is false. Therefore $Y'_\alpha(\tau) > 0$ for $\tau > 0$ and $Y'_\alpha(\tau) < 0$ for $\tau < 0$. Thus $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, to the right of $\mathcal{N}_\alpha$ after P, with $y < 0$ by Remark 2.1(i); it stays to the left of $\mathcal{N}_\alpha$ before P, and converges to $(0, 0)$ at $-\infty$ in $\mathcal{Q}_1$. Suppose that $S_w = \infty$. Then $\lim_{\tau\to\infty} |y| = \infty$, $\lim_{\tau\to\infty} \zeta = \alpha$, and $\lim_{\tau\to\infty} y_\alpha = L < 0$ by Proposition 2.9; thus $\lim_{\tau\to\infty} Y_\alpha = (\alpha L)^{p-1}$. As in Proposition 2.14, one finds that $Y''_\alpha(\tau) > 0$ for any $\tau > 0$, which is impossible. Thus $\mathcal{T}_{[P]}$ satisfies $S_w < \infty$, showing that $\lim_{\tau\to\ln S_w} Y/y = -1$. The corresponding w is of type (3).

Finally, let $\mathcal{R}$ be the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ delimited by $\mathcal{T}_r$ and containing $\mathcal{N}_\alpha$, and define the sets

$$\begin{aligned} \mathcal{A} &= \{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi < 0\} \neq \varnothing\}, \\ \mathcal{B} &= \{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \mathcal{N}_\alpha \neq \varnothing\}, \end{aligned} \tag{6–3}$$

corresponding to trajectories of type (1) or (3). These sets are nonempty and open, because here again the intersection with $\mathcal{N}_\alpha$ is transverse (recall that $\alpha \neq -p'$). Thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}$. There exists a trajectory in $\mathcal{R}$ disjoint from $\mathcal{N}_\alpha$, starting at $(0, 0)$ in $\mathcal{Q}_1$ and ending up in $\mathcal{Q}_2$. It cannot satisfy $\lim_{\tau\to\ln S_w} Y/y = -1$, so $S_w = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha$. Hence w is of type (2).

(ii) Suppose $\alpha = -p'$ (Figure 11, right). The regular solutions are given by (1–8), they have one zero, but $S_w = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha$. They satisfy $Y_{-p'} \equiv C$, thus $Y'_{-p'} \equiv 0$, thus $\mathcal{T}_r = \mathcal{M}_{-p'}$. Consider $\mathcal{T}_{[\bar{P}]}$; the solution passing through $\bar{P}$ at $\tau = 0$ satisfies and $Y_{-p'}(0) = 0$, thus $Y_{-p'}$ stays negative for $\tau > 0$ and $Y'_{-p'} < 0$. Suppose that $S_w = \infty$, then $\lim_{\tau\to\infty} y_\alpha = L > 0$, $\lim_{\tau\to\infty} Y_\alpha = -(|\alpha| L)^{p-1}$. But as in (2–46), $Y''_\alpha(\tau) < 0$ for any $\tau > 0$, which leads to a contradiction. Thus $S_w < \infty$, and w is of type (1). Finally suppose that there exists a trajectory $\mathcal{T} \neq \mathcal{T}_r$ staying in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. Then $Y_\alpha > 0$, $\lim_{\tau\to\infty} Y_\alpha = 0$, and it cannot meet $\mathcal{T}_r$, thus $S_w = \infty$, and $\lim_{\tau\to-\infty} Y_\alpha = \infty$, $\lim_{\tau\to\infty} Y_\alpha = C > 0$. As in Proposition 2.14, it is impossible. Thus there does not exist solution of type (2) or (3). □

Theorem 6.3. *Suppose $\varepsilon = -1$ and $\alpha < 0 < N = 1 < \delta$. Then any solution of (E_w) has still a finite number of zeros. Regular solutions have at least one zero, and precisely one if $-p' \leq \alpha$.*

If $-1 < \alpha < 0$, all regular solutions have a reduced domain ($S_w < \infty$). Moreover:

(1) *the solutions with $\lim_{r\to 0} w = a > 0$ and $\lim_{r\to 0} w' = b < 0$ have one zero and $S_w < \infty$;*

(2) *the solutions with $\lim_{r\to 0} w = 0$ and $\lim_{r\to 0} w' = b > 0$ are positive and $S_w < \infty$;*

(3) *there exist solutions with one zero and $\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b > 0$ and $S_w < \infty$;*

(4) *there exist positive solutions with $\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b > 0$ and $S_w < \infty$;*

(5) *for any $a > 0$ there exists $b > 0$ such that w is positive and $\lim_{r\to\infty} r^\alpha w = L \neq 0$.*

If $\alpha = -1$, for any $b > 0$, $w \equiv br$ is a solution. The other solutions such that $\lim_{r\to 0} w \neq 0$ have one zero, and satisfy $S_w < \infty$.

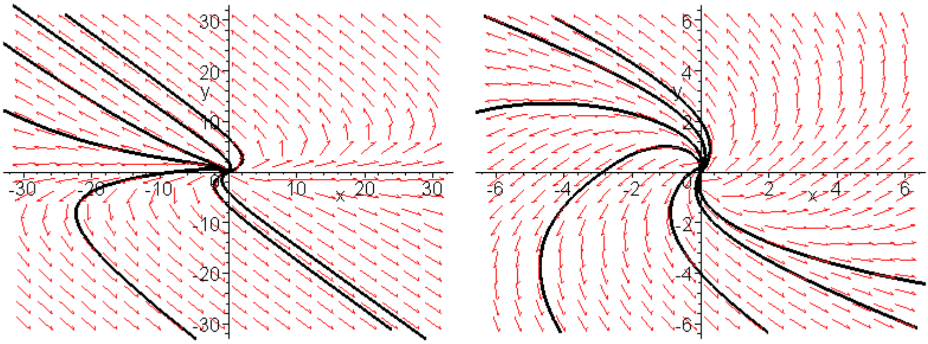
If $-p' < \alpha < -1$, then

(6) *there exist solutions with one zero, with $\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b < 0$, and $S_w < \infty$;*

(7) *the solutions with $\lim_{r\to 0} w = 0$ and $\lim_{r\to 0} w' = b > 0$ have one zero and $S_w < \infty$;*

(8) *there exist solutions with one zero, with $\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b > 0$ and $S_w < \infty$;*

(9) *there exist solutions with $\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b < 0$, with two zeros and $S_w < \infty$;*



$-1 < \alpha = -0.5 < N$ $\qquad$ $-p' = -3 < \alpha = -2.9 < -1 < N$

Figure 12. Theorem 6.3: $\varepsilon = -1$, $N = 1 < \delta = 3$.

(10) *for any* $a > 0$ *there exists* $b > 0$ *and a solution with* $\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b < 0$, *with one zero and* $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$.

