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By modifying Cole's example, we construct explicit Riemann surfaces with large bounds on corona solutions in an elementary way.

1. Introduction

For a given Riemann surface R, consider the algebra $H^\infty(R)$ of bounded analytic functions on R separating the points in R. The corona problem asks whether $\iota(R)$ is dense in the maximal ideal space $\mathcal{M}(R)$ of $H^\infty(R)$, where $\iota : R \to \mathcal{M}(R)$ is the natural inclusion defined by the point evaluation. If $\iota(R)$ is dense in $\mathcal{M}(R)$, we say that the *corona theorem holds* for R. Otherwise R is said to *have corona*.

The corona theorem holds for R if and only if the following statement is true (see [Gamelin 1978, Chapter 4] or [Garnett 1981, Chapter VIII]): for every collection $F_1, \dots, F_n \in H^\infty(R)$ and any $\delta \in (0, 1)$ with the property

$$\delta \le \max_j |F_j(\zeta)| \le 1 \quad \text{for all } \zeta \in R, \tag{1-1}$$

there exist $G_1, \dots, G_n \in H^\infty(R)$ such that

$$F_1 G_1 + F_2 G_2 + \cdots + F_n G_n = 1. \tag{1-2}$$

We refer to $G_1, \dots, G_n$ as *corona solutions*, $F_1, \dots, F_n$ as *corona data*, and $\max\{\|G_1\|, \dots, \|G_n\|\}$ as a *bound on the corona solutions* or *corona constant*. Here the notation $\|\cdot\|$ indicates the uniform norm. Throughout this paper, we assume that the corona data satisfies (1-1) for the given δ. The letter δ is reserved only for this use.

Theorem 1 (B. Cole; see [Gamelin 1978, Theorem 4.1, pp. 47–49]). *For any $\delta \in (0, 1)$ and $M > 0$, there exist a finite bordered Riemann surface R and corona data $F_1, F_2 \in H^\infty(R)$ such that any corona solutions $G_1, G_2 \in H^\infty(R)$ have a bound at least M; that is,* $\max\{\|G_1\|, \|G_2\|\} \ge M$.

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The purpose of this paper is to construct the Riemann surface R in Theorem 1 in an elementary way and describe it explicitly. Once Theorem 1 is proved, it is possible to construct a Riemann surface with corona.

Theorem 2 (B. Cole; see [Gamelin 1978, Theorem 4.2, pp. 49–52]). *There exists an open Riemann surface with corona.*

The basic idea of the proof of Theorem 2 is that if a Riemann surface R is obtained by connecting two Riemann surfaces R_1 and R_2 with a thin strip, then any holomorphic function on R behaves almost independently on R_1 and R_2.

The corona theorem holds for the unit disc [Carleson 1962], finitely connected domains in $\mathbb{C}$ [Gamelin 1970], Denjoy domains [Garnett and Jones 1985], and various other classes of planar domains and Riemann surfaces [Alling 1964; Behrens 1970; 1971; Jones and Marshall 1985; Stout 1965]. On the other hand, examples of Riemann surfaces with corona (other than Cole's) can be found in [Barrett and Diller 1998] and [Hayashi 1999]. Furthermore, by modifying the proof of Theorem 2, Cole's example can be used to obtain a Riemann surface R with corona that is of Parreau–Widom type [Nakai 1982]. (This means that $\sum_{z\in E} G(z,w) < +\infty$, where $G(\,\cdot\,,w)$ is the Green's function on R with the pole w and $E=\{z : \nabla G(z,w)=0\}$.)

The corona problem for a general domain in $\mathbb{C}$ is still open, and the answer is also unknown for a polydisc or a unit ball in $\mathbb{C}^n$, for $n \geq 2$.

2. Proof of Theorem 1

For given $\delta \in (0,1)$ and $M > 0$, we choose a natural number n such that $\delta^n \leq \min\{(16M)^{-1}, \frac{1}{4}\}$. Let $d = 4\delta^{n^2+n}$ and $c = 2\delta^{n^2}$. Since $2\delta^{n^2} < 2\delta^n \leq \frac{1}{2}$, we have

$$\frac{4\delta^{n+1}}{1-c} \leq 8\delta^{n+1} < 8\delta^n \leq \frac{1}{2M}. \tag{2-1}$$

Moreover,

$$\frac{d}{c-d} = \frac{4\delta^{n^2+n}}{2\delta^{n^2} - 4\delta^{n^2+n}} = \frac{2\delta^n}{1-2\delta^n} \leq 4\delta^n < \frac{1}{2M}. \tag{2-2}$$

