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A GENERALIZATION OF GAUCHMAN'S RIGIDITY THEOREM

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Dedicated to Professor Chaohao Gu on the occasion of his 80th birthday.

We generalize the well-known Gauchman theorem for closed minimal submanifolds in a unit sphere, and prove that if M is an n-dimensional closed submanifold of parallel mean curvature in S^{n+p} and if $\sigma(u) \le \frac{1}{3}$ for any unit vector $u \in TM$, where $\sigma(u) = \|h(u, u)\|^2$, and h is the second fundamental form of M, then either $\sigma(u) \equiv H^2$ and M is a totally umbilical sphere, or $\sigma(u) \equiv \frac{1}{3}$. Moreover, we give a geometrical classification of closed submanifolds with parallel mean curvature satisfying $\sigma(u) \equiv \frac{1}{3}$.

1. Introduction and statement of results

Let $S^m(r)$ be the m-dimensional sphere of radius r, with $S^m = S^m(1)$. By M we will always denote an n-dimensional connected and closed Riemannian manifold isometrically immersed in some S^{n+p}. We will be interested in the case when M has *parallel mean curvature*, meaning that the mean curvature vector ξ on M forms a parallel vector field in the normal bundle over M. (When ξ vanishes identically, M is a minimal submanifold; M is a hypersurface of constant mean curvature if $p = 1$ and the norm of ξ is constant.)

Our investigation contributes to the theory of geometrical invariants and structures of Riemannian manifolds and submanifolds, an important problem in global differential geometry. After the pioneering rigidity theorem for closed minimal submanifolds in a sphere due to Simons [1968], Lawson [1969], and Chern, do Carmo and Kobayashi [Chern et al. 1970], A. M. Li and J. M. Li [1992] improved Simons' pinching constant to $\max\{n/(2-1/p), 2n/3\}$.

Extending this rigidity result to submanifolds of parallel mean curvature in a sphere, we have the theorem below, first proved by Okumura [1965] and Yau [1974; 1975], then by Xu [1991], and finally by Alenca and do Carmo [1994] in codimension 1 and independently by Xu [1993; 1995] in codimension p.

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Theorem 1.1. *For given $H \geq 0$ and positive integers $n \geq 2$ and p, set*

$$C(n, p, H) = \begin{cases} \alpha(n, H) & \text{if } p = 1 \text{ or } p = 2 \text{ and } H \neq 0, \\ \min\bigl(\alpha(n, H), \frac{1}{3}(2n + 5nH^2)\bigr) & \text{if } p \geq 3 \text{ or } p = 2 \text{ and } H = 0, \end{cases}$$

where

$$\alpha(n, H) = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)H}{2(n-1)}\sqrt{n^2H^2 + 4(n-1)}.$$

If M^n is a closed submanifold in the standard unit sphere S^{n+p} of parallel mean curvature vector of norm H, and if the squared norm S of the second fundamental form satisfies

$$S \leq C(n, p, H),$$

then M is congruent to one of the following:

(1) $S^n_H := S^n\Bigl(\frac{1}{\sqrt{1+H^2}}\Bigr)$;

(2) *the isoparametric hypersurface $S^{n-1}\Bigl(\frac{1}{\sqrt{1+\lambda^2}}\Bigr) \times S^1\Bigl(\frac{\lambda}{\sqrt{1+\lambda^2}}\Bigr)$ in $S^{n+1}(1)$, where*

$$\lambda = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)};$$

(3) *one of the Clifford minimal hypersurfaces $S^k\Bigl(\sqrt{\frac{k}{n}}\Bigr) \times S^{n-k}\Bigl(\sqrt{\frac{n-k}{n}}\Bigr)$ in S^{n+1}, for $k = 1, \ldots, n-1$;*

(4) *the Clifford torus $S^1(r_1) \times S^1(r_2)$ in $S^3(r)$ with constant mean curvature H_0, where $0 \leq H_0 \leq H$,*

$$r_1, r_2 = \frac{1}{\sqrt{2(1+H^2) \pm 2H_0(1+H^2)^{1/2}}} \quad \text{and} \quad r = \frac{1}{\sqrt{1+H^2-H_0^2}};$$

(5) *the Veronese surface in $S^4_H = S^4\Bigl(\frac{1}{\sqrt{1+H^2}}\Bigr)$.*

Taking $H = 0$, we have:

Corollary 1.2 [Chern et al. 1970; An-Min and Jimin 1992]. *If M^n is a closed minimal submanifold in the standard unit sphere S^{n+p}, and if*

$$S \leq \max\Bigl(\frac{n}{2-1/p}, \frac{2}{3}n\Bigr),$$

then M is congruent to one of the following:

(1) S^n;

(2) *one of the Clifford minimal hypersurfaces $S^k\Bigl(\sqrt{\frac{k}{n}}\Bigr) \times S^{n-k}\Bigl(\sqrt{\frac{n-k}{n}}\Bigr)$ in S^{n+1}, for $k = 1, \ldots, n-1$;*

(3) *the Veronese surface in S^4.*

Since $\min_{H\geq 0}\alpha(n, H) = 2\sqrt{n-1}$, we get from Theorem 1.1:

Corollary 1.3. *Let M^n be a closed submanifold with parallel mean curvature in S^{n+p}. Suppose that $H \neq 0$ and that*

$$S \leq \begin{cases} 2\sqrt{n-1} & \text{if } p \leq 2 \text{ or } p \geq 3 \text{ and } n \geq 8, \\ \frac{2}{3}n & \text{if } p \geq 3 \text{ and } n \leq 7. \end{cases}$$

Then M is either a totally umbilical sphere in S^{n+p}, a Clifford isoparametric hypersurface in an $(n+1)$-dimensional sphere, or the Veronese surface in S^4_H.