Proof. The case $N = 1$ is still the more complex one, since some trajectories start in $\mathcal{Q}_2$ (or $\mathcal{Q}_4$), corresponding to the solutions such that $\lim_{r\to 0} w = a$ and $\lim_{r\to 0} w' = b$, with $b \neq 0$, $ab \geq 0$. Any solution has still a finite number of zeros, by Proposition 2.11.

(i) Suppose $-1 < \alpha < 0$ (Figure 12, left). By Proposition 2.5, any solution has at most one zero, so regular solutions have precisely one zero. Thus $\mathcal{T}_r$ meets the axis $y = 0$ at some point $(0, \xi_r)$.

Consider the trajectory $\mathcal{T}_s$ such that $\lim_{r\to 0} w = 0$ and $\lim_{r\to 0} w' = b < 0$ (which means $\lim_{\tau\to -\infty} \zeta = \eta = -1$), starting from $(0, 0)$ in $\mathcal{Q}_2$, so $w < 0$ near 0. For any $d \in (-1, \alpha)$, the function y_d satisfies $y_d(\tau) = be^{(d+1)\tau}(1 + o(1))$ near $-\infty$, so $\lim_{\tau\to -\infty} y_d = 0$. Then y_d has no zeros, because $|y_d|$ has no maximal point, by (2–14); thus $\mathcal{T}_s$ stays in $\mathcal{Q}_2$. If $\mathcal{T}_s$ satisfies $S_w = \infty$, then $\lim_{\tau\to\infty} y_\alpha = L < 0$, so $\lim_{\tau\to\infty} y_d = 0$, which is impossible; thus w is of type (2). The domain is reduced since $\mathcal{T}_r$ cannot meet $\mathcal{T}_s$.

For $\bar{P} = (\varphi, 0)$ with $\varphi < 0$, the trajectory $\mathcal{T}_{[\bar{P}]}$ does not meet $\mathcal{T}_s$, thus converges to $(0, 0)$ at $-\infty$ in $\mathcal{Q}_2$; then $\lim_{r\to 0}(-w) = a > 0$ and $\lim_{r\to 0}(-w)' = b > 0$, and $\mathcal{T}_{[\bar{P}]}$ ends up in $\mathcal{Q}_4$; thus y has one zero and $-w$ is of type (3).

For $P = (0, \xi)$, with $\xi \in (0, \xi_r)$, $\mathcal{T}_{[P]}$ has one zero and converges to $(0, 0)$ at $-\infty$ in $\mathcal{Q}_1$; hence $\lim_{r\to 0} w = a > 0$ and $\lim_{r\to 0} w' = b < 0$. The domain is reduced since $\mathcal{T}_{[P]}$ and $\mathcal{T}_s$ do not meet. Thus w is of type (1). Conversely, any solution such that $\lim_{r\to 0} w = a > 0$ and $\lim_{r\to 0} w' = b < 0$ has one zero and satisfies $S_w < \infty$.

Next consider a trajectory $\mathcal{T}$ such that $\lim_{r\to 0}(-w) = a > 0$ and $\lim_{r\to 0}(-w') = b > 0$; that is, $\mathcal{T}$ starts in $\mathcal{Q}_2$ below $\mathcal{T}_s$. Then $\zeta(\tau) = -(b/a)e^{\tau}(1 + o(1)$ near $-\infty$, so $\lim_{\tau\to-\infty}\zeta=0$. If ζ has an extremal point θ, we have

$$(p-1)\zeta''(\theta)=(2-p)(\zeta-\alpha)(\delta-\zeta)|\zeta y|^{2-p},$$

by (2–18); thus θ is a minimal point if $\zeta(\theta)>\alpha$, and maximal if $\zeta(\theta)<\alpha$. (Equality is impossible since it would require $\zeta\equiv\alpha$.) Thus either ζ has a first zero τ_1 and $\alpha<\zeta(\tau)<0$ for $\tau<\tau_1$, and $\mathcal{T}$ is one of the $\mathcal{T}_{[\bar{P}]}$; or ζ remains negative, in which case if $S_w=\infty$, then $\lim_{\tau\to\infty}\zeta=\alpha$, so ζ is necessarily decreasing, and $\alpha<\zeta(\tau)<0$ for any τ. In both cases, $\mathcal{T}$ stays below the curve

$$\mathcal{M}'=\left\{(y,Y)\in\mathbb{R}\times(0,\infty):\alpha y=Y^{1/(p-1)}\right\},$$

as long as it is in $\mathcal{Q}_2$. Hence, for any $P\in\mathcal{Q}_2$ such that P is on or above $\mathcal{M}'$, the trajectory $\mathcal{T}_{[P]}$ satisfies $S_w<\infty$; in particular on finds again $\mathcal{T}_s$. For any P between $\mathcal{M}'$ and $\mathcal{T}_s$, the solution has constant sign, $\mathcal{T}_{[P]}$ converges to $(0,0)$ at $-\infty$ and $\lim_{r\to0}(-w)=a>0$ and $\lim_{r\to0}(-w')=b>0$, and $\lim_{\tau\to\ln S_w}Y/y=-1$, so $\mathcal{T}_{[P]}$ meets $\mathcal{M}_\alpha$. Thus $-w$ is of type (4).

Finally, let $\mathcal{R}_1$ be the domain of $\mathcal{Q}_2$ delimited by $\mathcal{T}_s$ and the axis $Y=0$, and set

$$\mathcal{A}_1=\left\{P\in\mathcal{R}_1:\mathcal{T}_{[P]}\cap\{(\varphi,0):\varphi<0\}\neq\varnothing\right\},$$
$$\mathcal{B}_1=\left\{P\in\mathcal{R}_1:\mathcal{T}_{[P]}\cap\mathcal{N}_\alpha\neq\varnothing\right\}.$$

These sets are open, since the intersection is transverse (recall that $\alpha\neq-1$). They are also nonempty, so $\mathcal{A}_1\cup\mathcal{B}_1\neq\mathcal{R}_1$, and there exists a trajectory such that y is defined on $\mathbb{R}$ and $\lim_{\tau\to\infty}\zeta=\alpha$. By scaling, we can find for any $a>0$ at least one b such that the corresponding w has constant sign and $\lim_{r\to\infty}r^\alpha w=L\neq0$; thus $|w|$ is of type (5).

