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For an equation or system of equations $Lu = 0$, where L is a uniformly elliptic operator of order $2m$ and u is a map from $\mathbb{R}^n$ to $\mathbb{R}^N$, we prove that the dimension of the space of polynomial growth solutions of degree at most d is bounded by Cd^{2mnN}, where C is a constant. If the system is in divergence form, we prove that this dimension is in fact bounded by Cd^{mnN}.

Introduction

We consider an equation or a system of equations of the form

$$Lu = 0,$$

where L is a uniformly elliptic operator of order $2m$, with $m > 1$, defined on $\mathbb{R}^n$. We want to estimate the dimension of the following space of solutions to $Lu = 0$.

Definition 0.1. For each nonnegative number d we denote by

$$\mathcal{H}^L_d(\mathbb{R}^n) = \left\{ u \;\middle|\; Lu = 0 \text{ and } |u|(x) = O(r_p^d(x)) \right\}$$

the space of polynomial growth solutions of degree at most d, where $r_p(x)$ is the Euclidean distance from a fixed point p to x in $\mathbb{R}^n$. We denote the dimension of $\mathcal{H}^L_d(\mathbb{R}^n)$ by

$$h^L_d(\mathbb{R}^n) = \dim \mathcal{H}^L_d(\mathbb{R}^n).$$

When $L = \Delta$ is the Laplacian, this subject has been studied extensively for a variety of open manifolds M (meaning noncompact and without boundary). Let n be the dimension of M. Yau conjectured that $h^\Delta_d(M) < \infty$ for all $d \geq 1$. For $M = \mathbb{R}^n$ this is easy to see; in fact $h^\Delta_d(\mathbb{R}^n)$ equals

$$\text{(0-1)} \qquad \binom{n+d-1}{d} + \binom{n+d-2}{d-1} \sim \frac{2}{(n-1)!}\, d^{n-1} \quad \text{as } d \to \infty.$$

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Yau's conjecture was partially confirmed for the case $d = 1$ by Li and Tam [1989], who proved that under the same conditions, if the volume growth of M satisfies $V_p(r) = O(r_p^k)$ for some $k > 0$, then

$$h_d^\Delta(M) \le k + 1 = h_d^\Delta(\mathbb{R}^k).$$

The conjecture was then proved in full by Colding and Minicozzi [1997], who showed that for a complete open manifold M of nonnegative Ricci curvature, there exists $C > 0$ depending only on the dimension n and such that

$$h_d^\Delta(M) \le C\, d^{n-1}.$$

In view of the formula (0-1) for $h_d^\Delta(\mathbb{R}^n)$, this estimate is sharp in the order of d as $d \to \infty$. The authors also proved that if a complete open manifold M satisfies a Poincaré inequality and a volume doubling property, then $h_d^\Delta(M)$ is finite and can be estimated in terms of a constant depending on the manifold and d. However, in this case, the order in d is not sharp.

Soon thereafter, Li [1997] proved a more general estimate with a substantially simpler proof. Namely, if M (open, complete) satisfies a mean value inequality and a volume comparison condition, then

$$h_d^\Delta(M) \le C\, d^{n-1}.$$

Later Li and Wang [1999a] showed that the finiteness of $h_d^\Delta(M)$ is actually valid in a much bigger class of manifolds. In particular, they proved that if M satisfies a weak mean value inequality and has polynomial volume growth, then $h_d^\Delta(M)$ must be finite for all $d \ge 1$. However, in this case, the estimate on $h_d^\Delta(M)$ is exponential in d as $d \to \infty$.

Recently, Li and Wang [1999b] showed that if M is a complete manifold satisfying the Sobolev inequality $\mathcal{S}(B, \nu)$, the space $\mathcal{H}_d^\Delta(M)$ is finite-dimensional, and its dimension h_d^Δ satisfies

$$h_d^\Delta(M) \le C(B, \nu)\, d^\nu$$

for all $d \ge 1$. They proved that if M is a complete n-dimensional open manifold with nonnegative sectional curvature, then

$$\liminf_{d\to\infty} d^{-(n-1)}\, h_d^\Delta(M) \le \frac{2}{(n-1)!},$$

and the equality

$$\liminf_{d\to\infty} d^{-(n-1)}\, h_d^\Delta(M) = \frac{2}{(n-1)!}$$

holds if and only if $M = \mathbb{R}^n$.

In this note we extend some of the preceding results to higher-order operators. To simplify the presentation, we restrict ourselves to Euclidean space. So we assume that L is of higher order $2m$ with $m > 1$ and try to estimate $h_d^L(\mathbb{R}^n)$.

