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The convolution sum $\sum_{m<n/8}\sigma(m)\sigma(n-8m)$ is evaluated for all $n \in \mathbb{N}$. This evaluation is used to determine the number of representations of n by the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+2x_6^2+2x_7^2+2x_8^2$.

1. Introduction

Let $\mathbb{N}$ denote the set of natural numbers. For $n \in \mathbb{N}$ and $k \in \mathbb{N}$, we set

$$\sigma_k(n) = \sum_{d|n} d^k,$$

where d runs through the positive integers dividing n. If $n \notin \mathbb{N}$, we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. We define the convolution sum $W_k(n)$ by

$$W_k(n) := \sum_{m<n/k} \sigma(m)\sigma(n-km), \tag{1}$$

where m runs through the positive integers $< n/k$. The sum $W_k(n)$ has been evaluated for $k = 1, 2, 3, 4, 9$ for all $n \in \mathbb{N}$, and for $k = 5$ for $n \equiv 8 \pmod{16}$, $n \not\equiv 0 \pmod 5$; see [Besge 1862; Huard et al. 2002] for $k = 1$, [Huard et al. 2002; Melfi 1998a; 1998b] for $k = 2$, [Huard et al. 2002; Melfi 1998a; 1998b; Williams 2004] for $k = 3$, [Huard et al. 2002; Melfi 1998a; 1998b] for $k = 4$, [Melfi 1998a; 1998b; Williams 2004; 2005] for $k = 9$, and [Melfi 1998a; 1998b] for $k = 5$. In this paper, we evaluate $W_k(n)$ for $k = 8$ and all $n \in \mathbb{N}$.

Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of integers, real numbers, and complex numbers, respectively. For $q \in \mathbb{C}$ with $|q| < 1$, we set [Ramanujan 1916, eqn. (92), p. 151]

$$\Delta(q) := q \prod_{n=1}^{\infty} (1-q^n)^{24}. \tag{2}$$

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For $n \in \mathbb{N}$, we define $k(n) \in \mathbb{Z}$ by

$$(3) \qquad \bigl(\Delta(q^2)\,\Delta(q^4)\bigr)^{1/6} = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 := \sum_{n=1}^{\infty} k(n)\,q^n.$$

Clearly, $k(n) = 0$ for $n \equiv 0 \pmod 2$. By Euler's identity [Knopp 1970, Corollary 5, p. 37], we have

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2} = 1 - q - q^2 + q^5 + q^7 - q^{12} - \cdots .$$

Using MAPLE, we find

$$q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - \cdots ,$$

so that the first few values of $k(2n-1)$, for $n \in \mathbb{N}$, are

$$k(1) = 1, \quad k(3) = -4, \quad k(5) = -2, \quad k(7) = 24, \quad k(9) = -11.$$

The evaluation of $W_8(n)$ is given in Theorem 1 and proved in Section 2.

Theorem 1. *For $n \in \mathbb{N}$,*

$$\sum_{m<n/8} \sigma(m)\sigma(n-8m) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\Bigl(\frac{n}{2}\Bigr) + \frac{1}{16}\sigma_3\Bigl(\frac{n}{4}\Bigr) + \frac{1}{3}\sigma_3\Bigl(\frac{n}{8}\Bigr) + \Bigl(\frac{1}{24} - \frac{n}{32}\Bigr)\sigma(n) + \Bigl(\frac{1}{24} - \frac{n}{4}\Bigr)\sigma\Bigl(\frac{n}{8}\Bigr) - \frac{1}{64}k(n).$$

As an application of Theorem 1, we evaluate in Section 3 the number

$$N(n) = \operatorname{card}\bigl\{(x_1, x_2, \ldots, x_8) \in \mathbb{Z}^8 \bigm| n = x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+2x_6^2+2x_7^2+2x_8^2\bigr\}.$$

We prove:

Theorem 2. *For $n \in \mathbb{N}$,*

$$N(n) = 4\sigma_3(n) - 4\sigma_3\Bigl(\frac{n}{2}\Bigr) - 16\sigma_3\Bigl(\frac{n}{4}\Bigr) + 256\sigma_3\Bigl(\frac{n}{8}\Bigr) + 4k(n).$$

