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# A SPECTRAL DECOMPOSITION FOR SINGULAR-HYPERBOLIC SETS

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**We extend the Spectral Decomposition Theorem for hyperbolic sets to singular-hyperbolic sets on 3-manifolds. We prove that an attracting singular-hyperbolic set with dense periodic orbits and a unique equilibrium of a $C^r$ vector field, where $r \geq 1$, is a finite union of transitive sets; the union is disjoint or the set contains finitely many distinct homoclinic classes. If the vector field is $C^r$-generic, the union is in fact disjoint.**

## 1. Introduction and statement of the results

The *Spectral Decomposition Theorem for hyperbolic systems* plays a central role in dynamics [Smale 1967]. In the case of an attracting hyperbolic set in which the periodic orbits are dense it asserts that the set is a finite disjoint union of homoclinic classes. Here we present a version of this result in the context of *singular-hyperbolic systems* [Morales et al. 2004], proving that an attracting singular-hyperbolic set with dense periodic orbits and a unique equilibrium is a finite union of transitive sets. Moreover, the union is disjoint or the set contains finitely many distinct homoclinic classes. If the flow is $C^r$-generic, the union is in fact disjoint. Let us state our results in a precise way.

Throughout, $M$ denotes a compact 3-manifold and $X$ denotes a $C^r$ vector field in $M$, where $r \geq 1$. The flow of $X$ will be denoted by $X_t$, $t \in \mathbb{R}$. The *omega-limit set* of a point $p \in M$ is the set $\omega_X(p)$ defined by

$$\omega_X(p) = \left\{ x \in M : x = \lim_{n \to \infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty \right\}.$$

A compact invariant set $A$ is *transitive* if $A = \omega_X(p)$ for some $p \in A$. We say that $A$ is *attracting* if there is a compact neighborhood $U$ of $A$ such that

$$A = \bigcap_{t \geq 0} X_t(U).$$

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An *attractor* is a transitive attracting set. (Note that many authors, such as Milnor [1985], define an attractor to be what we call an attracting set.) A *homoclinic class* of $X$ is the closure of the transverse homoclinic points associated to a hyperbolic periodic orbit [Palis and Takens 1993]. It follows from the Birkhoff–Smale Theorem that any homoclinic class is a transitive set with dense periodic orbits.

**Definition 1.** A compact invariant set $\Lambda$ of $X$ is *partially hyperbolic* if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants $K, \lambda$ such that:

1. *$E^s$ is contracting:*

$$\|DX_t/E^s_x\| \le Ke^{-\lambda t} \quad \text{for all } x \in \Lambda \text{ and } t > 0.$$

2. *$E^s$ dominates $E^c$*; that is, $E^s_x \neq 0$, $E^c_x \neq 0$ and

$$\|DX_t/E^s_x\| \cdot \|DX_{-t}/E^c_{X_t(x)}\| \le Ke^{-\lambda t} \quad \text{for all } x \in \Lambda \text{ and } t > 0.$$

We say that the central subbundle $E^c$ of a partially hyperbolic set $\Lambda$ is *volume-expanding* if the constants $K, \lambda$ above satisfy

$$|J(DX_t/E^c_x)| \ge Ke^{\lambda t},$$

for every $x \in \Lambda$ and $t > 0$, where $J(\cdot)$ is the jacobian.

**Definition 2.** Let $\Lambda$ be a compact invariant set of a vector field $X$ on a 3-manifold. We say that $\Lambda$ is *singular-hyperbolic* if all its singularities are hyperbolic, and it is partially hyperbolic with a volume-expanding central subbundle [Bonatti et al. 2005; Morales et al. 2004].

A *singular-hyperbolic attractor* is an attractor that is also a singular-hyperbolic set. The most important examples of singular-hyperbolic attractors are nontrivial hyperbolic attractors and the *geometric Lorenz attractor* [Afraĭmovich et al. 1982; Guckenheimer and Williams 1979]. More examples can be found in [Morales et al. 2000; 2005; Morales and Pujals 1997]. See [Bonatti et al. 2005, Chapter 9] for background concerning singular-hyperbolic sets.

