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We prove various results on contact structures obtained by contact surgery on a single Legendrian knot in the standard contact three-sphere. Our main tools are the contact Ozsváth–Szabó invariants.

1. Introduction

According to a result of Ding and Geiges [2004], any closed contact three-manifold is obtained by contact surgery along a Legendrian link $\mathbb{L}$ in the standard contact three-sphere (S^3, ξ_{st}), where the surgery coefficients on the individual components of $\mathbb{L}$ can be chosen to be ± 1 relative to the contact framing. (For additional discussion on this theorem, see [Ding et al. 2004].) It is an intriguing question how to establish interesting properties of a contact structure from one of its surgery presentations. More precisely, we would like to find a way to determine whether the result of a certain contact surgery is tight or fillable. Recall that contact (-1)-surgery (also called *Legendrian surgery*) on a Legendrian link $\mathbb{L}$ produces a Stein fillable, hence tight contact three-manifold.

Given a Legendrian knot $K \subset (S^3, \xi_{st})$, the result of contact $(+1)$-surgery along K is denoted by (Y_K, ξ_K). Here is a first result, which has an elementary proof:

Theorem 1.1. *Let K be a Legendrian knot in the standard contact three-sphere. Assume that, for some orientation of K, a front projection of K contains the configuration of Figure 1, with an odd number of cusps from the strand U to the strand U' as the knot is traversed in the direction of the orientation. Then (Y_K, ξ_K) is overtwisted.*

Corollary 1.2. *Let K be a Legendrian knot in the standard contact three-sphere. If K is smoothly isotopic to a negative torus knot, then (Y_K, ξ_K) is overtwisted.*

Notice the contrast: when the Legendrian knot K satisfies $\mathrm{tb}(K) = 2g_s(K) - 1$ (where $\mathrm{tb}(K)$ is the Thurston–Bennequin invariant of K, and $g_s(K)$ denotes its slice genus) — for example, if K is a *positive* torus knot — then (Y_K, ξ_K) is tight

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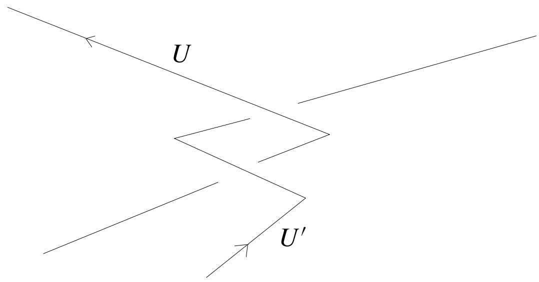

Figure 1. Configuration producing an overtwisted disk.

[Lisca and Stipsicz 2004a]. The tightness question for contact structures can be fruitfully attacked with the use of the contact Ozsváth–Szabó invariants [Ozsváth and Szabó 2005]. In fact, the nonvanishing of these invariants implies tightness, while their computation can sometimes be performed (see, e.g., [Lisca and Stipsicz 2004a; 2004b]) using a contact surgery presentation in conjunction with the surgery exact triangle established in Heegaard Floer theory by Peter Ozsváth and Zoltán Szabó [2003a]. Such ideas can be used to prove the next theorem.

Let $S^3_n(K)$ denote the three-manifold obtained by performing Dehn surgery along the knot $K \subset S^3$ with surgery coefficient n.

Theorem 1.3. *Let $K \subset S^3$ be a smooth knot. Suppose that, for some integer $n > 0$, the three-manifold $S^3_n(K)$ is a lens space. Let $L \subset (S^3, \xi_{st})$ be a Legendrian knot smoothly isotopic to K. Then L has Thurston–Bennequin invariant at most n.*

In the proof of Theorem 1.3 we will assume only that $S^3_n(K)$ is an L-space, a weaker condition specified in Section 2 and known to be satisfied by lens spaces. It should be noted that, in view of [Lisca and Stipsicz 2004a, Proposition 4.1], if $S^3_n(K)$ is an L-space for some $n > 0$, then $S^3_{2g_s(K)-1}(K)$ is an L-space as well, where $g_s(K)$ is the four-ball genus of K. Therefore, the upper bounds on the Thurston–Bennequin invariants of Legendrian knots coming from Theorem 1.3 are never strictly weaker than the ones coming from the slice Bennequin inequality [Rudolph 1993]. On the other hand, the authors do not know of an example for which such bounds are strictly stronger than the ones coming from the slice Bennequin inequality. We also observe that the same bounds easily follow from [Ozsváth and Szabó 2004a, Theorem 1.4], which requires a more involved machinery.

In our investigations we prove tightness by establishing the nonvanishing of the appropriate contact Ozsváth–Szabó invariant. Therefore, we are interested in cases when this invariant vanishes although overtwistedness does not obviously hold.

Proposition 1.4. *Let $L_1, L_2 \subset (S^3, \xi_{st})$ be two smoothly isotopic Legendrian knots whose Thurston–Bennequin invariants satisfy*

$$\mathrm{tb}(L_1) < \mathrm{tb}(L_2).$$

Then the result of contact $(+1)$*-surgery along* L_1 *has a vanishing contact Ozsváth–Szabó invariant. If* $\mathrm{tb}(L) \leq -2$, *the contact Ozsváth–Szabó invariant* $c^+(Y_L, \xi_L)$ *vanishes.*

Remark. The hypotheses of Proposition 1.4 do not imply that either L_1 or L_2 is a stabilization of other Legendrian knots. In fact, examples of Legendrian knots L_1 and L_2 satisfying the assumptions of Proposition 1.4 without being stabilizations were found by Etnyre and Honda [2005].

In many cases the contact invariants can be explicitly computed. We perform such computations for the *Chekanov–Eliashberg knots*, a subfamily of Legendrian knots; see [Epstein et al. 2001]. These knots are of particular interest because they have equal *classical invariants* (i.e., knot type, Thurston–Bennequin invariant and rotation number), but are not Legendrian isotopic. Our computation shows that, at least when combined with the particular surgery approach we adopt here, the contact Ozsváth–Szabó invariant is not strong enough to distinguish these knots up to Legendrian isotopy. For the precise formulation of this fact, see Section 4.

As a further application, we present examples where the contact Ozsváth–Szabó invariants distinguish contact structures defined on a fixed three-manifold. In particular, by a simple calculation we recover the main result of [Lisca and Matić 1997]:

Theorem 1.5 [Lisca and Matić 1997]. *The Brieskorn integral homology sphere* $-\Sigma(2, 3, 6n-1)$ *admits at least* $(n-1)$ *nonisotopic tight contact structures.*

Remark. O. Plamenevskaya [2004] obtained the same result in a more general form.