Gauchman [1986] proved that if M is an n-dimensional closed minimal submanifold in S^{n+p} and if $\sigma(u) \leq \frac{1}{3}$ for any unit vector $u \in TM$, where $\sigma(u) = \|h(u, u)\|^2$ for h the second fundamental form of M, then either $\sigma(u) \equiv 0$ and M is a totally geodesic sphere, or $\sigma(u) \equiv \frac{1}{3}$. Moreover, he gave a geometrical classification of closed minimal submanifolds satisfying $\sigma(u) \equiv \frac{1}{3}$.

A natural question is how to generalize this striking rigidity result to the case where M is an n-dimensional closed submanifold of parallel mean curvature in S^{n+p}. In this paper we provide such a generalization. To state our main result precisely, we start with some explicit examples of submanifolds with parallel mean curvature in a sphere, which extend Gauchman's examples for the minimal cases [Gauchman 1986; Sakamoto 1977].

Example 1.4. Let $S^q(r)$ be a q-dimensional sphere of radius r in $\mathbb{R}^{q+1}$, and let $1 \leq k \leq n-1$. We embed $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$ in $S^{n+1}(1)$ as follows. Let $u \in S^k(1/\sqrt{2})$ and $v \in S^{n-k}(1/\sqrt{2})$ be vectors of length $1/\sqrt{2}$ in $\mathbb{R}^{k+1}$ and $\mathbb{R}^{n-k+1}$, respectively. We can consider (u, v) as a unit vector in $\mathbb{R}^{n+2} = \mathbb{R}^{k+1} \times \mathbb{R}^{n-k+1}$. It is easy to see that $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$ is a submanifold in $S^{n+1}(1)$ of parallel mean curvature

$$H = \left|\frac{2k-n}{n}\right|.$$

In particular, M is minimal if $n = 2k$. The exact same construction yields an embedding of $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$ in $S^{n+2}(1)$.

Example 1.5. Denote by RP^2, CP^2, QP^2, and $CayP^2$ the projective plane over the real numbers, complex numbers, quaternions and octonions, and by $\psi_1 : RP^2 \to S^4(1)$, $\psi_2 : CP^2 \to S^7(1)$, $\psi_3 : QP^2 \to S^{13}(1)$ and $\psi_4 : CayP^2 \to S^{25}(1)$ the corresponding isometric embeddings. Let $\psi_1' : S^2(\sqrt{3}) \to S^4(1)$ be the isometric immersion defined by $\psi_1' = \psi_1 \circ \pi$, where $\pi : S^2(\sqrt{3}) \to RP^2$ is the canonical projection.

For $n \geq 2$, $m \geq 0$, let $S^n(1)$ be the great sphere in $S^{n+m}(1)$ given by

$$S^n(1) = \{(x_1, \dots, x_{n+m+1}) \in S^{n+m}(1) \mid x_{n+2} = \cdots = x_{n+m+1} = 0\},$$

and $\tau_{n,m} : S^n(1) \to S^{n+m}(1)$ the inclusion. We set

$$\begin{aligned} \phi_{1,p} &= \tau_{4,p-2} \circ \psi_1 : RP^2 \to S^{2+p}, && p \geq 2, \\ \phi_{2,p} &= \tau_{7,p-3} \circ \psi_2 : CP^2 \to S^{4+p}, && p \geq 3, \\ \phi_{3,p} &= \tau_{13,p-5} \circ \psi_3 : QP^2 \to S^{8+p}, && p \geq 5, \\ \phi_{4,p} &= \tau_{25,p-9} \circ \psi_4 : CayP^2 \to S^{16+p}, && p \geq 9, \\ \phi'_{1,p} &= \tau_{4,p-2} \circ \psi'_1 : S^2(\sqrt{3}) \to S^{2+p} && p \geq 2. \end{aligned}$$

Then $\phi_{i,p}$ is an isometric minimal embedding and $\phi'_{1,p}$ is an isometric minimal immersion.

Denote by UM the unit tangent bundle of M. Define

$$C(p, H) = \begin{cases} 1 & \text{for } p = 1 \text{ or } p = 2 \text{ and } H \neq 0; \\ \frac{1}{3} & \text{for } p \geq 3 \text{ or } p = 2 \text{ and } H = 0. \end{cases}$$

Main Theorem 1.6. *Let M be an n-dimensional compact submanifold of the unit sphere S^{n+p}, with parallel mean curvature vector field of norm H. If*

$$\sigma(u) \leq C(p, H) \quad \text{for any } u \in UM,$$

we are in one of the following cases:

(1) *M is the totally umbilical sphere $S^n_H = S^n\Big(\dfrac{1}{\sqrt{1+H^2}}\Big)$;*

(2) *M is one of the embeddings $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$, with $k = 1, 2, \dots, n$ and $k \neq \frac{1}{2}n$;*

(3) *the isometric immersion of M in S^{n+p} is either the totally umbilical sphere $S^n(\sqrt{3}/2) \to S^{n+p}$, or one of the embeddings $\phi_{i,p}$, $i = 1, 2, 3, 4$, or the immersion $\phi'_{1,p}$.*

The case $H = 0$ goes back to Gauchman [1986, p. 781].