In Section 1 we show that if $Lu = 0$ is a uniformly elliptic equation or a uniformly elliptic system of equations of order $2m$ in nondivergence form, then

$$h_d^L(\mathbb{R}^n) \leq C\, d^{2mnN},$$

where N is the number of equations in the system $Lu = 0$.

In Section 2 we consider the case where $Lu = 0$ is a uniformly elliptic equation or a uniformly elliptic system of equations of order $2m$ in divergence form. Then

$$h_d^L(\mathbb{R}^n) \leq C\, d^{mnN},$$

where N is the number of equations in the system $Lu = 0$.

1. Equations in nondivergence form

In Euclidean space $\mathbb{R}^n$ with rectangular coordinates $x_1, \dots, x_n$, we consider the differential operator

$$Lu \equiv \sum_{|\alpha|=2m} a_\alpha(x)\, D^\alpha u(x),$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha = \frac{\partial^{2m}}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Throughout the section, we impose the following condition on the operator L.

Condition L**.** *The coefficients a_α in the equation $Lu = 0$ are uniformly continuous and satisfy the uniform ellipticity condition*; *that is*, *there exists a constant $\Lambda > 0$ such that*

$$\Lambda |\Xi|^{2m} \geq \sum_{|\alpha|=2m} a_\alpha(x)\, \Xi^\alpha \geq \Lambda^{-1} |\Xi|^{2m}$$

for all x, $\Xi \in \mathbb{R}^n$.

The assumptions imply that there exists a constant $C > 0$ such that, for any function $w \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla^{2m} w|^2(x)\, dx \leq C \int_{\mathbb{R}^n} |Lw|^2(x)\, dx$$

(see [Agmon et al. 1959; 1964]). We establish some preliminary lemmas before we prove our first main result.

Lemma 1.1 [Li and Wang 1999b]. *Let V be a k-dimensional subspace of a vector space W. Assume that W is endowed with an inner product I and a bilinear form Φ. Then for any given linearly independent set of vectors $\{w_1, \dots, w_{k-1}\} \subset W$, there exists an orthonormal basis $\{v_1, \dots, v_k\}$ of V with respect to I such that $\Phi(v_i, w_j) = 0$ for all $1 \le j < i \le k$.*

Let ϕ be a positive function defined on a fixed geodesic ball $B_p(r)$. We introduce two inner products I_r and Φ_r on the space $W = L^2(B_p(r), dx) \cap L^2(B_p(r), \phi\, dx)$:

$$I_r(f, g) = \int_{B_p(r)} f(x)\, g(x)\, dx, \quad \Phi_r(f, g) = \int_{B_p(r)} f(x)\, g(x)\, \phi(x)\, dx.$$

For $i = 1, 2, \dots,$ let $\lambda_i(r)$ be the i-th Dirichlet eigenvalue of $B_p(r)$ arranged in nondecreasing order.

Lemma 1.2. *Let V be a k-dimensional subspace of $\mathcal{H}_d^L(\mathbb{R}^n)$. For any fixed number $\theta > 1$, let $\operatorname{tr}_{I_{\theta r}} I_r(V)$ denote the trace of the bilinear form I_r with respect to the inner product $I_{\theta r}$ on V. Then*

$$\operatorname{tr}_{I_{\theta r}} I_r(V) \le \sum_{i=1}^{k} \frac{C_8 \theta^{4m-2}}{\lambda_i(\theta r)(\theta - 1)^{4m} r^2},$$

where C_8 is a constant.

Proof. Set $\bar\theta = \frac{1}{2}(1+\theta)$ and let $\phi \in C_0^{2m}(B_p(\bar\theta r))$ be a nonnegative function defined on $B_p(\bar\theta r)$ satisfying $\phi = 1$ on $B_p(r)$, $0 \le \phi \le 1$ on $B_p(\bar\theta r)$, $\phi = 0$ on $\partial B_p(\bar\theta r)$, and

$$|\nabla^j \phi| \le \frac{C}{(\theta - 1)^j\, r^j}$$

for some constant $C = C(n, m)$, and $1 \le j \le 2m$. By unique continuation, $V \subset H^{2m}(B_p(\bar\theta r), dx) \cap H^{2m}(B_p(r), \phi\, dx)$ is a k-dimensional subspace. Applying Lemma 1.1 with $w_1, \dots, w_k$ the Dirichlet eigenfunctions of $B_p(\bar\theta r)$ corresponding to the eigenvalues $\lambda_1(\bar\theta r), \dots, \lambda_k(\bar\theta r)$, we get an orthonormal basis $\{v_1, \dots, v_k\}$ of V with respect to the inner product $I_{\bar\theta r}$ and