Section 2 is devoted to the necessary (and brief) recollection of background information about contact surgery and Ozsváth–Szabó invariants. Proofs of most of the statements announced in the Introduction are given in Section 3. Section 4 is devoted to the Legendrian Chekanov–Eliashberg knots. In Section 5 we prove Theorem 1.5.

2. Preliminaries

For the basics of contact geometry and topology we refer the reader to [Etnyre 2003; Geiges 2006].

Contact surgery. Let (Y, ξ) be a closed, contact three-manifold and $L \subset (Y, \xi)$ a Legendrian knot. The contact structure ξ can be extended from the complement of a neighborhood of L to the three-manifold obtained by (± 1)-surgery along L (with respect to the contact framing). In fact, by the classification of tight contact structures on the solid torus $S^1 \times D^2$ [Honda 2000], such an extension is uniquely specified by requiring that its restriction to the surgered solid torus be tight. The same uniqueness property holds for all surgery coefficients of the form $1/k$ with $k \in \mathbb{Z}$. For a general nonzero rational surgery coefficient, there is a finite number of choices for the tight extension. Consequently, a Legendrian knot $L \subset (S^3, \xi_{st})$ decorated with $+1$ or -1 gives rise to a well-defined contact three-manifold, which we denote by (Y_L, ξ_L) and (Y^L, ξ^L), respectively. For a more extensive discussion on contact surgery, see [Ding and Geiges 2004].

Heegaard Floer theory. In this subsection we recall the basics of the Ozsváth–Szabó homology groups. For a more detailed treatment, see [Ozsváth and Szabó 2004b; 2004c; 2006].

According to [Ozsváth and Szabó 2004c], one can associate to a closed, oriented spin^c three-manifold $(Y, \mathbf{t})$ a finitely generated abelian group $\widehat{HF}(Y, \mathbf{t})$ and a finitely generated $\mathbb{Z}[U]$-module $HF^+(Y, \mathbf{t})$. A spin^c cobordism $(W, \mathbf{s})$ between $(Y_1, \mathbf{t}_1)$ and $(Y_2, \mathbf{t}_2)$ gives rise to homomorphisms

$$\hat{F}_{W,\mathbf{s}} \colon \widehat{HF}(Y_1, \mathbf{t}_1) \to \widehat{HF}(Y_2, \mathbf{t}_2) \quad \text{and} \quad F^+_{W,\mathbf{s}} \colon HF^+(Y_1, \mathbf{t}_1) \to HF^+(Y_2, \mathbf{t}_2),$$

with $F^+_{W,\mathbf{s}}$ U-equivariant.

Let Y be a closed, oriented three-manifold and $K \subset Y$ a framed knot with framing f. Let $Y(K)$ denote the three-manifold given by surgery along $K \subset Y$ with respect to this framing. The surgery can be viewed at the four-manifold level as a two-handle addition. The resulting cobordism X induces a homomorphism

$$\hat{F}_X := \sum_{\mathbf{s} \in \mathrm{Spin}^c(X)} \hat{F}_{X,\mathbf{s}} \colon \widehat{HF}(Y) \to \widehat{HF}(Y(K)),$$

where

$$\widehat{HF}(Y) := \bigoplus_{\mathbf{t} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathbf{t}).$$

Similarly, there is a cobordism Z defined by adding a two-handle to $Y(K)$ along a normal circle N to K with framing -1 with respect to a normal disk to K. The boundary components of Z are $Y(K)$ and the three-manifold $Y'(K)$ obtained from Y by a surgery along K with framing $f+1$. As before, Z induces a homomorphism

$$\hat{F}_Z \colon \widehat{HF}(Y(K)) \to \widehat{HF}(Y'(K)).$$

The construction above can be repeated starting with $Y(K)$ and $N \subset Y(K)$ equipped with the framing specified above: we get Z (playing the role of X) and a new cobordism W starting from $Y'(K)$, given by attaching a four-dimensional two-handle along a normal circle C to N with framing -1 with respect to a normal disk. It is easy to check that this last operation yields Y at the three-manifold level.

Theorem 2.1 [Ozsváth and Szabó 2004b, Theorem 9.16]. *The homomorphisms* $\hat{F}_X$, $\hat{F}_Z$ *and* $\hat{F}_W$ *fit into an exact triangle*

$$\begin{tikzcd} \widehat{HF}(Y) \arrow[rr, "\hat{F}_X"] & & \widehat{HF}(Y(K)) \arrow[dl, "\hat{F}_Z"] \\ & \widehat{HF}(Y'(K)) \arrow[ul, "\hat{F}_W"] & \end{tikzcd}$$ □

For a torsion spin^c structure (i.e., a spin^c structure whose first Chern class is torsion), the homology theories $\widehat{HF}$ and HF^+ come with a relative $\mathbb{Z}$-grading that admits a lift to an absolute $\mathbb{Q}$-grading [Ozsváth and Szabó 2003a]. The action of U shifts this degree by -2.

For $a \in \mathbb{Q}$, define $\mathcal{T}_a^+ := \bigoplus_b (\mathcal{T}_a^+)_b$ as the graded $\mathbb{Z}[U]$-module such that, for every $b \in \mathbb{Q}$,

$$(\mathcal{T}_a^+)_b = \begin{cases} \mathbb{Z} & \text{for} \quad b \geq a \quad \text{and} \quad b - a \in 2\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

and the U-action $(\mathcal{T}_a^+)_b \to (\mathcal{T}_a^+)_{b-2}$ is an isomorphism for every $b \neq a$. The following proposition can be extracted from [Ozsváth and Szabó 2003a, Propositions 4.2 and 4.10; 2004b, Theorem 10.1].

