Pacific Journal of Mathematics

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Volume 228 No. 1 November 2006

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

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A basic result of Milnor and Scharlau determines the Witt ring of rational function fields $Wk(x)$ whenever char $k \neq 2$. An analogous result is obtained here for the Witt group of quadratic forms $W_q\mathcal{F}(x)$, where $\mathcal{F}$ is a field of characteristic 2. This generalizes earlier work by the authors where $\mathcal{F}$ was assumed to be perfect.

Milnor's determination [1970] of the Witt ring of a rational function field in terms of the Witt rings of the finite extensions of the base field is a fundamental result in the algebraic theory of quadratic forms, and was complemented by Scharlau's reciprocity law (see [Lam 1973] or [Scharlau 1972]). Here we give an analogue of these results for the Witt group of rational function fields in characteristic 2, extending earlier work where the base field was assumed to be perfect [Aravire and Jacob 2004].

All our fields will have characteristic 2. We use the notation $\mathcal{F}$ for the base field of our rational function field $F = \mathcal{F}(x)$. Whenever $p \in \mathcal{F}[x]$ is monic and irreducible, we denote by $\mathcal{F}(x)_p$ the completion at the discrete valuation $v_p : \mathcal{F}(x) \to \mathbb{Z}$ determined by p. Similarly, we denote by $\mathcal{F}(x)_{\frac{1}{x}}$ the completion at the $\frac{1}{x}$-adic (or infinite) valuation $v_{\frac{1}{x}} : \mathcal{F}(x) \to \mathbb{Z}$. We use W_qF and WF to denote the Witt group and Witt ring of F, and we follow the standard notation. In particular, $[a, b]$ denotes the Witt class of quadratic form $ax^2 + xy + by^2$. These classes form an additive set of generators for W_qF, and $\langle a\rangle$ denotes the 1-dimensional symmetric bilinear form $(x, y) \mapsto axy$. The symbol $[\ ,\ ]$ is biadditive and W_qF is a WF-module via the action $\langle a\rangle[c, d] = [ac, a^{-1}c]$. This means that W_qF is also generated by the forms $\langle a\rangle[1, b]$ and when considering such an element we will refer to a as being in the *multiplicative slot* and b as being in the *additive slot*. We use the standard notation I^nF for the n-th power of the fundamental ideal in WF, so that I^nW_qF is generated by the forms $\langle\langle a_1, a_2, \dots, a_n\rangle\rangle[1, b]$. Arason [1979, Satz 8] gave a generator-relation description of W_qF as a WF-module, and we use these relations throughout. We frequently use what we call the *fundamental relation*,

MSC2000: 11E04.
Keywords: Witt group, rational function fields, characteristic two.
Both authors acknowledge support from Fondecyt.

$\langle a+b\rangle [1,c] = \langle a\rangle [1, ac/(a+b)] + \langle b\rangle [1, bc/(a+b)]$, which shows how addition in the multiplicative slot can be distributed across a sum of forms.

A key component of the classical Milnor–Scharlau sequence is the second residue homomorphism $\partial_p : WF \to W\overline{F}_p$, where $\overline{F}_p$ is the residue class field of a discrete valuation $v_p : F \to \mathbb{Z}$. This map has proved to be of considerable importance in quadratic form theory. For example, if X is a variety defined over F, the kernel under all $\partial_p : WF(X) \to W\overline{F}_p$ is the *unramified* Witt group of X, which when X is a nonsingular curve coincides with the Witt group of X. This paper gives the appropriate version of ∂_p in characteristic two, and in a subsequent paper this work is applied to the study of Witt groups of curves in characteristic two.

Whenever $v_p : F \to \mathbb{Z}$ is a discrete valuation, we set

$$W_1 F_p := \operatorname{coker}(W_q \overline{F}_p \to W_q F_p),$$

where the map is induced by a Teichmüller lifting $\overline{F}_p \hookrightarrow F_p$. We show in Corollary 1.7 that the group $W_1 F_p$ is independent of the choice of a Teichmüller lifting. For such a lifting we define the second residue map $\partial_p : W_q F \to W_1 F_p$ to be the composite map induced by inclusion and projection:

$$W_q F \to W_q F_p \to W_1 F_p.$$

We are also able to identify a quotient of $W_1 F_p$ with $W_q \overline{F}_p$ where, when $p \in \mathcal{F}[x]$ is an irreducible polynomial, we have $\overline{F}_p \cong \mathcal{F}[x]/(p)$. Using this we obtain a version of Scharlau's transfer $s_p^* : W_1 F_p \to W_q \mathcal{F}$ as a composite of maps $W_1 F_p \to W_q \mathcal{F}[x]/(p) \to W_q \mathcal{F}$, where the latter map is the same transfer used by Scharlau. Both maps ∂_p and s_p^* are analogous to the classical maps, but as they depend upon choices of Teichmüller liftings and of subgroups of $W_1 F_p$, these selections must be made to meet certain compatibility requirements for our main result to hold.

With this notation, the main result of this paper is the following.

Theorem 6.2 (Analogue of the Milnor–Scharlau Sequence). *Suppose that $\mathcal{F}$ is a field of characteristic 2 and $F = \mathcal{F}(x)$ is a rational function field in one variable over $\mathcal{F}$. There exists a compatible collection of second residue and transfer maps that fit into an exact sequence*

$$0 \longrightarrow W_q \mathcal{F} \longrightarrow W_q F \xrightarrow{\bigoplus \partial_p} \bigoplus_{p, \frac{1}{x}} W_1 F_p \xrightarrow{\bigoplus s_p^*} W_q \mathcal{F} \longrightarrow 0,$$

where the direct sum is taken over discrete valuations on F.

We now provide an overview of the proof. As we do this we will recall the main features of the proof in the classical case in order to illustrate the similarities and differences. When char $F \neq 2$ and F is complete with respect to a discrete valuation $v : F \to \mathbb{Z}$, a well-known result of Springer shows that $WF \cong W\overline{F} \oplus$

$\langle\pi\rangle W\bar{F}$, where π is a uniformizing parameter for v. This decomposition enables one to construct the second residue map and the transfer in the Milnor–Scharlau sequence. When char $F = 2$ and F is complete it is first necessary for us to compute the Witt group $W_q F$. This is the main objective of Section 1. The main result proved there, Theorem 1.3, shows that $W_q F \cong W_q \bar{F} \oplus \mathcal{R} \oplus \langle\pi\rangle W_q \bar{F}$, where again π is a uniformizing parameter. The subgroup $\mathcal{R}$ is quite large and although its description depends upon choosing a 2-basis for F and a Teichmüller lifting $\bar{F} \hookrightarrow F$ it has adequate uniqueness properties. (When $\mathcal{F}$ is perfect, then there is a unique Teichmüller lifting, however in general, such lifts depend upon the choice of a 2-basis for $\mathcal{F}$. See [Schilling 1950, p. 236] for details.) This decomposition shows that $W_1 F \cong \mathcal{R} \oplus \langle\pi\rangle W_q \bar{F}$ and enables us to define both the second residue and Scharlau transfer maps needed for the main theorem.

After defining the second residue maps, Milnor's proof requires a filtration $L_0 \subset L_1 \subset L_2 \subset \cdots \subset WF$, where by $L_d \subset WF$ he considered the subgroup generated by all $\langle f\rangle$, where f is a polynomial of degree at most d. He then proves a key result, namely that the successive quotients L_d/L_{d-1} for $d \geq 1$ are isomorphic to the direct sum of groups $\bigoplus_{\deg p = d} W\bar{F}_p$. To do this he shows there is a well defined splitting of the sum of induced maps $\bigoplus_{\deg p=d} \partial_p : L_d/L_{d-1} \to \bigoplus_{\deg p=d} W\bar{F}_p$. In Section 2 we use the same idea and notation, except that our L_d are generated by the forms $\langle f\rangle[1, h/u^e]$, where now both f and u have degree at most d in $\mathcal{F}[x]$ and $h \in \mathcal{F}[x]$ is arbitrary. These forms are needed for two reasons. First $W_q F$ has as generators 2-dimensional forms, and second, the quotients h/u^e are needed to take into account all the extra stuff in $\mathcal{R}$. In the following section, Theorem 3.5 gives the exact analogue of Milnor's key result, namely that the map

$$\bigoplus_{\deg p=d} \partial_p : L_d/L_{d-1} \to \bigoplus_{\deg p=d} W_1 \bar{F}_p$$

is a split isomorphism.

To prove the latter result we must take several detours. First there is the complexity introduced by the existence of different ways to extend a 2-basis for $\mathcal{F}$ to a 2-basis for F and F_p. If p is separable, one can either add x or p, with x the natural choice for the rational function field F and with p the natural choice for F_p. When p is not separable, we have to specify which element we choose to omit from the 2-basis of $\mathcal{F}$ and then we must add both x and p to form a 2-basis for F_p. Since the ∂_p relate $W_q F$ to $W_1 F_p$, we need to be able to relate these choices. The bulk of Section 2 accomplishes this, by establishing the equivalence of different generating sets for the L_d in Lemma 2.5 and Proposition 2.8.

A second detour provides a generator-relation description of $W_q F$ (Theorem 3.3) needed to prove that the splitting maps are well defined (Lemma and Definition 3.4). The proof of the splitting is similar to that of the classical case, but is complicated again by the fact that 2-bases and Teichmüller liftings have to be selected carefully and in a compatible fashion. The details of these choices are set up in the discussion that follows Lemma 3.1. Finally one has to deal with the structure of L_0, which is just $W\mathcal{F}$ in the classical case. In our case it is generated by forms with polynomials $f, g \in \mathcal{F}[x]$ in the additive slots of binary forms, $[f, g]$. The result in Theorem 3.6 is that L_0 is described by an exact sequence $0 \to W_q\mathcal{F} \to L_0 \to W_1 F_{\frac{1}{x}} \to W_q\mathcal{F} \to 0$.

When Theorems 3.5 and 3.6 are combined with the definitions, we obtain a version of what Milnor did, namely that the sequence in Theorem 6.2 is exact if truncated to

$$0 \longrightarrow W_q\mathcal{F} \longrightarrow W_q F \xrightarrow{\bigoplus \partial_p} \bigoplus_{p,\text{finite}} W_1 F_p \oplus (W_1 F_{\frac{1}{x}}/\langle x\rangle W\mathcal{F}) \longrightarrow 0,$$

where the reciprocity law provided by the transfer is omitted. However, because the reciprocity law has important applications, we continue with its development in subsequent sections. Section 4 is devoted to defining the transfer maps. The subgroups of $W_1 F_p$ needed to define the maps are given in Definition 4.1 and are selected in a compatible way to ensure that the resulting s_p^* vanish on the subgroup $\mathcal{R} \subset W_1 F_p$. The definition of s_p^* when p fails to be separable, Definition 4.3(ii), is adjusted to take into account the change in the 2-basis resulting from the failure of the 2-basis of $\mathcal{F}$ to extend to F_p. In this case the exact terms necessary to make the reciprocity law work are added to the transfer of the residue form.

Having defined the s_p^*, we check the reciprocity law for elements of $L_0 + \langle p\rangle L_0$ (Theorem 5.4). This requires computing the ordinary transfer $t_p^* : W_q\mathcal{F}[x]/(p) \to W_q\mathcal{F}$ on generators $[\lambda_1 x^i, \lambda_2 x^j]$ of $W_q\mathcal{F}[x]/(p)$. There are quite a few cases to consider, but it is a straightforward computation. With this result, the main theorem, with the reciprocity law in general, is proved in Section 6, where the final stages of the proof consist of checking that the definitions involved in setting up $W_1 F_p$ and the s_p^* are arranged properly to ensure cancellation of the appropriate terms. Although the definition of s_p^* is based on the same linear functional as in the classical case, this portion of the paper differs from the approach in that case. Because of the additive nature of generators for $W_q F$ we are able to reduce to forms that vanish on all but two ∂_p's, and therefore we don't have to consider more complex transfers from algebras such as $\mathcal{F}[x]/(p_1 p_2 \cdots p_n)$, as did Scharlau.

1. Local information

If F has characteristic 2, a collection of elements $t_1, t_2, \ldots, t_n \in F$ is said to form a 2-*basis* of F if we have a strictly increasing sequence of subfields

$$F^2 \subsetneq F^2(t_1) \subsetneq F^2(t_1, t_2) \subsetneq \cdots \subsetneq F^2(t_1, t_2, \ldots, t_n) = F.$$

A field F can have many different 2-bases; if $[F : F^2] = 2^n$, every 2-basis has exactly n elements. We will assume that fields in this paper have finite 2-bases, since our main results are readily reduced to this case.

For fixed n we denote by T the set of n-tuples $I = (i_1, i_2, \dots, i_n)$, where $i_j \in \{0, 1\}$ for all j. We order T lexicographically, with minimal element $O := (0, 0, \dots, 0)$, then $(1, 0, \dots, 0)$, then $(0, 1, 0, \dots, 0)$, and so forth. It will be convenient to add elements of T as in the $\mathbb{Z}/2\mathbb{Z}$-vector space $(\mathbb{Z}/2\mathbb{Z})^n$ and let T_0 denote the nonzero elements of T. Whenever $t_1, t_2, \dots, t_n \in F$ and $I \in T$, we abbreviate $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$ by t^I. In this notation, whenever $t_1, t_2, \dots, t_n$ form a 2-basis for F and $f \in F$, there exist *unique* elements $x_I \in F$ indexed by $I \in T$ such that

$$f = \sum_{I \in T} t^I x_I^2.$$

For the remainder of this section we assume that $v : F \to \mathbb{Z}$ is a complete discrete-valued field of characteristic 2 with residue class field $\bar{F}$ and uniformizing parameter π. We assume that $t_1, t_2, \dots, t_{n-1}$ are units in F whose residues form a 2-basis for $\bar{F}$. Since v is complete and discrete, we know that $t_1, t_2, \dots, t_{n-1}, \pi$ form a 2-basis for F. We will use the notation t^I, where $I = (I_1, I_2, \dots, I_n) \in T$, to represent elements in this 2-basis:

$$t^I = t_1^{I_1} t_2^{I_2} \cdots t_{n-1}^{I_{n-1}} \pi^{I_n}.$$

From [Schilling 1950, pp. 230–238] we also know that there is a unique monomorphism of fields $\rho : \bar{F} \hookrightarrow F$ with $\rho(\bar{t}_i) = t_i$ for $i = 1, 2, \dots, n-1$. Since we will regard this map as an inclusion, we will drop the residue bars from the t_i and view $t_1, t_2, \dots, t_{n-1}$ as lying in $\bar{F} \subset F$. With these conventions, $v(t^J) = J_n \in \{0, 1\}$ for all $J \in T$.

Since F is complete, we can view $F = \bar{F}((\pi))$ as formal Laurent series in its fixed uniformizing parameter π. We let $R := \bar{F}[\pi^{-1}] \subset F$ be the "backwards" polynomial ring, and with this, if $f \in F$ then there exists a unique $r \in \pi^{-1} R$ with $v(f - r) \geq 0$. Moreover, every element $r \in \pi^{-1} R$ can be uniquely expressed as $r = \sum_{I \in T} t^I r_I^2$, where $r_I \in R$.