$$\Phi_{\bar\theta r}(v_i, w_j) = \int_{B_p(\bar\theta r)} v_i(x)\, w_j(x)\, \phi(x)\, dx = 0$$

for $0 \le j < i \le k$. Thus, for any $1 \le i \le k$, the variational principle implies that

$$\lambda_i(\bar\theta r) \int_{B_p(\bar\theta r)} (\phi\, v_i)^2 \le \int_{B_p(\bar\theta r)} \left|\nabla(\phi\, v_i)\right|^2.$$

Hence,

$$
\begin{aligned}
\text{(1-1)} \qquad \operatorname{tr}_{I_{\theta r}} I_r(V) &= \sum_{i=1}^{k} \int_{B_p(r)} v_i^2 \le \sum_{i=1}^{k} \frac{1}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \left|\nabla(\phi v_i)\right|^2 \\
&\le C \sum_{i=1}^{k} \frac{1}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \left|\nabla^2(\phi v_i)\right|^2 \bar\theta^2 r^2 \\
&\le C^{2m-1} \sum_{i=1}^{k} \frac{1}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \left|\nabla^{2m}(\phi v_i)\right|^2 \bar\theta^{4m-2} r^{4m-2} \\
&\le C_1 \sum_{i=1}^{k} \frac{\bar\theta^{4m-2} r^{4m-2}}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \left|L(\phi v_i)\right|^2 \\
&\le C_2 \sum_{i=1}^{k} \sum_{j=0}^{2m-1} \frac{\bar\theta^{4m-2} r^{4m-2}}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \frac{|\nabla^j v_i|^2}{(\bar\theta-1)^{4m-2j} r^{4m-2j}},
\end{aligned}
$$

where C_1 and C_2 are constants.

Let $\eta \in C_0^{2m}\big(B_p(\theta r)\big)$ be a nonnegative function defined on $B_p(\theta r)$ satisfying $\eta = 1$ on $B_p(\bar\theta r)$, $0 \le \eta \le 1$, $\eta = 0$ on $\partial B_p(\theta r)$, and

$$
|\nabla^j \eta| \le \frac{\bar C}{(\theta-1)^j r^j}
$$

for some constant $\bar C = \bar C(n,m)$, and $1 \le j \le 2m$. Note that

$$
\begin{aligned}
\text{(1-2)} \qquad \int_{B_p(\bar\theta r)} |\nabla^{2m} v_i|^2 &\le \int_{B_p(\theta r)} \left|\nabla^{2m}(\eta v_i)\right|^2 \le C_3 \int_{B_p(\theta r)} \left|L(\eta v_i)\right|^2 \\
&\le C_4 \int_{B_p(\theta r)} \sum_{j=0}^{2m-1} \frac{|\nabla^j v_i|^2}{(\bar\theta-1)^{4m-2j} r^{4m-2j}}.
\end{aligned}
$$

Introduce the weighted seminorms

$$
\Psi_k = \sup_{1<\bar\theta<\sigma<\theta} (\sigma-1)^{2k} r^{2k} \int_{B_p(\sigma r)} |\nabla^k v_i|^2
$$

for each $0 \le k \le 2m$. In terms of these seminorms, (1-2) implies that

$$
\text{(1-3)} \qquad \Psi_{2m} \le C_5 \sum_{k=0}^{2m-1} \Psi_k.
$$

For each $1 \le k \le 2m-1$, we apply an interpolation inequality to get

$$
\text{(1-4)} \qquad \Psi_k \le \epsilon \Psi_{2m} + C(k) \epsilon^{k/(k-2m)} \Psi_0
$$

for any $\epsilon > 0$, where $C(k)$ is a constant. Putting (1-4) into (1-3), and arranging $\epsilon > 0$ to be small, we conclude that $\Psi_{2m} \le C_6 \Psi_0$. In particular, we have

$$\int_{B_p(\bar\theta r)} |\nabla^{2m} v_i|^2 \le \frac{C_6}{(\bar\theta - 1)^{4m} r^{4m}} \int_{B_p(\theta r)} v_i^2.$$