2. Preliminaries

We make the following conventions on the range of indices:

$$1 \leq A, B, C \leq n + p, \quad 1 \leq i, j, k, l, m \leq n < \alpha, \beta, \gamma, \delta \leq n + p.$$

Choose a local orthonormal frame field $\{e_A\}$ on S^{n+p} such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of S^{n+p} respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\, \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \quad h = \sum_{\alpha,i,j} h^\alpha_{ij}\, \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha,i} h^\alpha_{ii}\, e_\alpha,$$

$$R_{ijkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk} + \sum_{\alpha} \big(h^{\alpha}_{ik}\,h^{\alpha}_{jl} - h^{\alpha}_{il}\,h^{\alpha}_{jk}\big), \tag{1}$$

$$R_{\alpha\beta kl} = \sum_{i} \Big(h^{\alpha}_{ik}\,h^{\beta}_{il} - h^{\alpha}_{il}\,h^{\beta}_{ik}\Big),$$

where h, ξ, R_{ijkl} and $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the curvature tensor and the normal curvature tensor of M. We set

$$S = \|h\|^2, \quad H = \|\xi\|, \quad H_\alpha = (h^{\alpha}_{ij})_{n\times n}.$$

Denoting the first and second covariant derivatives of h^{α}_{ij} by h^{α}_{ijk} and h^{α}_{ijkl} respectively, we have

$$\sum_k h^{\alpha}_{ijk}\,\omega_k = dh^{\alpha}_{ij} + \sum_k h^{\alpha}_{kj}\,\omega_{ik} + \sum_k h^{\alpha}_{ik}\,\omega_{jk} + \sum_\beta h^{\beta}_{ij}\,\omega_{\alpha\beta}, \tag{2}$$

$$\sum_l h^{\alpha}_{ijkl}\,\omega_l = dh^{\alpha}_{ijk} + \sum_l h^{\alpha}_{ljk}\,\omega_{il} + \sum_l h^{\alpha}_{ilk}\,\omega_{jl} + \sum_l h^{\alpha}_{ijl}\,\omega_{kl} + \sum_\beta h^{\beta}_{ijk}\,\omega_{\alpha\beta}. \tag{3}$$

The Laplacian of h is defined by $\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{ijkk}$. Following [Yau 1974; 1975], we have

$$\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{kkij} + \sum_{k,m} h^{\alpha}_{km}\,R_{mijk} + \sum_{k,m} h^{\alpha}_{mi}\,R_{mkjk} + \sum_{k,\beta} h^{\beta}_{ki}\,R_{\alpha\beta kj}.$$

From now on we assume that M is a submanifold of parallel mean curvature in S^{n+p}. Choose e_{n+1} such that e_{n+1} is parallel to ξ, $\operatorname{tr} H_{n+1} = nH$ and $\operatorname{tr} H_\beta = 0$, where $n+2 \le \beta \le n+p$. Again by the same work of Yau, we have

$$\Delta h^{n+1}_{ij} = \sum_{k,m} h^{n+1}_{km}\,R_{mijk} + \sum_{k,m} h^{n+1}_{im}\,R_{mkjk},$$

$$\Delta h^{\beta}_{ij} = \sum_{k,m} h^{\beta}_{mk}\,R_{mijk} + \sum_{k,m} h^{\beta}_{im}\,R_{mkjk} + \sum_{k,\alpha\neq n+1} h^{\alpha}_{ki}\,R_{\alpha\beta jk}, \quad \beta \neq n+1.$$

Since the Laplacian formulas for the special orthonormal frame field as above are not apply to our case, we will give the following Laplacian formula which holds for any orthonormal frame fields.

Proposition 2.1. *Let M be an n-dimensional submanifold of parallel mean curvature in S^{n+p}. Then*

$$\Delta h^{\alpha}_{ij} = \sum_{k,m} h^{\alpha}_{km}\,R_{mijk} + \sum_{k,m} h^{\alpha}_{mi}\,R_{mkjk} + \sum_{k,\beta} h^{\beta}_{ki}\,R_{\beta\alpha jk}, \tag{4}$$

$$\sum_\alpha R_{\alpha\beta kl}\,(\operatorname{tr} H_\alpha) = 0. \tag{5}$$

Proof. Putting $c_\alpha = (1/n)\,\mathrm{tr}\,H_\alpha$, we have $\xi = \sum c_\alpha\, e_\alpha$. Since ξ is parallel in the normal bundle over M, we have

$$\begin{aligned}0 = \nabla^\perp_X \xi &= \sum_\alpha X(c_\alpha)\, e_\alpha + \sum_\alpha c_\alpha\, \nabla^\perp_X e_\alpha \\ &= \sum_\alpha X(c_\alpha)\, e_\alpha + \sum_\alpha c_\alpha \Big(\sum_\beta \omega_{\beta\alpha}(X)\, e_\beta\Big) = \sum_\alpha \Big(X(c_\alpha) + \sum_\beta c_\beta\ \omega_{\alpha\beta}(X)\Big)\, e_\alpha\end{aligned}$$

for any tangent vector field X on M. It follows that

$$dc_\alpha + \sum_\beta c_\beta\, \omega_{\alpha\beta} = 0 \quad \text{for any } \alpha. \tag{6}$$