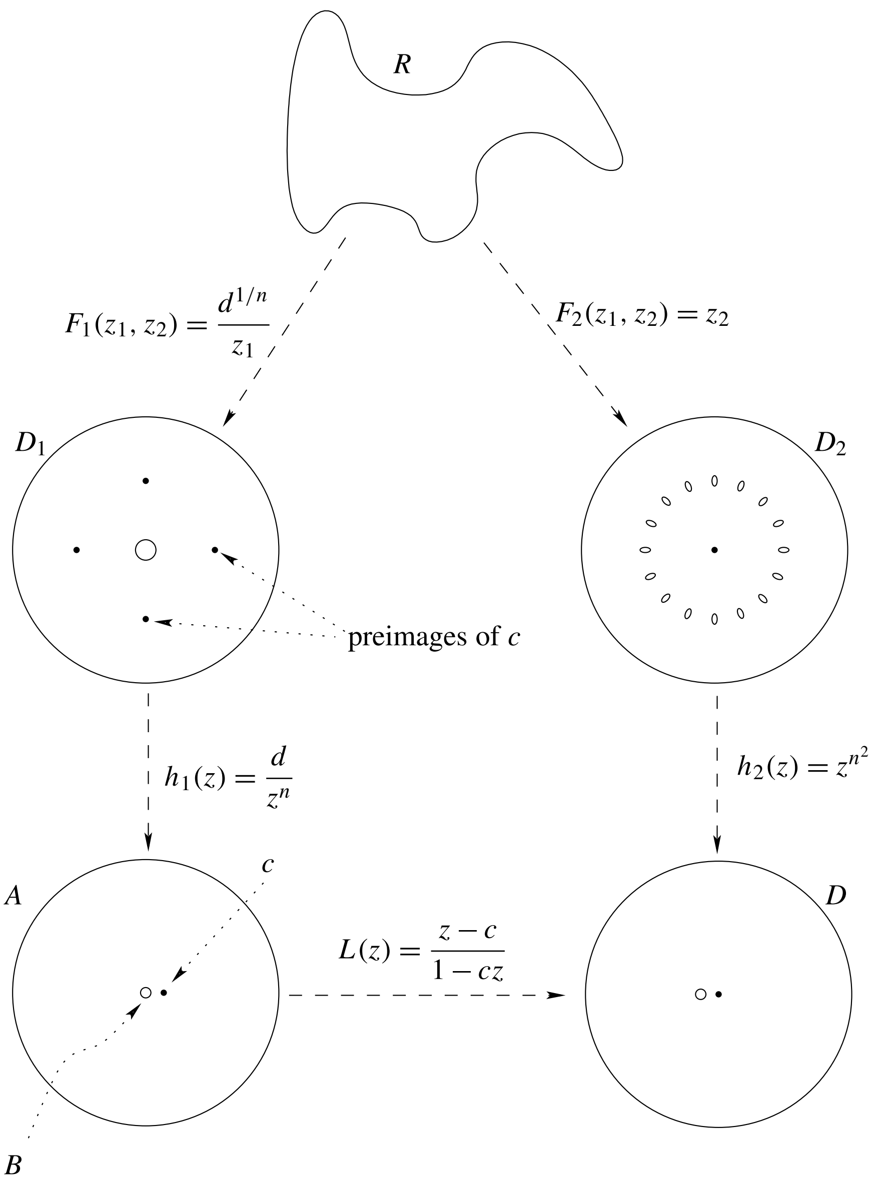
The important features in our choice of c, d and n are that $d^{1/n}$ is small (equation (2-1)), d/c is small (equation (2-2)), and $(d/c)^{1/n}$ is not small — say greater than δ.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$, $B := B(0,d) = \{z \in \mathbb{C} : |z| < d\}$, and $A := \mathbb{D}\backslash\bar{B}$. Further, define

$$\begin{aligned} D \ &:= \{z : (z+c)/(1+cz) \in A\}, \\ D_1 &:= \{z : z^n \in A\} = \{z : d/z^n \in A\}, \\ D_2 &:= \{z : z^{n^2} \in D\}. \end{aligned}$$

Thus D is the image of A under the Möbius transformation $L(z) := (z-c)/(1-cz)$, and D_1 and D_2 are preimages of A and D under $h_1(z) := d/z^n$ and $h_2(z) := z^{n^2}$. Finally we define the bordered Riemann surface

$$R := \left\{ (z_1, z_2) \in \mathbb{C}^2 : z_1 \in D_1, z_2 \in D_2 \text{ and } \frac{z_1^n - c}{1 - cz_1^n} = z_2^{n^2} \right\}. \tag{2-3}$$



Scheme of the construction of R, for $n = 4$.