Proposition 2.2. *Let Y be a rational homology sphere. Then, for each* $\mathbf{t} \in \mathrm{Spin}^c(Y)$,

$$HF^+(Y, \mathbf{t}) = \mathcal{T}_a^+ \oplus A(Y),$$

where $a \in \mathbb{Q}$, and $A(Y) = \bigoplus_d A_d(Y)$ is a graded, finitely generated abelian group. Moreover,

$$HF^+(-Y, \mathbf{t}) = \mathcal{T}_{-a}^+ \oplus A(-Y),$$

with $A_d(-Y) \cong A_{-d-1}(Y)$. If $b_1(Y) = 1$ and $\mathbf{t} \in \mathrm{Spin}^c(Y)$ is torsion then

$$HF^+(Y, \mathbf{t}) = \mathcal{T}_a^+ \oplus \mathcal{T}_{a'}^+ \oplus A'(Y),$$

where $a - a'$ is an odd integer, and $A'(Y) = \bigoplus_d A'_d(Y)$ is a graded, finitely generated abelian group. Moreover,

$$HF^+(-Y, \mathbf{t}) = \mathcal{T}_{-a}^+ \oplus \mathcal{T}_{-a'}^+ \oplus A'(Y),$$

with $A'_d(-Y) \cong A'_{-d-1}(Y)$. □

The two theories $\widehat{HF}$ and HF^+ are related by a long exact sequence, which takes the form

$$\text{(2–1)} \quad \cdots \to \widehat{HF}_a(Y, \mathbf{t}) \xrightarrow{f} HF_a^+(Y, \mathbf{t}) \xrightarrow{U} HF_{a-2}^+(Y, \mathbf{t}) \to \widehat{HF}_{a-1}(Y, \mathbf{t}) \to \cdots$$

for a torsion spin^c structure $\mathbf{t}$, where the map U denotes multiplication by U. All the gradings appearing in the sequence can be worked out from the definitions and the construction of the exact sequence [Ozsváth and Szabó 2003a, Section 2].

Corollary 2.3. *Let Y be a rational homology three-sphere. Then $HF^+(Y, \mathbf{t}) \cong \mathcal{T}_a^+$ if and only if $\widehat{HF}(Y, \mathbf{t}) \cong \mathbb{Z}$. If $b_1(Y) = 1$ and $\mathbf{t}$ is a torsion spin^c structure, then $HF^+(Y, \mathbf{t}) \cong \mathcal{T}_{a_1}^+ \oplus \mathcal{T}_{a_2}^+$ if and only if $\widehat{HF}(Y, \mathbf{t}) \cong \mathbb{Z}^2$.*

Proof. We sketch the proof of the statement for $b_1(Y) = 0$; the other case can be proved by similar arguments. Clearly, if $HF^+(Y, \mathbf{t}) \cong \mathcal{T}_a^+$ then it follows immediately from Exact Sequence (2–1) that $\widehat{HF}(Y, \mathbf{t}) = \widehat{HF}_a(Y, \mathbf{t}) \cong \mathbb{Z}$. Conversely, if $\widehat{HF}(Y, \mathbf{t}) \cong \mathbb{Z}$, then Exact Sequence (2–1) and Proposition 2.2 imply $HF^+(Y, \mathbf{t}) \cong \mathcal{T}_a^+$. □

Observe that, in view of Corollary 2.3, if Y is a rational homology three-sphere, the two conditions are equivalent:

(i) For each spin^c structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$, we have $HF^+(Y, \mathbf{t}) \cong \mathcal{T}_a^+$ for some a.

(ii) For each spin^c structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$, we have $\widehat{HF}(Y, \mathbf{t}) \cong \mathbb{Z}$.

Definition. A rational homology three-sphere satisfying these two equivalent conditions is called an *L-space*.

It follows from Proposition 2.2 that an oriented rational homology three-sphere Y is an L-space if and only if $-Y$ is an L-space. Moreover, lens spaces are L-spaces [Ozsváth and Szabó 2004b, Section 3].

We use the following fact regarding the maps connecting the Ozsváth–Szabó homology groups. Suppose that W is a cobordism defined by a single two-handle attachment.

Proposition 2.4 [Lisca and Stipsicz 2004a]. *Let W be a cobordism containing a smooth, closed, oriented surface Σ of genus g, with $\Sigma \cdot \Sigma > 2g - 2$. Then the induced maps $\hat{F}_{W,\mathbf{s}}$ and $F^+_{W,\mathbf{s}}$ vanish for all spin^c structures $\mathbf{s}$ on W.* □

Contact Ozsváth–Szabó invariants. Let (Y, ξ) be a closed, contact three-manifold. Then the contact Ozsváth–Szabó invariants

$$\hat{c}(Y, \xi) \in \widehat{HF}(-Y, \mathbf{t}_\xi)/\langle \pm 1\rangle \quad \text{and} \quad c^+(Y, \xi) \in HF^+(-Y, \mathbf{t}_\xi)/\langle \pm 1\rangle$$

are defined [Ozsváth and Szabó 2005], with $f(\hat{c}(Y, \xi)) = c^+(Y, \xi)$, where f is the homomorphism appearing in Exact Sequence (2–1) and $\mathbf{t}_\xi$ is the spin^c structure induced by the contact structure ξ.

To simplify notation, we will ignore the sign ambiguity in the definition of the contact invariants, and treat them as honest elements of the appropriate homology groups rather than equivalence classes. The reader should have no problem checking that there is no loss in making this abuse of notation. Alternatively, one could work with $\mathbb{Z}/2\mathbb{Z}$ coefficients to make the sign ambiguity disappear altogether. The relevant properties of $\hat{c}$ and c^+ can be summarized as follows.

Theorem 2.5 [Ozsváth and Szabó 2005]. *Let (Y, ξ) be a closed, contact three-manifold, and denote by $c(Y, \xi)$ either one of the contact invariants $\hat{c}(Y, \xi)$ and $c^+(Y, \xi)$.*

(i) *The class $c(Y, \xi)$ is an invariant of the isotopy class of the contact structure ξ on Y.*

(ii) *If (Y, ξ) is overtwisted then $c(Y, \xi) = 0$, while if (Y, ξ) is Stein fillable then $c(Y, \xi) \neq 0$.*

(iii) *Suppose that (Y_2, ξ_2) is obtained from (Y_1, ξ_1) by a contact $(+1)$-surgery. Then we have*
$$F_{-X}(c(Y_1, \xi_1)) = c(Y_2, \xi_2),$$
where $-X$ is the cobordism induced by the surgery with orientation reversed, and F_{-X} is the sum of $F_{-X,\mathbf{s}}$ over all spin^c structures $\mathbf{s}$ extending the spin^c structures induced on $-Y_i$ by ξ_i for $i = 1, 2$. In particular, if $c(Y_2, \xi_2) \neq 0$ then (Y_1, ξ_1) is tight.

