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Let M be an n-dimensional ($n \geq 3$) compact, oriented and connected submanifold in the unit sphere $\mathbb{S}^{n+\tilde{p}}(1)$, with scalar curvature $n(n-1)r$ and nowhere-zero mean curvature. Let S denote the squared norm of the second fundamental form of M and let $\alpha(n, r)$ denote a certain specific function of n and r. Using the Lawson–Simons formula for the nonexistence of stable k-currents, we obtain that, if $r \geq (n-2)/(n-1)$ and $S \leq \alpha(n, r)$, then either M is isometric to the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$ with $c^2 = (n-2)/(nr)$, or the fundamental group of M is finite. In the latter case, M is diffeomorphic to a spherical space form if $n = 3$, or homeomorphic to a sphere if $n \geq 4$.

1. Introduction

Let M be an n-dimensional hypersurface in the unit sphere $\mathbb{S}^{n+1}(1)$ of dimension $n+1$. If the scalar curvature $n(n-1)r$ of M is constant and $r \geq 1$, Cheng and Yau [1977] and Li [1996] obtained characterization theorems in terms of the sectional curvature, or the squared norm of the second fundamental form of M, respectively. Li obtained:

Theorem A [Li 1996]. *Let M be an n-dimensional ($n \geq 3$) compact hypersurface in the unit sphere $\mathbb{S}^{n+1}(1)$. If its constant scalar curvature $n(n-1)r$ satisfies $r \geq 1$, then M is isometric to either*

(1) *the totally umbilical sphere $\mathbb{S}^n(r)$,*

(2) *the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$ with $c^2 = \dfrac{n-2}{nr}$.*

The second case happens if

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

where S denotes the squared norm of the second fundamental form of M.

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We should notice that the condition $r \geq 1$ plays an essential role in the proof of Theorem A. On the other hand, by considering the standard immersions $\mathbb{S}^{n-1}(c) \subset \mathbb{R}^n$ and $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \subset \mathbb{R}^2$, for any $0 < c < 1$, and taking their Riemannian-product immersion

$$\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c) \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^n,$$

we obtain a compact hypersurface $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$ in $\mathbb{S}^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, where

$$r = \frac{n-2}{nc^2} > 1 - \frac{2}{n}.$$

Hence, some of the Riemannian products $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$ do not appear in the result of Li [1996]. From the assertion above, it is natural and interesting to generalize the result due to Li [1996] to the case when $r > 1 - 2/n$. Hence, Cheng asked this interesting question:

Problem 1 [Cheng 2001]. Let M be an n-dimensional ($n \geq 3$) complete hypersurface in $\mathbb{S}^{n+1}(1)$, with constant scalar curvature $n(n-1)r$. If

$$r > 1 - \frac{2}{n} \qquad \text{and} \qquad S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

then is M isometric to either a totally umbilical hypersurface or the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$?

Cheng [2003] tried to solve this problem. As it seems to be a very hard question, he solved it after adding a topological condition:

Theorem B [Cheng 2003]. *Let M be an n-dimensional compact hypersurface in $\mathbb{S}^{n+1}(1)$ with infinite fundamental group. If*

$$r \geq \frac{n-2}{n-1} \qquad \textit{and} \qquad S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

then M is isometric to the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$, where $n(n-1)r$ is the scalar curvature of M and $c^2 = (n-2)/(nr)$.

Notice that Theorem B only characterizes the compact hypersurfaces with infinite fundamental group. How about characterizing the hypersurfaces with finite fundamental group? This problem is also interesting.

On the other hand, it is natural and very important to study n-dimensional submanifolds in the unit sphere $\mathbb{S}^{n+\tilde{p}}(1)$ that have constant scalar curvature and higher codimension $\tilde{p}$.

In order to present the results that follow, we define a polynomial $R_r(x)$ by

$$R_r(x) = n^2r^2 - \big(3n - 5 + (n^2-n-1)(r-1)\big)x + \frac{(n-1)(5n-9)}{4n^2}x^2.$$

It can be easily checked, using the relation between roots and coefficients, that the equation $R_r(x)=0$ has two positive roots, which we denote by $x_1(r)$ and $x_2(r)$. These are

$$x_1(r)=\frac{2n^2}{(n-1)(5n-9)}\big((3n-5+(n^2-n-1)(r-1))-D(r)\big), \tag{1-1}$$

$$x_2(r)=\frac{2n^2}{(n-1)(5n-9)}\big((3n-5+(n^2-n-1)(r-1))+D(r)\big), \tag{1-2}$$

where

$$D(r)=(n-2)\big(4+2(3n-1)(r-1)+(n^2+2n-2)(r-1)^2\big)^{1/2}.$$

Obviously, $x_1(r)\le x_2(r)$. Hence, when $x\le x_1(r)$, we have $R_r(x)\ge 0$.