Therefore,

$$\begin{aligned}
\text{(1-5)}\quad & \int_{B_p(\bar\theta r)} \sum_{j=0}^{2m-1} \frac{|\nabla^j v_i|^2}{(\bar\theta - 1)^{4m-2j} r^{4m-2j}} \\
&\le \sum_{j=0}^{2m-1} \frac{1}{(\bar\theta - 1)^{4m-2j} r^{4m-2j}} \\
&\qquad\qquad \times \left(\epsilon_j \int_{B_p(\bar\theta r)} |\nabla^{2m} v_i|^2 + C(j) \epsilon_j^{j/(j-2m)} \int_{B_p(\bar\theta r)} v_i^2 \right) \\
&\le 2m \int_{B_p(\bar\theta r)} |\nabla^{2m} v_i|^2 + \frac{C}{(\bar\theta - 1)^{4m} r^{4m}} \int_{B_p(\bar\theta r)} v_i^2 \\
&\le \frac{C_7}{(\bar\theta - 1)^{4m} r^{4m}} \int_{B_p(\theta r)} v_i^2,
\end{aligned}$$

where we have set $\epsilon_j = (\theta - 1)^{4m-2j} r^{4m-2j}$. Substituting (1-5) into (1-1), we get

$$\begin{aligned}
\mathrm{tr}_{I_{\theta r}} I_r(V) &\le \sum_{i=1}^{k} \sum_{j=0}^{2m-1} \frac{C_2 \bar\theta^{4m-2} r^{4m-2}}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \frac{|\nabla^j v_i|^2}{(\bar\theta - 1)^{4m-2j} r^{4m-2j}} \\
&\le \sum_{i=1}^{k} \frac{C_2 C_7 \bar\theta^{4m-2} r^{4m-2}}{\lambda_i(\bar\theta r)(\bar\theta - 1)^{4m} r^{4m}} \int_{B_p(\theta r)} v_i^2 = \sum_{i=1}^{k} \frac{C_8 \theta^{4m-2}}{\lambda_i(\theta r)(\theta - 1)^{4m} r^2}. \qquad \square
\end{aligned}$$

Lemma 1.3 [Li 1997]. *Let K be a k-dimensional linear space of functions defined on $\mathbb{R}^n$. Suppose that each function $u \in K$ is of polynomial growth of at most degree d. Then for any $\theta > 1$, $\delta > 0$, and $r_0 > 0$, there exists $r > r_0$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of K with respect to the inner product*

$$I_{\theta r}(u, v) = \int_{B_p(\theta r)} u(x)\, v(x)\, dx,$$

then

$$\mathrm{tr}_{\theta r} I_r = \sum_{i=1}^{k} \int_{B_p(r)} u_i^2(x)\, dx \ge k \theta^{-(2d+n+\delta)}.$$

Proof. We reproduce Li's argument. Let $\mathrm{tr}_\rho I_r$ denote the trace of the bilinear form I_r with respect to I_ρ, and let $\det_\rho I_r$ be the determinant of I_r with respect to I_ρ. Assume that the lemma is false. Then, for $r > r_0$, we have

$$\mathrm{tr}_{\theta r} I_r < k \theta^{-(2d+n+\delta)}.$$

The arithmetic-geometric mean inequality asserts that

$$(\det_{\theta r} I_r)^{1/k} \leq k^{-1}(\mathrm{tr}_{\theta r} I_r).$$

This implies that

$$\det_{\theta r} I_r \leq \theta^{-k(2d+n+\delta)}$$

for all $r > r_0$. Setting $r = r_0 + 1$ and iterating the inequality j times yields

$$\det_{\theta^j r} I_r \leq \theta^{-jk(2d+n+\delta)}. \tag{1-6}$$

However, for a fixed I_r-orthonormal basis $\{u_i\}_{i=1}^k$ of K, the assumption on K implies that there exists a constant $C > 0$, depending on K, such that

$$\int_{B_p(r)} u_i^2(x)\,dx \leq C(1 + r^{2d+n})$$

for all $1 \leq i \leq k$. In particular, this implies that

$$\det_r I_{\theta^j r} \leq kC\,\theta^{jk(2d+n)}\, r^{k(2d+n)}.$$

This contradicts (1-6) as $j \to \infty$. □

Theorem 1.4. *Assume that Condition L holds. Let $n > 2$. Then the space $\mathcal{H}_d^L(\mathbb{R}^n)$ is finite-dimensional, and its dimension $h_d^L(\mathbb{R}^n)$ satisfies the estimate*

$$h_d^L(\mathbb{R}^n) \leq C_{10}\, d^{2mn}$$

for all $d \geq 1$, where C_{10} is a constant.