To prove (4), it is sufficient to show that $\sum_k h^\alpha_{kkij} = 0$ for any α, i, j. By (2), we get

$$\sum_{i,k} h^\alpha_{iik}\, \omega_k = d\Big(\sum_i h^\alpha_{ii}\Big) + 2\sum_{i,k} h^\alpha_{ik}\, \omega_{ik} + \sum_{\beta,i} h^\beta_{ii}\, \omega_{\alpha\beta} = n\Big(dc_\alpha + \sum_\beta c_\beta\, \omega_{\alpha\beta}\Big) = 0.$$

Therefore, $\sum_i h^\alpha_{iik} = 0$ for all k, α. Together with (3), this implies

$$\sum_{i,l} h^\alpha_{iikl}\, \omega_l = d\Big(\sum_i h^\alpha_{iik}\Big) + 2\sum_{i,l} h^\alpha_{ilk}\, \omega_{il} + \sum_{i,l} h^\alpha_{iil}\, \omega_{kl} + \sum_{i,\beta} h^\beta_{iik}\, \omega_{\alpha\beta} = 0.$$

Hence $\sum_i h^\alpha_{iikl} = 0$ for all k, l, α.

Taking the exterior derivative of (6) we get

$$\begin{aligned}0 &= d^2 c_\alpha + d\Big(\sum_\beta c_\beta\, \omega_{\alpha\beta}\Big) \\ &= \sum_\beta dc_\beta \wedge \omega_{\alpha\beta} + \sum_\beta c_\beta \Big(-\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \tfrac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\, \omega_k \wedge \omega_l\Big) \\ &= \sum_\beta \Big(dc_\beta + \sum_\gamma c_\gamma\, \omega_{\beta\gamma}\Big) \wedge \omega_{\alpha\beta} + \tfrac{1}{2}\sum_{\beta,k,l} c_\beta\, R_{\alpha\beta kl}\ \omega_k \wedge \omega_l \\ &= \tfrac{1}{2}\sum_{\beta,k,l} c_\beta\, R_{\alpha\beta kl}\ \omega_k \wedge \omega_l.\end{aligned}$$

Thus $\sum_\beta R_{\alpha\beta kl}\, (\mathrm{tr}\, H_\beta) = 0$ for all α, k, l, as desired. □

3. Maximal directions

Let $x \in M$. A vector $u \in UM_x$ is called a *maximal direction* at x if $\sigma(u) = \max_{v \in UM_x} \sigma(v)$.

Choose an orthonormal frame $\{e_1, \dots, e_{n+p}\}$ at x such that restricted to M, the vectors $e_1, \dots, e_n$ are tangent to M. Assume that e_1 is a maximal direction at x, $\sigma(e_1) \neq 0$, and $e_{n+1} = h(e_1, e_1)/\|h(e_1, e_1)\|$. Choose e_{n+2} such that

$$e_{n+2} = \frac{\xi - \langle \xi, e_{n+1}\rangle e_{n+1}}{\|\xi - \langle \xi, e_{n+1}\rangle e_{n+1}\|}$$

if ξ is not parallel to e_{n+1}. By our choices of e_{n+1} and e_{n+2}, we have

$$h^{\alpha}_{11} = 0 \quad \text{if } \alpha \neq n+1 \qquad \text{and} \qquad c_{\alpha} = 0 \quad \text{if } \alpha \neq n+1, n+2. \tag{7}$$

Since e_1 is a maximal direction, we have at the point x for any $t \in \mathbb{R}$

$$\|h(e_1 + te_i, e_1 + te_i)\|^2 \leq (1+t^2)^2\,(h^{n+1}_{11})^2. \tag{8}$$

Expanding in terms of t, we obtain

$$4\,t\,h^{n+1}_{11}\,h^{n+1}_{1i} + O(t^2) \leq 0.$$

It follows that

$$h^{n+1}_{1i} = 0, \quad i = 2, \ldots, n. \tag{9}$$

It is easy to see that e_1 is also an eigenvector of the Weingarten transformation A^{n+1}. Therefore, we can choose an adapted frame at $x \in M$ such that in addition to (7) and (9),

$$h^{n+1}_{ij} = 0, \quad i \neq j. \tag{10}$$

Once more expanding (8) in terms of t, we obtain

$$-2t^2\Big(h^{n+1}_{11}(h^{n+1}_{11} - h^{n+1}_{ii}) - 2\sum_{\alpha\neq n+1}(h^{\alpha}_{1i})^2\Big) + O(t^3) \leq 0.$$

It follows that

$$2\sum_{\alpha\neq n+1}(h^{\alpha}_{1i})^2 \leq h^{n+1}_{11}(h^{n+1}_{11} - h^{n+1}_{ii}) \quad \text{for } i = 2, \ldots, n. \tag{11}$$

Define a tensor field $T = (T_{ijkl})$ on M by

$$T_{ijkl} = \sum_{\alpha} h^{\alpha}_{ij}\,h^{\alpha}_{kl}.$$

It is obvious that $\sigma(u) = T(u, u, u, u)$.