Definition 1.1. We set $\mathcal{R}$ to be the subgroup of $W_q F$ of all forms

$$\sum_{I \in T} \langle t^I \rangle \Big[1, \sum_{J,\, J+I > I} t^J r_{I,J}^2 \Big] \in \mathcal{R} \qquad \text{with } r_{I,J} \in \pi^{-1} R.$$

Whenever $v(a) > 0$ (that is, $a \in \bar{F}((\pi))$), we must have $a \in \wp(F)$ since F is complete with respect to v; consequently, $[1, a] = 0 \in W_q F$. We will use this fact repeatedly. When $r \in \pi^{-1} R$, however, we are in the opposite situation, since then $v(r) < 0$; in this case, if $v(r)$ is odd or the lead coefficient is not a square in $\mathcal{F}$, then r cannot lie in $\wp(F)$. This is why the module $\mathcal{R}$ is of interest.

Lemma 1.2. *Every element $\phi \in W_q F$ can be expressed as $\phi = \sum_{I \in T} \langle t^I \rangle [1, A_I]$, where $A_I = \sum_{J, J+I > I} t^J r_{I,J}^2$ and $r_{I,J} \in F$.*

Proof. Applying the W_q-relations we know that every element of $W_q F$ is a sum of elements $\langle t^I \rangle [1, t^J b^2]$ for $b \in F$. Applying the W_q-relation $\langle t \rangle [1, a] = \langle ta \rangle [1, a]$ we find for I, J with $I + J < I$ that $\langle t^I \rangle [1, t^J b^2] = \langle t^{I+J} \rangle [1, t^J b^2]$, and when $I = J$ we find that $\langle t^I \rangle [1, t^I b^2] = \langle 1 \rangle [1, t^I b^2]$. Hence every element of $\phi \in W_q F$ can be expressed as $\phi = \sum_{I \in T} \langle t^I \rangle \big[1, \sum_{J, J+I > I} t^J b_{I,J}^2\big]$, with $b_{I,J} \in F$. (In fact this much is true for any 2-basis of any field F of characteristic 2.) The statement for $\mathcal{R}$ follows applying this same argument to that case. □

Theorem 1.3. *Suppose that $v : F \to \mathbb{Z}$ is a complete discrete valued field of characteristic 2 with residue field $\overline{F} \subset F$ and uniformizing parameter π. Then every class $\phi \in W_q F$ can be expressed uniquely as*

$$\phi = \phi_1 \perp \psi \perp \langle \pi \rangle \phi_2,$$

where $\phi_1, \phi_2 \in W_q \overline{F}$ and $\psi = \sum_{I \in T} \langle t^I \rangle [1, \sum_{J, J+I > I} t^J r_{I,J}^2] \in \mathcal{R}$ with $r_{I,J} \in \pi^{-1} R$. The classes of ϕ_1, ϕ_2 and the $r_{I,J}$ are uniquely determined by ϕ. In particular, there is a split exact sequence

$$0 \to W_q \overline{F} \to W_q F \to \big(\mathcal{R} \oplus \langle \pi \rangle W_q \overline{F}\big) \to 0.$$

Proof. Consider $\sum_{I \in T} \langle t^I \rangle [1, \sum_{J, J+I > I} t^J b_{I,J}^2] \in W_q F$. Since F is complete and discretely valued, we can express each $b_{I,J}$ as $r_{I,J} + \overline{f}_{I,J} + b'_{I,J}$, where $r_{I,J} \in \pi^{-1} R$, $\overline{f}_{I,J} \in \overline{F} \subset F$ and $v(b'_{I,J}) > 0$. Since $v(t^J) \geq 0$ we have $t^J {b'_{I,J}}^2 \in \wp(F)$ and hence $[1, t^J {b'_{I,J}}^2] = 0$. We observe:

• If $J_n = 0$ then $t^J \in \overline{F}$ and we have $t^J r_{I,J}^2 \in \pi^{-1} R$ while $t^J \overline{f}_{I,J}^2 \in \overline{F} \subset F$. When $I_n = 0$ we have $\langle t^I \rangle [1, t^J \overline{f}_{I,J}^2] \in W_q \overline{F}$, and when $I_n = 1$ we have $\langle t^I \rangle [1, t^J \overline{f}_{I,J}^2] \in \langle \pi \rangle W_q \overline{F}$. So

$$\langle t^I \rangle [1, t^J b_{I,J}^2] = \langle t^I \rangle [1, t^J r_{I,J}^2] + \langle t^I \rangle [1, t^J \overline{f}_{I,J}^2] \in W_q \overline{F} + \mathcal{R} + \langle \pi \rangle W_q \overline{F}$$

in this case.

• If $J_n = 1$, then we know $v(t^J) = 1$ and consequently $t^J r_{I,J}^2 \in \pi^{-1} R$ while $v(t^J \overline{f}_{I,J}^2) > 0$, so $[1, t^J \overline{f}_{I,J}^2] = 0$. Therefore, $\langle t^I \rangle [1, t^J b_{I,J}^2] = \langle t^I \rangle [1, t^J r_{I,J}^2] \in \mathcal{R}$ in this case.

Altogether this shows that every element of $\phi \in W_q F$ can be expressed as $\phi = \phi_1 + \psi + \langle \pi \rangle \phi_2$, with $\phi_1, \phi_2 \in W_q \overline{F}$ and $\psi \in \mathcal{R}$.

To prove the uniqueness assertions and the exactness of the sequence we need a bit more notation. We denote by n^s the set of all subsets of $\{1, 2, \ldots, n\}$ containing s elements. For any $I = (I_1, I_2, \ldots, I_n) \in T$ we use $\tilde{I}$ to denote $\{j \mid I_j \neq 0\}$. Note that $\tilde{I} \in n^s$ for some $s = 0, 1, \dots, n$, and that any subset $\tilde{I} \in n^s$ is determined by a unique $I \in T$. For any subset $S \subseteq \{1, 2, \dots, n\}$ we write

$$\langle\langle t_S \rangle\rangle = \bigotimes_{i \in S} \langle\langle t_i \rangle\rangle,$$

where, when $S = \varnothing$, $\langle\langle t_\varnothing \rangle\rangle = \langle 1 \rangle$ by convention. Whenever $I \in T$, we define $\ell(I) = \max(\tilde{I})$. Finally, we define $[\tilde{I}]_0 := \{J \in T \mid J \neq (0, 0, \dots, 0) \text{ and } \tilde{J} \subseteq \tilde{I}\}$ and $[\tilde{I}^c] := \{J \in T \mid \tilde{J} \cap \tilde{I} = \varnothing\}$.

Lemma 1.4 [Aravire and Jacob 1996, Lemma 1.6]. *Suppose that $t_1, t_2, \dots, t_n$ are 2-independent in a field F and let $a_{\tilde{I}} \in F$. Suppose*

$$q = \sum_{\tilde{I} \in n^s} \langle\langle t_{\tilde{I}} \rangle\rangle [1, a_{\tilde{I}}] \in I^{s+1} W_q F.$$

Then each

$$a_{\tilde{I}} \in \Big(\wp(F) + \sum_{J \in [\tilde{I}]_0 + [\tilde{I}^c]} t^J F^2 \Big) = \Big(\wp(F) + \sum_{J, \tilde{J} \cap \tilde{I} \neq \varnothing} t^J F^2 \Big) \qquad \square$$

The next result is a modification of [Aravire and Jacob 1996, Proposition 1.7].

Proposition 1.5. *Suppose that $v : F \to \mathbb{Z}$ is a complete discrete valued field and $t_1, t_2, \dots, t_n$ are as above. Suppose*

$$q = \sum_{\tilde{I} \in n^s} \langle\langle t_{\tilde{I}} \rangle\rangle [1, a_{\tilde{I}}] \in I^{s+1} W_q F,$$

where $a_{\tilde{I}} \in \sum_{J+I>I} t^J (\pi^{-1} R)^2$. Then $a_{\tilde{I}} = 0$ for each I. $\square$

Proof. Assume the contrary. Let M be the maximal index among the I with $\tilde{I} \in n^s$ and $a_{\tilde{I}} \neq 0$. We express $a_{\widetilde{M}}$ as a sum $\sum_{K+M>M} t^K A_{K,M}^2$, where each $A_{K,M} \in \pi^{-1} R$. Since $a_{\widetilde{M}} \neq 0$, there is some J with $J + M > M$ and $t^J A_{J,M}^2 \neq 0$. The result will be proved when we derive the contradiction that $t^J A_{J,M}^2 = 0$.

Since $J + M > M$ we have $\ell(J) \notin \widetilde{M}$. We denote by $t_1', t_2', \dots, t_n'$ the 2-basis obtained from $t_1, t_2, \dots, t_n$ by replacing $t_{\ell(J)}$ by t^J. Then, since $\ell(J) \notin \widetilde{M}$, we have $\langle\langle t_{\widetilde{M}} \rangle\rangle = \langle\langle t'_{\widetilde{M}} \rangle\rangle$. Also, we have

$$\langle\langle t'_{\ell(J)} \rangle\rangle = \langle\langle t^J \rangle\rangle \equiv \sum_{j \in \tilde{J}} \langle\langle t^j \rangle\rangle \pmod{I^2 F}.$$

We now suppose that $K \in T$, $\tilde{K} \in n^s$, and $\ell(J) \in \tilde{K}$. We express $\tilde{K}$ as $\{\ell(J)\} \cup \tilde{Q}$ for $\tilde{Q} \in n^{s-1}$. Computing in WF modulo $I^{s+1}F$ we have

$$\begin{aligned}\langle\!\langle t'_{\tilde{K}} \rangle\!\rangle &= \langle\!\langle t'_{\tilde{Q}} \rangle\!\rangle \langle\!\langle t'_{\ell(J)} \rangle\!\rangle = \langle\!\langle t_{\tilde{Q}} \rangle\!\rangle \langle\!\langle t^J \rangle\!\rangle \equiv \sum_{j \in \tilde{J}} \langle\!\langle t_{\tilde{Q}} \rangle\!\rangle \langle\!\langle t_j \rangle\!\rangle \\ &\equiv \langle\!\langle t_{\tilde{K}} \rangle\!\rangle + \sum_{\substack{j \in \tilde{J} \\ j \neq \ell(J)}} \langle\!\langle t_{\tilde{Q}} \rangle\!\rangle \langle\!\langle t_j \rangle\!\rangle \equiv \langle\!\langle t_{\tilde{K}} \rangle\!\rangle + \sum_{\substack{j \in \tilde{J}, j \neq \ell(J) \\ j \notin \tilde{Q}}} \langle\!\langle t_{\tilde{Q}} \rangle\!\rangle \langle\!\langle t_j \rangle\!\rangle .\end{aligned}$$

The conditions $j \in \tilde{J}$, $j \neq \ell(J)$, and $j \notin \tilde{Q}$, are equivalent to the single condition $j \in \tilde{J} - \tilde{K}$. So, as each such $j < \ell(J)$ we find $\langle\!\langle t_{\tilde{Q}} \rangle\!\rangle \langle\!\langle t_j \rangle\!\rangle = \langle\!\langle t'_{\tilde{Q}} \rangle\!\rangle \langle\!\langle t'_j \rangle\!\rangle = \langle\!\langle t'_{\tilde{L}} \rangle\!\rangle$ for some $L \in T$ with $L < K$. Altogether this shows that whenever $K \in T$, $\tilde{K} \in n^s$, and $\ell(J) \in \tilde{K}$,

$$\langle\!\langle t_{\tilde{K}} \rangle\!\rangle \in \langle\!\langle t'_{\tilde{K}} \rangle\!\rangle + \Bigg(\sum_{\substack{L \in T \\ \tilde{L} \in n^s,\, L < K}} \langle\!\langle t'_{\tilde{L}} \rangle\!\rangle WF + I^{s+1}F \Bigg). \tag{1}$$

Expanding using (1) we can rewrite q in terms of the new 2-basis involving the t'. We find

$$\begin{aligned} q = \sum_{\tilde{I} \in n^s} \langle\!\langle t_{\tilde{I}} \rangle\!\rangle [1, a_{\tilde{I}}] &= \Bigg(\sum_{\substack{K < M \\ \tilde{K} \in n^s}} \langle\!\langle t_{\tilde{K}} \rangle\!\rangle [1, a_{\tilde{K}}] \Bigg) + \langle\!\langle t_{\widetilde{M}} \rangle\!\rangle [1, a_{\widetilde{M}}] \\ &\equiv \Bigg(\sum_{\substack{K < M \\ \tilde{K} \in n^s}} \langle\!\langle t'_{\tilde{K}} \rangle\!\rangle [1, a'_{\tilde{K}}] \Bigg) + \langle\!\langle t'_{\widetilde{M}} \rangle\!\rangle [1, a_{\widetilde{M}}] \pmod{I^{s+1} W_q F}, \end{aligned}$$

where the $a'_{\tilde{K}}$ for $K < M$ are the elements of F that arise in the expansion using (1) repeatedly. Observe that $a_{\widetilde{M}}$ remains unchanged when passing to the 2-basis using the t'. We now apply Lemma 1.2, where the 2-basis used is the one with the t'. We find that

$$a_{\widetilde{M}} = \sum_{K+M>M} t^K A^2_{K,M} \in \wp(F) + \sum_{L, \tilde{L} \cap \widetilde{M} \neq \varnothing} t'^L F^2. \tag{2}$$

When constructing the 2-basis involving t' we replaced $t_{\ell(J)}$ by t^J, which means that $t^J = t'^{J'}$, where $\tilde{J}' = \{\ell(J)\} \in n^1$. Since $\ell(J) \notin \widetilde{M}$, this gives $\tilde{J}' \cap \widetilde{M} = \varnothing$. Therefore, moving all the other terms on the left side of (2) to the right we find

$$t^J A^2_{J,M} \in \wp(F) + \sum_{I \in T_0, I \neq J} t^I F^2.$$

We claim that this gives $A_{J,M} = 0$. As $t^J A^2_{J,M} \in \pi^{-1} R$, if $t^J A^2_{J,M} \neq 0$ we must have $v(t^J A^2_{J,M}) = s < 0$, where s is even if $J_n = 0$ and is odd if $J_n = 1$. For $K \in T$

let K' be such that $K'_n=0$ and $K'_i=K_i$ for $1\le i<n$. With this notation,

$$\overline{\pi^{-s}t^J A^2_{J,M}}\in t^{J'}\overline{F}^2.$$

Next, if $w=\wp(b)+\sum_{I\in T_0,I\neq J}t^I b^2_{I,J}\in\wp(F)+\sum_{I\in T_0,I\neq J}t^I F^2$ is such that $v(b)=s<0$, then

$$\overline{pi^{-s}w}\in\begin{cases}\overline{F}^2+\sum\limits_{\substack{I\in T_0,I\neq J\\ I_n=0}}t^I\overline{F}^2 & \text{when } s \text{ is even,}\\ \sum\limits_{\substack{I'\in T_0,I\neq J\\ I_n=1}}t^{I'}\overline{F}^2 & \text{when } s \text{ is odd.}\end{cases}$$

In either case, because $t_1, t_2, \ldots, t_{n-1}$ is a 2-basis for $\overline{F}$, we cannot have

$$\overline{\pi^{-s}t^J A^2_{J,M}}=\overline{\pi^{-s}w},$$

contrary to the assumption that $A_{J,M}\neq 0$. This proves the proposition. □

We may now complete the proof of Theorem 1.3. The main task is showing that if $\psi=\sum_{I\in T}\langle t^I\rangle\bigl[1,\sum_{J,J+I>I}t^J r^2_{I,J}\bigr]=0\in\mathcal{R}$ with $r_{I,J}\in\pi^{-1}R$, then each $r_{I,J}=0$. Assuming this temporarily for all such complete discrete valued fields, to prove the uniqueness statements we consider an expression $\phi=\phi_1\perp\psi\perp\langle\pi\rangle\phi_2=0$. Let L be a separable finite unramified extension of F chosen so that $(\phi_1)_L=0$ and $(\phi_2)_L=0$. Then L is still complete and discretely valued, the 2-basis is unchanged, $R\subset R_L$, and we have that $\phi_L=\psi_L$. So our temporary assumption applies to $\psi_L=0\in\mathcal{R}_L$, the $r_{I,J}$ vanish in this case, and we now have $\phi_1\perp\langle\pi\rangle\phi_2=0$. Now, by valuation theory, if both ϕ_1 and ϕ_2 are anisotropic over $\overline{F}$, then $\phi_1\perp\langle\pi\rangle\phi_2$ anisotropic as well, since π is a uniformizing parameter. So this gives $\phi_1=\phi_2=0$ and the uniqueness assertion follows.