2. Proof of Theorem 1

Let $q \in \mathbb{C}$ be such that $|q| < 1$. The Eisenstein series $L(q)$, $M(q)$, $N(q)$ are

$$L(q) := 1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n, \tag{4}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{5}$$

$$N(q) := 1 - 504\sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{6}$$

see for example [Berndt 1989, p. 318; Ramanujan 1916, eqn. (25), p. 140]. It was shown in [Ramanujan 1916, eqn. (44)] that

$$\Delta(q) = \frac{1}{1728}\big(M(q)^3 - N(q)^2\big). \tag{7}$$

First, we determine the generating function of $W_8(n)$ for $n \in \mathbb{N}$. By (4), we have

$$\begin{aligned}
&\big(1 - L(q)\big)\big(1 - L(q^8)\big)\\
&\qquad = \Big(24\sum_{l=1}^{\infty}\sigma(l)q^l\Big)\Big(24\sum_{m=1}^{\infty}\sigma(m)q^{8m}\Big) = 576\sum_{l,m=1}^{\infty}\sigma(l)\sigma(m)q^{l+8m}\\
&\qquad = 576\sum_{n=1}^{\infty}q^n\sum_{\substack{l,m=1\\ l+8m=n}}^{\infty}\sigma(l)\sigma(m) = 576\sum_{n=1}^{\infty}q^n\sum_{1\le m<n/8}\sigma(m)\sigma(n-8m),
\end{aligned}$$

that is,

$$\big(1 - L(q)\big)\big(1 - L(q^8)\big) = 576\sum_{n=1}^{\infty} W_8(n) q^n,$$

so that

$$\sum_{n=1}^{\infty} W_8(n) q^n = \frac{1}{576} - \frac{1}{576}L(q) - \frac{1}{576}L(q^8) + \frac{1}{576}L(q)L(q^8). \tag{8}$$

Next, we use (4) and (8) to determine the power-series expansion of $L(q)L(q^8)$ in powers of q. Letting $q \mapsto q^8$ in (4) yields

$$L(q^8) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^{8n} = 1 - 24\sum_{n=1}^{\infty}\sigma\Big(\frac{n}{8}\Big)q^n. \tag{9}$$

From (4), (8), and (9) we deduce

$$L(q)L(q^8) = 1 + \sum_{n=1}^{\infty}\Big(576\,W_8(n) - 24\sigma(n) - 24\sigma\Big(\frac{n}{8}\Big)\Big)q^n. \tag{10}$$

Our next objective is to determine the power series of $\bigl(L(q) - 8L(q^8)\bigr)^2$ in powers of q. The result,

$$L(q)^2 = 1 + \sum_{n=1}^{\infty} \bigl(240\sigma_3(n) - 288n\sigma(n)\bigr) q^n, \tag{11}$$

is classical, see for example [Glaisher 1885a; 1885b]. Letting $q \mapsto q^8$ in (11), we obtain

$$L(q^8)^2 = 1 + \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{8}\right) - 36n\sigma\left(\frac{n}{8}\right)\right) q^n. \tag{12}$$

Hence, by (10), (11), and (12), we obtain

$$\begin{aligned}
\bigl(L(q) - 8L(q^8)\bigr)^2 &= L(q)^2 + 64L(q^8)^2 - 16L(q)L(q^8) \\
&= 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n)) q^n \\
&\quad + 64\left(1 + \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{8}\right) - 36n\sigma\left(\frac{n}{8}\right)\right) q^n\right) \\
&\quad - 16\left(1 + \sum_{n=1}^{\infty} \left(576W_8(n) - 24\sigma(n) - 24\sigma\left(\frac{n}{8}\right)\right) q^n\right),
\end{aligned}$$

that is,

$$\begin{aligned}
&\bigl(L(q) - 8L(q^8)\bigr)^2 \\
&= 49 + \sum_{n=1}^{\infty} \left(240\sigma_3(n) + 15360\sigma_3\left(\frac{n}{8}\right) + (384 - 288n)\sigma(n)\right. \\
&\qquad\qquad \left. + (384 - 2304n)\sigma\left(\frac{n}{8}\right) - 9216W_8(n)\right) q^n.
\end{aligned} \tag{13}$$