(iv) *Suppose that $\mathbf{t}_\xi$ is torsion. Then $c(Y, \xi)$ is a homogeneous element of degree $-h(\xi) \in \mathbb{Q}$, where $h(\xi)$ is the Hopf-invariant of the two-plane field defined by the contact structure ξ.* □

Remark. The Hopf-invariant can be easily determined for a contact structure defined by a contact (± 1)-surgery diagram along the Legendrian link $\mathbb{L} \subset (S^3, \xi_{st})$ [Ding et al. 2004]. In fact, fix an orientation of $\mathbb{L}$ and consider the four-manifold X defined by the Kirby diagram specified by the surgery [Gompf and Stipsicz 1999]. Let $c \in H^2(X; \mathbb{Z})$ denote the cohomology class that evaluates as $\mathrm{rot}(L)$ on the homology class determined by a component L of the link $\mathbb{L}$. If $\mathbf{t}_\xi$ is torsion, then $c^2 \in \mathbb{Q}$ is defined, and $h(\xi)$ is equal to $(1/4)(c^2 - 3\sigma(X) - 2\chi(X) + 2) + q$, where q is the number of $(+1)$-surgeries made along $\mathbb{L}$ to get (Y, ξ).

3. Proofs

Now we can turn to the proofs of the statements announced in Section 1.

Proof of Theorem 1.1. Consider the Legendrian push-off K' of K drawn as a dotted line in Figure 2, left. The obvious annulus between K and K' induces framing $\mathrm{tb}(K)$ on both K and K'. Consider the modification K'' of K' illustrated in Figure 2, right. Since the total number of cusps of any front projection is even, it is easy to check that the parity assumption on the number of cusps between the strands U and U' ensures that the obvious surface S between K'' and K is oriented. Moreover, S has genus 1 and it induces framing $\mathrm{tb}(K) + 1$ on K and

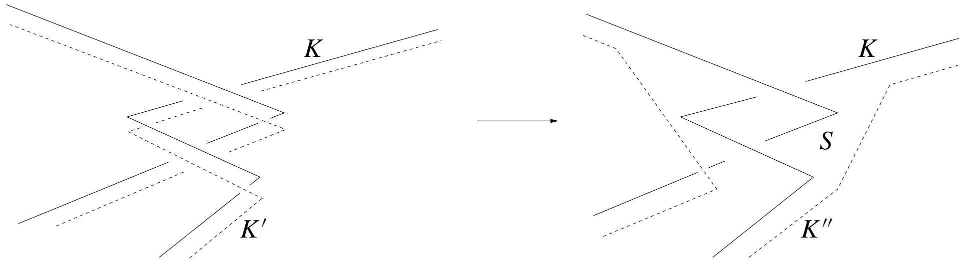

Figure 2. Modification of the Legendrian push-off.

K''. In particular, S extends to a meridian disk D inside the surgered solid torus. Since S induces framing $\mathrm{tb}(K)+1$ on K'', while $\mathrm{tb}(K'') = \mathrm{tb}(K')+3 = \mathrm{tb}(K)+3$, we have $\mathrm{tb}_{S\cup D}(K'') = 2$, that is, the Legendrian knot $K'' = \partial(S \cup D)$ violates the Bennequin–Eliashberg inequality with respect to the punctured torus $S \cup D$. We conclude that (Y_K, ξ_K) is overtwisted. □

To prove Theorem 1.3, Corollary 1.2 and Proposition 1.4, we shall need the following lemma (for a different proof of a more general result, see [Ozbagci 2005]).

Lemma 3.1. *Let K be a Legendrian knot in the standard contact three-sphere. If K is the stabilization of another Legendrian knot, then (Y_K, ξ_K) is overtwisted.*

Proof. By assumption, K admits a front projection containing one of the configurations of Figure 3. Without loss we may assume that we are in the situation of the left-hand side of the figure. Consider the Legendrian push-off K' of K drawn

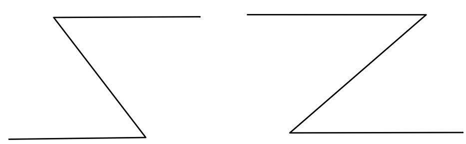

Figure 3. The two possible “zig-zags”.

as a dotted line in Figure 4, left. The obvious annulus between K and K' induces framing $\mathrm{tb}(K)$ on both K and K'. Consider the modification K'' of K' illustrated in Figure 4, right. There still is an obvious annulus A between K'' and K, except that now it induces framing $\mathrm{tb}(K'') = \mathrm{tb}(K)+1$ on K and K''. Since we perform contact $(+1)$-surgery on K, the annulus A extends to a meridian disk D inside the surgered solid torus. Therefore, $D \cup A$ is an overtwisted disk in (Y_K, ξ_K). □

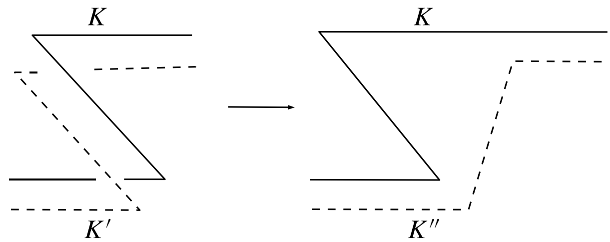

Figure 4. Modification of the Legendrian push-off.

The proof of Lemma 3.1 clearly applies to establish the following proposition, which shows that Lemma 3.1 holds if K is a Legendrian knot in a general contact three-manifold:

Proposition 3.2. *Suppose that the Legendrian link* $\mathbb{L} \subset (S^3, \xi_{st})$ *is obtained by stabilizing some components of another Legendrian link. Let* $(Y_{\mathbb{L}}, \xi_{\mathbb{L}})$ *be the result of contact* (± 1)*-surgeries along the components of* $\mathbb{L}$*. If the surgery coefficient on one of the stabilized components is* $(+1)$*, then* $(Y_{\mathbb{L}}, \xi_{\mathbb{L}})$ *is overtwisted.* □

Proof of Corollary 1.2. Examining [Etnyre and Honda 2001, Figure 8], we easily check that any Legendrian negative torus knot K with maximal Thurston–Bennequin invariant contains the configuration of Figure 1, with an odd number of cusps between the two strands U and U'. Therefore, by Theorem 1.1, (Y_K, ξ_K) is overtwisted. On the other hand, according to the results of [Etnyre and Honda 2001], any Legendrian negative torus knot K' with nonmaximal Thurston–Bennequin invariant is isotopic to the stabilization of one with maximal Thurston–Bennequin invariant. Thus, by Lemma 3.1, $(Y_{K'}, \xi_{K'})$ is overtwisted. □