Thus we are reduced to studying $\psi=\sum_{I\in T}\langle t^I\rangle\bigl[1,\sum_{J,J+I>I}t^J r^2_{I,J}\bigr]=0\in\mathcal{R}$, where we want to show that each $r_{I,J}=0$.

Lemma 1.6 [Aravire and Jacob 1996, Lemma 1.5]. *Suppose that $t_1, t_2, \ldots, t_n$ are 2-independent in a field F and $f\in F$. Then $\langle\langle t_1, t_2, \ldots, t_n\rangle\rangle[1, f]=0\in W_qF$ if and only if*

$$f\in\wp(F)+\sum_{J\in T_0}t^J F^2.$$

Applying the identity in WF (symmetric bilinear forms)

$$\langle\langle xy\rangle\rangle=\langle\langle x\rangle\rangle+\langle\langle y\rangle\rangle+\langle\langle x, y\rangle\rangle,$$

we obtain

$$\langle t^I\rangle=\sum_{K,\,\tilde{K}\subseteq\tilde{I}}\langle\langle t_{\tilde{K}}\rangle\rangle$$

(recall that $\langle\!\langle t_{\varnothing}\rangle\!\rangle = \langle 1\rangle$.) Abbreviating $a_I := \sum_{J, J+I>I} t^J r_{I,J}^2$ and rewriting ψ using this identity we obtain

$$\begin{aligned}0 = \psi = \sum_{I\in T}\langle t^I\rangle\,[1, a_I] &= \sum_{I\in T}\Big(\sum_{K,\,\tilde{K}\subseteq\tilde{I}}\langle\!\langle t_{\tilde{K}}\rangle\!\rangle\Big)[1, a_I]\\ &= \sum_{K\in T}\langle\!\langle t_{\tilde{K}}\rangle\!\rangle\Big[1, \sum_{I,\,\tilde{K}\subseteq\tilde{I}} a_I\Big] = \sum_{s=0,\dots,n}\Big(\sum_{\tilde{K}\in n^s}\langle\!\langle t_{\tilde{K}}\rangle\!\rangle\Big[1, \sum_{I,\,\tilde{K}\subseteq\tilde{I}} a_I\Big]\Big).\end{aligned}$$

By induction on s we shall show that

$$\sum_{I,\,\tilde{K}\subseteq\tilde{I}} a_I = 0$$

whenever $K \in n^s$. When $s = 0$, $\tilde{K} = \varnothing$ and since $\sum_{I\in T} a_I$ is the Arf invariant of q we find $\sum_{I\in T} a_I \in \wp(F)$. Since nonzero elements of $\pi^{-1}R$ have negative value, by valuation theory we find $\sum_{I\in T_0} t^I(\pi^{-1}R)^2 \cap \wp(F) = \{0\}$ and we are done if $s = 0$. Assuming the result for $1, 2, \dots, s-1$, we have

$$\sum_{\tilde{K}\in n^s}\langle\!\langle t_{\tilde{K}}\rangle\!\rangle\Big[1, \sum_{I,\,\tilde{K}\subseteq\tilde{I}} a_I\Big] \in I^{s+1}W_q F.$$

We observe that if $\tilde{K} \subseteq \tilde{I}$ and $J + I > I$, then $J + K > K$. Therefore Proposition 1.5 applies and we conclude for fixed $\tilde{K} \in n^s$ that

$$\sum_{I,\,\tilde{K}\subseteq\tilde{I}} a_I = 0.$$

Using $a_I = \sum_{I+J>I} t^J r_{I,J}^2$ we obtain

$$0 = \sum_{I,\,\tilde{K}\subseteq\tilde{I}} a_I = \sum_{I,\,\tilde{K}\subseteq\tilde{I}}\Big(\sum_{I+J>I} t^J r_{I,J}^2\Big) = \sum_{J\in T_0} t^J\Big(\sum_{\substack{I,\,\tilde{K}\subseteq\tilde{I}\\ I+J>I}} r_{I,J}^2\Big).$$

Since $t_1, t_2, \dots, t_n$ form a 2-basis of F, for fixed K, J we find

$$\sum_{\substack{I,\,\tilde{K}\subseteq\tilde{I}\\ I+J>I}} r_{I,J}^2 = 0. \tag{3}$$

We next show that $r_{I,J}^2 = 0$ for all I, J such that $I + J > I$. We proceed by reverse induction on card$(\tilde{I})$. If card$(\tilde{I}) = n$ we have $I = (1, \dots, 1)$ and $I + J > J$ is impossible, so the conclusion is vacuous. Now suppose the desired conclusion is known for all I with card$(\tilde{I}) > r$. Fix some K with card$(\tilde{K}) = r$ and some J with $J + K > K$. If $I \neq K$, and if $\tilde{K} \subset \tilde{I}$, we have card$(\tilde{I}) > r$. Our inductive hypothesis implies that $r^2_{I,J} = 0$ for these I, J, and since these are all but one summand of (3), we find that $r^2_{K,J} = 0$ as well. This completes the induction. The definitions give $a_I = 0$ and the proof of Theorem 1.3 is complete.

Since the ring $R = \bar{F}[\pi^{-1}] \subset F$ is simply a polynomial ring over the residue field $\bar{F}$ of F, the choice of the lift to F of the 2-basis $t_1, t_2, \ldots, t_{n-1}$ of $\bar{F}$ does not affect the isomorphism type of R or $\mathcal{R}$. This, together with the uniqueness results, implies:

Corollary 1.7. *Suppose that $v : F \to \mathbb{Z}$ is a complete discrete valued field of characteristic 2 with residue field $\bar{F}$ and uniformizing parameter π. Then up to isomorphism, the submodule $\mathcal{R}$ is independent of the choice of lift of the 2-basis $t_1, t_2, \ldots, t_{n-1}$ of $\mathcal{F}$. In particular, the cokernel*

$$W_1 F := \operatorname{coker}(W_q \bar{F} \to W_q F)$$

is independent of the choice of lift of this 2-basis.

Remark 1.8. Both residue forms ϕ_1 and ϕ_2 in Theorem 1.3 depend upon the choice of the uniformizing parameter π.

When F is complete and discretely valued, the group $W_1 F$ defined in Corollary 1.7 will play the role of the "second residue forms" in characteristic 2. The projection map $\partial_v : W_q F \to W_1 F$, is the analogue of the *second residue map*. It is an immediate consequence of this definition that

$$0 \longrightarrow W_q \bar{F} \longrightarrow W_q F \xrightarrow{\partial_v} W_1 F \longrightarrow 0$$

is split exact. (This definition also coincides with the second residue map away from characteristic 2, for in that case Springer's Theorem gives a group isomorphism $WF \cong W\bar{F} \oplus \langle p \rangle W\bar{F}$, so $W_1 F \cong W\bar{F}$.)

Remark 1.9. Arason [2003] has proved a result that captures all the information in Theorem 1.3. His proof uses the generator-relation structure of the Witt group. His description of $\mathcal{R}$ is different (it uses a filtration based on negative exponents of the uniformizing parameter) and his proof does not require powers of the fundamental ideal since he directly uses the generator relation structure for the Witt group.

2. The filtration of $W_q \mathcal{F}(x)$

We now denote by $\mathcal{F}$ a fixed field of characteristic 2 with 2-basis $t_1, t_2, \ldots, t_n$. We study the Witt group of the field of rational functions $F = \mathcal{F}(x)$. The results of the previous section will be applied to the completions of F at its discrete valuations, which are trivial on $\mathcal{F}$. Following Milnor's original approach away from characteristic 2, we also filter the Witt group $W_q F$ by degree. We denote by $\mathcal{F}[x]_{\leq d}$ the set of polynomials in $\mathscr{F}[x]$ of degree at most d, and by $\mathscr{F}[x]_{<d}$ those of degree less than d.

Definition 2.1. For $d \geq 1$, let L_d be the subgroup of $W_q\mathscr{F}(x)$ generated by all forms $\langle f\rangle[1, h/u^e]$, where $f, u \in \mathscr{F}[x]_{\leq d}$ and $h \in \mathscr{F}[x]$. When $d = 0$, let L_0 be the subgroup of $W_q\mathscr{F}(x)$ generated by the forms $[\lambda_1 x^i, \lambda_2 x^j]$, where $\lambda_1, \lambda_2 \in \mathscr{F}$ and $i, j \in \mathbb{N}$.

Lemma 2.2. (i) *For any polynomials* $p, g, h \in \mathscr{F}[x]$ *we have* $\langle p\rangle[g, ph] \in L_0$.

(ii) *For* $d \geq 1$, L_d *is generated by the forms* $\langle ax^\epsilon\rangle[1, h/u^e]$, *where* $a \in \mathscr{F}$, $\epsilon \in \{0, 1\}$, $h \in \mathscr{F}[x]$ *and* u *factors as a product of elements in* $\mathscr{F}[x]_{\leq d}$.

(iii) *If* $f, u \in \mathscr{F}[x]_{<d}$ *and* $p \in \mathscr{F}[x]$ *then* $\langle pf\rangle[1, pg/u^e] \in L_{d-1}$.

Proof. (i) The first statement follows from the identity $\langle a\rangle[b, c] = [ab, c/a]$. For then $\langle p\rangle[g, ph] = [pg, h]$, and using the biadditivity of the symbol [,] this can be expressed as a sum of generators for L_0.

(ii) For the second statement, since the u's used as generators in this version are products of elements in $\mathscr{F}[x]_{\leq d}$, we can use apply partial fractions to h/u^e together with the additivity of [,] to express $\langle ax^\epsilon\rangle[1, h/u^e]$ as a sum of generators of the type in specified in Definition 2.1. Conversely, given $\langle f\rangle[1, h/u^e]$ as in Definition 2.1, where $f, u \in \mathscr{F}[x]_{\leq d}$, and given $h \in \mathscr{F}[x]$, we write $f = \sum_{i=0}^{d-1} a_i x^i$ with $a_i \in \mathscr{F}$ and use the fundamental relation to express $\langle f\rangle[1, h/u^e]$ in the form $\sum_{i=0}^{d-1}\langle a_i x^i\rangle[1, ha_i x^i/fu^e]$. Since $\langle x^i\rangle = \langle x^{\epsilon_i}\rangle$, where $\epsilon_i \in \{0, 1\}$ and $i \equiv \epsilon_i \pmod 2$, and since $ha_i x^i/fu^e = ha_i x^i f^{e-1}/(fu)^e$, we have a generator of the desired type.

(iii) We apply the fundamental relation, expressing pf as $\sum_{i=1}^{n-1} a_i x^i$ with $a_i \in \mathscr{F}$, so $\langle pf\rangle[1, pg/u^e] = \sum_{i=1}^{n-1}\langle a_i x^i\rangle[1, p_i x^i pg/pfu^e] = \sum_{i=1}^{n-1}\langle a_i x^i\rangle[1, p_i x^i g/fu^e] = \sum_{i=1}^{n-1}\langle a_i x^i\rangle[1, p_i x^i gf^{e-1}/(fu)^e] \in L_{d-1}$ by part (ii). □

In particular, by the lemma, for any $\lambda \in \mathscr{F}$ and $h \in \mathscr{F}[x]$, both $\langle\lambda\rangle[1, h]$ and $\langle\lambda x\rangle[1, hx]$ lie in L_0. This will be used frequently.

Lemma 2.3. *Suppose that* p *is a monic irreducible polynomial of degree* d. *If* $r \in F$ *is a* v_p*-unit, and if* s *is* v_p*-integral, then* $\partial_p(\langle r\rangle[1, s]) = 0$. *Consequently, if* $\deg p = d$ *and* $\phi \in L_{d-1}$, *we have* $\partial_p(\phi) = 0$.

Proof. Since r is a v_p-unit we can write $r = r_0 + pr'$ with $0 \neq r_0 \in \overline{F}_p$, where r' is v_p-integral. Next, in F_p we can write $s(r_0/r) = s_0 + ps'$, where $s_0 \in \overline{F}_p$ and s' is v_p-integral in F_p. Since $v_p(s(pr'/r)) > 0$ and $v_p(ps') > 0$, we know that both $s(pr'/r)$ and ps' lie in $\wp(F_p)$. Computing in $W_q F_p$ we find that

$$\begin{aligned}\langle r\rangle[1, s] &= \langle r_0 + pr'\rangle[1, s] = \langle r_0\rangle[1, s(r_0/r)] + \langle pr'\rangle[1, s(pr'/r)] \\ &= \langle r_0\rangle[1, s_0 + ps'] = \langle r_0\rangle[1, s_0].\end{aligned}$$

Since each of r_0 and s_0 lie in $\overline{F}_p$ we see $\langle r\rangle[1,s] \in \operatorname{im}(W_q\overline{F}_p \to W_qF_p)$ and $\partial_p(\langle r\rangle[1,s]) = 0$ follows.

Now consider a generator $\langle f\rangle[1, h/u^e]$ for L_{d-1}. As $f, u \in \mathcal{F}[x]_{<d}$, we know that h/u^e is v_p-integral and f is a v_p-unit, so $\partial_p(\langle f\rangle[1, h/u^e]) = 0$ and we are done in this case. Next consider a generator $[\lambda_1 x^i, \lambda_2 x^j] = \langle \lambda_1 x^i\rangle[1, \lambda_1\lambda_2 x^{i+j}]$ of L_0. If $p \neq x$ then $\lambda_1 x^i$ is a p-adic unit and $\lambda_1\lambda_2 x^{i+j}$ is v_p-integral. If $p = x$ and $i = 0$, then again λ_1 is an x-adic unit and $\lambda_1\lambda_2 x^j$ is v_p-integral. Otherwise if $p = x$ and $i > 0$ we know that $v_x(\lambda_1\lambda_2 x^{i+j}) > 0$ and so $\lambda_1\lambda_2 x^{i+j} \in \wp(F_{v_x})$, giving $[\lambda_1 x^i, \lambda_2 x^j] = 0 \in W_qF_{v_x}$. This proves the lemma. □

Definition 2.4. Whenever p is a monic irreducible polynomial we define S_p to be the subgroup of $W_q\mathcal{F}(x)$ generated by all forms $\langle r\rangle[1, h/p^s]$, where $r \in \mathcal{F}$, $h \in \mathcal{F}[x]$, and $s \geq 0$. When $p = \frac{1}{x}$ we denote by $S_{\frac{1}{x}}$ the subgroup generated by all forms generated by $\langle r\rangle[1, hx]$, where $r \in \mathcal{F}$ and $h \in \mathcal{F}[x]$.