Cheng [2002] generalized the result of Li [1996] to submanifolds with higher codimension $\tilde{p}$, and obtained the following:

Theorem C [Cheng 2002]. *Let M be an n-dimensional $(n\ge 3)$ compact submanifold in the unit sphere $\mathbb{S}^{n+\tilde{p}}(1)$, with constant scalar curvature $n(n-1)r$ satisfying $r>1$. Take the function*

$$\alpha(n,r)=\begin{cases}(n-1)\dfrac{n(r-1)+2}{n-2}+\dfrac{n-2}{n(r-1)+2}, & \text{for } \tilde{p}\le 2,\\ n(r-1)+x_1(r), & \text{for } \tilde{p}\ge 3,\end{cases} \tag{1-3}$$

with $x_1(r)$ as defined above. If $S\le\alpha(n,r)$, then M is isometric to a totally umbilical sphere or the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big)\times\mathbb{S}^{n-1}(c)$ with $c^2=(n-2)/(nr)$.

Remark 1. The statement of Professor Q. M. Cheng is certainly correct. We merely remark that the upper bound on S can be improved (as is done in the present paper) by choosing the smaller root of $R_r(x)$ as the essential ingredient of the upper bound.

In the same paper, it is stated that, when $r>1$,

$$\sum_\alpha\sum_{i,j,k}(h^\alpha_{ijk})^2\ge\|\mathrm{grad}(nH)\|^2 \qquad\text{and}\qquad H\ne 0.$$

Hence, the condition $r>1$ plays an essential role in Theorem C's proof. Since, for $0<c<1$, the Riemannian products $\mathbb{S}^1\big(\sqrt{1-c^2}\big)\times\mathbb{S}^{n-1}(c)$ in $\mathbb{S}^{n+1}(1)$ are compact hypersurfaces with constant scalar curvature $n(n-1)r$ satisfying

$$r=\frac{n-2}{nc^2}>1-\frac{2}{n},$$

it is natural and interesting to generalize the result of Cheng [2002] to the case $r>1-2/n$. We should ask this:

Problem 2. Let M be an n-dimensional ($n \geq 3$) compact submanifold in the unit sphere $\mathbb{S}^{n+\tilde{p}}(1)$, with constant scalar curvature $n(n-1)r$ and nowhere-zero mean curvature. Take the function $\alpha(n,r)$ defined in (1-3). The question is: If

$$r > 1 - \frac{2}{n} \qquad \text{and} \qquad S \leq \alpha(n,r),$$

then is M isometric to either a totally umbilical sphere or the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$ with $c^2 = (n-2)/(nr)$?

In this paper, we try to solve these problems. We shall give a topological answer that relies on the Lawson–Simons formula for the nonexistence of stable k-currents [Lawson and Simons 1973]. The latter enables us to eliminate the homology groups and show M to be a homology sphere. Our result is:

Main Theorem. *Let M be an n-dimensional ($n \geq 3$) compact, oriented and connected submanifold in the unit sphere $\mathbb{S}^{n+\tilde{p}}(1)$, with scalar curvature $n(n-1)r$ and nowhere-zero mean curvature. Take the squared norm S of the second fundamental form of M, and $\alpha(n,r)$ as defined in (1-3). If*

$$r \geq \frac{n-2}{n-1} \qquad \textit{and} \qquad S \leq \alpha(n,r),$$

then either:

(1) *the fundamental group of M is finite, and M is diffeomorphic to a spherical space form if $n = 3$, or homeomorphic to a sphere if $n \geq 4$;*
(2) *M is isometric to the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big) \times \mathbb{S}^{n-1}(c)$ with $c^2 = (n-2)/(nr)$.*

Remark 2. We do not assume that the scalar curvature is constant. Note that the condition $H \neq 0$ on M is necessary for proving the theorem.

2. Preliminaries

Let M be an n-dimensional submanifold in a unit sphere $\mathbb{S}^{n+\tilde{p}}(1)$, and take a local orthonormal frame field $e_1, \dots, e_{n+\tilde{p}}$ on $\mathbb{S}^{n+\tilde{p}}(1)$ so that, when restricted to M, $e_1, \dots, e_n$ are tangent to M. Let $\omega_1, \dots, \omega_{n+\tilde{p}}$ be the dual coframe field on $\mathbb{S}^{n+\tilde{p}}(1)$. We make the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+\tilde{p} \quad \text{and} \quad 1 \leq i, j, k, \dots \leq n, \; n+1 \leq \alpha, \beta, \gamma, \dots \leq n+\tilde{p}.$$

The structure equations of $\mathbb{S}^{n+\tilde{p}}(1)$ are

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$
$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \tfrac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC},$$

where K_{ABCD} are the components of the curvature tensor of $\mathbb{S}^{n+\tilde{p}}(1)$. On M, we then have

$$\omega_\alpha = 0 \qquad \text{for } \alpha = n+1, \dots, n+\tilde{p}.$$

It follows from Cartan's lemma that

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \qquad h^\alpha_{ij} = h^\alpha_{ji}.$$

The second fundamental form B and the mean curvature vector ξ of M are

$$B = \sum_\alpha \sum_{i,j} h^\alpha_{ij}\omega_i\omega_j e_\alpha \qquad \text{and} \qquad \xi = \frac{1}{n}\sum_\alpha \Big(\sum_i h^\alpha_{ii}\Big) e_\alpha.$$

The mean curvature of M is

$$H = \frac{1}{n}\sqrt{\sum_\alpha \Big(\sum_i h^\alpha_{ii}\Big)^2}.$$

The structure equations of M are given by

$$d\omega_i = -\sum_j \omega_{ij}\wedge\omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2-1}$$

$$d\omega_{ij} = -\sum_k \omega_{ik}\wedge\omega_{kj} + \tfrac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \tag{2-2}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2-3}$$

where R_{ijkl} are the components of the curvature tensor of M.