Proof of Theorem 1.3. By contradiction, suppose that $S^3_n(K)$ is an L-space (recall that lens spaces are L-spaces), and $L_1 \subset (S^3, \xi_{st})$ is a Legendrian knot smoothly isotopic to K with $tb(L_1) > n$. Let L be obtained by stabilizing L_1 $\mathrm{tb}(L_1) - n$ times, so that $\mathrm{tb}(L) = n$. Denote by (Y_L, ξ_L) the result of contact $(+1)$-surgery along L. By Lemma 3.1 (Y_L, ξ_L) is overtwisted, hence $\hat{c}(Y_L, \xi_L) = 0$. On the other hand, we can compute $\hat{c}(Y_L, \xi_L)$ using Theorem 2.5, getting $\hat{c}(Y_L, \xi_L) = \hat{F}_{-X}(c(S^3, \xi_{st}))$, where X is the appropriate cobordism. The map $\hat{F}_{-X}$ fits into the exact triangle

$$\begin{tikzcd}
\widehat{HF}(S^3) \arrow[rr, "\hat{F}_{-X}"] & & \widehat{HF}(S^3_{-n-1}(\overline{K})) \arrow[dl] \\
& \widehat{HF}(S^3_{-n}(\overline{K})) \arrow[ul, "\hat{F}_W"] &
\end{tikzcd}$$

where $\overline{K}$ is the mirror image of K, and $S^3_r(K)$ denotes the result of r-surgery along K. Since $S^3_{-n}(\overline{K}) = -S^3_n(K)$ is an L-space, we have

$$\operatorname{rk} \widehat{HF}(S^3_{-n}(\overline{K})) = \left|H_1(S^3_{-n}(\overline{K}))\right| = n,$$

while by Proposition 2.2

$$\operatorname{rk} \widehat{HF}(S^3_{-n-1}(\overline{K})) \geq \left|H_1(S^3_{-n-1}(\overline{K}))\right| = n+1.$$

Exactness of the triangle immediately implies $\hat{F}_W = 0$, therefore $\hat{F}_{-X}$ must be injective. Since $\hat{c}(S^3, \xi_{st}) \neq 0$, this shows $\hat{c}(Y_L, \xi_L) \neq 0$, which contradicts the fact that (Y_L, ξ_L) is overtwisted. □

Proof of Proposition 1.4. Consider a Legendrian knot L' obtained by stabilizing L_2 until $tb(L_1) = tb(L')$. Since L' and L_1 are smoothly isotopic and have the same contact framing, the cobordisms associated with the contact (+1)-surgeries along L_1 and L' can be identified. Since $c(Y_{L_1}, \xi_{L_1})$ and $c(Y_{L'}, \xi_{L'})$ are images of $c(S^3, \xi_{st})$ under the same map, $c(Y_{L_1}, \xi_{L_1}) = 0$ if and only if $c(Y_{L'}, \xi_{L'}) = 0$. Lemma 3.1 gives $c(Y_{L'}, \xi_{L'}) = 0$, and the first statement follows.

For the second statement consider the exact triangle in the HF^+-theory provided by the surgery along L. (The Thurston–Bennequin invariant $\operatorname{tb}(L)$ is denoted by t.) After reversing orientation the triangle takes the shape

$$\begin{tikzcd}
HF^+(S^3) \arrow[rr, "F^+_{-X}"] & & HF^+(S^3_{-t-1}(\overline{L})) \arrow[dl, "F^+_V"] \\
& HF^+(S^3_{-t}(\overline{L})) \arrow[ul, "F^+_{-W}"] &
\end{tikzcd}$$

Now the assumption $t < -1$, or $-t-1 > 0$, implies the cobordism $-X$ inducing the first map is positive definite. It is known that the map F^∞_{-X} on the HF^∞-theory vanishes if $b_2^+(-X) > 0$ [Ozsváth and Szabó 2004b]. Since for S^3 the natural map $HF^\infty(S^3) \to HF^+(S^3)$ is onto, this implies that $F^+_{-X} = 0$. Since

$$c^+(Y_L, \xi_L) = F^+_{-X}(c^+(S^3, \xi_{st})),$$

the vanishing of the contact invariant $c^+(Y_L, \xi_L)$ follows. □

4. Examples

Given a Legendrian knot $L \subset (S^3, \xi_{st})$, denote by (Y_L, ξ_L), respectively (Y^L, ξ^L), the contact three-manifold obtained by contact (+1)-, respectively (−1)-surgery.

Let $L_i = L_i(n)$, where $i = 1, \ldots, n-1$, be the Legendrian knot given by Figure 5, right. The knots $L_i(n)$ (n fixed and ≥ 2) were considered in [Epstein et al. 2001].

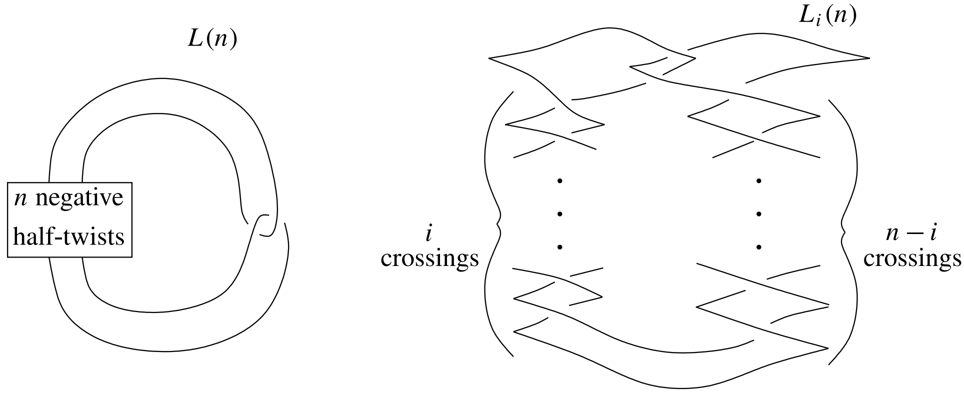

Figure 5. The n-twist knot and its Legendrian realizations.

They are all smoothly isotopic to the n-twist knot of Figure 5, left (having n negative half-twists). The knots L_i were the first examples of smoothly isotopic Legendrian knots having equal classical invariants (i.e., Thurston–Bennequin invariants and rotation numbers), but are not Legendrian isotopic [Chekanov 2002; Epstein et al. 2001]. The reader should be aware that our convention for representing a Legendrian knot via its front projection differs from the one used in [Epstein et al. 2001]: we use the contact structure given by the one-form $dz+x\,dy$ rather than the one-form $-dz+y\,dx$ used in that paper. However, the contactomorphism between the two contact structures given by sending (x, y, z) to $(y, -x, z)$ induces a one-to-one correspondence between the corresponding front projections, and under this correspondence Figure 1 from [Epstein et al. 2001] is sent to our Figure 5, right.