We observe that $W_q\mathcal{F}(x) = \bigcup_{d=0}^{\infty} L_d$. This is because the usual additive generators for $W_q\mathcal{F}(x)$ are included in some L_d for large enough d. The next lemma describes several generating sets for the L_d.

Lemma 2.5. (i) $L_d = \sum_{p,\,\deg p\leq d}(S_p + \langle x\rangle S_p)$.

(ii) $W_q\mathcal{F}(x) = \sum_p (S_p + \langle x\rangle S_p)$.

(iii) *When* $d \geq 1$, L_d *is generated by* L_{d-1} *and* $S_p \cup \langle x\rangle S_p$, *where* $\deg p = d$.

Proof. Part (i) follows from Lemma 2.2(ii) and partial fractions. Part (ii) follows from (i) since $W_q\mathcal{F}(x) = \bigcup_{d=0}^{\infty} L_d$. Part (iii) follows using partial fractions and (i). □

We next establish a result from linear algebra needed to relate $\langle x\rangle S_p$ and $\langle p\rangle S_p$. For $p = x^d + p_1x^{d-1} + \cdots + p_{d-1}x + p_d$ we express each p_i as $\sum_{K\in T} t^K p_{i,K}^2$, where $p_{i,J} \in \mathcal{F}$. For each $K \in T$ we define $P_K \in \overline{\mathcal{F}(x)_p}$ by

$$P_K := \begin{cases} \overline{t^K(p_{1,K}^2x^{d-1} + p_{3,K}^2x^{d-3} + \cdots + p_{d-1,K}^2x)} & \text{when } d \text{ is even,} \\ \overline{t^K(x^d + p_{2,K}^2x^{d-2} + \cdots + p_{d-1,K}^2x)} & \text{when } d \text{ is odd.} \end{cases} \tag{4}$$

Next let M be the $2^n \times 2^n$-matrix with entries indexed by the group T and with (I, J)-th entry P_{I+J}. We show that M is invertible:

Lemma 2.6. *Assume* T *is an elementary abelian* 2*-group with* 2^n *elements and* P_K *are elements of a field of characteristic* 2 *indexed by* $K \in T$. *Suppose that* M *is the* $(2^n\times 2^n)$*-matrix with* (I, J)*-th entry* P_{I+J}. *If* $\sum_{K\in T} P_K \neq 0$, *then* M *is invertible.*

Proof. Let $\operatorname{Perm} T$ denote the set of permutations of T. We know that

$$\det M = \sum_{\sigma\in\operatorname{Perm} T}\left(\prod_{\tau\in T} P_{\tau+\sigma(\tau)}\right)$$

For each $s \in T$ we define $\sigma_s \in \operatorname{Perm} T$ by $\sigma_s(\tau) = \tau + s$. In this case $\tau + \sigma_s(\tau) = s$ for all τ and we have $\prod_{\tau \in T} P_{\tau+\sigma_s(\tau)} = P_s^{2^n}$.

Now let T act on $\operatorname{Perm} T$ via $\sigma^\epsilon(\tau) = \sigma(\tau+\epsilon)+\epsilon$ for all $\epsilon \in T$. (That this is an action is readily checked using the fact that T is abelian.) If $\sigma^\epsilon = \sigma$ for all $\epsilon \in T$, then $\sigma(\epsilon) = \sigma^\epsilon(\epsilon) = \sigma(\epsilon+\epsilon)+\epsilon = \sigma(0)+\epsilon$ for all ϵ, and we see that $\sigma = \sigma_{\sigma(0)}$ in this case. In particular, if $\sigma \neq \sigma_s$ for some $s \in T$ then the orbit of σ under T has more than one element. We next note that for any $\sigma \in \operatorname{Perm} T$ and $\epsilon \in T$ we have

$$\prod_{\tau \in T} P_{\tau+\sigma^\epsilon(\tau)} = \prod_{\tau \in T} P_{\tau+\sigma(\tau+\epsilon)+\epsilon} = \prod_{(\tau+\epsilon)\in T} P_{(\tau+\epsilon)+\sigma(\tau+\epsilon)} = \prod_{\tau \in T} P_{\tau+\sigma(\tau)},$$

and consequently for any σ different from the σ_s we have

$$\sum_{\theta \in \operatorname{Orbit}(\sigma)} \left(\prod_{\tau \in T} P_{\tau+\theta(\tau)} \right) = \operatorname{card}(\operatorname{Orbit}(\sigma)) \prod_{\tau \in T} P_{\tau+\sigma^\epsilon(\tau)} = 0,$$

since card(Orbit(σ)) is a proper power of 2. Decomposing the sum in the determinant over the orbits in $\operatorname{Perm} T$ shows that

$$\det M = \sum_{K \in T} P_K^{2^n} = \left(\sum_{K \in T} P_K \right)^{2^n}.$$

If M fails to be invertible, we have $\det M = 0$, which implies $\sum_{K\in T} P_K = 0$, contrary to our hypothesis. The lemma is proved. □

Corollary 2.7. *If p is irreducible and separable and if the P_K are defined as in (4), then M is invertible as a matrix over $\mathcal{F}(p) := \mathcal{F}[x]/(p)$.*

Proof. Suppose that $\det M = 0$. Then $\sum_{K\in T} P_K = 0$. Each P_K lies in $\overline{t^K x \mathcal{F}(x)_p^2}$, and since the $t_1, t_2, \ldots, t_n$ remain 2-independent in $\overline{\mathcal{F}(x)_p}$ (because p is separable), we see that each $P_K = 0$. Now, since $1, x, x^2, \ldots x^{d-1}$ are linearly independent over $\mathcal{F}$, we find for all K that each $p_{i,K}$ vanishes, where i is odd when d is even and even when d is odd. So the same follows for the p_i. The first case contradicts the separability of p and the second case contradicts the irreducibility of p. □

We are now able to apply Corollary 2.7 and relate S_p, $\langle x\rangle S_p$ and $\langle p\rangle S_p$. Whenever p is not separable we choose i so that $t_i \in \mathcal{F}(p)^2(t_1, \ldots, t_{i-1})$ and then we denote by $\tilde{S}_p$ the subgroup of $S_p + \langle x\rangle S_p$ generated by the elements $\langle t^I\rangle[1, h/p^e]$, where t^I is a product of $t_1, t_2, \ldots t_{j-1}, t_{j+1}, \ldots, t_n, x$.

Proposition 2.8. (i) *For all p, we have $S_p + \langle p\rangle S_p \subseteq S_p + \langle x\rangle S_p$.*

(ii) *If p is separable, $S_p + L_0 + \langle x\rangle S_p = S_p + L_0 + \langle p\rangle S_p$.*

(iii) *If p is not separable, $S_p + L_0 + \langle x\rangle S_p = \tilde{S}_p + L_0 + \langle p\rangle \tilde{S}_p$.*

Proof. Part (i) essentially follows from Lemma 2.2(ii), but we give a direct calculation here because it is necessary for part (ii). Consider $\langle p\rangle[1, h/p^s] \in \langle p\rangle S_p$ and apply the fundamental relation in W_q to obtain

$$\langle p\rangle\Big[1, \frac{h}{p^s}\Big] = \sum_{j=0}^{d} \langle p_j x^{d-j}\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big].$$

Whenever $d-j$ is even, we have

$$\langle p_j x^{d-j}\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big] = \langle p_j\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big] \in S_p,$$

from which we find, modulo S_p,

$$\begin{aligned}\langle p\rangle\Big[1, \frac{h}{p^s}\Big] &\equiv \sum_{d-j \text{ odd}} \langle p_j x^{d-j}\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big] \equiv \sum_{d-j \text{ odd}} \Big(\sum_{K\in T} \langle t^K x\rangle\Big[1, \frac{t^K p_{j,K}^2 x^{d-j} h}{p^{s+1}}\Big]\Big) \\ &\equiv \sum_{K\in T} \langle t^K x\rangle\Big[1, \frac{t^K\big(\sum_{d-j \text{ odd}} p_{j,K}^2 x^{d-j}\big) h}{p^{s+1}}\Big] \equiv \sum_{K\in T} \langle t^K x\rangle\Big[1, \frac{P_K h}{p^{s+1}}\Big],\end{aligned}$$

where the second equivalence uses $p_j = \sum_{K\in T} t^K p_{j,K}^2$, the third changes order of summation, and the fourth uses the definition of the P_K. Part (i) follows from this.

For (ii) we consider the problem of reversing this process when p is separable. Namely, we must express an element $\langle x\rangle[1, g/p^{s+1}] \in W_q\mathscr{F}(x)$ as a sum, modulo $S_p + L_0$, of elements of the form $\sum_J \langle t^J p\rangle[1, h_J/p^s]$. For this we denote by $S_{p,s}$ the subgroup of S_p generated by the generators of S_p, where the exponent of p doesn't exceed s. We can then proceed by backwards induction on s and calculate in $\overline{F(x)_{v_p}}$. Multiplying the equivalence in the previous paragraph by $\langle t^J\rangle$ gives a system of 2^n such equivalences, one for each $J \in T$:

$$\langle t^J p\rangle\Big[1, \frac{h_J}{p^s}\Big] \equiv \sum_{K\in T} \langle t^{K+J} x\rangle\Big[1, \frac{P_K h_J}{p^{s+1}}\Big] \pmod{S_p}.$$

Taking the sum gives, again modulo S_p,

$$\sum_J \langle t^J p\rangle\Big[1, \frac{h_J}{p^s}\Big] \equiv \sum_J \sum_{K\in T} \langle t^{K+J} x\rangle\Big[1, \frac{P_K h_J}{p^{s+1}}\Big] \equiv \sum_L \langle t^L x\rangle\Big[1, \frac{\sum_J P_{L+J} h_J}{p^{s+1}}\Big],$$

where in the second sum the variable L is introduced to collect terms with like $K+J$. Since we want the latter sum to equal $\langle x\rangle[1, g/p^{s+1}]$, we obtain for the h_J the equations

$$\sum_J P_J h_J = g \quad \text{and} \quad \sum_J P_{L+J} h_J = 0 \ \text{ when } \ L \neq O$$

in $\overline{\mathcal{F}(x)_p}$. In matrix form this system is

$$M \cdot (h_J) = \begin{pmatrix} g & 0 & \cdots & 0 \end{pmatrix}^T,$$

where (h_J) means the column with 2^n entries aligned with corresponding entries of M, whose (I, J)-th entry is P_{I+J}. Since M is invertible over $\overline{\mathcal{F}(x)_p}$ by Corollary 2.7, we can find elements $h_J \in \mathcal{F}[x]_{<\deg p}$ such that

$$\langle x\rangle\Big[1, \frac{g}{p^{s+1}}\Big] \equiv \sum_J \langle t^J p\rangle\Big[1, \frac{h_J}{p^s}\Big] \pmod{S_p + \langle x\rangle S_{p,s}}.$$

By backwards induction on s we can reduce to $s = 0$. When $s = 0$, the error terms are sums $\sum_J \langle t^J p\rangle[1, h_J p]$ lying in L_0 by Lemma 2.2(i). The result in (ii) follows.

For part (iii) we write $p = \sum_j p_j x^{d-j}$ and note that since p is not separable, each $d-j$ is even, and we have a 2-dependence between $t_1, t_2, \ldots t_{n-1}, t_n, p$. Reordering $t_1, t_2, \ldots, t_n$ we can assume that $t_n \in \mathcal{F}(x)^2(t_1, t_2, \ldots, t_{n-1}, p)$. This relabeling guarantees that $t_1, t_2, \ldots t_{n-1}, x, p$ is a basis for $\mathcal{F}(x)$ as well as $\mathcal{F}(x)_p$. We express each p_j as $p_{0,j} + p_{1,j} t_n$, where $p_{i,j} \in \mathcal{F}(x)^2(t_1, t_2, \ldots, t_{n-1})$, and further express each $p_{1,j}$ as $\sum_J t^J p^2_{1,j,J}$; here each J_n vanishes. Then we can form

$$\tilde{P}_K = t^K (p^2_{1,0,K} x^d + p^2_{1,2,K} x^{d-2} + \cdots + p^2_{1,d,K})$$

and note that $\sum_K \tilde{P}_K = \partial p / \partial t_n \neq 0 \in \mathcal{F}^2(t_1, t_2, \ldots, t_{n-1})[x^2]$.

We can write

$$\langle p\rangle\Big[1, \frac{h}{p^e}\Big] = \sum_i \langle p_i\rangle\Big[1, \frac{p_i x^{d-i} h}{p^{e+1}}\Big] = \sum_i \langle p_{0,i} + p_{1,i} t_n\rangle\Big[1, \frac{p_i x^{d-i} h}{p^{e+1}}\Big],$$

and so, modulo $\tilde{S}_p$,

$$\begin{aligned}\langle p\rangle\Big[1, \frac{h}{p^e}\Big] &\equiv \sum_i \langle p_{1,i} t_n\rangle\Big[1, \frac{p_{1,i} t_n x^{d-i} h}{p^{e+1}}\Big] \\ &\equiv \sum_{i,K} \langle t^K t_n\rangle\Big[1, \frac{t^K p^2_{1,i,K} t_n x^{d-i} h}{p^{e+1}}\Big] \equiv \sum_K \langle t^K t_n\rangle\Big[1, \frac{\tilde{P}_K t_n h}{p^{e+1}}\Big].\end{aligned}$$

What we must do is reverse this process and solve, modulo $\tilde{S}_p + L_0$, the congruence

$$\begin{aligned}\langle t_n\rangle\Big[1, \frac{g}{p^{e+1}}\Big] &\equiv \sum_L \langle t^L p\rangle\Big[1, \frac{h_L}{p^e}\Big] \equiv \sum_{K,L} \langle t^{L+K} t_n\rangle\Big[1, \frac{\tilde{P}_K t_n h_L}{p^{e+1}}\Big] \\ &\equiv \sum_J \langle t^J t_n\rangle\Big[1, \frac{\sum_L \tilde{P}_{J+L} t_n h_L}{p^{e+1}}\Big],\end{aligned}$$

for appropriate polynomials h_L. As in part (ii) it suffices to reduce the exponent e by 1. This system is equivalent to solving the system in 2^{n-1} variables in $\mathcal{F}(p)$, $g = \sum_L \tilde{P}_L t_n h_L$ and $0 = \sum_L \tilde{P}_{K+L} t_n h_L$, where $K \neq O$. This can be written in matrix form as

$$\tilde{M} \cdot (h_L) = \begin{pmatrix} g & 0 & \cdots & 0 \end{pmatrix}^T,$$

for the $h_L \in \mathcal{F}(p)$. Here $\tilde{M}$ is the matrix with (K, L)-th entry $\tilde{P}_{K+L} t_n \in \mathcal{F}(p)$. However, we have noted that $\sum_K \tilde{P}_K \neq 0$, so the invertibility of $\tilde{M}$ follows from Lemma 2.6. This gives what is needed. □

3. The maps ∂_p and their splitting

From now on, unless stated otherwise, p denotes either a monic irreducible polynomial in $\mathcal{F}[x]$ or $\frac{1}{x}$. Then v_p denotes the associated valuation and we continue to use F_p to denote the completion of $F = \mathcal{F}(x)$ at v_p. We use F_p to denote the completion of $F = \mathcal{F}(x)$ at v_p. We continue to assume that $\mathcal{F}$ has a finite 2-basis $t_1, t_2, \ldots, t_n$. To apply the results from Section 1 we will need to specify a 2-basis for $\mathcal{F}(p) := \mathcal{F}[x]/(p) = \overline{F}_p$. So in this section we will have to be careful and keep track of separability conditions. We recall a well-known result, whose proof is embedded in the subsequent discussion, where we set up notation.