Let R_{ij} denote the components of the Ricci curvature, and let $n(n-1)r$ be the scalar curvature of M. From (2-3), we have

$$R_{jk} = (n-1)\delta_{jk} + \sum_\alpha \Big(\sum_i h^\alpha_{ii}h^\alpha_{jk} - \sum_i h^\alpha_{ik}h^\alpha_{ji}\Big), \tag{2-4}$$

$$n(n-1)r = n(n-1) + n^2H^2 - S, \tag{2-5}$$

where $S = \sum_\alpha \sum_{i,j}(h^\alpha_{ij})^2$ is the squared norm of M's second fundamental form. We also have

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} + \tfrac{1}{2}\sum_{i,j} R_{\alpha\beta ij}\omega_i\wedge\omega_j, \tag{2-6}$$

$$R_{\alpha\beta ij} = \sum_l (h^\alpha_{il}h^\beta_{lj} - h^\alpha_{jl}h^\beta_{li}). \tag{2-7}$$

The Codazzi equation and the Ricci identities are

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik}, \tag{2-8}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{2-9}$$

where $\{h_{ijk}\}$ and $\{h_{ijkl}\}$ denote the first and second covariant derivatives of h_{ij}. These are defined by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{2-10}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{2-11}$$

We also need the next lemmas.

Lemma 1 [Cai 1987; Leung 1992]. *Let $A = (a_{ij})_{i,j=1,\dots,n}$ be a symmetric $n \times n$ matrix ($n \geq 2$), and set $A_1 = \operatorname{tr} A$ and $A_2 = \sum_{i,j}(a_{ij})^2$. We have*:

$$\begin{aligned}\sum_i (a_{in})^2 - A_1 a_{nn} \\ \leq \frac{1}{n^2}\Big(n(n-1)A_2 + (n-2)\sqrt{n-1}\,\big|A_1\big|\sqrt{nA_2-(A_1)^2} - 2(n-1)(A_1)^2\Big).\end{aligned} \tag{2-12}$$

Equality holds if and only if either $n = 2$, or $n > 2$ and (a_{ij}) is of the form

$$\begin{pmatrix} a & & & 0 \\ & \ddots & & \\ & & a & \\ 0 & & & A_1-(n-1)a \end{pmatrix}$$

with $(na - A_1)A_1 \geq 0$.

A simple and direct method proves this algebraic lemma:

Lemma 2. *Let $A = (a_{ij})_{i,j=1,\dots,n}$ be a symmetric $n \times n$ matrix, and p, q positive integers ≥ 2 with $p + q = n$. Setting*

$$A_1 = \sum_{s=1}^p a_{ss} + \sum_{t=p+1}^n a_{tt} \qquad \textit{and} \qquad \tilde{A}_2 = \sum_{i=1}^n (a_{ii})^2,$$

we have

$$\begin{aligned}\Big(\sum_{s=1}^p a_{ss}\Big)^2 - A_1\Big(\sum_{s=1}^p a_{ss}\Big) \\ \leq \frac{1}{n^2}\Big(pqn\tilde{A}_2 - 2pq(A_1)^2 + |p-q|\sqrt{pq}\,\big|A_1\big|\sqrt{n\tilde{A}_2-(A_1)^2}\Big).\end{aligned} \tag{2-13}$$

Proof. From the Cauchy–Schwarz inequality,

$$\begin{aligned}\tilde{A}_2 &= \sum_{s=1}^p (a_{ss})^2 + \sum_{t=p+1}^n (a_{tt})^2 \geq \frac{1}{p}\Big(\sum_{s=1}^p a_{ss}\Big)^2 + \frac{1}{q}\Big(\sum_{t=p+1}^n a_{tt}\Big)^2 \\ &= \frac{n}{pq}\Big(\sum_{s=1}^p a_{ss}\Big)^2 - \frac{2}{q}A_1\Big(\sum_{s=1}^p a_{ss}\Big) + \frac{1}{q}(A_1)^2.\end{aligned} \tag{2-14}$$

Hence,

$$\Big(\sum_{s=1}^{p} a_{ss}\Big)^2 - \frac{2p}{n} A_1\Big(\sum_{s=1}^{p} a_{ss}\Big) + \frac{p}{n}(A_1)^2 - \frac{pq}{n}\tilde{A}_2 \le 0,$$