(ii) Suppose $\alpha=-1$. Then $\mathcal{T}_s$ is given explicitly by $w\equiv br$, so $Y\equiv-y^{p-1}$, or equivalently $Y_{-1}\equiv b$; hence $\mathcal{T}_s=\mathcal{N}_{-1}$. For any other solution, one finds $Y''_{-1}=Y'_{-1}(1+e^{2\tau}|Y_{-1}|^{(2-p)/(p-1)})$, so Y_{-1} is strictly monotone, by uniqueness, and Y''_{-1} has the sign of Y'_{-1}. Any trajectory such that $\lim_{r\to0}w=a>0$ and $\lim_{r\to0}w'=b<0$, starting in $\mathcal{Q}_1$, satisfies $Y'_{-1}>0$, and Y_{-1} is convex. Thus Y_{-1} cannot have a finite limit, $S_w<\infty$, and the trajectory ends up in $\mathcal{Q}_2$, so y has a zero. Any trajectory such that $\lim_{r\to0}(-w)=a>0$ and $\lim_{r\to0}(-w)'=b>0$, starting in $\mathcal{Q}_2$, satisfies $Y'_{-1}<0$, so Y_{-1} has a zero and the trajectory ends up in $\mathcal{Q}_4$. Hence, apart from $\mathcal{T}_s$, all trajectories satisfy $S_w<\infty$, and y has one zero.

(iii) Suppose $-p'<\alpha<-1$ (Figure 12, right). Then $\mathcal{T}_r$ starts in $\mathcal{Q}_1$, y has one zero from Proposition 2.14, and $\mathcal{T}_r$ ends up in $\mathcal{Q}_2$, with $S_w<\infty$. Any solution has at most two zeros.

Consider $\mathcal{T}_s$: we claim that it cannot stay in $\mathcal{Q}_2$. Suppose that it stays in it, thus $y<0<Y$. Then $\zeta<0$, and $\lim_{\tau\to-\infty}\zeta=\eta=-1$, and ζ is monotone near $-\infty$; if $\zeta'\leq0$, then $\zeta\leq-1$ near $-\infty$, and we reach a contradiction from (2–9). Then

$\zeta' \geq 0$ near $-\infty$; but any extremal point of ζ is a minimal point by (2–18). Hence ζ remains increasing, is defined on $\mathbb{R}$ and has a limit $\lambda \in [-1, 0]$; but $\lambda = \alpha$, by Proposition 2.8, again leading to a contradiction. Therefore $\mathcal{T}_s$ enters $\mathfrak{Q}_3$ at some point $(\varphi_s, 0)$ with $\varphi_s < 0$, then enters $\mathfrak{Q}_4$, and y has precisely one zero; and w is of type (7).

Any solution such that $\lim_{r\to 0}(-w) = a > 0$ and $\lim_{r\to 0}(-w)' = b > 0$ also has one zero, since its trajectory stays under $\mathcal{T}_s$ in $\mathfrak{Q}_2$; thus w is of type (8).

As in the case $N \geq 2$, for any $P = (\varphi, \xi) \in \mathcal{N}_\alpha$ with $\varphi \leq 0$, $\mathcal{T}_{[P]}$ stays in $\mathfrak{Q}_1 \cup \mathfrak{Q}_2$ and $S_w < \infty$. In particular for $P_0 = (0, \xi_0)$, where $\xi_0 = ((p-1)(-1-\alpha))^{(p-1)/(2-p)}$, the trajectory $\mathcal{T}_{[P_0]}$ starts from $\mathfrak{Q}_1$, so $\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b_0(a) > 0$; also w has one zero, and $S_w < \infty$. Thus w is of type (6).

The sets $\mathcal{A}, \mathcal{B}$ defined as in (6–3) are still open in this case, and $\mathcal{B}$ contains $\mathcal{T}_{[P_0]}$. Also, $\mathcal{A}$ contains $\mathcal{T}_s$; hence $\mathcal{A}$ contains any $\mathcal{T}_{[P]}$, where $P = (\varphi, 0)$ with $\varphi < \varphi_s$. Such a trajectory satisfies $\lim_{r\to 0} w = a > 0$ and $\lim_{r\to 0} w' = b < 0$, and w is of type (9). Moreover $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}$; thus for any $a > 0$ there exists $b < 0$ such that the corresponding w has one zero and $\lim_{r\to\infty} r^\alpha w = L \neq 0$, so w is of type (10). □

$\alpha < \delta < N$

As in the case $\varepsilon = 1$, $\delta < \min(\alpha, N)$ of page 246, here two kinds of periodic trajectories can appear, and the study is delicate. Here also $N \geq 2$, and we still have three stationary points, and $(0, 0)$ is a saddle point. M_ℓ is a source if $N/2 \leq \delta$ or $\delta < N/2$ and $\alpha^* < \alpha$, and a sink if $\delta < N/2$ and $\alpha < \alpha^*$; notice that $\alpha^* < -p' < 0$, by (2–32). Also M_ℓ is a node whenever $\alpha \leq \alpha_1$ or $\alpha_2 \leq \alpha$, where α_1, α_2 are defined in (2–48), and α_2 can be greater or less than $-p'$. We begin with the simplest case.

Theorem 6.4. *Assume $\varepsilon = -1$ and $0 < \alpha < \delta < N$. All regular solutions have constant sign and a reduced domain ($S_w < \infty$). The function $w \equiv \ell r^{-\delta}$ is a solution. There exist solutions satisfying any one of these characterizations*:

(1) *w is positive*, $\lim_{r\to 0} r^\delta w = \ell$ *and* $S_w < \infty$;

(2) *w has one zero*, $\lim_{r\to 0} r^\delta w = \ell$ *and* $S_w < \infty$;

(3) *w is positive*, $\lim_{r\to 0} r^\delta w = \ell$ *and* $\lim_{r\to\infty} r^\eta w = c > 0$;

(4) *w is positive*, $\lim_{r\to 0} r^\delta w = \ell$ *and* $\lim_{r\to\infty} r^\alpha w = L > 0$.

Up to symmetry, all solutions are as above.

Proof. Since $\alpha > 0$, regular solutions have constant sign and satisfy $S_w < \infty$, by Propositions 2.5 and 2.14. Here $\mathcal{T}_r$ starts in $\mathfrak{Q}_4$ and stays in it, by Remark 2.3 (Figure 13). Any solution has at most one zero by Proposition 2.5. The point M_ℓ is a source, and a node point, by Remark 2.17, and $0 < \lambda_1 < \delta < \lambda_2$. The eigenvectors $u_1 = (\nu(\alpha), \lambda_1 - \delta)$ and $u_2 = (-\nu(\alpha), \delta - \lambda_2)$ form a positively oriented basis, where now $\nu(\alpha) < 0$; thus u_1 points toward $\mathcal{Q}_3$ and u_2 toward $\mathcal{Q}_4$. There are two particular trajectories $\mathcal{T}_1, \mathcal{T}_2$ starting from M_ℓ at $-\infty$, with respective tangent vectors u_2 and $-u_2$. All other trajectories $\mathcal{T}$ approaching M_ℓ at $-\infty$ do so along u_1; and y is monotone at the extremities, by Proposition 2.7, since $\mathcal{T}$ cannot meet $\mathcal{T}_1, \mathcal{T}_2$.