Lemma 3.1. *Let u be a maximal direction at $x \in M$. Assume that $\sigma(u) \neq 0$. Let $e_1, \ldots, e_{n+p}$ be an adapted frame at x such that*

$$e_1 = u, \quad e_{n+1} = \frac{h(e_1, e_1)}{\|h(e_1, e_1)\|},$$

$h^{n+1}_{ij} = 0$ for $i \neq j$, and $e_{n+2} = (\xi - \langle\xi,\ e_{n+1}\rangle e_{n+1})/\|\xi - \langle\xi,\ e_{n+1}\rangle e_{n+1}\|$ if ξ is not parallel to e_{n+1}. At the point x,

(i) *if $p = 1$, or $p = 2$ and $H \neq 0$, then*

$$\tfrac{1}{2}(\Delta T)_{1111} \geq h^{n+1}_{11}\Big(n\big(h^{n+1}_{11} + c_{n+1}(h^{n+1}_{11})^2 - c_{n+1}\big) - h^{n+1}_{11}\sum_{k}(h^{n+1}_{kk})^2\Big); \tag{12}$$

(ii) *if $p \geq 3$, or $p = 2$ and $H = 0$, then*

$$\tag{13} \begin{aligned} &\tfrac{1}{2}(\Delta T)_{1111} \\ &\quad \geq h_{11}^{n+1}\Big(n(h_{11}^{n+1} + 3c_{n+1}(h_{11}^{n+1})^2 - c_{n+1} - (h_{11}^{n+1})^3) - 2h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2\Big), \end{aligned}$$

and equality holds if and only if

$$\tag{14} (h_{11}^{n+1} - h_{kk}^{n+1})\Big(h_{11}^{n+1}(h_{11}^{n+1} - h_{kk}^{n+1}) - 2\sum_{\alpha\neq n+1}(h_{1k}^{\alpha})^2\Big) = 0$$

and $h_{11k}^{\alpha} = 0$, for all k and α.

Proof. We have

$$\tag{15} \tfrac{1}{2}(\Delta T)_{1111} = h_{11}^{n+1}\Delta h_{11}^{n+1} + \sum_{i,\alpha}(h_{11i}^{\alpha})^2.$$

From Proposition 2.1 and equations (7) and (10), we have

$$\begin{aligned} \Delta h_{11}^{n+1} &= \sum_{k,m} h_{km}^{n+1}R_{m11k} + \sum_{k,m} h_{m1}^{n+1}R_{mk1k} + \sum_{k,\alpha} h_{1k}^{\alpha}R_{\alpha n+11k} \\ &= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1})R_{1k1k} + \sum_{k,\alpha} h_{1k}^{\alpha}\Big(\sum_l (h_{l1}^{\alpha}h_{lk}^{n+1} - h_{lk}^{\alpha}h_{l1}^{n+1})\Big) \\ &= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1})\Big(1 - (\delta_{1k})^2 + \sum_\alpha (h_{11}^{\alpha}h_{kk}^{\alpha} - (h_{1k}^{\alpha})^2)\Big) \\ &\qquad\qquad + \sum_{k,\alpha}(h_{1k}^{\alpha})^2(h_{kk}^{n+1} - h_{11}^{n+1}) \\ &= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1}) + \sum_k (h_{11}^{n+1} - h_{kk}^{n+1})h_{11}^{n+1}h_{kk}^{n+1} - 2\sum_{k,\alpha}(h_{1k}^{\alpha})^2(h_{11}^{n+1} - h_{kk}^{n+1}) \\ &= n\big(h_{11}^{n+1} + c_{n+1}(h_{11}^{n+1})^2 - c_{n+1}\big) - h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2 \\ &\qquad\qquad - 2\sum_{k,\alpha\neq n+1}(h_{1k}^{\alpha})^2(h_{11}^{n+1} - h_{kk}^{n+1}). \end{aligned}$$

If $p = 1$, the last term above vanishes. If $p = 2$ and $H \neq 0$, we have $R_{(n+1)(n+2)kl} = 0$ for any k, l, by (5) and (7); hence the last term above vanishes again. If $p \geq 3$, or if or $p = 2$ and $H = 0$, we obtain by (11)

$$\begin{aligned} \Delta h_{11}^{n+1} &\geq n\big(h_{11}^{n+1} + c_{n+1}(h_{11}^{n+1})^2 - c_{n+1}\big) - h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2 - \sum_k h_{11}^{n+1}(h_{11}^{n+1} - h_{kk}^{n+1})^2 \\ &= n\big(h_{11}^{n+1} + 3c_{n+1}(h_{11}^{n+1})^2 - c_{n+1} - (h_{11}^{n+1})^3\big) - 2h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2. \end{aligned}$$

Substituting this into (15), we obtain

$$\tfrac{1}{2}(\Delta T)_{1111} \geq h_{11}^{n+1}\Big(n(h_{11}^{n+1} + c_{n+1}(h_{11}^{n+1})^2 - c_{n+1}) - h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2\Big)$$

if $p=1$ or $p=2$ and $H\neq 0$, and

$$\tfrac{1}{2}(\Delta T)_{1111}\geq h_{11}^{n+1}\Big(n(h_{11}^{n+1}+3c_{n+1}(h_{11}^{n+1})^2-c_{n+1}-(h_{11}^{n+1})^3)-2h_{11}^{n+1}\sum_k(h_{kk}^{n+1})^2\Big)$$

if $p\geq 3$ or $p=2$ and $H=0$. □

Lemma 3.2. *Let $\{e_1,\dots,e_{n+p}\}$ be an adapted frame at $x\in M$ as in Lemma 3.1. Suppose that*

$$\sigma(u)\leq\begin{cases}1 & \text{if } p=1 \text{ or } p=2 \text{ and } H\neq 0,\\ \frac{1}{3} & \text{if } p\geq 3 \text{ or } p=2 \text{ and } H=0,\end{cases}$$

for all $u\in UM$. Then $(\Delta T)_{1111}\geq 0$. If equality holds, i.e., if $(\Delta T)_{1111}=0$, then