From this follows

$$\frac{pA_1}{n} - \frac{\sqrt{pq}}{n}\sqrt{n\tilde{A}_2 - (A_1)^2} \le \sum_{s=1}^{p} a_{ss} \le \frac{pA_1}{n} + \frac{\sqrt{pq}}{n}\sqrt{n\tilde{A}_2 - (A_1)^2}, \tag{2-15}$$

and also

$$\Big(\sum_{s=1}^{p} a_{ss}\Big)^2 - A_1\Big(\sum_{s=1}^{p} a_{ss}\Big) \le \frac{pq}{n}\tilde{A}_2 - \frac{p}{n}(A_1)^2 + \frac{p-q}{n} A_1\Big(\sum_{s=1}^{p} a_{ss}\Big). \tag{2-16}$$

From (2-15) we have

$$\begin{aligned}\Big(\sum_{s=1}^{p} a_{ss}\Big)^2 &- A_1\Big(\sum_{s=1}^{p} a_{ss}\Big) \\ &\le \frac{pq}{n}\tilde{A}_2 - \frac{p}{n}(A_1)^2 + \frac{(p-q)p}{n^2}(A_1)^2 + \Big|\frac{p-q}{n}A_1\Big|\frac{\sqrt{pq}}{n}\sqrt{n\tilde{A}_2 - (A_1)^2}.\end{aligned}$$

Hence (2-13) holds and Lemma 2 is proved. □

Lemma 3 [Lawson and Simons 1973]. *Let M be a compact n-dimensional submanifold of the unit sphere $\mathbb{S}^{n+\tilde{p}}(1)$, with second fundamental form B. Take positive integers p, q such that $1 < p, q < n-1$ and $p+q = n$. If the inequality*

$$\sum_{s=1}^{p}\sum_{t=p+1}^{n}\Big(2\big|B(e_s, e_t)\big|^2 - \big\langle B(e_s, e_s), B(e_t, e_t)\big\rangle\Big) < pq, \tag{2-17}$$

holds for any point of M and any local orthonormal frame field $\{e_s, e_t\}$ on M, then

$$H_p(M, \mathbb{Z}) = H_q(M, \mathbb{Z}) = 0,$$

where $H_k(M, \mathbb{Z})$ denotes the k-th homology group of M with integer coefficients.

Lemma 4 [Aubin 1998]. *If the Ricci curvature of a compact Riemannian manifold is non-negative and positive somewhere, then the manifold carries a metric with positive Ricci curvature.*

Lemma 5 [Ôtsuki 1970]. *Let M be a hypersurface in a unit sphere $\mathbb{S}^{n+1}(1)$. If the multiplicities of the principal curvatures are constant, then the distribution of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of principal vectors.*

3. Proof of the Main Theorem

Let M be a compact, oriented and connected submanifold, with scalar curvature $n(n-1)r$ and nowhere-zero mean curvature H. We know that $e_{n+1} = \xi/H$ is a normal vector field defined globally on M. Define S_1 and S_2 by

$$S_1 = \sum_{i,j}(h_{ij}^{n+1} - H\delta_{ij})^2 \qquad \text{and} \qquad S_2 = \sum_{\alpha \geq n+2}\sum_{i,j}(h_{ij}^{\alpha})^2.$$

These functions are globally defined on M and do not depend on the choice of orthonormal frame $e_1, \ldots, e_n$. Since we chose $e_{n+1} = \xi/H$, we have $S - nH^2 = S_1 + S_2$. Further,

$$\sum_i h_{ii}^{n+1} = nH \qquad \text{and} \qquad \sum_i h_{ii}^{\alpha} = 0 \quad \text{for } n+2 \leq \alpha \leq n+\tilde{p}. \tag{3-1}$$

For any point p and any unit vector $v \in T_pM$, choose a local orthonormal frame field $e_1, \ldots, e_n$ such that $e_n = v$. From Gauss' equation (2-3) it follows that the Ricci curvature $\operatorname{Ric}(v, v)$ of M with respect to v is

$$\operatorname{Ric}(v, v) = (n-1) + \sum_{\alpha}\Big((\operatorname{tr} H_{\alpha})h_{nn}^{\alpha} - \sum_i (h_{in}^{\alpha})^2\Big), \tag{3-2}$$

where H_{α} is the $n \times n$ matrix (h_{ij}^{α}). Setting

$$T_{\alpha} = \operatorname{tr} H_{\alpha} \qquad \text{and} \qquad S_{\alpha} = \sum_{i,j}(h_{ij}^{\alpha})^2,$$

we have

$$n^2H^2 = \sum_{\alpha} T_{\alpha}^2 \qquad \text{and} \qquad S = \sum_{\alpha} S_{\alpha}.$$