First consider $\mathcal{T}_1$. The function y is nondecreasing near $-\infty$ and remains so as long as $\mathcal{T}_1$ stays in $\mathcal{Q}_1$; indeed, Y is nonincreasing near $-\infty$, so $Y(\tau) < (\delta\ell)^{p-1}$. If y has a maximal point τ, then $y(\tau) > \ell$ by (2–16), and $Y^{1/(p-1)} = \delta y$; hence $Y(\tau) > (\delta\ell)^{p-1}$, so Y has a minimal point τ_1 in $\mathcal{Q}_1$; therefore $Y(\tau_1) < (\delta\ell)^{p-1}$ by (E_Y); and $Y'(\tau_1) = 0$, so $\alpha\ell < \alpha y(\tau_1) < (N-\delta)\alpha Y(\tau_1)/(\delta-\alpha)$, a contradiction. If $\mathcal{T}_1$ stays in $\mathcal{Q}_1$, then $\lim_{\tau\to-\infty}\zeta = \alpha > 0$ by Proposition 2.8, which is also contradictory. Thus $\mathcal{T}_1$ enters $\mathcal{Q}_4$ at some point $(\varphi_1, 0)$ and stays in it; $S_w < \infty$ because $\mathcal{T}_1$ and $\mathcal{T}_r$ don't meet, so w is of type (1).

Next consider $\mathcal{T}_2$. Near $-\infty$, the function Y is nondecreasing, and y is nonincreasing; y is monotone as long as $y > 0$: if there existed a minimal point τ, we would have $y(\tau) > \ell$ by (2–16). Also Y is nondecreasing as long as $Y > 0$: if Y has a maximal point τ, then $Y(\tau) > (\delta\ell)^{p-1}$ by (E_Y); and

$$\alpha\ell > \alpha y(\tau) > (N-\delta)\alpha Y(\tau)/(\delta-\alpha),$$

which is again impossible. Thus $\mathcal{T}_2$ cannot stay in $\mathcal{Q}_1$; it enters $\mathcal{Q}_2$ at some point $(0, \xi_2)$ and stays in $\mathcal{Q}_2$, since it does not meet $-\mathcal{T}_r$. Hence $S_w < \infty$, and w is of type (2).

There exists also a unique trajectory $\mathcal{T}_3$ converging to $(0,0)$ at ∞, ending up in $\mathcal{Q}_1$, since $(0,0)$ is a saddle point. It stays in the domain of $\mathcal{Q}_1$ delimited by $\mathcal{T}_1, \mathcal{T}_2$, because $\mathcal{Q}_1$ is backward invariant. Thus $\mathcal{T}_3$ converges to M_ℓ at $-\infty$, tangentially to u_1. And y is increasing on $\mathbb{R}$: indeed $y' < 0$ near $\pm\infty$, and y cannot have two extremal points. Then w is of type (3).

For any point $P = (\varphi, 0)$ with $\varphi > \varphi_1$, the trajectory $\mathcal{T}_{[P]}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_4$, by Remark 2.1(i). It does not meet $\mathcal{T}_r$ or $\mathcal{T}_1$; hence it stays in $\mathcal{Q}_4$ after P, and

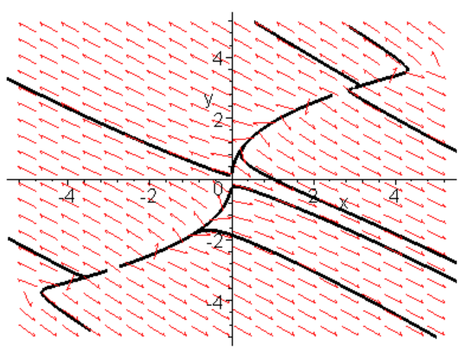


Figure 13. Theorem 6.4: $\varepsilon = -1,\ 0 < \alpha = 2 < \delta = 3 < N = 4$.

$S_w < \infty$. Before P, it stays in $\mathcal{Q}_1$ because it does not meet $\mathcal{T}_1$ or $\mathcal{T}_2$, by the same remark. By Proposition 2.8, either $\lim_{\tau\to-\infty} \zeta = \alpha < \delta$, so $y' = y(\delta - \zeta) > 0$ near $-\infty$, and $\lim_{\tau\to-\infty} y = \infty$, which is impossible; or (necessarily) $\mathcal{T}_{[P]}$ converges to M_ℓ, tangentially to u_1, and $\mathcal{T}_{[P]}$ is of type (2). Similarly, for any $P' = (0, \xi)$ with $\xi > \xi_2$, the trajectory $\mathcal{T}_{[P']}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_2$; it remains there after P (so $S_w < \infty$) and remains in $\mathcal{Q}_1$ before P, converging to M_ℓ at $-\infty$, tangentially to $-u_1$. Thus w is still of type (2).

The sets

$$\mathcal{A} = \{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \varnothing\},$$

$$\mathcal{B} = \{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(0, \xi) : \xi > 0\} \neq \varnothing\},$$

are open and nonempty, so $\mathcal{A} \cup \mathcal{B} \neq \mathcal{Q}_1$. There is at least one trajectory $\mathcal{T}_4$ in $\mathcal{Q}_1$ converging to M_ℓ at $-\infty$ and such that $\lim_{\tau\to\infty} \zeta = \alpha$; thus w is of type (4).

For any point P in the bounded domain $\mathcal{R}'$ of $\mathcal{Q}_1$ delimited by $\mathcal{T}_2$ and $\mathcal{T}_3$, the trajectory $\mathcal{T}_{[P]}$ is confined to $\mathcal{R}'$ before P, and y has no maximal point; thus y is monotone, and $\mathcal{T}$ converges to M_ℓ at $-\infty$. It cannot stay in $\mathcal{Q}_1$ since it cannot converge to $(0, 0)$. Thus it goes from $\mathcal{Q}_1$ into $\mathcal{Q}_2$ and stays there, because it does not meet $-\mathcal{T}_r$. Thus $S_w < \infty$, and w is again of type (2).

For any P in the domain of $\mathcal{Q}_1$ delimited by $\mathcal{T}_1$ and $\mathcal{T}_3$, the trajectory $\mathcal{T}_{[P]}$ converges to M_ℓ at $-\infty$, tangentially to u_1; it enters $\mathcal{Q}_4$ and stays there. Thus $S_w < \infty$ and w is of type (1). No trajectory can stay in $\mathcal{Q}_4(\mathcal{Q}_2)$ except $\mathcal{T}_r(-\mathcal{T}_r)$; thus all the solutions have been described, up to a symmetry. □

Now we come to the case $\alpha < 0$, and discuss according to the sign of $\alpha - p'$. This situation is different from the case $\varepsilon = 1$, $\delta < \min(\alpha, N)$ discussed on page 246, by Remark (i) on page 249 and part (i) of the next remark.