$$h_{11}^{n+1}=\big|h_{22}^{n+1}\big|=\cdots=\big|h_{nn}^{n+1}\big|. \tag{16}$$

Proof. Since e_1 is a maximal direction at $x\in M$,

$$-h_{11}^{n+1}\leq h_{kk}^{n+1}\leq h_{11}^{n+1},\quad k=2,\dots,n. \tag{17}$$

It is clear that the convex function $f(h_{22}^{n+1},\dots,h_{nn}^{n+1})=\sum_{k=2}^{n}(h_{kk}^{n+1})^2$ subject to the constraint (17) attains its maximal value when

$$\big|h_{22}^{n+1}\big|=\cdots=\big|h_{nn}^{n+1}\big|=h_{11}^{n+1}.$$

Therefore, by inequalities (12) and (13),

$$\tfrac{1}{2}(\Delta T)_{1111}\geq\begin{cases}nh_{11}^{n+1}(h_{11}^{n+1}-c_{n+1})(1-\sigma(e_1)) & \text{if } p=1 \text{ or } p=2 \text{ and } H\neq 0,\\ nh_{11}^{n+1}(h_{11}^{n+1}-c_{n+1})(1-3\sigma(e_1)) & \text{if } p\geq 3 \text{ or } p=2 \text{ and } H=0,\end{cases}$$

where $c_{n+1}=(1/n)\sum_{i=1}^{n}h_{ii}^{n+1}\leq h_{11}^{n+1}$. □

Let $L(x)$ be a function on M defined by $L(x)=\max_{u\in UM_x}\sigma(u)$. By a similar argument as in [Gauchman 1986], we get:

Lemma 3.3. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. If*

$$\sigma(u)\leq\begin{cases}1, & \text{for } p=1, \text{ or } p=2 \text{ and } H\neq 0\\ \frac{1}{3}, & \text{for } p\geq 3, \text{ or } p=2 \text{ and } H{=}0,\end{cases}$$

for all $u\in UM$, then $L(x)$ is a constant function on M.

4. Rigidity of submanifolds of parallel mean curvature

This section is devoted to the proof of the Main Theorem 1.6, through a series of intermediate results.

Lemma 4.1. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. Suppose that*

$$\sigma(u) < \begin{cases} 1 & \text{if } p = 1 \text{ or } p = 2 \text{ and } H \neq 0, \\ \frac{1}{3} & \text{if } p \geq 3 \text{ or } p = 2 \text{ and } H = 0, \end{cases}$$

for all $u \in UM$. Then M is the totally umbilical sphere S^n_H.

Proof. Let e_1 be a maximal direction at $x \in M$. Assume $\sigma(e_1) \neq 0$. By Lemmas 3.2 and 3.3, we have $(\Delta T)_{1111} = 0$ on M. From the proof of Lemma 3.2, we see that

$$h^{n+1}_{11} = c_{n+1}.$$

Thus the average value of the $\{h^{n+1}_{ii}\}^n_{i=1}$ equals their maximum. This possibility occurs if and only if

$$h^{n+1}_{11} = \cdots = h^{n+1}_{nn}.$$

This and (11) yield $h^\alpha_{1i} = 0$, for $\alpha \neq n+1$ and $i = 2, \ldots, n$. Since each of the vectors e_i, for $i = 1, \ldots, n$, is a maximal direction, we have

$$h^\alpha_{ij} = 0 \quad \text{for } i, j = 1, 2, \ldots, n \text{ and } i \neq j.$$

From $\|h(e_i, e_i)\|^2 \leq (h^{n+1}_{11})^2$, we obtain

$$h^\alpha_{ii} = 0 \quad \text{for } \alpha \neq n+1 \text{ and } i = 1, 2, \ldots, n.$$

The last three displayed equations say that M is a totally umbilical sphere. □

For convenience, we establish a convention on indices $a, b, \ldots, r, s, \ldots$:

$$1 \leq a, b, c, d \leq k < r, s, t, w \leq n,$$

where k is a fixed integer in the range $1, \ldots, n$.

Here is the rigidity theorem for hypersurfaces with constant mean curvature in a sphere:

Theorem 4.2. *Let M be an n-dimensional compact hypersurface with constant mean curvature in a unit sphere $S^{n+1}(1)$.*

(i) *If $\sigma(u) < 1$ for any $u \in UM$, then M is the totally umbilical sphere S^n_H.*

(ii) *If $\max_{u \in UM} \sigma(u) = 1$, M is one of the embeddings $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$, with $k = 1, 2, \ldots, n$.*

Proof. Assertion (i) follows from Lemma 4.1. We prove (ii). As in the proof of Lemma 4.1, $(\Delta T)_{1111} = 0$. By (16), we may assume after a suitable renumbering of $e_1, \ldots, e_n$ that

$$h^{n+1}_{aa} = -h^{n+1}_{rr} = 1 \quad \text{for } a = 1, \ldots, k \text{ and } r = k+1, \ldots, n.$$

By Lemma 3.1, h^{n+1}_{11k} vanishes for $k = 1, \ldots, n$. It follows that $h^{n+1}_{iik} = 0$. By polarization, h^{n+1}_{ijk} vanishes for all i, j, k. By (2) and (10), we have

$$0 = \sum_l h^{n+1}_{il}\omega_{lj} + \sum_l h^{n+1}_{lj}\omega_{li} = (h^{n+1}_{ii} - h^{n+1}_{jj})\omega_{ij}.$$