From Lemma 1 follows that

$$\begin{aligned}
\operatorname{Ric}(v, v) &\geq (n-1) - \sum_{\alpha}\frac{1}{n^2}\Big(n(n-1)S_{\alpha} + (n-2)\sqrt{n-1}\,|T_{\alpha}|\sqrt{nS_{\alpha} - T_{\alpha}^2} \\
&\qquad\qquad - 2(n-1)T_{\alpha}^2\Big) \\
&= (n-1) - \frac{n-1}{n}S - \frac{n-2}{n}\sqrt{\frac{n-1}{n}}\sum_{\alpha}|T_{\alpha}|\sqrt{S_{\alpha} - \frac{T_{\alpha}^2}{n}} + \frac{2(n-1)}{n^2}\sum_{\alpha}T_{\alpha}^2 \\
&\geq (n-1) - \frac{n-1}{n}S - \frac{n-2}{n}\sqrt{\frac{n-1}{n}}\sqrt{\Big(\sum_{\alpha}T_{\alpha}^2\Big)\Big(\sum_{\alpha}\Big(S_{\alpha} - \frac{T_{\alpha}^2}{n}\Big)\Big)} \\
&\qquad\qquad + \frac{2(n-1)}{n^2}\sum_{\alpha}T_{\alpha}^2 \\
&= \frac{n-1}{n}\Big(n + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{S - nH^2}\Big) \\
&= \frac{n-1}{n}\Big(n + nH^2 - f^2 - n|H|\frac{n-2}{\sqrt{n(n-1)}}f\Big),
\end{aligned} \tag{3-3}$$

where f is a nonnegative function defined globally on M by $f^2 = S - nH^2$.

Define

$$P_H(f)=n+nH^2-f^2-n|H|\frac{n-2}{\sqrt{n(n-1)}}f. \tag{3-4}$$

From (2-5) we know that

$$f^2=\frac{n-1}{n}\big(S-n(r-1)\big),$$

and so we write $P_H(f)$ as

$$\begin{aligned}P_r(S)=n+n(r-1)-\frac{n-2}{n}\big(S-n(r-1)\big)\\ -\frac{n-2}{n}\sqrt{\big(n(n-1)(r-1)+S\big)\big(S-n(r-1)\big)}.\end{aligned} \tag{3-5}$$

Hence (3-3) becomes

$$\mathrm{Ric}(v,v)\geq\frac{n-1}{n}P_r(S). \tag{3-6}$$

On the other hand, from (3-3) we have

$$\begin{aligned}\mathrm{Ric}(v,v)&\geq\frac{n-1}{n}\Big(n+nH^2-f^2-n|H|\frac{n-2}{\sqrt{n(n-1)}}f\Big)\\ &\geq\frac{n-1}{n}\Big(n+nH^2-\frac{3}{2}f^2-n|H|\frac{n-2}{\sqrt{n(n-1)}}f\Big).\end{aligned} \tag{3-7}$$

Define

$$Q_H(f)=n+nH^2-\frac{3}{2}f^2-n|H|\frac{n-2}{\sqrt{n(n-1)}}f. \tag{3-8}$$

By (2-5), $Q_H(f)$ can be rewritten as

$$\begin{aligned}Q_r(S)=n+n(r-1)-\frac{3n-5}{2n}\big(S-n(r-1)\big)\\ -\frac{n-2}{n}\sqrt{\big(n(n-1)(r-1)+S\big)\big(S-n(r-1)\big)}.\end{aligned} \tag{3-9}$$

Hence (3-7) becomes

$$\mathrm{Ric}(v,v)\geq\frac{n-1}{n}P_r(S)\geq\frac{n-1}{n}Q_r(S). \tag{3-10}$$

If $S\leq\alpha(n,r)$ and $\tilde{p}\leq 2$, then from (1-3) follows that the inequality

$$S\leq(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}$$

is equivalent to

$$\begin{aligned}\Big(n+n(r-1)-\frac{n-2}{n}\big(S-n(r-1)\big)\Big)^2\\ \geq\frac{(n-2)^2}{n^2}\big(n(n-1)(r-1)+S\big)\big(S-n(r-1)\big).\end{aligned} \tag{3-11}$$

Since $r \geq (n-2)/(n-1)$, we get

$$r-1 \geq -\frac{1}{n-1} \qquad \text{and} \qquad n(r-1)+2 \geq \frac{n-2}{n-1}.$$

Hence, we have

$$\begin{aligned}
n+n(r-1)-\frac{n-2}{n}\bigl(S-n(r-1)\bigr)&\\
\geq n+2(n-1)(r-1)-\frac{n-2}{n}\Bigl((n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}\Bigr)&\\
=\frac{n^2-2(n-1)}{n}+(n-1)(r-1)-\frac{(n-2)^2}{n}\frac{1}{n(r-1)+2}&\\
\geq \frac{n^2-2(n-1)}{n}-1-\frac{(n-2)^2}{n}\frac{n-1}{n-2}=0.&
\end{aligned}$$

Obviously, from (2-5) and $f^2=((n-1)/n)\bigl(S-n(r-1)\bigr)$, we have

$$n(n-1)(r-1)+S>0 \qquad \text{and} \qquad S-n(r-1)\geq 0.$$

Hence, from (3-11) follows that

$$n+n(r-1)-\frac{n-2}{n}\bigl(S-n(r-1)\bigr) \geq \frac{n-2}{n}\sqrt{\bigl(n(n-1)(r-1)+S\bigr)\bigl(S-n(r-1)\bigr)}, \tag{3-12}$$

that is,

$$P_r(S)\geq 0. \tag{3-13}$$

Hence, from (3-6) we have $\operatorname{Ric}(v,v)\geq 0$.