Remark 6.5. Assume $\varepsilon = -1$ and $\alpha < 0$.

(i) The regular trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. There exists a unique trajectory $\mathcal{T}_s$ converging to $(0, 0)$, lying in $\mathcal{Q}_1$ for large τ, having an infinite slope at $(0, 0)$, and satisfying $\lim_{r\to 0} r^\eta w = c > 0$. If $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$, then $\mathcal{T}_r$ does stay in it, and it is bounded and contained in the domain delimited by $\mathcal{Q}_1 \cap \mathcal{T}_s$, by Remark 2.1(i). If $\mathcal{T}_r$ is homoclinic, it stays in $\mathcal{Q}_1$.

Conversely, if $\mathcal{T}_s$ stays in $\mathcal{Q}_1$ and is not homoclinic, $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, for the following reason. $\mathcal{T}_s$ either converges to M_ℓ at $-\infty$ or has a limit cycle around it; if $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, either the corresponding y is increasing, so $\lim_{\tau\to\ln S_w} Y/y = -1$; or $\lim_{\tau\to\infty} \zeta = \alpha < 0$, by Propositions 2.15 and 2.8, so $\mathcal{T}_r$ enters $\mathcal{Q}_4$ and we reach a contradiction; or y oscillates around ℓ near ∞, by Proposition 2.7, so it meets $\mathcal{T}_s$, which is impossible.

(ii) Any trajectory $\mathcal{T}$ is bounded near $-\infty$ from Propositions 2.8 and 2.10. Any trajectory $\mathcal{T}$ bounded at $\pm\infty$ converges to $(0, 0)$ or $\pm M_\ell$, or its limit set $\Gamma_\pm$ at $\pm\infty$ is a limit cycle; or $\mathcal{T}_r$ is homoclinic and $\Gamma_\pm = \overline{\mathcal{T}}_r$.

(iii) If there exists a limit cycle around $(0, 0)$, it also surrounds $\pm M_\ell$, by (2–42) and (2–43).

Next we study the case $-p' \leq \alpha$, where there is no cycle and no homoclinic orbit in $\mathcal{Q}_1$, by Theorem 2.20.

Theorem 6.6. (i) *Assume $\varepsilon = -1$ and $-p' < \alpha < 0 < \delta < N$. Then all regular solutions have precisely one zero, and $S_w < \infty$. The function $w \equiv \ell r^{-\delta}$ is a solution. There exist solutions satisfying any one of these characterizations*:

(1) *w is positive*, $\lim_{r\to 0} r^\delta w = \ell$ *and* $\lim_{r\to\infty} r^\eta w = c > 0$;

(2) *w has one zero*, $\lim_{r\to 0} r^\delta w = \ell$, *and* $\lim_{r\to\alpha} r^\alpha w = L < 0$;

(3) *w has one zero*, $\lim_{r\to 0} r^\delta w = \ell$, *and* $S_w < \infty$;

(4) *w has two zeros*, $\lim_{r\to 0} r^\delta w = \ell$, *and* $S_w < \infty$.

(ii) *Assume $\alpha = -p'$. Then the regular solutions, given by* (1–8), *have one zero, and* $\lim_{r\to\alpha} r^\alpha w = L < 0$. *There exist solutions of type* (1) *and* (4).

Up to symmetry, all solutions are as above.

Proof. (i) Assume $-p' < \alpha < 0$ (Figure 14, left). By Proposition 2.5, any solution y has at most two zeros, and Y has at most one zero.

First consider $\mathcal{T}_s$. The function Y_α defined by (2–3) with $d = \alpha$ satisfies $Y_\alpha = O(e^{(\alpha-\eta)\tau})$ near ∞, thus $\lim_{\tau\to\infty} Y_\alpha = 0$. Then from Remark 2.6, Y_α is decreasing, thus $Y_\alpha > 0$, and $\mathcal{T}_s$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. In fact it stays in $\mathcal{Q}_1$, by Remark 2.1(i). From Propositions 2.8, 2.7, 2.11, and Theorem 2.20, $\mathcal{T}_s$ converges to M_ℓ at $-\infty$. Indeed if $\lim y = \infty$, then $\lim_{\tau\to\infty} \zeta = \alpha < 0$; if $S_w < \infty$, then $\lim Y/y = -1$; which contradicts $Y > 0$. Then w is of type (1).

The trajectory $\mathcal{T}_r$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, and y has precisely one zero, and $S_w < \infty$, so $\lim_{\tau\to\ln S_w} Y/y = -1$. We claim that $\mathcal{T}_r$ cannot stay in $\mathcal{Q}_1$. Indeed, it cannot

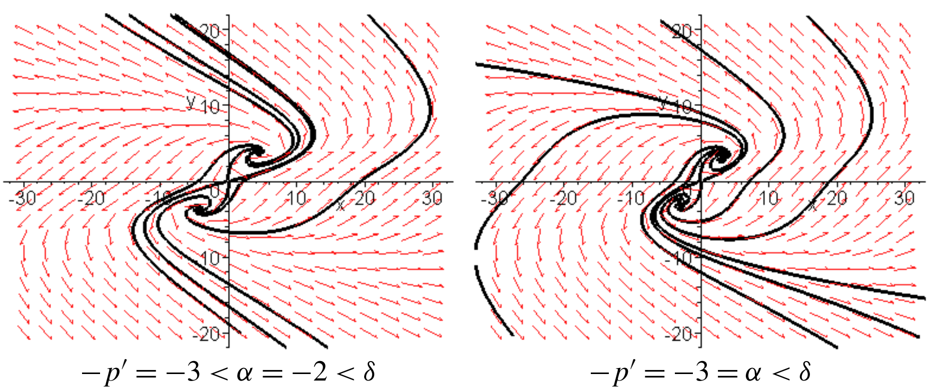


Figure 14. Theorem 6.6: $\varepsilon = -1$, $\delta = 3 < N/2 < N = 9$.

converge to M_ℓ, which is a source, or oscillate around $\mathcal{Q}_1$, because it does not meet $\mathcal{T}_s$, or tend to ∞, or satisfy $S_w < \infty$ with $Y > 0$. Thus y has precisely one zero, $\mathcal{T}_r$ enters $\mathcal{Q}_2$ and stays in it. Moreover the corresponding Y_α satisfies $Y'_\alpha > 0$, or equivalently (6–1). Consider again the curve $\mathcal{N}_\alpha$ defined in (6–2). Here $\mathcal{T}_r$ stays strictly to the right of $\mathcal{N}_\alpha$, and $\mathcal{T}_s$ to the left of $\mathcal{N}_\alpha$.