Hence, $\omega_{ar} = 0$. It follows that the two distributions defined by $\omega_1 = \cdots = \omega_k = 0$ and $\omega_{k+1} = \cdots = \omega_n = 0$ are integrable and give a local decomposition of M. Then every point of M has a neighborhood U which is a Riemannian product $V_1 \times V_2$ with $\dim V_1 = k$ and $\dim V_2 = n-k$. The curvatures of V_1 and V_2 are

$$R_{abcd} = 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \quad \text{for } 1 \le a, b, c, d \le k,$$
$$R_{rstw} = 2(\delta_{rt}\delta_{sw} - \delta_{rw}\delta_{st}) \quad \text{for } k+1 \le r, s, t, w \le n.$$

Thus V_1 and V_2 are spaces of constant curvature 2. The compactness of M allows us to complete the proof. □

For the case of codimension two:

Theorem 4.3. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+2}(1)$, $H \neq 0$.*

(i) *If $\sigma(u) < 1$ for any $u \in UM$, then M is the totally umbilical sphere S^n_H.*

(ii) *If $\max_{u\in UM} \sigma(u) = 1$, M is one of the embeddings $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$, with $k = 1, \ldots, n$, $k \neq \frac{1}{2}n$.*

Proof. Assertion (i) follows from Lemma 4.1. We prove (ii). As in the proof of Lemma 4.1, $(\Delta T)_{1111} = 0$. By (16), we have

$$h^{n+1}_{aa} = -h^{n+1}_{rr} = 1 \quad \text{for } a = 1, \ldots, k \text{ and } r = k+1, \ldots, n.$$

From (7) and (11) we obtain $h^{n+2}_{1a} = 0$ for $a = 1, \ldots, k$. Since each of vectors e_i, for $i = 1, \ldots, n$, is a maximal direction, we get

$$h^{n+2}_{ab} = 0 \quad \text{for } a, b = 1, \ldots, k.$$

Similarly,

$$h^{n+2}_{rs} = 0 \quad \text{for } r, s = k+1, \ldots, n.$$

As in the proof of Lemma 3.1, we have $R_{(n+1)(n+2)kl} = 0$. Hence

$$h^{n+2}_{kl}(h^{n+1}_{kk} - h^{n+1}_{ll}) = 0,$$

which implies $h^{n+2}_{ar} = 0$ for $a = 1, \ldots, k$ and $r = k+1, \ldots, n$. Thus

$$h^{n+2}_{ij} = 0 \quad \text{for } i, j = 1, \ldots, n. \tag{18}$$

By a similar argument as in the proof of Theorem 4.2, we have $h_{ijk}^{n+1} = 0$ for all i, j, k. By (2), (10) and (18), we have

$$0 = \sum_l h_{il}^{n+1}\omega_{lj} + \sum_l h_{jl}^{n+1}\omega_{li} = (h_{ii}^{n+1} - h_{jj}^{n+1})\omega_{ij}.$$

Therefore, $\omega_{ar} = 0$. Then M is a locally Riemannian product $V_1 \times V_2$, with $\dim V_1 = k$ and $\dim V_2 = n-k$. The curvature of V_1 is

$$\begin{aligned} R_{abcd} &= \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + \sum_{\alpha=n+1}^{n+2} (h_{ac}^{\alpha}h_{bd}^{\alpha} - h_{ad}^{\alpha}h_{bc}^{\alpha}) \\ &= \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + h_{ac}^{n+1}h_{bd}^{n+1} - h_{ad}^{n+1}h_{bc}^{n+1} = 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \end{aligned}$$

(see (1)), where the second equality follows from (18). A similar argument applies to V_2. In conclusion, V_1 and V_2 are spaces of constant curvature 2. The compactness of M allows us to complete the proof. □

Remark 4.4. In assertion (ii) of Theorem 4.3, we exclude the case of $n = 2m$ even and $k = m$, in that it results in $H = 0$, contradicting the theorem's assumption.

Let F be the real numbers, the complex numbers, or the quaternions, and let d be the dimension of F as a real vector space (1, 2, or 4). Let FP^m denote the projective space over F, $M^n(c)$ the n-dimensional Riemannian manifold with constant curvature c.

Lemma 4.5 [Sakamoto 1977]. *Let $f : M^n \to S^{n+p}(\bar{c})$ be an isotropic immersion of parallel second fundamental tensor. Except for the totally umbilical case, f is a composition of a minimal isotropic immersion $\eta : M^n \to S^{n+q}(\tilde{c})$ $(q \le p)$ of parallel second fundamental tensor, and a totally umbilical $\tau : S^{n+q}(\tilde{c}) \to S^{n+p}(\bar{c})$, where $n = md$ and M must be one of $S^n(c)$, FP^m and $CayP^2$. Assume that the isotropic constants of f and η are λ and μ respectively. Then*

$$c = \frac{m}{2(m+1)}\tilde{c}, \quad q = \frac{(m-1)(md+2)}{2}, \quad \mu^2 = \frac{m-1}{m+1}\tilde{c},$$

where $m = n$ if $M = S^n(c)$ and $m = 2$ if $M = CayP^2$.