On the other hand, since

$$S\leq n(r-1)+x_1(r), \tag{3-14}$$

when $\tilde{p}\geq 3$, from (1-3) we have

$$\begin{aligned}
R_r\bigl(S-n(r-1)\bigr)&=n^2r^2-\bigl(3n-5+(n^2-n-1)(r-1)\bigr)\bigl(S-n(r-1)\bigr)\\
&\quad+\frac{(n-1)(5n-9)}{4n^2}\bigl(S-n(r-1)\bigr)^2\\
&\geq 0,
\end{aligned} \tag{3-15}$$

that is,

$$\Bigl(n+n(r-1)-\frac{3n-5}{2n}\bigl(S-n(r-1)\bigr)\Bigr)^2 \geq \frac{(n-2)^2}{n^2}\Bigl(\bigl(S-n(r-1)\bigr)+n^2(r-1)\Bigr)\bigl(S-n(r-1)\bigr). \tag{3-16}$$

When $r \geq (n-2)/(n-1)$, it is directly checked from (3-14) that

$$n + n(r-1) - \frac{3n-5}{2n}\big(S - n(r-1)\big) \geq 0.$$

Hence, we have

$$n + n(r-1) - \frac{3n-5}{2n}\big(S - n(r-1)\big) \geq \frac{n-2}{n}\sqrt{\big(n(n-1)(r-1)+S\big)\big(S-n(r-1)\big)}, \tag{3-17}$$

that is,

$$Q_r(S) \geq 0. \tag{3-18}$$

Hence, by (3-10) we also have $\mathrm{Ric}(v, v) \geq 0$.

To sum up, we know that, if $S \leq \alpha(n, r)$, then $\mathrm{Ric}(v, v) \geq 0$. If $\mathrm{Ric}(v, v) \geq 0$, we have the following cases:

***Case* 1: *When, at some point and every v,* $\mathbf{Ric}(v, v) > 0$.** When $\mathrm{Ric}(v, v) > 0$ holds for all v at all points of M, then, according to Myers' theorem, the fundamental group is finite. When $\mathrm{Ric}(v, v) > 0$ holds for all v at some point of M, then, from Aubin's Lemma 4, there exists a metric on M such that the Ricci curvature is positive on M. Hence, according to Myers' theorem, we again know that the fundamental group is finite.

When the fundamental group of M is finite, the proof of the Main Theorem in the case when $n = 3$ follows directly from the theorem of Hamilton [1982] which states that a compact and connected Riemannian 3-manifold with positive Ricci curvature is diffeomorphic to a spherical space form.

Now, we consider the case when $n \geq 4$. Take any positive integers p, q such that $p + q = n$ and $1 < p, q < n-1$. We have

$$pq = n + (p-1)n - p^2 \geq n + (p-1)(p+2) - p^2 = n + (p-2) \geq n.$$

Let

$$T_\alpha = \operatorname{tr} H_\alpha = \sum_{s=1}^{p} h^\alpha_{ss} + \sum_{t=p+1}^{n} h^\alpha_{tt},$$
$$\tilde{S}_\alpha = \sum_i (h^\alpha_{ii})^2, \qquad S_\alpha = \sum_{i,j} (h^\alpha_{ij})^2,$$

so that

$$S = \sum_\alpha S_\alpha \qquad \text{and} \qquad n^2 H^2 = \sum_\alpha T_\alpha^2.$$

We have

$$2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h^\alpha_{st})^2 + \frac{pq}{n}\tilde{S}_\alpha \leq \frac{pq}{n}\Big(2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h^\alpha_{st})^2 + \tilde{S}_\alpha\Big) \leq \frac{pq}{n}S_\alpha. \tag{3-19}$$

On one hand, when $p \geq q$, we have

$$|p-q| = p-q = n-2q < n-2.$$

On the other hand, when $p < q$, we have

$$|p-q| = q-p = n-2p < n-2.$$

Therefore $|p-q| < n-2$ always, and

$$\sqrt{pq} \geq \sqrt{n} > \sqrt{n-1}.$$

From Lemma 2 and the inequalities (3-19) and $\tilde{S}_\alpha \leq S_\alpha$, and by making use of the same calculation as in [Shiohama and Xu 1997], we get, when $S \leq \alpha(n,r)$,