Lemma 3.1. *A 2-basis for $\mathcal{F}$ is a 2-basis for $\mathcal{F}(p)$ if and only if p is separable.*

Since $t_1, t_2, \ldots, t_n$ is the fixed 2-basis for $\mathcal{F}$, $t_1, t_2, \ldots, t_n, x$ is a 2-basis for $F = \mathcal{F}(x)$. We express the monic irreducible $p \in \mathcal{F}[x]$ as

$$p = \sum_{I \in T} t^I (p_I(x))^2,$$

where $p_I(x) \in \mathcal{F}[x]$ and the multiindices t^I refer to the 2-basis for F (which includes x). Since $\overline{F}_p \cong \mathcal{F}[x]/(p)$, we find that $t_1, t_2, \ldots, t_n$ remain 2-independent in $\overline{F}_p$ if and only if for some I with $I_{n+1} \neq 0$ we also have $p_I(x) \neq 0$. But this happens if and only if $p(x)$ has a nonzero summand of odd degree, i.e., if and only if p is separable. Now, when p is separable, if $I_{n+1} \neq 0$ for some $p_I(x) \neq 0$ we take $t_1, t_2, \ldots, t_n$ as our 2-basis for $\overline{F}_p \subset F_p$ and then we can use $t_1, t_2, \ldots, t_n, p$ as our 2-basis for F_p.

Otherwise, when p is not separable, we choose j maximal with $I_j \neq 0$ for some $p_I(x) \neq 0$ and we note that in this case $t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n, \bar{x}$ is a 2-basis for $\overline{F}_p$. We then take $t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n, \bar{x}, p$ as our 2-basis for F_p. In this case we use the lifting to $t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n-1}, x \in F$ to define the embedding $\overline{F}_p \hookrightarrow F_p$ needed to define $\partial_p : W_q F \to W_1 F_p$.

We will need to keep track of products of elements of these various 2-bases. This will be accomplished by using three different notations for multiindex sets:

• We use $\{t^I \mid I \in T\}$ to denote products of elements of the original 2-basis for $\mathcal{F}$.

• We use $\{t^I \mid I \in T_p\}$ to denote products of elements of the residue 2-basis of $\mathcal{F}(p)$ as described above (which vary depending upon whether p is separable).

• We use $\{t^I \mid I \in \tilde{T}_p\}$ to denote products of elements of the local 2-basis of F_p as described above.

In our study of $W_q F$ we will need to understand how $W_q F$ elements map to elements of $W_1 F_p$, so we need to represent $W_1 F_p$ classes in a special way. The next result is a consequence of Theorem 1.3. Whenever $s \in \mathcal{F}[x]$ we denote by $\bar{s} \in \mathcal{F}[x]_{<\deg p}$ the unique polynomial with $s \equiv \bar{s} \pmod{p}$. When applying Theorem 1.3, the Teichmüller lifting used is the one associated with the 2-basis $\tilde{T}_p$.

Theorem 3.2. (i) *If p is monic and irreducible, every class ϕ in $W_1 F_p$ can be expressed uniquely as*

$$\phi = \psi \perp \langle p \rangle \phi_2,$$

where

$$\psi = \sum_{I \in \tilde{T}_p} \langle t^I \rangle \Big[1, \sum_{J, J+I > I} t^J u_{I,J} \Big],$$

$u_{I,J} = \sum_{r \geq 1} \overline{s^2_{I,J,r}} / p^{2r}$ with $s_{I,J,r} \in \mathcal{F}[x]_{<\deg p}$ and $\phi_2 \in W_q \bar{v} F_p$, and where $s_{I,J,r}$ and the Witt classes of ψ in $W_q F_p$ and ϕ_2 in $W_q \bar{v} F_p$ are uniquely determined by the class of ϕ.

(ii) *Every class ϕ in $W_1 F_{\frac{1}{x}}$ can be expressed as*

$$\phi = \psi \perp \Big\langle \frac{1}{x} \Big\rangle \phi_2,$$

where $\psi = \sum_{I \in \tilde{T}_{\frac{1}{x}}} \langle t^I \rangle \big[1, \sum_{J, J+I > I} t^J u^2_{I,J} \big]$ with $u_{I,J} \in x \cdot \mathcal{F}[x]$ and $\phi_2 \in W_q \mathcal{F}$, and where the Witt classes of ψ in $W F_p$ and ϕ_2 in $W_q \mathcal{F}$ are uniquely determined by the class of ϕ. In this expression we note that $\sum_{J, J+I > I} t^J u^2_{I,J} \in x \cdot \mathcal{F}[x]$.

Proof. In (i), since $\overline{F_p} = \mathcal{F}(p)$, applying valuation theory we conclude that every element $f \in \overline{F_p} \subset F_p$ can be expressed as $f_1 + f_2$, where $f_1 \in \mathcal{F}[x]_{<\deg p}$, $f_2 \in F_p$, and $v_p(f_2) > 0$. Consequently, any element $r \in R = \overline{\mathcal{F}(x)_p}[p^{-1}]$ can be expressed in the form $r = s_1 + s_2$, where $s_1 \in \mathcal{F}[x]^2_{<\deg p}[p^{-1}]$ and $v_p(s_2) > 0$. Part (i) now follows, interpreting Theorem 1.3 in this setting and making the appropriate substitutions. Part (ii) is a direct consequence of Theorem 1.3, since there is no ambiguity about viewing the residue field as a subfield of $F_{\frac{1}{x}}$. □

We next digress slightly and give a generator-relation structure for the Witt group. A very similar characterization was found by Arason [2003].

Theorem 3.3. *For any field F of characteristic 2 the Witt group $W_q F$ is isomorphic to $(F^+ \otimes F^+)/\mathcal{W}$, where $\mathcal{W}$ is the subgroup generated by*

(i) *$a \otimes b$ for $ab \in \wp(F)$,*

(ii) *$a \otimes b + b \otimes a$ for all $a, b \in F$, and*

(iii) *$a \otimes b + c \otimes ab/c$ whenever $c \in D_F[a, b]$.*

Proof. We show that the epimorphism $F^+ \otimes F^+ \to W_q F$ defined by $a \otimes b \mapsto [a, b]$ has kernel $\mathcal{W}$. First we note that the generators of $\mathcal{W}$ map to trivial elements of $W_q F$. For type (i) generators, the Arf invariant of $ax^2 + xy + by^2$ is ab, and therefore $[a, b] = 0 \in W_q F$ if and only if $ab \in \wp(F)$. Type (ii) generators vanish by the symmetry of $[a, b]$. For type (iii) generators, since $c \in D_F[a, b]$, we know the form $[a, b] \perp \langle c\rangle[1, ab] = \langle a, c\rangle[1, ab]$ is isotropic, hence hyperbolic. So $[a, b] = \langle c\rangle[1, ab] = [c, ab/c]$, which is what we want.

We next note that whenever $[a, b] = [c, d] \in W_q F$, and since $c \in D_F[a, b]$, we find $[a, b] = [c, ab/c] = [c, d]$, the later equality being equivalent to $[c, ab/c + d] = 0$. By our observations about (i) this is equivalent to $ab + cd \in \wp(F)$. This shows that all equalities $[a, b] = [c, d] \in W_q F$ are a consequence of multilinearity and the relations (i), (ii), and (iii).

Next, given an isomorphism $q = [a_1, b_1] \perp \psi \cong [c, d] \perp \chi$, we must show that it follows from the relations defining $\mathcal{W}$. For this we view each representation q as having the same underlying vector space $V = F^{2n}$ and we let $v, w \in V$ denote the first two symplectic basis elements in the second expression. Hence $q(v) = c$, $q(w) = d$ and the inner product $(v, w)_q$ equals 1. We suppose $\psi = [a_2, b_2] \perp \cdots \perp [a_n, b_n]$. If we view $v \in V$ according to the decomposition given by the first form, we can express c as a sum $c_1 + c_2 + \cdots + c_n$, where $c_i \in D_F[a_i, b_i]$. Applying (iii) at each summand, we can write $[a_1, b_1] \perp [a_2, b_2] \perp \cdots \perp [a_n, b_n] \cong [c_1, e_1] \perp [c_2, e_2] \perp \cdots \perp [c_n, e_n]$ for $e_i = a_i b_i / c_i \in F$. Using the bilinearity of the symbol $[\ ,\ ]$, and since $c = c_1 + c_2 + \cdots + c_n$, we have $[c_1, e_1] \perp [c_2, e_2] \perp \cdots \perp [c_n, e_n] \cong [c, e_1] \perp [c_2, e_1 + e_2] \perp \cdots \perp [c_n, e_1 + e_n]$. If $v_1 = v, w_1, v_2, w_2, \ldots, v_n, w_n$ is the symplectic basis corresponding to this new decomposition $q = [c, e_1] \perp [c_2, e_1 + e_2] \perp \cdots \perp [c_n, e_1 + e_n]$, we can express w as $z_1 + z_2 + \cdots + z_n$ with each z_i a linear combination of v_i and w_i. This means that if $d_i = q(z_i)$ for $1 \le i \le n$, then $d = q(w) = d_1 + d_2 + \cdots + d_n$. Since $(v, w)_q = 1$ while $(v, z_i)_q = 0$ for $2 \le i \le n$, we see that $(v, z_1)_q = 1$. Since $\operatorname{span}(v, w_1) = \operatorname{span}(v, z_1)$, restricting our attention to this subspace we see that in fact $[c, e_1] \cong [c, d_1]$. We now apply relation (iii) to the other summands to obtain $[c_i, e_1 + e_i] \cong [c'_i, d_i]$ for $c'_i = c_i(e_1 + e_i)/d_i \in F$. Using bilinearity again we find $[c, e_1] \perp [c_2, e_1 + e_2] \perp \cdots \perp [c_n, e_1 + e_n] \cong [c, d_1] \perp [c'_2, d_2] \perp \cdots \perp [c'_n, d_n] \cong [c, d] \perp [c + c'_2, d_2] \perp \cdots \perp [c + c'_n, d_n]$. Altogether, using only bilinearity and the rules (i), (ii), (iii), we have shown that our original $[a_1, b_1] \perp \psi$ is Witt equivalent to $[c, d] \perp \psi'$ for some ψ'. By Witt cancellation we now have $\psi' \cong \chi$. By induction on n we can reduce to the case $[a, b] \cong [c, d]$ already considered. This proves the theorem. $\square$

We now define the Milnor splittings.

Lemma and Definition 3.4. *Suppose p is a monic irreducible polynomial of degree $d \geq 1$ and that $\phi \in W_1 F_p$ is of the form*

$$\phi = \psi \perp \langle p \rangle \phi_2,$$

where

$$\psi = \sum_{I \in T_p} \langle t^I \rangle \Big[1, \sum_{J, J+I > I} t^J r_{I,J}^2 \Big]$$

with $r_{I,J} \in p^{-1}\mathcal{F}[x]_{< \deg p}[p^{-1}]$ and $\phi_2 \in W_q \bar{v} F_p$. Here the t^I depend upon the 2-basis for $\mathcal{F}(x)_p$, which has last element p and will include x in the case where p is not separable. We further write $\phi_2 = \sum_i [r_i(\bar{x}), s_i(\bar{x})]$, where $\bar{x}$ denotes x modulo $p(x)$ and the $r_i(x), s_i(x)$ lie in $\mathcal{F}[x]_{<\deg p}$. Then the map τ_p defined by

$$\tau_p(\phi) = \psi \perp \langle p \rangle \left(\sum_i [r_i(x), s_i(x)] \right) \pmod{L_{d-1}} \in L_d / L_{d-1}$$

is a well defined homomorphism $\tau_p : W_1 F_p \to L_d / L_{d-1}$.

Proof. According to Theorem 3.2(i) every class ϕ in $W_1 F_p$ can be expressed as stated, and the Witt classes of $\psi \in W_q F_p$ and $\phi_2 \in W_q \overline{F_p}$ are uniquely determined. Further, the Witt class of ψ uniquely determines the $r_{I,J}$ as elements of $p^{-1}\mathcal{F}[x]_{< \deg p}[p^{-1}]$.

The expression of ϕ_2 as $\sum_i [r_i(\bar{x}), s_i(\bar{x})]$ need not be unique so we suppose also that $\phi_2 = \sum_j [u_j(\bar{x}), v_j(\bar{x})] \in W_q \mathcal{F}(p)$, where each $u_j(x)$ and $v_j(x)$ lies in $\mathcal{F}[x]_{<\deg p}$. By Theorem 3.3, using the biadditivity of the symbol [,] we have the expansion

$$\sum_i [r_i(\bar{x}), s_i(\bar{x})] + \sum_j [u_j(\bar{x}), v_j(\bar{x})] = \sum_k [a_k(\bar{x}), b_k(\bar{x})],$$

where the latter sum is a sum of relations of the form given in Theorem 3.3(i), (ii) or (iii). Since we only used the biadditivity of [,] in the expansion, we know that each $a_k(x), b_k(x) \in \mathcal{F}[x]_{<\deg p}$ and we also have $\sum_i [r_i(x), s_i(x)] + \sum_j [u_j(x), v_j(x)] = \sum_k [a_k(x), b_k(x)] \in W_q \mathcal{F}(x)$. Checking for each of the types of relations given in Theorem 3.3 we will show that $\langle p \rangle$ times this sum lies in L_{d-1}.