For any $\bar{P} = (\varphi, 0)$ with $\varphi < 0$, the trajectory $\mathcal{T}_{[\bar{P}]}$ enters $\mathcal{Q}_3$ after $\bar{P}$, by Remark 2.1(i). The solution going through $\bar{P}$ at $\tau = 0$ satisfies $Y_\alpha(0) = 0$; thus Y_α stays positive as before, and $Y'_\alpha < 0$, since Y_α has no maximal point, by Remark 2.6. Thus $\mathcal{T}_{[\bar{P}]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ before $\bar{P}$, to the left of $\mathcal{N}_\alpha$. It cannot stay in $\mathcal{Q}_2$, by Propositions 2.7 and 2.8. As τ decreases, it enters $\mathcal{Q}_1$, and converges to M_ℓ, by Theorem 2.20. If $S_w = \infty$, then $\lim |y| = \infty$ and $\lim_{\tau\to\infty} \zeta = \alpha < 0$; this is impossible, since $\mathcal{T}_{[\bar{P}]}$ does not meet $-\mathcal{T}_r$. Thus $S_w < \infty$, $\lim Y/y = -1$, $\mathcal{T}_{[\bar{P}]}$ goes from $\mathcal{Q}_3$ into $\mathcal{Q}_4$ and stays in it, and w is of type (4). The solution y has precisely two zeros.

Next consider $\mathcal{T}_{[P]}$ for any $P = (\varphi, \xi) \in \mathcal{N}_\alpha$ with $\varphi < 0$. The solution passing through P at $\tau = 0$ satisfies $Y'_\alpha(0) = 0$ and $Y_\alpha(0) > 0$, and 0 is a minimal point. Therefore $Y''_\alpha(0) > 0$; indeed, if $Y''_\alpha(\tau) = 0$, we conclude from uniqueness that Y_α is constant on $\mathbb{R}$; then (2–6) yields $Y_\alpha \equiv 0$, since $\alpha \neq -p'$. But this cannot be. Therefore $Y'_\alpha(\tau) > 0$ for $\tau > 0$, $Y'_\alpha(\tau) < 0$ for $\tau < 0$, and $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, to the right of $\mathcal{N}_\alpha$ after P, with $y < 0$ by Remark 2.1(i), and to the left of $\mathcal{N}_\alpha$ before P. As above it cannot stay in $\mathcal{Q}_2$ near $-\infty$, and converges to M_ℓ. Suppose that it satisfies $S_w = \infty$. Then $\lim |y| = \infty$, $\lim_{\tau\to\infty} \zeta = \alpha$, and $\lim_{\tau\to\infty} y_\alpha = L < 0$ by Proposition 2.9; hence $\lim_{\tau\to\infty} Y_\alpha = (\alpha L)^{p-1}$. As in Proposition 2.5(iii), we find $Y''_\alpha(\tau) > 0$ for any $\tau > 0$, which is impossible. Then $S_w < \infty$, so $\lim_{\tau\to\ln S_w} Y/y = -1$ and w is of type (3).

Finally consider the domain $\mathcal{R}$ of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ delimited by $\mathcal{T}_r$ and $\mathcal{T}_s$ and containing $\mathcal{N}_\alpha$. Form the sets

$$\mathcal{A} = \{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \mathcal{N}_\alpha \neq \varnothing\},$$
$$\mathcal{B} = \{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \{(\xi, 0) : \xi > 0\} \neq \varnothing\},$$

corresponding to trajectories of type (3) or (4). They are nonempty and open, since here again the intersection with $\mathcal{N}_\alpha$ is transverse ($\alpha \neq -p'$). Thus $\mathcal{A} \cup \mathcal{B}$ is distinct from $\mathcal{R}$: there exists a trajectory in $\mathcal{R}$ that does not meet $\mathcal{N}_\alpha$; it converges to M_ℓ at $-\infty$ or oscillates around it, and it is located below $\mathcal{N}_\alpha$ in $\mathcal{Q}_2$. It cannot satisfy $\lim_{\tau\to\ln S_w} Y/y = -1$, so $S_w = \infty$ and we have $\lim_{\tau\to\infty} \zeta = \alpha$. Hence w is of type (2).

(ii) Assume $\alpha = -p'$ (Figure 14, right). Then regular solutions have a different behavior: they are given explicitly by (1–8). They satisfy $Y_{-p'} \equiv C$, thus $Y'_{-p'} \equiv 0$, thus $\mathcal{T}_r = \mathcal{M}_{-p'}$. Here y has a zero, and $S_w = \infty$, and $\lim_{\tau\to\infty} \zeta = -p'$. As above $\mathcal{T}_s$ stays in $\mathfrak{Q}_1$ and w is of type (1).

Next consider again $\mathcal{T}_{[\bar{P}]}$. The solution going through $\bar{P}$ at $\tau = 0$ satisfies $Y_{-p'}(0) = 0$, thus $Y_{-p'}$ stays negative for $\tau > 0$ and $Y'_{-p'} < 0$. Suppose that $S_w = \infty$, and $\lim_{\tau\to\infty} \zeta = -p'$, then $\lim_{\tau\to\infty} y_\alpha = L > 0$, $\lim_{\tau\to\infty} Y_\alpha = -(|\alpha|L)^{p-1}$. But as in (2–46), $Y''_\alpha(\tau) < 0$ for any $\tau > 0$, which leads to a contradiction. Then $S_w < \infty$ and w is of type (4).

Finally suppose that there exists a trajectory $\mathcal{T} \neq \mathcal{T}_r$ staying in $\mathfrak{Q}_1 \cup \mathfrak{Q}_2$. Then it converges to M_ℓ, thus $Y_\alpha > 0$, $S_w = \infty$, and $\lim_{\tau\to-\infty} Y_\alpha = \infty$, $\lim_{\tau\to\infty} Y_\alpha = C > 0$. If $\mathcal{T}$ has a minimal point, then it has an inflection point where $Y'_\alpha > 0$, which as above is impossible. Then $Y'_\alpha < 0$; (2–6) yields

$$(p-1)Y''_{-p'} = Y'_{-p'}\big(e^{p'\tau}Y_{-p'}^{(2-p)/(p-1)} - N(p-1)\big) = Y'_{-p'}(Y - N(p-1)),$$

and $\lim_{\tau\to\infty} Y = \infty$, so $Y''_{-p'} < 0$ for large τ, which is impossible. Thus there exist no solutions of type (2) or (3). □

We now come to the most difficult case: $\alpha < -p'$.