Note that R is an n-sheeted covering of D_2 and an n^2-sheeted branched covering of D_1. This is because D_1 is an n-sheeted covering of A and D_2 is an n^2-sheeted branched covering of D.

We claim that the Riemann surface R, together with the holomorphic functions

$$F_1(z_1, z_2) = \frac{d^{1/n}}{z_1} \quad \text{and} \quad F_2(z_1, z_2) = z_2,$$

satisfies the conditions in Theorem 1.

First, note that F_1 and F_2 have values in D_1 and D_2, respectively. Thus we have $\max\{\|F_1\|, \|F_2\|\} \leq 1$. Furthermore, if $|F_2(z_1, z_2)| = |z_2| < \delta$, we have

$$|z_1^n - c| = |z_2|^{n^2} |1 - cz_1^n| < 2\delta^{n^2},$$

and hence $|z_1|^n < c + 2\delta^{n^2} = 4\delta^{n^2}$. Therefore

$$|F_1(z_1, z_2)| = \frac{d^{1/n}}{|z_1|} > \frac{4^{1/n}\delta^{n+1}}{4^{1/n}\delta^n} = \delta,$$

and the inequality $\max\{|F_1(z_1, z_2)|, |F_2(z_1, z_2)|\} \geq \delta$ holds for all $(z_1, z_2) \in R$; i.e., (F_1, F_2) becomes a pair of corona data for the given δ.

It remains to show that $\max\{\|G_1\|, \|G_2\|\} \geq M$ for any corona solutions G_1, G_2 such that

$$F_1 G_1 + F_2 G_2 = 1. \tag{2-4}$$

In fact, we will show that $\|G_1\| \geq M$. To prove this claim, we assume, without loss of generality, that G_1 and G_2 are holomorphic across the boundary of R. Then we define for all $z \in A$,

$$f(z) := \frac{1}{n^3} \sum F_1(z_1, z_2)\, G_1(z_1, z_2),$$

where the summation is over all the points $(z_1, z_2) \in R$ such that $z_1^n = z$, counting multiplicity. (Note that the map $(z_1, z_2) \mapsto z_1^n$ is an n^3-sheeted branched covering from R to A.) Then f is analytic in (a neighborhood of) A.

Since $F_2(z_1, z_2) = z_2 = 0$ when $z_1^n = c$, it is easy to see from (2-4) that $f(c) = 1$. On the other hand, $|f(z)| \leq \|G_1\|$ for all $z \in A$ since $\|F_1\| \leq 1$, and $|f(z)| \leq 4\delta^{n+1}\|G_1\|$ for $|z| = 1$ since on $\{|z_1| = 1\}$ we have

$$|F_1(z_1, z_2)| = \frac{d^{1/n}}{|z_1|} = 4^{1/n}\delta^{n+1} \leq 4\delta^{n+1}.$$

Therefore, by Cauchy's integral formula,

$$1 = |f(c)| = \left| \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)}{\xi - c} d\xi - \frac{1}{2\pi i} \int_{|\xi|=d} \frac{f(\xi)}{\xi - c} d\xi \right|$$
$$\leq \frac{4\delta^{n+1}\|G_1\|}{1-c} + \frac{2\pi d\|G_1\|}{2\pi(c-d)}.$$

This inequality, together with (2-1) and (2-2), proves the claim. This completes the proof.

3. Further remarks

1. In the construction of R, one can take F_1 as the projection map $(z_1, z_2) \mapsto z_1$, but then it is necessary to modify the definition (2-3) of R to

$$R := \left\{ (z_1, z_2) \in \mathbb{C}^2 : z_1 \in D_1, z_2 \in D_2 \text{ and } \frac{d/z_1^n - c}{1 - cd/z_1^n} = z_2^{n^2} \right\}$$

because we want to make the pair (F_1, F_2) a set of corona data satisfying (1-1).

2. Consider the function $h(z) := z^n$ defined on D_1. It is not difficult to see that the Riemann surface R constructed in Section 2 is nothing but the Riemann surface of the multivalued function

$$h^{-1} \circ L^{-1} \circ h_2(z) = \left(\frac{z^{n^2} + c}{1 + cz^{n^2}} \right)^{1/n}$$

defined on D_2. This function takes values in D_1. Similarly, one can consider R as a Riemann surface of the multivalued function

$$h_2^{-1} \circ L \circ h(z) = \left(\frac{z^n - c}{1 - cz^n} \right)^{1/n^2}$$

defined on D_1.