Next, we use Ramanujan's evaluation of $L(q)$ [Berndt 1991, p. 129] and the principle of duplication [Berndt 1991, p. 125] to determine the power-series expansion of $\bigl(L(q) - 8L(q^8)\bigr)^2$ in another way. For $z \in \mathbb{C}$ with $|z| < 1$, we set

$$w(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{z^n}{n!}, \tag{14}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol, see for example [Copson 1935, p. 247; Rainville 1971, p. 45]. Clearly, $w(0) = 1$. The infinite series (14) diverges at $z = 1$ [Copson 1935, p. 249], so that

$w(1)=+\infty$. For $x\in\mathbb{R}$ with $0\le x<1$, we have

$$w(x)=1+\sum_{n=1}^{\infty}\frac{(2n!)^2}{(n!)^4 2^{4n}}x^n\ge 1$$

so that

$$w(x)\ne 0,\qquad 0\le x<1. \tag{15}$$

The derivative with respect to x of the function

$$y(x):=\pi\frac{w(1-x)}{w(x)},\qquad 0<x<1, \tag{16}$$

is

$$y'(x)=\frac{-1}{x(1-x)w(x)^2},\qquad 0<x<1, \tag{17}$$

see [Berndt 1989, p. 87]. Thus, by (15) and (17), we have

$$y'(x)<0,\qquad 0<x<1. \tag{18}$$

Hence, as x increases from 0 to 1, $y(x)$ strictly decreases

$$\text{from}\qquad y(0)=\pi\frac{w(1)}{w(0)}=+\infty\qquad\text{to}\qquad y(1)=\pi\frac{w(0)}{w(1)}=0.$$

Now, restrict q so that $q\in\mathbb{R}$ and $0<q<1$. Thus, $0<-\log q<+\infty$. Hence, there is a unique value of x between 0 and 1 such that

$$y(x)=-\log q. \tag{19}$$

Ramanujan gave in his notebooks [Ramanujan 1957] the following formulae for $L(q)$, $M(q)$ and $N(q)$, which are proved in [Berndt 1991, pp. 126–129]:

$$L(q)=(1-5x)w^2+12x(1-x)w\frac{dw}{dx}, \tag{20}$$

$$M(q)=(1+14x+x^2)w^4, \tag{21}$$

$$N(q)=(1+x)(1-34x+x^2)w^6. \tag{22}$$

From (7), (21), and (22), we obtain

$$\Delta(q)=\frac{x(1-x)^4w^{12}}{2^4}. \tag{23}$$

Applying the principle of duplication [Berndt 1991, p. 125]

$$q\mapsto q^2,\qquad x\mapsto\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2,\qquad w\mapsto\left(\frac{1+\sqrt{1-x}}{2}\right)w$$

to (20), (21), and (23), we obtain

$$L(q^2) = (1-2x)w^2 + 6x(1-x)w\frac{dw}{dx}, \tag{24}$$

$$M(q^2) = (1-x+x^2)w^4, \tag{25}$$

$$\Delta(q^2) = \frac{x^2(1-x)^2 w^{12}}{2^8}. \tag{26}$$

Formulae (24) and (25) are given in [Berndt 1991, pp. 122, 126]. Applying the principle of duplication to (24), (25), and (26), we obtain

$$L(q^4) = \Big(1 - \frac{5}{4}x\Big)w^2 + 3x(1-x)w\frac{dw}{dx}, \tag{27}$$

$$M(q^4) = \Big(1 - x + \frac{1}{16}x^2\Big)w^4, \tag{28}$$

$$\Delta(q^4) = \frac{x^4(1-x)\,w^{12}}{2^{16}}. \tag{29}$$

Formulae (27) and (28) can be deduced from [Berndt 1991, pp. 122, 127]. From (3), (26), and (29), we obtain

$$x\sqrt{1-x}\; w^4 = 16\sum_{n=1}^{\infty} k(n)q^n. \tag{30}$$

Applying the duplication principle to (27) and (28), we have

$$L(q^8) = \Big(\frac{5}{8} - \frac{11}{16}x + \frac{3}{8}\sqrt{1-x}\Big)w^2 + \frac{3}{2}x(1-x)\,w\frac{dw}{dx}, \tag{31}$$

$$M(q^8) = \Big(\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1-x} - \frac{15}{64}x\sqrt{1-x}\Big)w^4, \tag{32}$$

see [Cheng and Williams 2004, p. 564]. From (20) and (31), we deduce

$$L(q) - 8L(q^8) = \Big(-4 + \frac{1}{2}x - 3\sqrt{1-x}\Big)w^2. \tag{33}$$

Squaring both sides of (33), we have

$$\big(L(q) - 8L(q^8)\big)^2 = \Big(25 - 13x + \frac{1}{4}x^2 + 24\sqrt{1-x} - 3x\sqrt{1-x}\Big)w^4. \tag{34}$$