As already mentioned, an attracting hyperbolic set with dense periodic orbits is a finite *disjoint* union of homoclinic classes. A natural candidate for a singular-hyperbolic version of this result can be obtained replacing hyperbolic by singular-hyperbolic in its statement. However, the resulting version is false; we describe a counterexample. Start with the modification of the geometric Lorenz attractor [Guckenheimer and Williams 1979] obtained by adding two singularities to the flow located at $W^u(\sigma)$, as indicated in Figure 1. This modification is done in such a way that the new flow restricted to the cross section $S$ has a $C^\infty$ invariant stable foliation and the quotient map in the leaf space is piecewise expanding with a single discontinuity $c$, as in the Lorenz case [Guckenheimer and Williams 1979, p. 63]. Then the resulting attracting set can be proved to be a homoclinic class, just as

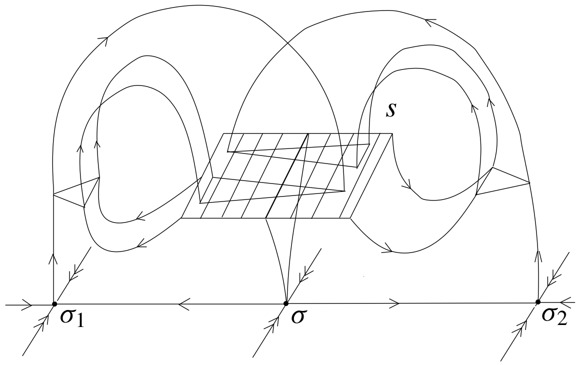


**Figure 1**

in the Lorenz case recently done in [Bautista 2004]. In particular, such a set is transitive with *dense periodic orbits* and is also singular-hyperbolic by construction. Afterward we glue together in a $C^\infty$ fashion two copies of this flow along the unstable manifold of the singularity $\sigma$, thus obtaining the flow depicted in Figure 2. In this way we obtain an attracting singular-hyperbolic set with dense periodic orbits and three equilibria which is not the *disjoint* union of homoclinic classes (although it is the union of two transitive sets). This completes the construction of the counterexample.

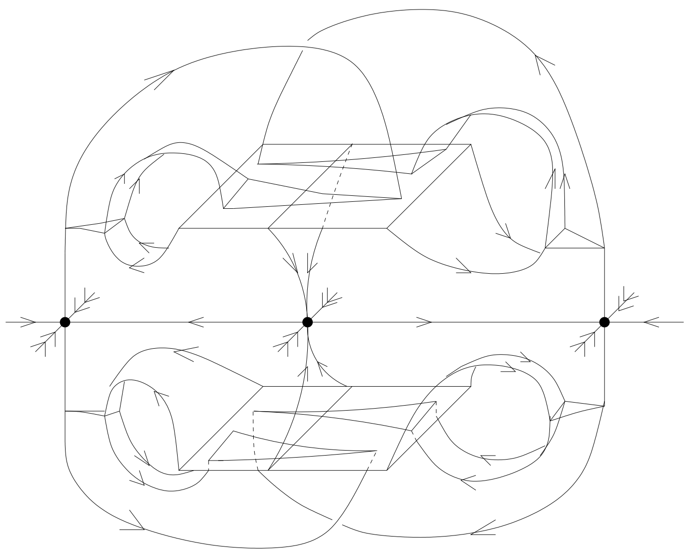

**Figure 2**

Although this counterexample has *three* equilibria, it is possible to construct one with a *unique* equilibrium. The construction in this case is much more elaborate than the one just described above [Bautista et al. 2005].

These counterexamples illustrate the situations that can appear when we consider the spectral decomposition for singular-hyperbolic sets instead of hyperbolic sets. In particular, it is possible to obtain a *finite union of transitive sets* rather than a finite disjoint union of homoclinic classes. It turns out that the former situation always occurs in the presence of a sole equilibrium. More precisely, we shall prove the following result.

**Theorem 3.** *An attracting singular-hyperbolic set with dense periodic orbits and a unique singularity is a finite union of transitive sets.*

The most common example of a transitive set is a homoclinic class. Every known example of a singular-hyperbolic attractor is a homoclinic class [Bautista 2004], and it has been conjectured that this is always the case [Morales 2004; 2005]. If this conjecture were true, we would be able to strengthen Theorem 3, obtaining a finite union of homoclinic classes instead of transitive sets.