Suppose first we have a summand $[a, b]$ with $a, b \in \mathcal{F}[x]_{<\deg p}$, where $\bar{a}\bar{b} \in \wp(\mathcal{F}(p))$. Then we can write $ab = \wp(z) + pg$ in $\mathcal{F}[x]$ and we find that $[a, b] = \langle a \rangle [1, ab] = \langle a \rangle [1, \wp(z) + pg] = \langle a \rangle [1, pg]$ in $W_q F$. By Lemma 2.2(iii) the form $\langle p \rangle \langle a \rangle [1, pg]$ lies in L_{d-1} since $a \in \mathcal{F}[x]_{<\deg p}$. Next, any pair in the sum of the form $[a, b] + [b, a]$ is zero in $W_q F$ as well. Finally, suppose we have a pair in the sum $[a, b] + [c, d]$, where $\bar{c} \in D_{\mathscr{F}(p)}[\bar{a}, \bar{b}]$ and $\bar{d} = \overline{ab/c}$ where $a, b, c, d \in \mathscr{F}[x]_{<\deg p}$. Then we can write $c = ar^2 + rs + bs^2 + pg$ and $ab = cd + ph$ in $\mathscr{F}[x]$ where $r, s \in \mathscr{F}[x]_{<\deg p}$ also. Since $(s/r + bs^2/r^2)b \in \wp(F)$, we obtain $[s/r + bs^2/r^2, b] = 0$. Rewriting the expression for c, we find $a = c/r^2 + s/r + bs^2/r^2 + pg/r^2$, so in $W_q F$ we have

$$\begin{aligned}[a, b] + [c, d] &= [c/r^2, b] + [s/r + bs^2/r^2, b] + [pg/r^2, b] + [c, d]\\ &= \langle c\rangle[1/r^2, bc] + [pg/r^2, b] + \langle c\rangle[1, cd]\\ &= \langle c\rangle[1, bc/r^2] + \langle c\rangle[1, cd] + [pg/r^2, b].\end{aligned}$$

Next, substituting $bc/r^2 = ab + bs/r + b^2s^2/r^2 + bpg/r^2$ and $cd = ab + ph$ we find

$$\begin{aligned}[a,b]+[c,d] &= \langle c\rangle[1, ab+bs/r+b^2s^2/r^2+pgb/r^2]+\langle c\rangle[1, ab+ph]+[pg/r^2, b]\\ &= \langle c\rangle[1, pgb/r^2+ph]+[pg/r^2, b],\end{aligned}$$

since $bs/r+b^2s^2/r^2 \in \wp(F)$. Applying Lemma 2.2(iii) to each of these latter forms we find that $\langle p\rangle([a, b] + [c, d])$ lies in L_{d-1}. It follows that τ_p is well defined. It is clear from the defining formula that τ_p is additive in ψ, and the proof that the lift of ϕ_2 is well defined shows that τ_p is additive in that term as well. Hence τ_p is a homomorphism. □

Our goal is to prove the surjectivity of the Milnor splitting. This requires the information provided in Proposition 2.8 and is given next.

Theorem 3.5. *The map* $\bigoplus \tau_p : \bigoplus_{p,\, \deg p = d} W_1 F(x)_{v_p} \to L_d/L_{d-1}$ *is an isomorphism for* $d \geq 1$.

Proof. It suffices to show the map is surjective, since for any p the composite $W_1 F(x)_{v_p} \to L_d/L_{d-1} \to W_1 F(x)_{v_p}$ is the identity. (Here, the first map is τ_p and the second map is ∂_p, which vanishes on L_{d-1} by Lemma 2.3.) By the definition of τ_p, every element $\psi = \sum_{I \in T_p} \langle t^I\rangle\big[1, \sum_{J, J+I>I} t^J r_{I,J}^2\big]$ with $r_{I,J} \in p^{-1}\mathscr{F}[x]_{<\deg p}[p^{-1}]$ lies in the image. When p is separable, these elements generate $S_p + \langle p\rangle S_p$ so we have $S_p + \langle p\rangle S_p \subseteq \operatorname{im}(\tau_p)$, and when p is not separable, these elements generate $\tilde{S}_p + \langle p\rangle \tilde{S}_p$ and we have $\tilde{S}_p + \langle p\rangle \tilde{S}_p \subseteq im(\tau_p)$. Further, if p is separable, then $S_p + \langle p\rangle S_p + L_0 = S_p + \langle x\rangle S_p + L_0$ by Proposition 2.8(ii) and in case p is not separable we have $\tilde{S}_p + \langle p\rangle \tilde{S}_p + L_0 = S_p + \langle x\rangle S_p + L_0$ by Proposition 2.8(iii). However, L_d is generated by $S_p \cup \langle x\rangle S_p$ for p with $\deg p = d$ together with L_{d-1} by Lemma 2.5. From this the theorem is proved. □

The definition of L_0 combined with Theorem 3.2(ii) gives:

Theorem 3.6. *There is an exact sequence*

$$0 \to W_q\mathscr{F} \to L_0 \to W_1 F_{\frac{1}{x}} \to W_q\mathscr{F} \to 0,$$

where the first two maps are induced by inclusion and the last map is $\phi \mapsto \phi_2$, *where* ϕ_2 *is as given in Theorem 3.2(ii).*

Proof. Since L_0 is generated by the forms $[\lambda_1 x^i, \lambda_2 x^j] = \langle \lambda_1 x^i \rangle [1, \lambda_1 \lambda_2 x^{i+j}]$, applying the relations in $W_q F$ we see that every element $\phi \in L_0$ can be expressed as $\phi = \phi_1 \perp \psi$, where $\phi_1 \in W_q \mathcal{F}$ and $\psi = \sum_{I+J>I} \langle t^I \rangle [1, t^J r_{I,J}^2]$ with $r_{I,J} \in x \cdot \mathcal{F}[x]$. Moreover, the expression of ψ as such a sum is unique, according to the local theory at the $v_{\frac{1}{x}}$-adic valuation as given in Theorem 3.2(ii). This means that the natural map from L_0 to $W_1 F_{\frac{1}{x}}$ has kernel $W_q \mathcal{F}$ and cokernel the elements in $\langle \frac{1}{x} \rangle W_q \mathcal{F}$. The result follows. □

4. The transfer maps s_p^*

We continue to use the 2-bases for $\mathcal{F}(p)$ and $\mathcal{F}(x)_p$ defined in the discussion following Lemma 3.1, as well as the notation T, T_p and $\tilde{T}_p$. When p fails to be separable and $I \in T_p$, we denote by I_x the entry corresponding to the exponent of x (so x occurs in t^I if and only if $I_x = 1$.) In the next definition, we define subgroups $S_{p,r}$ of $S_p + \langle p \rangle S_p$, for each $r \geq 1$. We include $p = \frac{1}{x}$ in our list. We set $d = \deg p$ and $d = 1$ when $p = \frac{1}{x}$. As in the last section, when considering elements of S_p of the form $\langle t^I \rangle [1, h/p^r]$ for $h \in \mathcal{F}[x]$, we write $\bar{h} \in \mathcal{F}[x]_{<\deg p}$ for the unique element with $h \equiv \bar{h} \pmod{p}$.

Definition 4.1. (i) Suppose p is separable or is $\frac{1}{x}$. We define $S_{p,r} \subset S_p + \langle p \rangle S_p$ as the subgroup generated by elements of two types: those of the form

$$\langle t^I \rangle [1, t^J \overline{s_{I,J}^2} / p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, and where $I + J > I$ whenever r is even, and those of the form

$$\langle t^I p \rangle [1, t^J \overline{s_{I,J}^2} / p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, r is even, and where $I + J > I$.

(ii) Suppose p is not separable. We define $S_{p,r} \subset \tilde{S}_p + \langle p \rangle \tilde{S}_p$ as the subgroup generated by elements of two types: those of the form

$$\langle t^I \rangle [1, t^J \overline{s_{I,J}^2} / p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, and where $I + J > I$ whenever r is even, and those of the form

$$\langle t^I p \rangle [1, t^J \overline{s_{I,J}^2} / p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, r is even, and where $I + J > I$.

(iii) Suppose p is not separable. We define $S^0_{p,r} \subset S_{p,r}$ as the subgroup generated by elements of two types: those of the form

$$\langle t^I \rangle [1, t^J \overline{s^2_{I,J}}/p^r],$$

where $I_x = 0,\ I,\ J \in T_p,\ s_{I,J} \in \mathscr{F}[x]_{<d}$, and where $I + J > I$ whenever r is even, and those of the form

$$\langle t^I p \rangle [1, t^J \overline{s^2_{I,J}}/p^r],$$

where $I_x = 0,\ I,\ J \in T_p,\ r$ is even, $s_{I,J} \in \mathscr{F}[x]_{<d}$, and where $I + J > I$.

(iv) We define $U_p := \sum_{r \geq 1} S_{p,r}$ for all p and $U^0_p := \sum_{r \geq 1} S^0_{p,r}$ when p is not separable.

The definitions in (i) and (ii) are formally the same, except that the T_p's differ according to whether p is separable or not, which also accounts for using S_p or $\tilde{S}_p$. In part (iii) the listed elements are a subcollection of those listed in (ii), and are precisely those without an x in the t^I. The reason for the restrictions on I, J, r in the definition will become clear in the proof of the next lemma, where we apply Theorem 3.2.

Lemma 4.2. (i) *For each p, $S_p + \langle p \rangle S_p \subseteq U_p + L_0 + \langle p \rangle L_0$.*

(ii) *The group*

$$U_p = \bigoplus_{r \geq 1} S_{p,r} \subset S_p + \langle p \rangle S_p$$

is a direct sum.

(iii) *Let the image of this group in $W_1 F_p$ also be denoted by U_p. Then*

$$W_1 F_p / U_p \cong \langle p \rangle \cdot W_q \mathscr{F}(p).$$

Thus every element in $W_1 F_p / U_p$ can be represented by an element of $\langle p \rangle L_0$.

Proof. (i) This follows from the additive property of the symbol $[1, a]$, expanding elements of S_p into sums of $S_{p,r}$ elements one power of p at a time, leaving an element of the form $[1, g]$ where $g \in \mathscr{F}[x]$. The last summands lie in L_0.

(ii) The summands from ψ in Theorem 3.2 can be uniquely expressed as a sum of elements of the form

$$\langle t^I \rangle [1, t^J \overline{s^2_{I,J,r}}/p^{2r}],$$

where $I,\ J \in \tilde{T}_p$ satisfy $I + J > I$ and $s_{I,J,r} \in \mathscr{F}[x]_{<\deg p}$. Given this, we claim that the generators identified in Definition 4.1 are equivalent to those required to apply Theorem 3.2. There are four cases, depending upon whether $I,\ J \in T_p$ or not. In case both $I,\ J \in T_p$, then the condition that $I + J > I$ is the same for T_p elements as for $\tilde{T}_p$ elements, and this is recorded in Definition 4.1 in the first type where r is even. In case $I \in T_p$ but $J \notin T_p$, then $I + J > I$ is automatic and this corresponds to the first type in Definition 4.1 where r is odd. In case, $I \notin T_p$ but $J \in T_p$ then $I + J > I$ in $\tilde{T}_p$ is equivalent to $I' + J > I'$ in T_p where I' is the same as I except the p component is deleted, and this is the case of the second type of generator in Definition 4.1. Finally, if both $I, J \notin T_p$, then $I + J > I$ is impossible as elements of $\tilde{T}_p$, and this case is ignored by Definition 4.1. So we are ready to apply Theorem 3.2.

First, every sum of elements of $S_{p,r}$ can be represented as

$$\sum_{r\geq 1}\sum_{I+J>I} \langle t^I\rangle\left[1, \frac{t^J\overline{s_{I,J,r}^2}}{p^r}\right] + \sum_{r\geq 2,\ \text{even}}\ \sum_{I+J>I} \langle t^I p\rangle\left[1, \frac{t^J\overline{s'^2_{I,J,r}}}{p^r}\right],$$

where $s_{I,J,r}, s'_{I,J,r}$ lie in $\mathscr{F}[x]_{<d}$ and as elements of T_p we have $I + J > I$ in each summand. These expressions can be rewritten as

$$\sum_{I+J>I} \langle t^I\rangle\left[1, \sum_{r\geq 1}\frac{t^J\overline{s_{I,J,r}^2}}{p^r}\right] + \sum_{I+J>I} \langle t^I p\rangle\left[1, \sum_{r\geq 2,\ \text{even}} \frac{t^J\overline{s'^2_{I,J,r}}}{p^r}\right].$$

Now, in Theorem 3.2 the 2-basis $\hat{T}_p$ is used, which is in this case consists of the elements of T_p and $p \cdot T_p$. So each of the sums in the additive slots of the last expression,

$$\sum_{r\geq 1}\frac{t^J\overline{s_{I,J,r}^2}}{p^r} \quad \text{and} \quad \sum_{r\geq 2,\ \text{even}} \frac{t^J\overline{s'^2_{I,J,r}}}{p^r},$$

correspond uniquely to elements listed as $t^J s_{I,J,r}^2$ in the statement of Theorem 3.2(i). The directness of the sum is follows. The statement in (iii) is also a consequence of Theorem 3.2, proving the lemma. □

We are now able to define the transfer maps s_p^*.

Definition 4.3. For $\psi \in W_1 F_p$ we define $\theta_p(\psi) \in W_q\mathscr{F}(p)$ to be the unique element of $W_q \bar{v} F_p$ for which $\psi \equiv \langle p\rangle\theta_p(\psi) \pmod{U_p}$. We denote by $t_p^* : \mathscr{F}(p) \to \mathscr{F}$ the Scharlau transfer associated to the linear functional t_p for which $t_p(x^{d-1}) = 1$ but $t_p(x^i) = 0$ when $0 \leq i < d-1$. Finally, whenever $f \in \mathscr{F}[x]$ is a polynomial we denote by f_c its constant term.

(i) Suppose p is separable or $\frac{1}{x}$. Then we define $s_p^*(\psi) := t_p^*(\theta_p(\psi)) \in W_q\mathscr{F}$.

(ii) If p is not separable we express $\psi - \theta_p(\psi)$ modulo U_p^0 as

$$\sum_{I_x=1,r} \langle t^I\rangle\left[1, \frac{t^J\overline{s_{I,J,r}^2}}{p^r}\right] + \sum_{I_x=1,r} \langle (t^I)p\rangle\left[1, \frac{t^J\overline{s'^2_{I,J,r}}}{p^r}\right].$$

Then we define $s^*_p(\psi) \in\in W_q\mathscr{F}$ by

$$s^*_p(\psi) := t^*_p(\theta_p(\psi)) + \sum_{I_x=1,r} \langle t^I/x\rangle \left[1, \frac{(t^J s^2_{I,J,r})_c}{p^r_c}\right] + \sum_{I_x=1,r} \langle t^I p_c/x\rangle \left[1, \frac{(t^J s'^2_{I,J,r})_c}{p^r_c}\right].$$

(Note that $t^I/x \in \mathscr{F}$ since $I_x = 1$.)

To prove reciprocity, we must check it for each $\langle p\rangle L_0$ and each U_p. This will be done in the next two sections.

5. The reciprocity law for $L_0 + \langle p\rangle L_0$

In this section we prove the reciprocity law in a critical special case. We assume $p = x^d + p_1x^{d-1} + \cdots + p_d \in \mathscr{F}[x]$ is a monic irreducible polynomial of degree $d = 2e$ when d is even, and of degree $d = 2e + 1$ when d is odd. Since we will be calculating in both $\mathscr{F}[x]$ and $\mathscr{F}[x]/(p)$ we will use x to represent the variable in $\mathscr{F}[x]$ as well as its residue in $\mathscr{F}[x]/(p)$, since no confusion will arise. For $h \in \{0, 1\}$, $\lambda_1, \lambda_2 \in \mathscr{F}$, and $k \geq 0$ we will compute the transfer $t^*_p([\lambda_1x^h, \lambda_2x^k]) = \phi \in W_q\mathscr{F}$ described in Definition 4.3. We note that in case $h' > 1$ we can write $h' = h+2h_0$ for $h \in \{0, 1\}$ and as $[\lambda_1x^{h'}, \lambda_2x^k] = \langle\lambda_1x^{h'}\rangle[1, \lambda_1\lambda_2x^{k+h'}] = \langle\lambda_1x^h\rangle[1, \lambda_1\lambda_2x^{k+h'}] = [\lambda_1x^h, \lambda_2x^{k+h'-h}]$ we see there is no loss of generality in our restriction on h.