Proposition 4.1. *For every* $1 \le i, j \le n-1$ *we have*

$$\hat{c}(Y_{L_i}, \xi_{L_i}) = \hat{c}(Y_{L_j}, \xi_{L_j}).$$

Proof. The statement follows easily from basic properties of the contact invariant: by the surgery formula for contact $(+1)$-surgeries, we have $\hat{c}(Y_{L_i}, \xi_{L_i}) = F_{-X}(\hat{c}(S^3, \xi_{st}))$, where X is the cobordism induced by the four-dimensional handle attachment dictated by the surgery. Since X depends only on the smooth isotopy class of the Legendrian knot and its Thurston–Bennequin invariant, and is therefore independent of i, the claim trivially follows. □

According to the main result of this section, Theorem 4.2, the same equality holds if we perform Legendrian surgeries along $L_i(n)$; that is, the contact Ozsváth–Szabó invariants of the results of contact (± 1)-surgeries do not distinguish the Chekanov–Eliashberg knots.

Theorem 4.2. *Let $n \geq 2$ be an even integer, and let $1 \leq i, j \leq n-1$ be both odd. Then*

$$\hat{c}(Y^{L_i}, \xi^{L_i}) = \hat{c}(Y^{L_j}, \xi^{L_j}).$$

The proof of Theorem 4.2 rests on the following two lemmas.

Lemma 4.3 [Ozsváth and Szabó 2003b]. *Let $n \geq 2$ be an even integer, and denote by $\overline{L}(n)$ the mirror image of $L(n)$. Then*

$$HF^+\big(S^3_0\big(\overline{L}(n)\big)\big) \cong \mathcal{T}^+_{1/2} \oplus \mathcal{T}^+_{3/2} \oplus \mathbb{Z}^{(n/2)-1}_{(1/2)}.$$

Proof. Let $k = n/2$. Choosing a suitable oriented basis for an obvious Seifert surface for $L(n)$, one can easily compute the Seifert matrix

$$\begin{pmatrix} -k & k-1 \\ k & -k \end{pmatrix},$$

with eigenvalues $-k \pm \sqrt{k^2-k} < 0$. This immediately gives signature $\sigma(L(n)) = -2$ and Alexander polynomial

$$\Delta_{L(n)}(t) = kt^{-1} - (2k-1) + kt.$$

Since $L(n)$ is an alternating knot with genus $g(L(n)) = 1$, applying [Ozsváth and Szabó 2003b, Theorem 1.4] we get

$$\begin{cases} HF^+\big(S^3_0\big(L(n)\big), \mathbf{s}\big) \cong \mathcal{T}^+_{-1/2} \oplus \mathcal{T}^+_{-3/2} \oplus \mathbb{Z}^{(n/2)-1}_{(-3/2)} & \text{if} \quad c_1(\mathbf{s}) = 0, \\ HF^+\big(S^3_0\big(L(n)\big), \mathbf{s}\big) = 0 & \text{if} \quad c_1(\mathbf{s}) \neq 0. \end{cases}$$

By Proposition 2.2 this implies the result. □

Lemma 4.4. *Let $k \geq 0$ be an integer, and let $V(k)$ be the oriented three-manifold defined by the surgery diagram of Figure 6. Then*

$$\widehat{HF}(V(k)) \cong \mathbb{Z}^{2k+2} \quad \textit{and} \quad HF^+(V(k)) = \bigoplus_{i=1}^{2k+2} \mathcal{T}^+_{a_i} \ \textit{for some } a_i \in \mathbb{Q}.$$

Proof. In order to compute $\widehat{HF}(V(k))$ we will use the exact triangle defined by the $(k+1)$-framed unknot of Figure 6. It is easy to see that this unknot bounds a punctured torus smoothly embedded in the complement of the knot K. Thus, the cobordism we get by attaching this last two-handle contains a torus with self-intersection $(k+1)$, and the induced map in the surgery triangle vanishes by Proposition 2.4. Consequently, the surgery triangle is actually a short exact sequence. Notice that K is the (left-handed) trefoil knot, hence $\widehat{HF}(S^3_0(K)) = \mathbb{Z}^2$ [Ozsváth and Szabó 2003b, Theorem 1.4]. Arguing by induction we get

$$\widehat{HF}(V(k+1)) \cong \widehat{HF}(V(k)) \oplus \mathbb{Z}^2$$

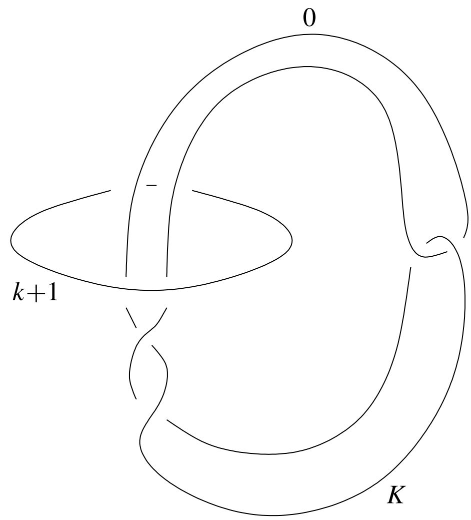

Figure 6. Surgery diagram for $V(k)$.

for every $k \geq 0$. On the other hand, for $k = 0$ the unknot can be blown down, showing that $V(0) \cong S^1 \times S^2$. This fact immediately implies

$$\widehat{HF}(V(k)) \cong \mathbb{Z}^{2k+2} \tag{4–1}$$

for every $k \geq 0$. Using the surgery presentation of Figure 6 it is easy to check that

$$H_1(V(k); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/(k+1)\mathbb{Z},$$

therefore $V(k)$ admits $(k+1)$ different torsion spin^c structures. By Proposition 2.2 and Exact Sequence (2–1) we have

$$\operatorname{rk} \widehat{HF}(V(k), \mathbf{t}) \geq 2$$

if $\mathbf{t}$ is a torsion spin^c structure. Therefore, using (4–1), we see that $\widehat{HF}(V(k), \mathbf{t}) \cong \mathbb{Z}^2$ for each torsion spin^c structure $\mathbf{t}$ and

$$\widehat{HF}(V(k), \mathbf{t}) = 0$$

if $\mathbf{t}$ is not torsion. The statement now follows from Proposition 2.2 and Corollary 2.3. □

Proof of Theorem 4.2. The idea of the proof is this: First we find a contact three-manifold (Y, ξ) such that contact $(+1)$-surgery along some Legendrian knot $K \subset (Y, \xi)$ gives (Y^{L_i}, ξ^{L_i}) and $A(Y) \subset HF^+(Y, \mathbf{t}_\xi)$ (as it is defined in Proposition 2.2) vanishes. Therefore $c^+(Y, \xi)$ is an element of some $\mathcal{T}_a^+$. The U-equivariance of the map induced by the surgery will then show that $c^+(Y^{L_i}, \xi^{L_i}) \in \mathcal{T}_a^+ \subset HF^+(Y^{L_i}, \mathbf{t}_{\xi^{L_i}})$, from which the conclusion easily follows.