Proof. It is well known that the k-th Dirichlet eigenvalue of $B_p(r) \subset \mathbb{R}^n$ satisfies

$$\lambda_k(r) \geq C\, r^{-2}\, k^{2/n}$$

for all k and $r > 0$, where C is a constant depending only on n. In particular,

$$\sum_{i=1}^k \lambda_i^{-1}(\theta r) \leq C\,\theta^2\, r^2\, k^{1-(2/n)}.$$

Lemma 1.3 yields that for any k-dimensional subspace V of $\mathcal{H}_d^L(\mathbb{R})$ and any $\theta > 1$, there exists $R > 0$ such that

$$\mathrm{tr}_{I_{\theta R}}\, I_R(V) \geq k\theta^{-(2d+n+1)}.$$

Applying Lemma 1.2, we conclude that

$$\begin{aligned} k\theta^{-(2d+n+1)} &\leq \mathrm{tr}_{I_{\theta R}}\, I_R(V) \\ &\leq \frac{C_8\theta^{4m-2}}{(\theta-1)^{4m}\, R^2} \sum_{i=1}^k \lambda_i^{-1}(\theta R) \leq C_9\theta^{4m}\,(\theta-1)^{-4m}\, k^{1-(2/n)}. \end{aligned}$$

Choosing $\theta = 1 + d^{-1}$, we obtain $k \le C_{10}\, d^{2mn}$. This shows that $h_d^L(\mathbb{R}^n) \le C_{10}\, d^{2mn}$ for all $d \ge 1$. □

Consider the system of partial differential equations

$$(\mathcal{L}u)_i \equiv \sum_{j=1}^{N} \sum_{|\alpha|=2m} a_{ij}^{\alpha}(x)\, D^{\alpha} u_j(x),$$

where $i = 1, \ldots, N$, and $u = (u_1, \ldots, u_N) : \mathbb{R}^n \to \mathbb{R}^N$.

Condition $\mathcal{L}$**.** *The coefficient matrix $a_{ij}^{\alpha}(x)$ is uniformly continuous and satisfies the ellipticity condition that there exists a constant $\Lambda > 0$ such that*

$$\Lambda |\Xi|^{2mN} |\eta|^2 \ge \sum_{|\alpha|=2m} \sum_{i,j=1}^{N} a_{ij}^{\alpha}(x)\, \Xi^{\alpha} \eta_i \eta_j \ge \Lambda^{-1} |\Xi|^{2mN} |\eta|^2$$

for all x, $\Xi \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^N$.

Definition 1.5. For each nonnegative number d we denote by

$$\mathcal{H}_d^{\mathcal{L}}(\mathbb{R}^n) = \left\{ u \mid \mathcal{L}u = 0 \text{ and } |u(x)| = O(r_p^d(x)) \right\}$$

the space of polynomial growth $\mathcal{L}$-harmonic functions of degree at most d.

By modifying our previous argument, we have the following theorem.

Theorem 1.6. *Assume that Condition $\mathcal{L}$ holds. Then the space $\mathcal{H}_d^{\mathcal{L}}(\mathbb{R}^n)$ is finite-dimensional, and its dimension $h_d^{\mathcal{L}}(\mathbb{R}^n)$ satisfies*

$$h_d^{\mathcal{L}}(\mathbb{R}^n) \le C_{11}\, d^{2mnN}$$

for all $d \ge 1$, where C_{11} is a constant.

2. Equations in divergence form

In this section, we consider the differential equation

$$Lu \equiv \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|} D^{\alpha}\big(a_{\alpha\beta}(x)\, D^{\beta} u(x)\big)$$

on $\mathbb{R}^n$. We assume that the coefficients of L satisfy the following condition.

Condition L_d**.** *The coefficients $a_{\alpha\beta}$ are measurable, bounded, and there exists a constant $\delta_0 > 0$ such that for all $u \in \mathbb{H}_0^m(\mathbb{R}^n)$,*

$$\mathfrak{Q}(u, u) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^{\beta} u(x)\, D^{\alpha} u(x)\, dx \ge \delta_0 \|\nabla^m u\|_2^2.$$

It is easy to see that if L satisfies the uniform ellipticity condition (meaning that $\Lambda|\Xi|^{2m} \geq \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\,\Xi^\alpha\,\Xi^\beta \geq \Lambda^{-1}|\Xi|^{2m}$ for some $\Lambda > 0$ and all x, $\Xi \in \mathbb{R}^n$), then Condition L_d holds.