3. We can construct R by cutting and pasting. For example, we can construct the Riemann surface of $h^{-1} \circ L^{-1} \circ h_2$ over D_2 in the following way: we make n^2 cuts on D_2 radially so that each cut connects a hole to the outer boundary of D_2 (i.e., to the unit circle). We denote this region (D_2 minus cuts) by $D(1)$, and enumerate the cuts by $e(1, k, l)$, $k = 1, \ldots, n^2$, $l = 1, 2$ so that $e(1, k, 1) = e(1, k, 2)$ as sets, and as z approaches $e(1, k, 1)$ the argument of z increases. Let $D(j)$, $j = 1, \ldots, n$, be the copies of $D(1)$ with the corresponding cuts $e(j, k, l)$, $j = 1, \ldots, n$, $k = 1, \ldots, n^2$, $l = 1, 2$. For all j (mod n), paste $D(j)$ and $D(j+1)$ by identifying $e(j, k, 1)$ with $e(j+1, k, 2)$, $k = 1, 2, \ldots, n^2$. The resulting surface is conformally equivalent to R with the natural projection map $\pi \approx F_2$. By analytic continuation, the map

$h^{-1} \circ L^{-1} \circ h_2 \circ \pi$ is well-defined on R, hence analytic. We leave the details to the reader.

4. One can recover the same Riemann surface R via interpolation problems. Fix $\epsilon \in (0, \frac{1}{2})$ and let $D_1' := \{z : \epsilon < |z| < 1\}$. Choose a natural number n sufficiently large so that $2^{-n} < \epsilon$, and let E_n be the set of n-th roots of 2^{-n}. Note that $|z| = \frac{1}{2}$ for all $z \in E_n$.

We consider two interpolation problems:

(1) Find $G_1 \in H^\infty(D_1')$ (with the smallest uniform norm) such that $G_1(z) = \bar{z}$ for all $z \in E_n$.

(2) Find $F_2 \in H^\infty(D_1')$ (with the largest $\delta_0 := \min_{z \in D_1'}\{|F_2(z)|, |z|\}$) such that $\|F_2\| = 1$ and $F_2(z) = 0$ for all $z \in E_n$.

Any solution G_1 of (1) has uniform norm greater than C/ϵ, for some absolute constant C. To see this, one can repeat the argument in Section 2; thus, for w such that $\epsilon^n < |w| < 1$, define

$$f(w) = \frac{1}{n} \sum zG_1(z),$$

where the summation is over all $z \in D_1'$ such that $(\epsilon/z)^n = w$. Note that $f(2^n\epsilon^n) = \frac{1}{4}$ since $zG_1(z)$ equals $\frac{1}{4}$ for $z \in E_n$, and then Cauchy's integral formula gives a lower bound estimate $\|G_1\| \geq C/\epsilon$.

On the other hand, any solution F_2 of (2) should yield a small $\delta_0 = o(1)$ as $\epsilon \to 0$. To see this, let $F_1 = z$ and F_2 be the solution of (2). Now if δ_0 were not $o(1)$, the pair (F_1, F_2) would become a set of corona data on D_1' with corresponding $0 < \delta \leq \liminf_{\epsilon \to 0} \delta_0$. But then any corona solutions G_1 and G_2 such that $F_1G_1 + F_2G_2 = 1$ would have a bound $\geq C/\epsilon$, because $G_1/4$ should be a solution of (1). This violates the corona theorem on annuli [Scheinberg 1963; Stout 1965]. (In fact, it violates a statement slightly stronger than the corona theorem, which is true for annuli; namely, for *any* annulus D_1' and corona data defined on D_1', there always exist corona solutions with bound $\leq M = M(\delta)$, where M does not depend on D_1'. See [Gamelin 1978, p. 47] for details.) Therefore to make F_1 and F_2 corona data, or to get a solution for (2) with large δ_0, we take a number N such that the multivalued function

$$F(z) = \left(\frac{z^n - 2^{-n}}{1 - 2^{-n}z^n} \right)^{1/N},$$

has modulus $\geq \frac{1}{4}$ for $|z| < \frac{1}{4}$. (Such an N should be asymptotically greater than a fixed multiple of n^2 as $n \to \infty$, as we have seen in Section 2. Also note that F^N is a solution for (2).) Now since F is not analytic on D_1', we consider the Riemann surface of F over D_1', which gives us the Riemann surface R constructed in Section 2 (with $\delta = \frac{1}{4}$).

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