Lemma 6.7. *Assume $\varepsilon = -1$ and $\alpha < -p'$. If $\delta < N/2$ and $\alpha^* < \alpha$, either $\mathcal{T}_r$ has a limit cycle in $\mathfrak{Q}_1$, or is homoclinic, or all regular solutions have at least two zeros. If $N/2 \le \delta < N$, they have at least two zeros.*

Proof. In any case M_ℓ is a source. Suppose that $\mathcal{T}_r$ has no limit cycle in $\mathfrak{Q}_1$, or is not homoclinic (in particular it happens when $N/2 \le \delta < N$, by Proposition 2.11), and stays in $\mathfrak{Q}_1 \cup \mathfrak{Q}_2$, thus Y stays positive. Then from Propositions 2.8, 2.9 and 2.15, either $\lim_{\tau\to-\infty} y = \infty$, $\lim_{\tau\to\infty} y_\alpha = L \neq 0$, $\lim_{\tau\to\infty} Y_\alpha = (\alpha L)^{p-1}$, or $S_w < \infty$. In any case, for any $d \in (\alpha, -p')$, the function $Y_d = e^{(d-\alpha)\tau} Y_\alpha$ satisfies $\lim_{\tau\to\ln S_w} Y_d = \infty = \lim_{\tau\to\infty} Y_d$. Then it has a minimum point, and this contradicts (2–15). Thus $\mathcal{T}_r$ enters $\mathfrak{Q}_3$. If it stays in it, it has a limit cycle; then $-\mathcal{T}_r$ has a limit cycle in $\mathfrak{Q}_1$. But $-\mathcal{T}_r$ does not meet $\mathcal{T}_r$, and M_ℓ is in the domain of $\mathfrak{Q}_1$ delimited by $\mathcal{T}_r$, since $\mathcal{T}_r$ meets $\mathcal{M}$ to the right of M_ℓ, by (2–16); this is impossible. Then $\mathcal{T}_r$ enters $\mathfrak{Q}_4$, and y has at least two zeros. □

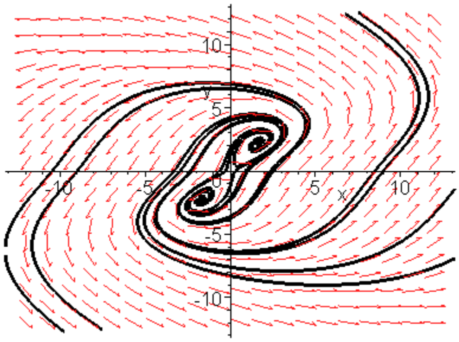
Theorem 6.8. *Assume $\varepsilon = -1$ and $\delta < N/2$, $\alpha < -p'$. Then $w(r) = \ell r^{-\delta}$ is still a solution.*

(i) *There exists a (minimal) critical value α^{crit} of α, such that*

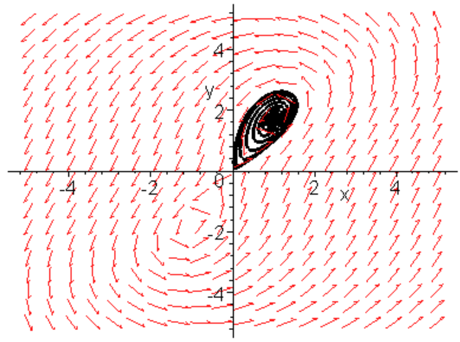
$$\alpha^* < \alpha^{\text{crit}} < \min(-p', \alpha_2) < 0,$$

and $\mathcal{T}_r$ is homoclinic: all regular solutions have constant sign and satisfy

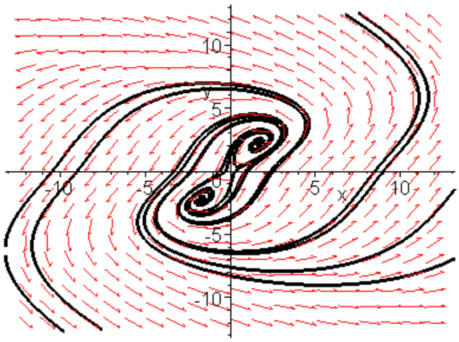
$$\lim_{r\to\infty} r^\eta w = c \neq 0.$$



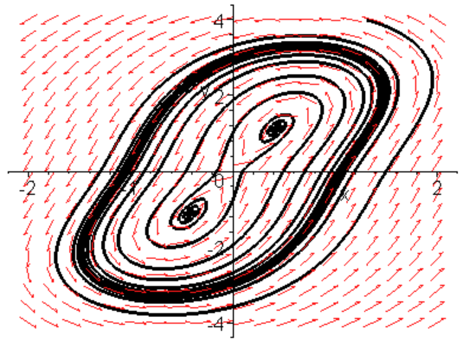
$\alpha = -5 < -p' = -3 < 0 < \delta$



$\alpha = -7.4 < 0 < \delta$



$\alpha^* = -9 < \alpha = -8 < \delta$



$\alpha = -13 < \alpha^* = -9 < \delta$

Figure 15. Theorem 6.8: $\varepsilon = -1,\ \delta = 3 < N/2 < N = 9$.

(ii) *For any $\alpha \in (\alpha^*, \alpha^{\text{crit}})$ there does exist a cycle in $\mathcal{Q}_1$; equivalently there exist solutions such that $r^\delta w$ is periodic in $\ln r$. All regular solutions have constant sign and $r^\delta w$ is asymptotically periodic in $\ln r$. There exist positive solutions such that $\lim_{r\to 0} r^\delta w = \ell$ and $r^\delta w$ is asymptotically periodic in $\ln r$.*

(iii) *For any $\alpha \le \alpha^*$, there does not exist such a cycle, regular solutions have constant sign, and $\lim_{r\to\infty} r^\delta |w| = \ell$.*

(iv) *For any $\alpha < \alpha^{\text{crit}}$, there exists also a cycle surrounding $(0,0)$ and $\pm M_\ell$, thus w is changing sign and $r^\delta w$ is periodic in $\ln r$. There exist solutions oscillating near 0, and $r^\delta w$ is asymptotically periodic in $\ln r$, and $\lim_{r\to\infty} r^\eta w = c \neq 0$. There exist solutions oscillating near 0, and $r^\delta w$ is asymptotically periodic in $\ln r$, and $S_w < \infty$ or $\lim_{r\to\infty} r^\alpha w = L \neq 0$.*

Proof. (i) For any $\alpha \in (\alpha_1, \alpha_2)$, such that $\alpha \le -p'$ we have three possibilities, by Remark 6.5:

- $\mathcal{T}_s$ converges to M_ℓ at $-\infty$, spiraling around this point, since α is a spiral point, or it has a limit cycle around M_ℓ. Then $\mathcal{T}_s$ meets the set $\mathcal{E} = \{(\ell, Y) :$

$Y > (\delta\ell)^{p-1}\}$ at a first point $(\ell, Y_s(\alpha))$; and $\mathcal{T}_r$ meets $\mathcal{E}$ at a last point $(\ell, Y_r(\alpha))$ such that $Y_r(\alpha) - Y_s(\alpha) > 0$. Moreover $\mathcal{T}_r$ enters $\mathcal{Q}_2$, by Proposition 2.8. See Figure 15, top left.