Lemma 2.1. *Let V be a k-dimensional subspace of $\mathcal{H}_d^L(\mathbb{R}^n)$. For any fixed number $\theta > 1$, let $\operatorname{tr}_{I_{\theta r}} I_r(V)$ denote the trace of the bilinear form I_r with respect to the inner product $I_{\theta r}$ on V. Then for any fixed integer m, we have*

$$\operatorname{tr}_{I_{\theta r}} I_r(V) \leq C_{20} \sum_{i=1}^{k} \frac{\theta^{2m-2}}{\lambda_i(\theta r)(\theta-1)^{2m} r^2},$$

where C_{20} is a constant.

Proof. Set $\bar\theta = \frac{1}{2}(1+\theta)$ and let $\phi \in C_0^m\big(B_p(\bar\theta r)\big)$ be a nonnegative function defined on $B_p(\bar\theta r)$ satisfying $\phi = 1$ on $B_p(r)$, $0 \leq \phi \leq 1$ on $B_p(\bar\theta r)$, $\phi = 0$ on $\partial B_p(\bar\theta r)$, and

$$|\nabla^j \phi| \leq \frac{C}{(\theta-1)^j\, r^j}$$

for some constant $C = C(n, m)$, and $1 \leq j \leq m$. Observe that by unique continuation, $V \subset H^m\big(B_p(\bar\theta r), dx\big) \cap H^m\big(B_p(r), \phi\, dx\big)$ is a k-dimensional subspace. Applying Lemma 1.1, with $w_1, \ldots, w_k$ the Dirichlet eigenfunctions of $B_p(\bar\theta r)$ corresponding to the eigenvalues $\lambda_1(\bar\theta r), \ldots, \lambda_k(\bar\theta r)$, we get an orthonormal basis $\{v_1, \ldots, v_k\}$ of V with respect to the inner product $I_{\bar\theta r}$, and

$$\Phi_{\bar\theta r}(v_i, w_j) = \int_{B_p(\bar\theta r)} v_i(x)\, w_j(x)\, \phi(x)\, dx = 0$$

for $0 \leq j < i \leq k$. Thus, for any $1 \leq i \leq k$, the variational principle implies that

$$\lambda_i(\bar\theta r) \int_{B_p(\bar\theta r)} (\phi\, v_i)^2 \leq \int_{B_p(\bar\theta r)} \big|\nabla(\phi\, v_i)\big|^2.$$

Hence,

$$\begin{aligned}
\text{(2-1)}\quad \operatorname{tr}_{I_{\theta r}} I_r(V) &= \sum_{i=1}^{k} \int_{B_p(r)} v_i^2 \leq \sum_{i=1}^{k} \frac{1}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \big|\nabla(\phi v_i)\big|^2 \\
&\leq C \sum_{i=1}^{k} \frac{1}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \big|\nabla^2(\phi v_i)\big|^2 \bar\theta^2 r^2 \\
&\leq C^{m-1} \sum_{i=1}^{k} \frac{\bar\theta^{2m-2} r^{2m-2}}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \big|\nabla^m(\phi v_i)\big|^2 \\
&\leq C^{m-1} 2^m \sum_{i=1}^{k} \sum_{j=0}^{m} \frac{\bar\theta^{2m-2} r^{2m-2}}{\lambda_i(\bar\theta r)} \int_{B_p(\bar\theta r)} \frac{|\nabla^j v_i|^2}{(\theta-1)^{2m-2j} r^{2m-2j}}.
\end{aligned}$$

Let $\eta \in C_0^m\big(B_p(\theta r)\big)$ be a nonnegative function defined on $B_p(\theta r)$ satisfying $\eta = 1$ on $B_p(\bar{\theta} r)$, $0 \le \eta \le 1$, $\eta = 0$ on $\partial B_p(\theta r)$, and

$$|\nabla^j \eta| \le \frac{\bar{C}}{(\theta - 1)^j r^j}$$

for some constant $\bar{C} = \bar{C}(n, m)$, and $1 \le j \le 2m$. Note that

$$\begin{aligned} \text{(2-2)}\quad & \int_{B_p(\theta r)} \big|\nabla^m(\eta v_i)\big|^2 \\ &\le \frac{1}{\delta_0} \int_{B_p(\theta r)} a_{\alpha\beta} D^\alpha(\eta v_i)\, D^\beta(\eta v_i) \\ &= \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\big(\langle D^\alpha(\eta v_i) - \eta D^\alpha v_i,\, D^\beta(\eta v_i)\rangle + \langle \eta D^\alpha v_i,\, D^\beta(\eta v_i)\rangle\big) \\ &= \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\big\langle D^\alpha(\eta v_i) - \eta D^\alpha v_i,\, D^\beta(\eta v_i)\big\rangle \\ &\qquad + \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\big\langle D^\alpha v_i,\, \eta D^\beta(\eta v_i) - D^\beta(\eta^2 v_i)\big\rangle, \end{aligned}$$