To this end, consider the contact structure $\eta_i(n)$ defined by Legendrian surgery along the two-component link of Figure 7. One of the knots in the link is topologically the unknot, while the other one is $L_i(n)$. According to the Kirby moves indicated in Figure 8, it follows that this contact structure lives on the three-manifold $Y(n) := -V(n/2)$, where $V(k)$ is defined by Figure 6. According to [Ding and Geiges 2001], the effect of a contact (± 1)-surgery along a Legendrian knot can be canceled by contact (∓ 1)-surgery along a Legendrian push-off of the knot. Therefore, doing contact $(+1)$-surgery along the push-off of the unknot in Figure 7, we get (Y^{L_i}, ξ^{L_i}). On the other hand, denoting by X_n the cobordism induced by the contact $(+1)$-surgery, we have

$$\hat{F}_{-X_n}\big(\hat{c}\big(Y(n), \eta_i(n)\big)\big) = \hat{c}(Y^{L_i}, \xi^{L_i}).$$

A simple computation shows that $h(\xi^{L_i}) = -1/2$, therefore by Theorem 2.5(iv) we have

$$\hat{c}(Y^{L_i}, \xi^{L_i}) \in \widehat{HF}_{1/2}(-Y^{L_i}).$$

Moreover, $\hat{c}(Y^{L_i}, \xi^{L_i})$ is primitive [Plamenevskaya 2004]. Thus, to prove the statement it will be enough to verify that there is a rank-1 subgroup of $\widehat{HF}_{1/2}(-Y^{L_i})$

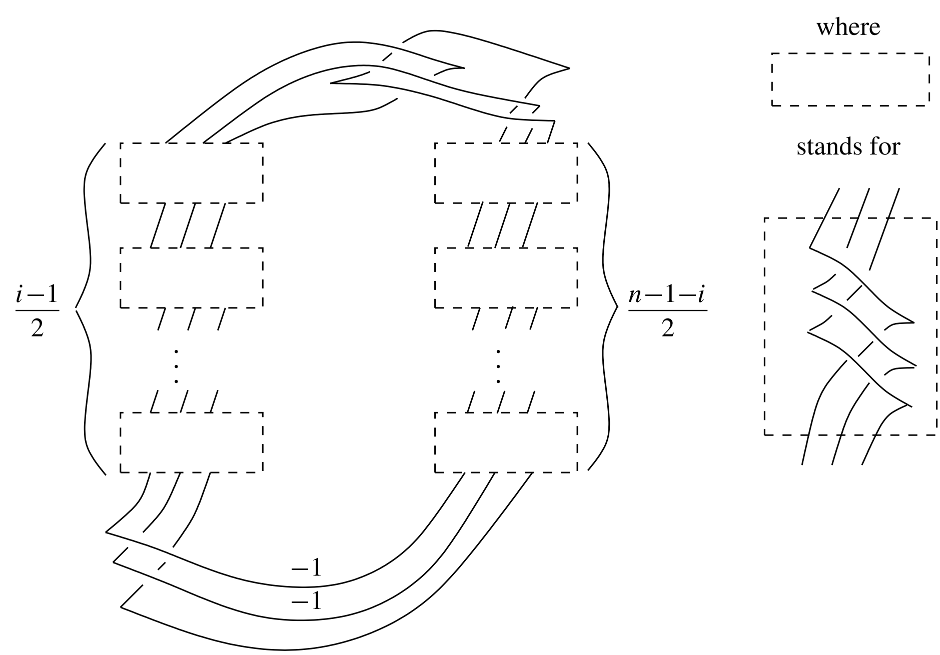

Figure 7. Contact surgery diagram defining $(Y(n), \eta_i(n))$.

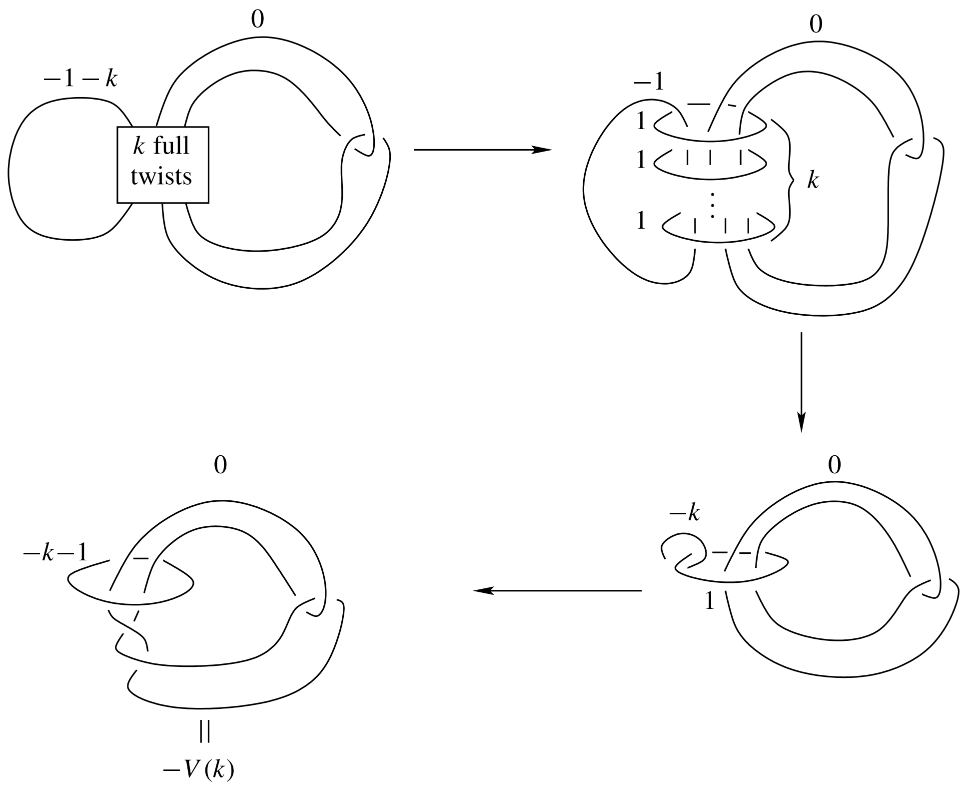

Figure 8. Kirby moves for $Y(n)$.