$$\begin{aligned}
&\sum_{s=1}^{p}\sum_{t=p+1}^{n}\left(2\left|B(e_s,e_t)\right|^2-\left\langle B(e_s,e_s),\,B(e_t,e_t)\right\rangle\right)\\
&\quad= 2\sum_\alpha\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st}^\alpha)^2-\sum_\alpha\sum_{s=1}^{p}\sum_{t=p+1}^{n}h_{ss}^\alpha h_{tt}^\alpha\\
&\quad= \sum_\alpha\Bigl(2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st}^\alpha)^2+\bigl(\sum_{s=1}^{p}h_{ss}^\alpha\bigr)^2-T_\alpha\bigl(\sum_{s=1}^{p}h_{ss}^\alpha\bigr)\Bigr)\\
&\quad\leq \sum_\alpha\Bigl(2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st}^\alpha)^2+\frac{pq}{n}\tilde{S}_\alpha-\frac{2pq}{n^2}T_\alpha^2+\frac{|p-q|}{n^2}\sqrt{pq}\,|T_\alpha|\sqrt{n\tilde{S}_\alpha-T_\alpha^2}\Bigr)\\
&\quad\leq \sum_\alpha\Bigl(\frac{pq}{n}S_\alpha-\frac{2pq}{n^2}T_\alpha^2+\frac{|p-q|}{n^2}\sqrt{pq}\,|T_\alpha|\sqrt{nS_\alpha-T_\alpha^2}\Bigr)\\
&\quad\leq \frac{pq}{n}S-2pqH^2+\frac{|p-q|}{n^2}\sqrt{pq}\sqrt{\sum_\alpha T_\alpha^2\sum_\alpha(nS_\alpha-T_\alpha^2)}\\
&\quad= \frac{pq}{n}\Bigl(S-2nH^2+\frac{\sqrt{n}\,|p-q|}{\sqrt{pq}}|H|\sqrt{S-nH^2}\,\Bigr)\\
&\quad< \frac{pq}{n}\Bigl(S-2nH^2+\frac{\sqrt{n}(n-2)}{\sqrt{n-1}}|H|\sqrt{S-nH^2}\Bigr)\\
&\quad= -\frac{pq}{n}\Bigl(n+nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H|f-f^2\Bigr)+pq.
\end{aligned}$$

On one hand, if $\tilde{p} \leq 2$, then (3-13) holds. By (3-4) or (3-5), we have

$$\sum_{s=1}^{p}\sum_{t=p+1}^{n}\left(2\left|B(e_s,e_t)\right|^2-\left\langle B(e_s,e_s),\,B(e_t,e_t)\right\rangle\right) < -\frac{pq}{n}P_r(S)+pq < pq. \tag{3-20}$$

On the other hand, if $\tilde{p} \geq 3$, then (3-18) holds. By (3-8) or (3-9), we have

$$\sum_{s=1}^{p}\sum_{t=p+1}^{n}\left(2\left|B(e_s,e_t)\right|^2-\left\langle B(e_s,e_s),\,B(e_t,e_t)\right\rangle\right) < -\frac{pq}{n}Q_r(S)+pq < pq. \tag{3-21}$$

Therefore, from Lemma 3 we have

$$H_p(M, Z) = H_q(M, Z) = 0$$

for all $1 < p, q < n-1$ with $p+q = n$. Since $H_{n-2}(M, Z) = 0$, from the universal coefficient theorem and following the same argument as in [Leung 1983], we get that $H^{n-1}(M, Z)$ has no torsion and consequently, by Poincaré duality, $H_1(M, Z)$ has no torsion. From our assumptions, since the fundamental group $\pi_1(M)$ of M is finite, we have $H_1(M, Z) = 0$ and thus M is a homology sphere.

The above arguments can then be applied to the universal covering $\tilde{M}$ of M. Since $\tilde{M}$ is a homology sphere which is simply connected, that is, $\pi_1(\tilde{M}) = 0$, it is also a homotopy sphere. By the generalized Poincaré conjecture (proved by S. Smale for $n \geq 5$ and M. Freedman for $n = 4$) $\tilde{M}$ is homeomorphic to a sphere. Hence the homotopy sphere M is covered by a sphere $\tilde{M}$. By a result of Sjerve [1973], $\pi_1(M) = 0$ and hence M is itself homeomorphic to a sphere.

***Case 2: When at every point there is some v such that* Ric$(v, v) = 0$.** First of all, we can prove that this does not occur when $\tilde{p} \geq 3$: suppose that at every point, there exists a unit vector v such that $\operatorname{Ric}(v, v) = 0$. Since $S \leq n(r-1) + x_1(r)$, (3-18) holds, that is, $Q_r(S) \geq 0$. Hence the equalities in (3-10) hold. Therefore, $P_r(S) = Q_r(S) = 0$. From (3-5) we have

$$S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

while from (3-9) we also have $S = n(r-1)$. This is a contradiction, because if $r \geq (n-2)/(n-1)$ then

$$(n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} > (n-1)\frac{n(r-1)+2}{n-2} > n(r-1).$$

Therefore, we know that Case 2 can occur only when $\tilde{p} \leq 2$.

We thus assume $\tilde{p} \leq 2$. From (3-6), we have $P_r(S) \leq 0$, while from (3-13), we have $P_r(S) \geq 0$. We get $P_r(S) = 0$, that is, $P_H(f) = 0$. Therefore equalities hold in the inequalities (3-3) and (2-12) of Lemma 1.