where we have used the Gårding inequality from the first to the second line, and inserted the term $\mathfrak{Q}(v_i, \eta^2 v_i) = 0$ into the last equality. For any $\epsilon_1 > 0$, we have

$$\begin{aligned} \text{(2-3)}\quad & \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\big\langle D^\alpha(\eta v_i) - \eta D^\alpha v_i,\, D^\beta(\eta v_i)\big\rangle \\ &\le \frac{C_{12}}{\delta_0}\bigg(\frac{1}{2\epsilon_1} \int_{B_p(\theta r)} \big|\nabla^m(\eta v_i) - \eta \nabla^m v_i\big|^2 + \frac{\epsilon_1}{2} \int_{B_p(\theta r)} \big|\nabla^m(\eta v_i)\big|^2\bigg) \\ &\le C_{13}\bigg(\frac{\epsilon_1}{2} \int_{B_p(\theta r)} \big|\nabla^m(\eta v_i)\big|^2 + \frac{1}{2\epsilon_1} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta-1)^{2m-2j} r^{2m-2j}}\bigg). \end{aligned}$$

Also,

$$\begin{aligned} & \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\big\langle D^\alpha v_i,\, \eta D^\beta(\eta v_i) - D^\beta(\eta^2 v_i)\big\rangle \\ &\le C_{14} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} |\nabla^m v_i|\, |\nabla^{m-j} \eta^2|\, |\nabla^j v_i| \\ &\le C_{15} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^m v_i|\, \eta\, |\nabla^j v_i|}{(\theta-1)^{m-j} r^{m-j}} = C_{15} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\eta \nabla^m v_i|\, |\nabla^j v_i|}{(\theta-1)^{m-j} r^{m-j}} \end{aligned}$$

$$\leq \frac{\epsilon_1}{2}\int_{B_p(\theta r)} |\eta \nabla^m v_i|^2 + \frac{C_{15}^2}{2\epsilon_1}\int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta-1)^{2m-2j} r^{2m-2j}}$$

$$\leq \frac{\epsilon_1}{2}\int_{B_p(\theta r)} \left|\nabla^m (\eta v_i)\right|^2 + \frac{C_{16}}{2\epsilon_1}\int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta-1)^{2m-2j} r^{2m-2j}}.$$

Choosing ϵ_1 to be sufficiently small and substituting this estimate and (2-3) into (2-2), we get

$$\text{(2-4)} \quad \int_{B_p(\bar\theta r)} |\nabla^m v_i|^2 \leq \int_{B_p(\theta r)} \left|\nabla^m (\eta v_i)\right|^2 \leq C_{17}\int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta-1)^{2m-2j} r^{2m-2j}}.$$

In terms of the weighted seminorms

$$\Psi_k = \sup_{1<\bar\theta<\sigma<\theta} (\sigma-1)^{2k} r^{2k} \int_{B_p(\sigma r)} |\nabla^k v_i|^2,$$

Equation (2-4) can be written into

$$\text{(2-5)} \qquad \Psi_m \leq C_{17} \sum_{k=0}^{m-1} \Psi_k.$$

For each $1 \leq k \leq 2m-1$, we have the interpolation inequality

$$\text{(2-6)} \qquad \Psi_k \leq \epsilon \Psi_{2m} + C(k)\epsilon^{k/(k-2m)} \Psi_0$$

for any $\epsilon > 0$. Thus, by substituting (2-6) into (2-5), with a properly chosen ϵ, we get $\Psi_m \leq C_{18}\Psi_0$. In particular, we conclude that

$$\int_{B_p(\bar\theta r)} |\nabla^m v_i|^2 \leq \frac{C_{18}}{(\bar\theta-1)^{2m} r^{2m}} \int_{B_p(\theta r)} v_i^2.$$