containing

$$\hat{F}_{-X_n}\big(\hat{c}\big(Y(n), \eta_i(n)\big)\big)$$

for every i. An easy computation shows that (since we assumed n to be even) the Thurston–Bennequin numbers of the knots $L_i(n)$ are all equal to 1, thanks to [Epstein et al. 2001]. Hence each of the three-manifolds Y^{L_i} is diffeomorphic to $S^3_0(L(n))$. By Lemma 4.3,

$$HF^+\big(-S^3_0(L(n))\big) \cong \mathcal{T}^+_{1/2} \oplus \mathcal{T}^+_{3/2} \oplus A,$$

where A is a finitely generated abelian group, while by Lemma 4.4 we have

$$HF^+(-Y(n)) = \bigoplus_{i=1}^{n+2} \mathcal{T}^+_{a_i}$$

for some $a_i \in \mathbb{Q}$. Since $F^+_{-X_n}$ is U-equivariant and for sufficiently large h the action of U^h vanishes on A, we have

$$\mathrm{Im}(F^+_{-X_n}) \subseteq \mathcal{T}^+_{1/2} \oplus \mathcal{T}^+_{3/2} \subseteq HF^+\big(-S^3_0\big(L(n)\big)\big).$$

Therefore, up to sign, there is a unique primitive element in $\mathrm{Im}(F^+_{-X_n})$ of degree $1/2$, implying that $c^+(Y^{L_i}, \xi^{L_i}) = c^+(Y^{L_j}, \xi^{L_j})$ for i, j as in the statement of Theorem 4.2. Since

$$HF^+_{-1/2}\big(-S^3_0\big(L(n)\big)\big) = 0,$$

it follows that the homomorphism

$$f\colon\ \widehat{HF}_{1/2}\big(-S^3_0\big(L(n)\big)\big) \to HF^+_{1/2}\big(-S^3_0\big(L(n)\big)\big)$$

from Exact Sequence (2–1) is injective. Since

$$f(\hat{c}(Y^{L_i}, \xi^{L_i})) = c^+(Y^{L_i}, \xi^{L_i}) \in \mathrm{Im}(F^+_{-X_n})$$

for every i, this concludes the proof. □

5. Distinguishing tight contact structures

Definition. Let ξ_i, for $i = 1, \dots, n-1$, denote the contact structure on the Brieskorn sphere $-\Sigma(2, 3, 6n-1)$ defined by the contact surgery specified by Figure 9.

Theorem 5.1. *The contact invariants $c^+(\xi_1), \dots, c^+(\xi_{n-1})$ are linearly independent over $\mathbb{Z}$.*

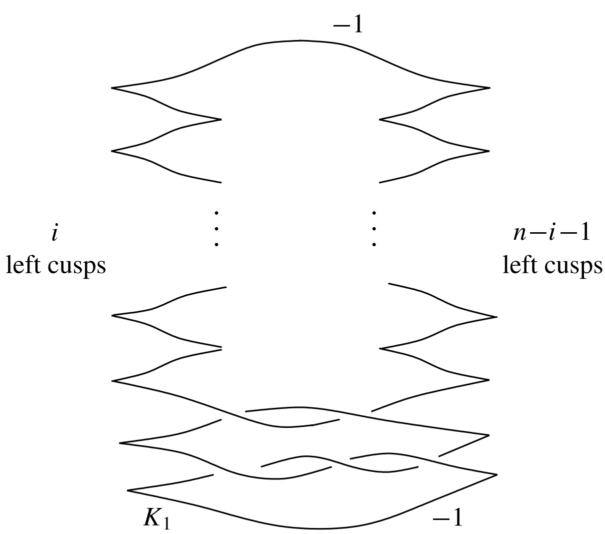

Figure 9. Contact structures on the three-manifold $-\Sigma(2, 3, 6n-1)$.

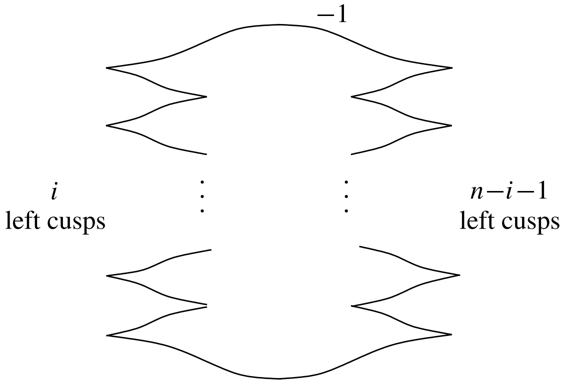

Figure 10. The contact structure η_i on $L(n, 1)$.

Proof. Consider the Legendrian push-off $\tilde{K}_1$ of the Legendrian trefoil K_1 of the figure. Attach a four-dimensional two-handle along $\tilde{K}_1$ to $-\Sigma(2, 3, 6n-1)$ with framing equal to the contact framing $+1$. Since contact $(+1)$-surgery along a Legendrian push-off cancels contact (-1)-surgery, we get a cobordism W such that $F_{-W}(c^+(\xi_i)) = c^+(\eta_i)$, where η_i is the contact structure on $L(n, 1)$ defined by Figure 10. The contact invariants $c^+(\eta_i)$ are linearly independent because they belong to groups corresponding to different spin^c structures on the same lens space $L(n, 1)$. Therefore, the invariants $c^+(\xi_i)$ are also linearly independent, concluding the proof. □

Corollary 5.2. *The contact structures $\xi_1, \dots, \xi_{n-1}$ are pairwise nonisotopic.* □

This was first proved by Lisca and Matić [1997] using Seiberg–Witten theory. For a different Heegaard Floer theoretic proof (of a more general statement), see [Plamenevskaya 2004].

Remark. It is known [Ozsváth and Szabó 2003a] that $HF^+(-\Sigma(2, 3, 6n-1)) = \mathcal{T}^+_{-2} \oplus \mathbb{Z}^{n-1}_{(-2)}$, therefore by Proposition 2.2, $HF^+(\Sigma(2, 3, 6n-1)) = \mathcal{T}^+_2 \oplus \mathbb{Z}^{n-1}_{(1)}$. It follows from Theorem 5.1 that the elements $c^+(\xi_i)$ $(i = 1, \dots, n-1)$ span $HF^+_1(\Sigma(2, 3, 6n-1))$.

If the trefoil knot of Figure 9 is replaced by any Legendrian knot L, the statement of Theorem 5.1 holds with the same proof. If $\mathrm{tb}(L) = 1$ and $\mathrm{rot}(L) = 0$, then the contact resulting structures $\xi_1, \dots, \xi_{n-1}$ are all homotopic as two-plane fields.

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