For $0 \leq j \leq e$ we define the polynomials $f_j = x^{e+j} + p_1x^{e+j-1} + \cdots + p_{2j}x^{e-j}$ and $g_j = x^{e+j} + p_1x^{e+j-1} + \cdots + p_{2j+1}x^{e-j-1}$. Thus $g_j = f_j + p_{2j+1}x^{e-j-1}$. We next define $\gamma_i \in \mathscr{F}$ by expressing $x^{d+i-1} = \gamma_i x^{d-1} + G_i \in \mathscr{F}[x]/(p)$ where $G_i \in \mathscr{F}[x]$ is a polynomial of degree at most $d-2$. Note that this means that $t_p(x^{d+i-1}) = \gamma_i$. Clearly, $\gamma_0 = 1$, and using the equation $x^d = p_1x^{d-1} + \cdots + p_d \in \mathscr{F}(p) := \mathscr{F}[x]/(p)$ we see that $\gamma_1 = p_1$ and by induction for $i \geq 1$,

$$\begin{aligned} x^{d+i-1} &= p_1x^{d+i-2} + p_2x^{d+i-3} + \cdots + p_dx^{i-1} \\ &= (p_1\gamma_{i-1}p_2\gamma_{i-2} + \cdots + p_i\gamma_0)x^{d-1} + p_1G_{i-1}p_2G_{i-2} + \cdots + p_iG_0. \end{aligned}$$

This shows that in general the γ_i satisfy the recurrence relation

$$\gamma_i = \gamma_{i-1}p_1 + \gamma_{i-2}p_2 + \cdots \gamma_0 p_i \qquad \text{for } i \geq 1.$$

In fact, this recurrence relation is the same as the relation that guarantees that, as power series, $(1 + p_1X + p_2X^2 + \cdots)(1 + \gamma_1X + \gamma_2X^2 + \cdots) = 1 \in \mathscr{F}((X))$, and which we use below in proving Lemma 5.3.

Lemma 5.1. (i) *Suppose $d = 2e$. We have $t_p(f_0^2) = \gamma_1 = p_1$ and $t_p(xf_0^2) = \gamma_2 = p_1^2 + p_2$. For all j with $1 \leq j \leq e$ we have $t_p(f_j^2) = p_{2j+1}$, $t_p(xf_j^2) = p_{2j+2}$. For all $k \geq 0$ we have $t_p(x^kg_0^2) = \gamma_{k+1} + p_1^2\gamma_{k-1}$.*

(ii) *Suppose* $d = 2e + 1$. *We have* $t_p(f_0^2) = \gamma_0 = 1$ *and* $t_p(xf_0^2) = \gamma_1 = p_1$. *For all* j *with* $1 \le j \le e$ *we have* $t_p(f_j^2) = p_{2j}$, $t_p(xf_j^2) = p_{2j+1}$. *For all* $k \ge 0$ *we have and* $t_p(x^k g_0^2) = \gamma_k + p_1^2 \gamma_{k-2}$.

Proof. Since $f_0 = x^e$ we have $t(x^h f_0^2) = t(x^{h+2e})$ so when $d = 2e$ we find $t(x^h f_0^2) = \gamma_{h+1}$ and when $d = 2e+1$ we find $t(x^h f_0^2) = \gamma_h$ as required. When $j > 0$ and $d = 2e$ we have in $\mathcal{F}(p)$

$$\begin{aligned} f_j^2 &= (x^{e+j} + p_1 x^{e+j-1} + \cdots + p_{2j} x^{e-j})^2 \\ &= (x^{2e} + p_1 x^{2e-1} + \cdots + p_{2j} x^{2e-2j})(x^{2j} + p_1 x^{2j-1} + \cdots + p_{2j}) \\ &= (p_{2j+1} x^{d-2j-1} + \cdots + p_d)(x^{2j} + p_1 x^{2j-1} + \cdots + p_{2j}) \\ &= p_{2j+1} x^{d-1} + (p_{2j+1} p_1 + p_{2j+2}) x^{d-2} + \cdots + p_d p_{2j}. \end{aligned}$$

We find that $t_p(f_j^2) = t_p(p_{2j+1} x^{d-1}) = p_{2j+1}\gamma_0 = p_{2j+1}$ and that $t_p(xf_j^2) = t_p(p_{2j+1} x^d + (p_{2j+1} p_1 + p_{2j+2}) x^{d-1}) = p_{2j+1}\gamma_1 + (p_{2j+1} p_1 + p_{2j+2})\gamma_0 = p_{2j+2}$ as $\gamma_0 = 1$ and $\gamma_1 = p_1$.

When $j > 0$ and $d = 2e + 1$ we have

$$\begin{aligned} f_j^2 &= (x^{e+j} + p_1 x^{e+j-1} + \cdots + p_{2j} x^{e-j})^2 \\ &= (x^{2e+1} + p_1 x^{2e} + \cdots + p_{2j} x^{2e-2j+1})(x^{2j-1} + p_1 x^{2j-2} + \cdots + p_{2j-1}) \\ &\qquad + (x^{2e} + p_1 x^{2e-1} + \cdots + p_{2j} x^{2e-2j}) p_{2j} \\ &= (p_{2j+1} x^{d-2j-1} + \cdots + p_d)(x^{2j-1} + p_1 x^{2j-2} + \cdots + p_{2j-1}) \\ &\qquad + (x^{2e} + p_1 x^{2e-1} + \cdots + p_{2j} x^{2e-2j}) p_{2j} \\ &= p_{2j} x^{d-1} + (p_{2j+1} + p_1 p_{2j}) x^{d-2} + \cdots + p_d p_{2j-1}. \end{aligned}$$

So we find $t_p(f_j^2) = p_{2j}$ and $t_p(xf_j^2) = p_{2j}\gamma_1 + (p_{2j} p_1 + p_{2j+1})\gamma_0 = p_{2j+1}$, as $\gamma_0 = 1$ and $\gamma_1 = p_1$.

Finally, as $g_0 = x^e + p_1 x^{e-1}$, we find that $x^k g_0^2 = x^{2e+k} + p_1^2 x^{2e+k-2}$ and therefore $t_p(x^k g_0^2) = \gamma_{k+1} + p_1^2 \gamma_{k-1}$ when $d = 2e$ and $t_p(x^k g_0^2) = \gamma_k + p_1^2 \gamma_{k-2}$ when $d = 2e+1$. This proves the lemma. □

The next lemma calculates the transfers.

Lemma 5.2. *Let* $\lambda_1, \lambda_2 \in \mathcal{F}$, $h \in \{0, 1\}$, *and* $k \ge 0$. *When* $d = 2e$, *the transfer is given by*

$$t_p^*([\lambda_1 x^h, \lambda_2 x^k]) = [\lambda_1 h, \lambda_2(\gamma_{k+1} + p_1^2 \gamma_{k-1})] + \sum_{i=1}^{e-1} [\lambda_1 p_{2(e-i)+h+1}, \lambda_2 \gamma_{k+2i-d-1}] + [\lambda_1 \gamma_{h+1}, \lambda_2 \gamma_{k-1}]$$

and when $d = 2e+1$ by

$$t_p^*([\lambda_1 x^h, \lambda_2 x^k]) = \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)+h}, \lambda_2 \gamma_{k+2i-d+1}].$$

Proof. We first give a symplectic basis for $t_p^*([\lambda_1 x^h, \lambda_2 x^k])$. For $(r, s) \in \mathcal{F}[x]/(p) \times \mathcal{F}[x]/(p)$, applying the quadratic form we have

$$t_p^*([\lambda_1 x^h, \lambda_2 x^j])(r, s) = t_p(\lambda_1 x^h r^2 + rs + \lambda_2 x^j s^2).$$

If $d = 2e$ is even we consider the basis

$$\{(1, 0), (0, g_{e-1});\ (x, 0), (0, g_{e-2});\ \ldots;\ (x^{e-1}, 0), (0, g_0);\\ (f_{e-1}, 0), (0, 1);\ (f_{e-2}, 0), (0, x);\ \ldots;\ (f_0, 0), (0, x^{e-1})\}.$$

If $d = 2e+1$ is odd we consider the basis

$$\{(1, 0), (0, g_e);\ (x, 0), (0, g_{e-1});\ \ldots;\ (x^{e-1}, 0), (0, g_1);\\ (f_e, 0), (0, 1);\ (f_{e-1}, 0), (0, x);\ \ldots;\ (0, x^e), (f_0, 0)\}.$$

In each case we claim the basis is symplectic. When $d = 2e$, since each polynomial $x^{i-1} f_{e-i}$ and $x^{i-1} g_{e-i}$ is monic of degree $d-1$ when $1 \le i \le e$ we see that the inner products $((x^{i-1}, 0), (0, g_{e-i}))$ and $((0, x^{i-1}), (f_{e-i}, 0))$ equal 1. Likewise, when $d = 2e+1$ each of $((x^i, 0), (0, g_{e-i}))$ and $((0, x^i), (f_{e-i}, 0))$ equal 1. Next, whenever $i + j \le d-2$ we have $((x^i, 0), (0, x^j)) = 0$, showing that such pairs are always orthogonal. Since the product $x^i f_{e-j}$ has degree $i + 2e - j$, when $d = 2e$ and $i < j$ we have $t_p(x^{i-1} f_{e-j}) = 0$, and when $d = 2e+1$ with $i < j$ we have $t_p(x^i f_{e-j}) = 0$. For $i > j$, when $d = 2e$, we have

$$\begin{aligned} x^{i-1} f_{e-j} &= x^{i-1}(x^{2e-j} + \cdots + p_{2(e-j)} x^j) \\ &= (x^d + \cdots + p_{2(e-j)} x^{2j}) x^{i-j-1} = (p_{2(e-j)+1} x^{2j-1} + \cdots + p_d) x^{i-j-1}, \end{aligned}$$

this last having degree $i + j - 2 < d - 1$. For $i > j$, when $d = 2e+1$, we have $x^i f_{e-j} = x^{i-1}(x^{2e-j} + \cdots + p_{2(e-j)} x^j) = (x^d + \cdots + p_{2(e-j)} x^{2j}) x^{i-j-1} = (p_{2(e-j)+1} x^{2j-1} + \cdots + p_d) x^{i-j-1}$, this last having degree $i + j - 2 < d - 1$. So we see that the inner products $((f_{e-j}, 0), (0, x^{i-1}))$ vanish whenever $i \ne j$ and $d = 2e$, while $((f_{e-j}, 0), (0, x^i))$ vanish whenever $i \ne j$ and $d = 2e+1$. This also shows that when $i \ne j$ the inner product $((0, g_{e-j}), (x^{i-1}, 0))$ vanishes as well when $d = 2e$:

$$\begin{aligned} ((0, g_{e-j}), (x^{i-1}, 0)) &= ((0, f_{e-j} + p_{2(e-j)+1} x^{j-1}), (x^{i-1}, 0)) \\ &= t_p(p_{2(e-j)+1} x^{i+j-2}) = 0. \end{aligned}$$

Similarly $((0, g_{e-j}), (x^i, 0)) = 0$ when $d = 2e+1$.

So to check the required orthogonality we must calculate $t_p(g_i f_j)$ where $0 \leq i, j \leq e-1$ and $j = e$ as well as when $d = 2e+1$. When $d = 2e$ and $j > i$ or $d = 2e+1$ and $j > i+1$ we calculate

$$\begin{aligned} t_p(g_i f_j) &= t_p((x^{e+i} + \cdots + p_{2i+1}x^{e-i-1})(x^{e+j} + \cdots + p_{2j}x^{e-j})) \\ &= t_p((x^{2i+1} + \cdots + p_{2i+1})(x^{2e+j-i-1} + \cdots + p_{2j}x^{2e-j-i-1})) \\ &= t_p((x^{2i+1} + \cdots + p_{2i+1})(p_{2j+1}x^{2e-j-i-2} + \cdots + p_d x^{j-i-1})) = 0, \end{aligned}$$

because the latter polynomial has degree $2e+i-j-1 < d-1$. When $d = 2e$ and $j \leq i$ or $d = 2e+1$ and $i \leq j+1$ we calculate

$$\begin{aligned} t_p(g_i f_j) &= t_p((x^{e+i} + \cdots + p_{2i+1}x^{e-i-1})x^{e-j}(x^{2j} + \cdots + p_{2j})) \\ &= t_p((x^{2e+i-j} + \cdots + p_{2i+1}x^{2e-i-j-1})(x^{2j} + \cdots + p_{2j})) \\ &= t_p((p_{2i+2}x^{2e-i-j-2} + \cdots + p_d x^{i-j})(x^{2j} + \cdots + p_{2j})) = 0, \end{aligned}$$

because the latter polynomial has degree $2e-i+j-2 < d-1$. This shows that the inner products $((f_i, 0), (0, g_j)) = 0$ and therefore both bases listed are symplectic.

We are now able to compute the transfer $t_p^*([\lambda_1 x^h, \lambda_2 x^k])$, where $h \in \{0, 1\}$. When $d = 2e$, since $h \leq 1$ the vectors $(x^i, 0)$ are isotropic as long as $0 \leq i < e-1$. So we can apply the previous lemma and we only need to use the portion of the symplectic basis that involves g_0 and the f_j. We find

$$\begin{aligned} t_p^*([\lambda_1 x^h, \lambda_2 x^k]) &= [t_p(\lambda_1 x^h x^{2(e-1)}), t_p(\lambda_2 x^k g_0^2)] + \sum_{i=1}^{e} [t_p(\lambda_1 x^h f_{e-i}^2), t_p(\lambda_2 x^k x^{2(i-1)})] \\ &= [\lambda_1 h, \lambda_2(\gamma_{k+1} + p_1^2 \gamma_{k-1})] \\ &\qquad + \sum_{i=1}^{e-1} [\lambda_1 p_{2(e-i)+h+1}, \lambda_2 \gamma_{k+2i-d-1}] + [\lambda_1 \gamma_{h+1}, \lambda_2 \gamma_{k-1}], \end{aligned}$$

where in this last summand we have used $t(x^h f_0^2) = \gamma_{h+1}$. When $d = 2e+1$ we note that, since $h \leq 1$, each of the vectors $(x^i, 0)$, where $0 \leq i \leq e-1$, is isotropic. Hence we need only consider the part of the symplectic basis involving the f_j and we find

$$\begin{aligned} t_p^*([\lambda_1 x^h, \lambda_2 x^k]) &= \sum_{i=0}^{e} [t_p(\lambda_1 x^h f_{e-i}^2), t_p(\lambda_2 x^k x^{2i})] \\ &= \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)+h}, \lambda_2 \gamma_{k+2i-d+1}], \end{aligned}$$

where in the latter sum we have used $\gamma_h = p_h$ for $h = 0, 1$ when $i = e$. This proves the lemma. □

We next have to compute $\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1 x^h, \lambda_2 x^k]) = \partial_{\frac{1}{x}}(\langle px^h\rangle[\lambda_1, \lambda_2 x^{h+k}])$. There are four cases, depending upon the parity of h and d:

Lemma 5.3.