From (21), (25), and (28), we obtain

$$w^4 = \frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4), \tag{35}$$

$$xw^4 = \frac{1}{15}M(q) - \frac{1}{15}M(q^2), \tag{36}$$

$$x^2w^4 = \frac{16}{15}M(q^2) - \frac{16}{15}M(q^4). \tag{37}$$

Using (30), (35), (36), and (37) in (32), we obtain

$$M(q^8) = -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}\,w^4 - \frac{15}{4}\sum_{n=1}^{\infty} k(n)q^n.$$

Thus,

$$\sqrt{1-x}\,w^4 = \frac{1}{15}M(q^2) - \frac{6}{5}M(q^4) + \frac{32}{15}M(q^8) + 8\sum_{n=1}^{\infty} k(n)q^n. \tag{38}$$

Now, using (30), (35), (36), (37), and (38) in (34), we deduce

$$\begin{aligned}(L(q) - 8L(q^8))^2 \\ = \frac{4}{5}M(q) - \frac{3}{5}M(q^2) - \frac{12}{5}M(q^4) + \frac{256}{5}M(q^8) + 144\sum_{n=1}^{\infty} k(n)q^n.\end{aligned} \tag{39}$$

Appealing to (5) and (39), we obtain

$$\begin{aligned}(L(q) - 8L(q^8))^2 = 49 + \sum_{n=1}^{\infty}\Big(192\sigma_3(n) - 144\sigma_3\Big(\frac{n}{2}\Big) - 576\sigma_3\Big(\frac{n}{4}\Big) \\ + 12288\sigma_3\Big(\frac{n}{8}\Big) + 144k(n)\Big)q^n.\end{aligned} \tag{40}$$

Equating coefficients of q^n in (13) and (40), we deduce that

$$\begin{aligned}240\sigma_3(n) + 15360\sigma_3\Big(\frac{n}{8}\Big) + (384-288n)\sigma(n) + (384-2304n)\sigma\Big(\frac{n}{8}\Big) - 9216\,W_8(n) \\ = 192\sigma_3(n) - 144\sigma_3\Big(\frac{n}{2}\Big) - 576\sigma_3\Big(\frac{n}{4}\Big) + 12288\sigma_3\Big(\frac{n}{8}\Big) + 144k(n),\end{aligned}$$

from which the asserted formula for $W_8(n)$ follows. □

3. The number of representations of n by the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+2x_6^2+2x_7^2+2x_8^2$

Proof of Theorem 2. Let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For $l \in \mathbb{N}_0$, we set

$$r_4(l) = \operatorname{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = l\},$$

so that $r_4(0) = 1$. The number $N(n)$ of representations of n by the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+2x_6^2+2x_7^2+2x_8^2$ is

$$
\begin{aligned}
N(n) &= \operatorname{card}\{(x_1,\dots,x_8)\in\mathbb{Z}^8 \mid x_1^2+x_2^2+x_3^2+x_4^2+2x_5^2+2x_6^2+2x_7^2+2x_8^2=n\}\\
&= \sum_{\substack{l,m\in\mathbb{N}_0\\ l+2m=n}} \Bigg(\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\ x_1^2+x_2^2+x_3^2+x_4^2=l}} 1\Bigg)\Bigg(\sum_{\substack{(x_5,x_6,x_7,x_8)\in\mathbb{Z}^4\\ x_5^2+x_6^2+x_7^2+x_8^2=m}} 1\Bigg) = \sum_{\substack{l,m\in\mathbb{N}_0\\ l+2m=n}} r_4(l)r_4(m)\\
&= r_4(0)r_4\Big(\frac{n}{2}\Big)+r_4(n)r_4(0)+\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}} r_4(l)r_4(m)\\
&= r_4(n)+r_4\Big(\frac{n}{2}\Big)+\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}} r_4(l)r_4(m).
\end{aligned}
$$