Hence,

$$\int_{B_p(\bar\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\bar\theta-1)^{2m-2j} r^{2m-2j}}$$

$$\leq \sum_{j=0}^{m-1} \frac{1}{(\bar\theta-1)^{2m-2j} r^{2m-2j}} \left(\epsilon_j \int_{B_p(\bar\theta r)} |\nabla^m v_i|^2 + C(j)\epsilon_j^{j/(j-m)} \int_{B_p(\bar\theta r)} v_i^2\right)$$

$$\leq m\delta^2 \int_{B_p(\bar\theta r)} |\nabla^m v_i|^2 + \frac{C(m)}{(\bar\theta-1)^{2m} r^{2m}} \int_{B_p(\bar\theta r)} v_i^2$$

$$\leq \frac{C_{19}}{(\bar\theta-1)^{2m} r^{2m}} \int_{B_p(\theta r)} v_i^2,$$

where $\epsilon_j = (\theta-1)^{2m-2j} r^{2m-2j}$. Substituting this inequality into (2-1), we get

$$\begin{aligned}\operatorname{tr}_{I_{\theta r}} I_r(V) &\le C_p^{m-1} 2^m \sum_{i=1}^{k} \sum_{j=0}^{m} \frac{\bar{\theta}^{2m-2} r^{2m-2}}{\lambda_i(\bar{\theta} r)} \int_{B_p(\bar{\theta} r)} \frac{|\nabla^j v_i|^2}{(\theta-1)^{2m-2j} r^{2m-2j}} \\ &\le C_{20} \sum_{i=1}^{k} \frac{\theta^{2m-2}}{\lambda_i(\theta r)(\theta-1)^{2m} r^2} \int_{B_p(\theta r)} v_i^2 \\ &\le C_{20} \sum_{i=1}^{k} \frac{\theta^{2m-2}}{\lambda_i(\theta r)(\theta-1)^{2m} r^2}. \end{aligned}$$

□

Now the next theorem may be proved in a similar fashion to Theorem 1.4 by using Lemmas 1.3 and 2.1.

Theorem 2.2. *Assume that Condition L_d holds. Then the space $\mathcal{H}_d^L(\mathbb{R}^n)$ is finite-dimensional, and its dimension $h_d^L(\mathbb{R}^n)$ satisfies the estimate*

$$h_d^L(\mathbb{R}^n) \le C\, d^{mn}$$

for all $d \ge 1$.

Theorem 2.2 can be generalized to the case of systems of partial differential equations. More specifically, for the system

$$(\mathcal{L}u)_i \equiv \sum_{|\alpha|=m} (-1)^m D^\alpha \left(\sum_{j=1}^{N} \sum_{|\beta|=m} a_{\alpha\beta}^{ij}(x)\, D^\beta u_j(x) \right),$$

where $1 \le i, j \le N$ and $u = (u_1, \dots, u_N) : \mathbb{R}^n \to \mathbb{R}^N$, assume that the coefficients of $\mathcal{L}$ satisfy the following condition.

Condition $\mathcal{L}_d$. *The coefficient matrix $\left(a_{\alpha\beta}^{ij}(x)\right)$ is measurable, bounded, and there exists a constant $\delta_0 > 0$ such that for all $u \in \mathbb{H}_0^m(\mathbb{R}^n, \mathbb{R}^N)$,*

$$\int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} \sum_{i,j=1}^{N} a_{\alpha\beta}^{ij}(x) D^\beta u_i(x)\, D^\alpha u_j(x)\, dx \ge \delta_0 \sum_{j=1}^{N} \|\nabla^m u_j\|_2^2.$$

It is easy to see that if L satisfies the uniform ellipticity condition, that is, there exists a constant $\Lambda > 0$ such that

$$\Lambda |\Xi|^{2mN} |\eta|^2 \ge \sum_{|\alpha|=|\beta|=m} \sum_{i,j=1}^{N} a_{ij}^{\alpha\beta}(x)\, \Xi^\alpha\, \Xi^\beta \eta_i \eta_j \ge \Lambda^{-1} |\Xi|^{2mN} |\eta|^2$$

for all x, $\Xi \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^N$, then Condition $\mathcal{L}_d$ holds.

Theorem 2.3. *Assume that Condition $\mathscr{L}_d$ holds. Then the space $\mathscr{H}_d^{\mathscr{L}}(\mathbb{R}^n)$ is finite dimensional, and its dimension $h_d^{\mathscr{L}}(\mathbb{R}^n)$ satisfies the estimate*

$$h_d^{\mathscr{L}}(\mathbb{R}^n) \leq C\, d^{mnN}$$

for all $d \geq 1$.

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