- $\mathcal{T}_s$ enters $\mathcal{Q}_4$; hence $\mathcal{T}_r$ converges to M_ℓ at ∞ and spirals around this point, or it has a limit cycle around M_ℓ. Then $\mathcal{T}_s$ meets $\mathcal{E}$ at a last point $(\ell, Y_s(\alpha))$, $\mathcal{T}_r$ meets $\mathcal{E}$ at a first point $(\ell, Y_r(\alpha))$ such that $Y_r(\alpha) - Y_s(\alpha) < 0$. See Figure 15, bottom row.
- $\mathcal{T}_r$ is homoclinic, or equivalently $Y_r(\alpha) - Y_s(\alpha) = 0$. See Figure 15, top right.

Now the function $\alpha \mapsto h(\alpha) = Y_r(\alpha) - Y_s(\alpha)$ is continuous. If $-p' < \alpha_2$, then $h(-p')$ is defined and $h(-p') > 0$, by Theorem 6.6. If $\alpha_2 \le -p'$, we observe that for $\alpha = \alpha_2$, by Theorem 2.18, $\mathcal{T}_r$ must leave $\mathcal{Q}_1$ (because α_2 is a source) and does so transversally; thus the same holds for $\alpha = \alpha_2 - \gamma$ if $\gamma > 0$ is small enough. Therefore $\mathcal{T}_s$ stays in $\mathcal{Q}_1$ by Remark 6.5, so $h(\alpha_2 - \gamma) > 0$. If $\alpha \le \alpha^*$, then M_ℓ is a sink or a weak sink, by Theorem 2.16; therefore $\mathcal{T}_s$ cannot converge to M_ℓ at $-\infty$. By Theorem 2.19, there are no cycles in $\mathcal{Q}_1$ and no homoclinic orbits. By Remark 6.5, $\mathcal{T}_s$ cannot stay in $\mathcal{Q}_1$; hence $\mathcal{T}_r$ stays in $\mathcal{Q}_1$ and is bounded and converges at ∞ to M_ℓ. Thus $h(\alpha) < 0$ for $\alpha_1 < \alpha \le \alpha^*$, so there exists at least an $\alpha^{\text{crit}} \in (\alpha^*, \min(-p', \alpha_2)$ such that $h(\alpha^{\text{crit}}) = 0$. If it is not unique, we choose the smallest one.

(ii) Let $\alpha > \alpha^*$. The existence and uniqueness of such a cycle in $\mathcal{Q}_1$ follows from Theorem 2.16 if $\alpha - \alpha^*$ is small enough (Figure 15, lower left). For any $\alpha \in (\alpha^*, \alpha^{\text{crit}})$, we still have existence: indeed, $h(\alpha) < 0$ on this interval, so $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and $\mathcal{T}_r$ cannot converge to M_ℓ at ∞, hence it has a limit cycle around M_ℓ at ∞. Since M_ℓ is a source, there also exist trajectories converging to M_ℓ at $-\infty$, with a limit cycle at ∞. And $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$, and it is bounded at $-\infty$. Thus it has a limit cycle at $-\infty$ surrounding $(0, 0)$ and $\pm M_\ell$.

(iii) Let $\alpha \le \alpha^*$ (Figure 15, lower right). Then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, is bounded on $\mathbb{R}$, and converges to M_ℓ at ∞, while $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$ as above. Thus $\mathcal{T}_s$ has a limit cycle at $-\infty$, containing the three stationary points.

(iv) For any $\alpha < \alpha^{\text{crit}}$ apart from $\mathcal{T}_r$ and the cycles, all trajectories have a limit cycle at $-\infty$ containing the three stationary points. By Theorem 2.21, all the cycles are contained in a ball B of $\mathbb{R}^2$. Take any point P exterior to B. By Remark 6.5, $\mathcal{T}_{[P]}$ has a limit cycle at $-\infty$ contained in B and cannot have a limit cycle at ∞. Therefore y has constant sign near $\ln S_w$. By Proposition 2.8, either $S_w < \infty$ or y is defined near ∞ and $\lim_{\tau\to\infty} \zeta = L$, $\lim_{r\to\infty} r^\alpha w = L$. $\square$

Finally we consider the case $N/2 \le \delta$, where no cycle can exist.

Theorem 6.9. *Assume $\varepsilon = -1$ and $\alpha < 0 < N/2 \leq \delta < N$. Then all solutions of (E_w) have a finite number of zeros, and $w(r) = \ell r^{-\delta}$ is a solution. If $-p' \leq \alpha$, Theorem 6.6 applies. If $\alpha < -p'$, there exist positive solutions such that $\lim_{r\to 0} r^\delta w = \ell$ and $\lim_{r\to\infty} r^\eta w = c > 0$. All regular solutions have the same number $m \geq 2$ of zeros. All other solutions satisfy $\lim_{r\to -\infty} r^\delta w = \pm\ell$, and have m or $m+1$ zeros; there exist solutions with $m+1$ zeros.*

Proof. By Proposition 2.11, all solutions have a finite number of zeros, and any solution is monotone near 0 and $\ln S_w$, or converges to $\pm M_\ell$. By Remark 6.5, apart from $\mathcal{T}_r$, all trajectories converge to $\pm M_\ell$ at $-\infty$. The functions V and W are nonincreasing. The trajectory $\mathcal{T}_s$ satisfies $\lim_{\tau\to\infty} V = \lim_{\tau\to\infty} W = 0$, so $V \geq 0$, $W \geq 0$. If y has a zero at some point τ, then $W(\tau) = -|Y(\tau)|^{p'}/p'$, which is impossible. If Y has a zero at some point θ, then $V(\theta) = -Y'(\theta)^2/2$, also a contradiction. Thus $\mathcal{T}_s$ stays in $\mathcal{Q}_1$. By Remark 6.5 and Proposition 2.11, $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, but enters $\mathcal{Q}_2$. By Lemma 6.7, $\mathcal{T}_r$ enters $\mathcal{Q}_4$, and y has at least two zeros. Let m be the number of its zeros. Then $\mathcal{T}_r$ cuts the axis $y = 0$ at points $(0, \xi_1), \ldots, (0, \xi_m)$. Consider any trajectory $\mathcal{T}_{[P]}$ with $P = (0, \xi)$, where $\xi > |\xi_i|$ for $1 \leq i \leq m$. It cannot intersect $\mathcal{T}_r$ or $-\mathcal{T}_r$, so y has $m+1$ zeros. Any trajectory has m or $m+1$ zeros, because it does not meet $\mathcal{T}_r$ or $-\mathcal{T}_r$ or $\mathcal{T}_{[P]}$. And $S_w < \infty$ or $\lim_{r\to\infty} r^\alpha w = L \neq 0$. □

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