If $\tilde{p} = 1$ and setting $h_{ij} = h_{ij}^{n+1}$, from (3-2) we have

$$\operatorname{Ric}(v, v) = (n-1) + nHh_{nn} - \sum_i (h_{in})^2. \tag{3-22}$$

Since $n \geq 3$, when equality holds in inequality (2-12) of Lemma 1, it follows that

$$h_{11} = h_{22} = \cdots = h_{n-1n-1}, \quad h_{ij} = 0 \quad \text{for } i \neq j, \quad \text{and} \quad h_{nn} = nH - (n-1)h_{11}.$$

It is clear that $P_r(S)=0$ is equivalent to

$$S=(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}.$$

Thus, M is not totally umbilical. This is because, when $r\geq (n-2)/(n-1)$,

$$S=(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}>n(r-1),$$

that is,

$$f^2=S-nH^2=\frac{n-1}{n}\big(S-n(r-1)\big)\neq 0.$$

Hence, $h_{nn}\neq h_{11}$. Therefore, M has only two distinct principal curvatures, one of which is simple. Without loss of generality, we can assume them to be $\lambda=\lambda_1=\cdots=\lambda_{n-1}$ and $\mu=\lambda_n$. By (3-22), we get

$$\operatorname{Ric}(v,v)=(n-1)+(\lambda_1+\cdots+\lambda_{n-1}+\lambda_n)\lambda_n-\lambda_n^2=(n-1)(1+\lambda\mu)=0.$$

Hence,

$$1+\lambda\mu=0. \tag{3-23}$$

From (2-5), we have

$$\mu=\frac{n(r-1)}{2\lambda}-\frac{n-2}{2}\lambda. \tag{3-24}$$

Hence, by (3-23) and (3-24), we have

$$\lambda^2=\frac{n(r-1)+2}{n-2}\qquad\text{and}\qquad\mu^2=\frac{n-2}{n(r-1)+2}. \tag{3-25}$$

Following the argument of [Cheng 2003], consider the integral submanifold for the distribution of principal vectors corresponding to the principal curvature λ. Since the multiplicity of the principal curvature λ is greater than 1, from Lemma 5 we know that the principal curvature λ is constant on this integral submanifold [Ôtsuki 1970]. From (3-25), the scalar curvature $n(n-1)r$ and the principal curvature μ must be constant. Thus, M is isoparametric. Since

$$S=(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2},$$

M is isometric to the Riemannian product $\mathbb{S}^1\big(\sqrt{1-c^2}\big)\times\mathbb{S}^{n-1}(c)$.

If $\tilde{p}=2$, we have:

Lemma 6 [Cheng 2002, (3.21)]. *If M is an n-dimensional submanifold in $\mathbb{S}^{n+\tilde{p}}(1)$, $\tilde{p}=2$, and M has nowhere-zero mean curvature, then*

$$\tfrac{1}{2}\Delta S_2\geq\sum_{\alpha\geq n+2}\sum_{i,j,k}(h^{\alpha}_{ijk})^2+\Big(n+nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{S_1}-(S_1+S_2)\Big)S_2.$$

From this lemma, for $f^2 = S - nH^2$ we have

$$\begin{aligned}\tag{3-26} \tfrac{1}{2}\triangle S_2 &\geq \sum_{\alpha\geq n+2}\sum_{i,j,k}(h^{\alpha}_{ijk})^2 + \Big(n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{f^2 - S_2} - f^2\Big)S_2 \\ &\geq \sum_{\alpha\geq n+2}\sum_{i,j,k}(h^{\alpha}_{ijk})^2 + \Big(n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H| f - f^2\Big)S_2 \\ &= \sum_{\alpha\geq n+2}\sum_{i,j,k}(h^{\alpha}_{ijk})^2 + P_H(f)\,S_2,\end{aligned}$$

On one hand, in our Case 2, when $\tilde{p} \leq 2$, we have $P_H(f) = 0$. On the other hand, since M is compact, from Hopf's lemma we have $\triangle S_2 = 0$. Hence, the equalities in (3-26) hold, and we conclude that

$$\tag{3-27} \sum_{\alpha\geq n+2}\sum_{i,j,k}(h^{\alpha}_{ijk})^2 = 0,$$

and $\sqrt{f^2 - S_2} = f$, that is, $S_2 = 0$.
From (2-10), we have

$$\tag{3-28} \sum_{i,k} h^{n+2}_{iik}\omega_k = -nH\omega_{n+2,n+1}.$$

As the mean curvature H is nowhere-zero on M, from (3-27) we have $\omega_{n+2,n+1} = 0$. Thus, e_{n+1} is parallel on the normal bundle $T^{\perp}(M)$ of M. From [Yau 1974, Theorem 1], M is a hypersurface in the totally geodesic submanifold $\mathbb{S}^{n+1}(1)$ of $\mathbb{S}^{n+\tilde{p}}(1)$, and satisfies

$$S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}.$$

Applying the result for the case $\tilde{p} = 1$, we conclude that our theorem is valid.

***Case* 3: *When, at some point,* Ric$(v, v) = 0$ *for all* v.** In this case, $r = 0$ at that point. This is contradictory, because we assumed $r \geq (n-2)/(n-1)$. This completes the proof of the Main Theorem.

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