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1,\lambda_2x^k]))=\begin{cases}\sum_{j=0}^{e-1}[\lambda_1p_{2j+1},\lambda_2\gamma_{k-2j-1}] & \textit{if } d \textit{ is even},\\ \sum_{j=0}^{e}[\lambda_1p_{2j},\lambda_2\gamma_{k-2j}] & \textit{if } d \textit{ is odd};\end{cases}$$

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle px\rangle[\lambda_1,\lambda_2x^{k+1}]))=\begin{cases}\sum_{j=0}^{e}[\lambda_1p_{2j},\lambda_2\gamma_{k-2j+1}] & \textit{if } d \textit{ is even},\\ \sum_{j=0}^{e}[\lambda_1p_{2j+1},\lambda_2\gamma_{k-2j}] & \textit{if } d \textit{ is odd}.\end{cases}$$

Proof. As $p=x^d+p_1x^{d-1}+\cdots+p_d$ we are able to express $\langle px^h\rangle[1,x^{h+k}]$ as

$$\langle x^{d+h}\rangle[1,x^{h+k+d}p^{-1}]+\langle p_1x^{d+h-1}\rangle[1,p_1x^{h+k+d-1}p^{-1}]+\cdots+\langle p_dx^h\rangle[1,p_dx^{h+k}p^{-1}].$$

Since $p=x^d(1+p_1x^{-1}+p_2x^{-2}+\cdots p_dx^{-d})$, inside the completion $\mathcal{F}(x)_{\frac{1}{x}}$ we can write $p^{-1}=x^{-d}(1+\gamma_1x^{-1}+\gamma_2x^{-2}+\cdots)$. When $d+h-i$ is even we have $s^*_{\frac{1}{x}}(\langle p_ix^{d+h-i}\rangle[\lambda_1,\lambda_2p_ix^{h+k+d-i}p^{-1}])=0$ and when $d+h-i$ is odd we have

$$\begin{aligned}s^*_{\frac{1}{x}}(\langle p_ix^{d+h-i}\rangle[\lambda_1,\lambda_2p_ix^{h+k+d-i}p^{-1}])&\\ &=s^*_{\frac{1}{x}}(\langle p_ix^{-1}\rangle[\lambda_1,\lambda_2p_ix^{h+k-i}(1+\gamma_1x^{-1}+\cdots)])\\ &=\langle p_i\rangle[\lambda_1,\lambda_2p_i\gamma_{h+k-i}].\end{aligned}$$

So when $h=0$ and d is even we have

$$\begin{aligned}s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1,\lambda_2x^k]))&=\langle p_1\rangle[\lambda_1,\lambda_2p_1\gamma_{k-1}]+\langle p_3\rangle[\lambda_1,\lambda_2p_3\gamma_{k-3}]\\ &\qquad+\cdots+\langle p_{d-1}\rangle[\lambda_1,\lambda_2p_{d-1}\gamma_{k-d+1}]\\ &=\sum_{j=0}^{e-1}[\lambda_1p_{2j+1},\lambda_2\gamma_{k-2j-1}].\end{aligned}$$

Similarly, if $h=1$ and d is even we find

$$\begin{aligned}s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle px\rangle[\lambda_1,\lambda_2x^{k+1}]))&=\langle p_0\rangle[\lambda_1,\lambda_2p_0\gamma_{k+1}]+\langle p_2\rangle[\lambda_1,\lambda_2p_2\gamma_{k-1}]\\ &\qquad+\cdots+\langle p_d\rangle[\lambda_1,\lambda_2p_d\gamma_{k-d+1}]\\ &=\sum_{j=0}^{e}[\lambda_1p_{2j},\lambda_2\gamma_{k-2j+1}].\end{aligned}$$

Next, if $h=0$ and d is odd,

$$\begin{aligned}s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1,\lambda_2x^{k+1}]))&=\langle p_0\rangle[\lambda_1,\lambda_2p_0\gamma_k]+\langle p_2\rangle[\lambda_1,\lambda_2p_2\gamma_{k-2}]\\ &\qquad+\cdots+\langle p_{2e}\rangle[\lambda_1,\lambda_2p_{2e}\gamma_{k-2e}]\\ &=\sum_{j=0}^{e}[\lambda_1p_{2j},\lambda_2\gamma_{k-2j}].\end{aligned}$$

Finally, if $h = 1$ and d is odd,

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle px\rangle[\lambda_1, \lambda_2 x^k])) = \langle p_1\rangle[\lambda_1, \lambda_2 p_1\gamma_k] + \langle p_3\rangle[\lambda_1, \lambda_2 p_3\gamma_{k-2}] + \cdots + \langle p_{d-1}\rangle[\lambda_1, \lambda_2 p_{d-1}\gamma_{k-d+1}]$$
$$= \sum_{j=0}^{e-1}[\lambda_1 p_{2j+1}, \lambda_2\gamma_{k-2j}]. \qquad \square$$

Theorem 5.4. *The reciprocity law* $\sum_q s^*_q(\partial_q(\phi)) = 0$ *holds for all* $\phi \in L_0 + \langle p\rangle L_0$.

Proof. We first consider a generator $\phi = [\lambda_1 x^i, \lambda_2 x^j]$ of L_0. By Lemma 2.3, $\partial_p(\phi)$ vanishes for all $p \neq \frac{1}{x}$. When $p = \frac{1}{x}$, Theorem 3.6 shows that $s^*_{\frac{1}{x}}(\phi) = 0$. So the reciprocity law holds for elements of L_0. We next note that for any generator $\phi = \langle p\rangle[\lambda_1 x^h, \lambda_2 x^k]$ of $\langle p\rangle L_0$ we have $s^*_q(\partial_q(\phi)) = 0$ as long as $q \neq p, \frac{1}{x}$. So we must check that $s^*_p(\partial_p(\phi)) = s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\phi))$ for all such generators.

When $d = 2e$ and $h = 0$, we have by Lemma 5.2

$$t^*_p([\lambda_1, \lambda_2 x^k]) = \sum_{i=1}^{e-1}[\lambda_1 p_{2(e-i)+1}, \lambda_2\gamma_{k+2i-d-1}] + [\lambda_1\gamma_1, \lambda_2\gamma_{k-1}].$$

Also in this case by Lemma 5.3 we have

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k])) = \sum_{j=0}^{e-1}[\lambda_1 p_{2j+1}, \lambda_2\gamma_{k-2j-1}],$$

But $p_1 = \gamma_1$; therefore the terms in these sums match exactly, which shows that $s^*_p(\partial_p(\langle p\rangle[\lambda_1, \lambda_2 x^k])) = t^*_p([\lambda_1, \lambda_2 x^k]) = s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k]))$ in this case.

When $d = 2e$ and $h = 1$, we have by Lemma 5.2

$$t^*_p([\lambda_1 x, \lambda_2 x^k]) = [\lambda_1, \lambda_2(\gamma_{k+1} + p_1^2\gamma_{k-1})] + \sum_{i=1}^{e-1}[\lambda_1 p_{2(e-i)+2}, \lambda_2\gamma_{k+2i-d-1}] + [\lambda_1\gamma_2, \lambda_2\gamma_{k-1}].$$

Also in this case by Lemma 5.3 we have

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1 x, \lambda_2 x^k])) = \sum_{j=0}^{e}[\lambda_1 p_{2j}, \lambda_2\gamma_{k-2j+1}].$$

These two expressions will be equal provided we can show

$$[\lambda_1, \lambda_2(\gamma_{k+1} + p_1^2\gamma_{k-1})] + [\lambda_1\gamma_2, \lambda_2\gamma_{k-1}] = [\lambda_1, \lambda_2\gamma_{k+1}] + [\lambda_1 p_2, \lambda_2\gamma_{k-1}]$$

since the summands $\sum_{i=1}^{e-1}[\lambda_1 p_{2(e-i)+2}, \lambda_2\gamma_{k+2i-d-1}]$ correspond exactly to the summands $\sum_{j=2}^{e}[\lambda_1 p_{2j}, \lambda_2\gamma_{k-2j+1}]$. So we need that

$$[\lambda_1, \lambda_2 p_1^2\gamma_{k-1}] + [\lambda_1\gamma_2, \lambda_2\gamma_{k-1}] = [\lambda_1 p_2, \lambda_2\gamma_{k-1}]$$

which follows because $[\lambda_1, \lambda_2 p_1^2 \gamma_{k-1}] = \langle p_1^2\rangle \cdot [\lambda_1 p_1^2, \lambda_2 \gamma_{k-1}]$ and $\gamma_2 = p_1^2 + p_2$.

When $d = 2e+1$ and $h = 0$, we have by Lemma 5.2

$$t_p^*([\lambda_1, \lambda_2 x^k]) = \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)}, \lambda_2 \gamma_{k+2i-d+1}].$$

Also in this case by Lemma 5.3 we have

$$s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k])) = \sum_{j=0}^{e} [\lambda_1 p_{2j}, \lambda_2 \gamma_{k-2j}],$$

which shows that $t_p^*([\lambda_1, \lambda_2 x^k]) = s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k]))$, since the summations are the same apart from indexing.

When $d = 2e+1$ and $h = 1$, we have by Lemma 5.2

$$t_p^*([\lambda_1 x, \lambda_2 x^k]) = \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)+1}, \lambda_2 \gamma_{k+2i-d+1}].$$

Also in this case by Lemma 5.3 we have

$$s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1 x, \lambda_2 x^k])) = \sum_{j=0}^{e} [\lambda_1 p_{2j+1}, \lambda_2 \gamma_{k-2j}],$$

which shows that $t_p^*([\lambda_1, \lambda_2 x^k]) = s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k]))$ again since the summations are the same. This gives the reciprocity law for $\langle p\rangle L_0$. □

6. The reciprocity law and the analogue of Milnor's theorem

We next turn to the reciprocity law for $W_q\mathscr{F}(x)$.

Theorem 6.1. *The composite* $W_q F \xrightarrow{\oplus \partial_p} \bigoplus_{p,\frac{1}{x}} W_1 F_p \xrightarrow{\oplus s_p^*} W_q\mathscr{F}$ *is zero.*

Proof. According to Lemma 2.5(ii), $W_q\mathscr{F}(x) = \sum_p (S_p + \langle x\rangle S_p)$. So it suffices to check the composite vanishes on $S_p + \langle x\rangle S_p$ for each p. In case $p = \frac{1}{x}$ then $S_{\frac{1}{x}} + \langle x\rangle S_{\frac{1}{x}} \subset L_0 + \langle \frac{1}{x}\rangle L_0 = L_0 + \langle x\rangle L_0$. Since the reciprocity law holds for $L_0 + \langle x\rangle L_0$ by Theorem 5.4, we can assume that p is monic and irreducible. By Lemma 4.2, we know that $S_p + \langle x\rangle S_p \subset U_p + L_0 + \langle p\rangle L_0$. By Theorem 5.4 we know the composite vanishes on $L_0 + \langle p\rangle L_0$. Since $U_p = \sum_r S_{p,r}$, therefore, it suffices to verify the composite vanishes on each generator of $S_{p,r}$. If q is not one of $p, x, \frac{1}{x}$, we know by Lemma 2.3 that ∂_p vanishes on $S_{p,r}$, so we only need to worry about those three primes.

We consider first the generators of $S_{p,r}$ of the form $\phi = \langle t^I \rangle[1, h/p^r] \in S_p$ or $\phi = \langle t^I p\rangle[1, h/p^r] \in \langle p\rangle S_p$ where $\deg h < \deg p$ and $r \geq 1$. Since $t^I \in \mathcal{F}$ we can assume $I = 0$ by Frobenius reciprocity. If $p \neq x$ then p is both an x-unit and a $\frac{1}{x}$-adic unit. This means $v_x(h/p^r) \geq 0$ and since $\deg h < \deg p$ we must have $v_{\frac{1}{x}}(h/p^r) > 0$. So by Lemma 2.3 we find that $\partial_x(\phi) = \partial_{\frac{1}{x}}(\phi) = 0$ in these cases. However by definition we know that $s_p^*(\partial_p(\phi)) = 0$ for these particular generators of U_p so reciprocity is established in these cases. In case $p = x$ then $v_{\frac{1}{x}}(h/x) > 0$ so $\partial_x(\phi) = \partial_{\frac{1}{x}}(\phi) = 0$ in this case as well. In case p is separable we know that all generators for U_p are of the form just considered so we are done when p is separable. When p is not separable, the generators just considered are the generators in U_p^0, so we are done in that case as well.

Finally, when p is not separable, we must consider generators of U_p that don't lie in U_p^0. These have the form $\langle t^I x\rangle[1, h/p^r] \in \tilde{S}_p$ or $\langle t^I px\rangle[1, h/p^r] \in \langle p\rangle \tilde{S}_p$ where $\deg h < \deg p$ and $r \geq 1$. Again, since $t^I \in \mathcal{F}$ we can by Frobenius reciprocity assume $I = 0$. In these cases we have by Definition 4.3(ii) that $s_x^*(\partial_x(\langle x\rangle[1, h/p^r])) = [1, h_c/p_c^r]$, and $s_x^*(\partial_x(\langle px\rangle[1, h/p^r])) = \langle p_c\rangle[1, h_c/p_c^r]$. Since $v_{\frac{1}{x}}(h/p^r) > 0$ we have $s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle x\rangle[1, h/p^r])) = 0$, and $s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle px\rangle[1, h/p^r])) = 0$. By the definition of s_p^*, we know that $s_p^*(\partial_p(\langle x\rangle[1, h/p^r]) = [1, h_c/p_c^r]$ and $s_p^*(\partial_p(\langle px\rangle[1, h/p^r]) = \langle p_0\rangle[1, h_c/p_c^r]$, giving the reciprocity law in this case. □

Putting everything together gives the main result of the paper.

Theorem 6.2 (Analogue of the Milnor–Scharlau Sequence). *Suppose that $\mathcal{F}$ is a field of characteristic 2 and $F = \mathcal{F}(x)$ is a rational function field in one variable over $\mathcal{F}$. There exists a compatible collection of second residue and transfer maps that fit into an exact sequence*

$$0 \longrightarrow W_q\mathcal{F} \longrightarrow W_qF \xrightarrow{\bigoplus \partial_p} \bigoplus_{p,\frac{1}{x}} W_1F_p \xrightarrow{\bigoplus s_p^*} W_q\mathcal{F} \longrightarrow 0,$$

where the direct sum is taken over discrete valuations on F.

Proof. Everything completed previously applies when $\mathcal{F}$ has a finite 2-basis. In that case Theorem 3.5 shows that

$$W_qF/L_0 \cong \bigoplus_{d\geq 1} L_d/L_{d-1} \to \bigoplus_p W_1F_p$$

is an isomorphism. Theorem 3.6 shows that

$$0 \to W_q\mathcal{F} \to L_0 \to W_1F_{\frac{1}{x}} \to W_q\mathcal{F} \to 0$$

is an exact sequence. Theorem 6.1 shows we can patch the two sequences together and obtain the result. When $\mathcal{F}$ does not have a finite 2-basis, the result follows from the finite 2-basis case because any element in any group in the sequence lies in the same sequence defined for a finitely generated subfield of $\mathcal{F}$. This proves the theorem. □

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Received February 25, 2005. Revised September 15, 2005.

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