Lemma 4.6. *Let $f : M^n \to S^{n+p}(1)$ be a λ-isotropic immersion of parallel second fundamental tensor. If $\lambda^2 \le (m-1)/(m+1)$, then f is totally umbilical, or minimal with $\lambda^2 = (m-1)/(m+1)$.*

Proof. Assume that f is not totally umbilical. Following Lemma 4.5, f can be considered as composition of a minimal μ-isotropic immersion $\eta : M^n \to S^{n+q}(\tilde{c})$ and a totally umbilical sphere $\tau : S^{n+q}(\tilde{c}) \to S^{n+p}(1)$, where μ and $\tilde{c}$ satisfy

$$\mu^2 = \frac{m-1}{m+1}\tilde{c}.$$

On the other hand, if H is the mean curvature of immersion f, it is easy to see

$$\mu^2 + H^2 = \lambda^2, \quad \tilde{c} =\overset{-}{\rightarrow} c + H^2.$$

Substituting into the preceding equation, we get

$$\lambda^2 - \frac{m-1}{m+1} = \frac{2m}{m+1} H^2 \geq 0. \tag{19}$$

The assumption $\lambda^2 \leq (m-1)/(m+1)$ and (19) together give

$$\lambda^2 = \frac{m-1}{m+1} \quad \text{and} \quad H = 0. \qquad \square$$

Theorem 4.7. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. Assume that $p \geq 3$, or $p = 2$ and $H = 0$.*

(i) *If $\sigma(u) < \frac{1}{3}$ for any $u \in UM$, then M is the totally umbilical sphere S^n_H.*

(ii) *If $\max_{u \in UM} \sigma(u) = \frac{1}{3}$, then $\sigma(u) \equiv \frac{1}{3}$ on UM, and the isometric immersion of M into S^{n+p} is either the totally umbilical sphere $S^n(\sqrt{3}/2) \to S^{n+p}(1)$, one of the embeddings $\phi_{i,p}$, $i = 1, 2, 3, 4$, or one of the immersions $\phi'_{1,p}$ described above.*

Proof. We need only consider the case $\max_{v \in UM_x} \sigma(v) = \sigma(u)$. As in the proof of Lemma 4.1, we obtain $(\Delta T)_{1111} = 0$. By (16), we have, after a suitable renumbering of $e_1, \ldots, e_n$,

$$h^{n+1}_{aa} = -h^{n+1}_{rr} = \frac{\sqrt{3}}{3} \quad \text{for } a = 1, \ldots, k \text{ and } r = k+1, \ldots, n. \tag{20}$$

Since $\|h(e_a, e_a)\|^2 \leq \frac{1}{3}$ and $\|h(e_r, e_r)\|^2 \leq \frac{1}{3}$, we obtain

$$h^\alpha_{aa} = h^\alpha_{rr} = 0 \quad \text{for } \alpha \neq n+1,\ a = 1, \ldots, k \text{ and } r = k+1, \ldots, n. \tag{21}$$

Still from (11),

$$h^\alpha_{ab} = h^\alpha_{rs} = 0 \quad \text{for } \alpha \neq n+1,\ a, b = 1, \ldots, k \text{ and } r, s = k+1, \ldots, n. \tag{22}$$

By (14), $\sum_{\alpha \neq n+1} (h^\alpha_{1r})^2 = \frac{1}{3}$. Since each vector e_i, for $i = 1, \ldots, n$, is a maximal direction,

$$\sum_{\alpha \neq n+1} (h^\alpha_{ar})^2 = \tfrac{1}{3} \quad \text{for } a = 1, \ldots, k \text{ and } r = k+1, \ldots, n. \tag{23}$$

For $x^2, \ldots, x^n$ and $t \in \mathbb{R}$, using (20)–(23) and (7)–(10), expanding the inequality

$$\left\| h\left(e_1 + t \sum_{i=2}^n x^i e_i,\ e_1 + t \sum_{i=2}^n x^i e_i \right) \right\|^2 \leq \left(1 + t^2 \sum_{i=2}^n (x^i)^2 \right)^2 (h^{n+1}_{11})^2 \tag{24}$$

in terms of t, we obtain

$$4t^2 \sum_{\alpha} \sum_{r,s} h^{\alpha}_{1r} h^{\alpha}_{1s} x^r x^s + O(t^3) \leq 0.$$

It follows that $\sum_{\alpha} h^{\alpha}_{1r} h^{\alpha}_{1s} = 0$ if $r \neq s$. Since each vector e_i is a maximal direction, we have

$$\sum_{\alpha} h^{\alpha}_{ar} h^{\alpha}_{as} = 0 \quad \text{if } r \neq s, \qquad \sum_{\alpha} h^{\alpha}_{ar} h^{\alpha}_{br} = 0 \quad \text{if } a \neq b.$$

Once more expand (24) to obtain

$$2t^3 \sum (h^{\alpha}_{1r} h^{\alpha}_{bs} + h^{\alpha}_{1s} h^{\alpha}_{br}) x^a x^r x^s + O(t^4) \leq 0.$$

It follows that

$$\sum_{\alpha} (h^{\alpha}_{ar} h^{\alpha}_{bs} + h^{\alpha}_{as} h^{\alpha}_{br}) = 0 \quad \text{if } a \neq b \text{ or } r \neq s. \tag{25}$$

Using (10) and (20)–(25), we obtain by direct computation that $\sigma(u) \equiv \frac{1}{3}$ for any $u \in UM$. It is easy to see that $h^{\alpha}_{ijk} = 0$ for all α, i, j, k. Therefore, M is a $(\sqrt{3}/3)$-isotropic submanifold in a unit sphere of parallel second fundamental tensor. By Lemmas 4.5 and 4.6 we know that M is either totally umbilical or minimal. This, together with a [Gauchman 1986, Theorem 3], completes the proof. □

Theorems 4.2, 4.3 and 4.7 together imply the Main Theorem 1.6.

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