It is a classical result of Jacobi — see for example [Spearman and Williams 2000] — that

$$
r_4(n) = 8\sum_{\substack{d\mid n\\ 4\nmid d}} d = 8\sigma(n) - 32\sigma\Big(\frac{n}{4}\Big), \qquad n\in\mathbb{N}.
$$

Hence,

$$
\begin{aligned}
N(n) = 8\sigma(n) - 32\sigma\Big(\frac{n}{4}\Big) + 8\sigma\Big(\frac{n}{2}\Big) - 32\sigma\Big(\frac{n}{8}\Big)&\\
+\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\Big(8\sigma(l)-32\sigma\Big(\frac{l}{4}\Big)\Big)\Big(8\sigma(m)-32\sigma\Big(\frac{m}{4}\Big)\Big)&.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&N(n) - 8\sigma(n) - 8\sigma\Big(\frac{n}{2}\Big) + 32\sigma\Big(\frac{n}{4}\Big) + 32\sigma\Big(\frac{n}{8}\Big)\\
&\qquad = 64\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma(m) - 256\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma\Big(\frac{l}{4}\Big)\sigma(m)\\
&\qquad\qquad -256\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma(l)\sigma\Big(\frac{m}{4}\Big) + 1024\sum_{\substack{l,m\in\mathbb{N}\\ l+2m=n}}\sigma\Big(\frac{l}{4}\Big)\sigma\Big(\frac{m}{4}\Big)\\
&\qquad = 64\sum_{m<n/2}\sigma(m)\sigma(n-2m) - 256\sum_{\substack{l,m\in\mathbb{N}\\ 4l+2m=n}}\sigma(l)\sigma(m)\\
&\qquad\qquad -256\sum_{\substack{l,m\in\mathbb{N}\\ l+8m=n}}\sigma(l)\sigma(m) + 1024\sum_{\substack{l,m\in\mathbb{N}\\ 4l+8m=n}}\sigma(l)\sigma(m)\\
&\qquad = 64W_2(n) - 256\sum_{l<n/4}\sigma(l)\sigma\Big(\frac{n}{2}-2l\Big)\\
&\qquad\qquad -256\sum_{m<n/8}\sigma(m)\sigma(n-8m) + 1024\sum_{m<n/8}\sigma(m)\sigma\Big(\frac{n}{4}-2m\Big),
\end{aligned}
$$

that is,

$$(41)\quad N(n) = 8\sigma(n) + 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{4}\right) - 32\sigma\left(\frac{n}{8}\right) + 64\,W_2(n) - 256\,W_2\left(\frac{n}{2}\right) - 256\,W_8(n) + 1024\,W_2\left(\frac{n}{4}\right).$$

From [Huard et al. 2002, Theorem 2], we have

$$(42)\qquad W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{2}\right).$$

Appealing to (41), (42), and Theorem 1, we obtain

$$N(n) = 4\sigma_3(n) - 4\sigma_3\left(\frac{n}{2}\right) - 16\sigma_3\left(\frac{n}{4}\right) + 256\sigma_3\left(\frac{n}{8}\right) + 4k(n),$$

as claimed. □

The values of $N(n)$, $\sigma_3(n)$ and $k(n)$ for $n = 1, 2, \ldots, 10$ are as follows:

n	1	2	3	4	5	6	7	8	9	10
$N(n)$	8	32	96	240	496	896	1472	2160	2984	4032
$\sigma_3(n)$	1	9	28	73	126	252	344	585	757	1134
$k(n)$	1	0	−4	0	−2	0	24	0	−11	0

Note added in proof

Since this paper was written, the sums $W_k(n)$ have been evaluated for $k = 5, 6, 7, 12, 16, 18$ and 24. See M. Lemire and K. S. Williams, *Bull. Austral. Math. Soc.* **73** (2006), 107–115; A. Alaca, S. Alaca and K. S. Williams, *Adv. Th. App. Math.* **1** (2006), 27–48, *Int. Math. Forum* **2** (2007), 45–68, *Math. J. Okayama Univ.* (in press), *Canad. Math. Bull.* (to appear); and S. Alaca and K. S. Williams, *J. Number Th.* (in press).

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