It is natural to ask whether the union in Theorem 3 is disjoint. To answer this question we recall that a vector field is *Kupka–Smale* if all its closed orbits are hyperbolic and their associated invariant manifolds are in general position [Smale 1967].

**Theorem 4.** *For a Kupka–Smale vector field, an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity is a finite* ***disjoint*** *union of transitive sets.*

Theorem 4 implies that the union in Theorem 3 is disjoint for most vector fields on closed 3-manifolds. Indeed, denote by $\chi^r(M)$ the set of all $C^r$ vector fields on a compact 3-manifold $M$ endowed with the $C^r$-topology, $r \geq 1$. A subset of $\chi^r(M)$ is *residual* if it is a countable intersection of open, dense subsets of $\chi^r(M)$.

Denote by $\mathcal{R}^r(M)$ the subset of all vector fields $X \in \chi^r(M)$ for which every attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of $X$ is a finite *disjoint* union of transitive sets. Standard $C^1$-generic arguments [Bonatti et al. 2005] imply that $\mathcal{R}^r(M)$ is residual in $\chi^r(M)$ when $r = 1$. The following corollary proves this assertion for all $r \geq 1$. The proof follows combining Theorem 4 with the classical Kupka–Smale Theorem [Palis and de Melo 1982].

**Corollary 5.** $\mathcal{R}^r(M)$ *is residual in* $\chi^r(M)$ *for every* $r \geq 1$.

Next we investigate what can happen outside the residual subset $\mathcal{R}^r(M)$ in Corollary 5. If $\Lambda$ is a compact invariant set of a vector field $X$ we define

$$\mathcal{H}_X(\Lambda) = \{H : H \text{ is a homoclinic class of } X \text{ contained in } \Lambda\}.$$

An interesting question is to give sufficient conditions for

$$\#\mathcal{H}_X(\Lambda) < \infty,$$

where # denotes cardinality. For instance, $\mathcal{H}_X(\Lambda)$ is finite if

- $\Lambda$ is a homoclinic class and $X \in \chi^r(M)$ is $C^1$-generic [Bonatti et al. 2005], or if
- $\Lambda$ is hyperbolic.

Problem 9.32 (p. 283) in [Bonatti et al. 2005] asks whether $\mathcal{H}_X(\Lambda)$ is finite for *every* singular-hyperbolic set $\Lambda$. The next result give a partial positive answer for this question.

**Theorem 6.** *Let $\Lambda$ be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of $X \in \chi^r(M)$. If $\Lambda$ is **not** a disjoint union of transitive sets, then $\#\mathcal{H}_X(\Lambda) < \infty$.*

To conclude this section we point out that Theorem 3 applies to the class of singular-hyperbolic vector fields introduced in [Bautista 2005]. By definition, a vector field $X$ is *singular-hyperbolic* if its nonwandering set $\Omega(X)$ is the closure of its closed orbits and, denoting by $S(X)$ the union of the attracting and repelling closed orbits, there is a *disjoint* union

$$\Omega(X) \setminus S(X) = \Omega_1(X) \cup \Omega_2(X),$$

where $\Omega_1(X)$ is a singular-hyperbolic set for $X$ and $\Omega_2(X)$ is a singular-hyperbolic set for $-X$.

The class of singular-hyperbolic vector fields contains Axiom A vector fields and the geometric Lorenz attractor. If the conjecture in [Morales 2004] mentioned above were true, this class would contain also the singular-Axiom A vector fields defined in [Morales and Pacifico 2003]. In any case there are many singular-hyperbolic vector fields which are also Kupka–Smale. An example of a singular-hyperbolic vector field in $S^3$ which is not Kupka–Smale can be derived from the example described before. An example of a singular-hyperbolic vector field in $S^3$ satisfying the hypotheses of the next corollary can be found in [Morales and Pacifico 2003].

The following is a direct consequence of Theorems 3 and 4.

**Corollary 7.** *Let $X$ be a singular-hyperbolic vector field with a unique singularity on a compact 3-manifold. If $\Omega_1(X)$ is attracting and $\Omega_2(X)$ is repelling, then $\Omega(X)$ is a finite union of transitive sets. If $X$ is Kupka–Smale, then such an union is disjoint. In particular, the union is disjoint for a residual subset of vector fields in $\chi^r(M)$, $r \geq 1$.*

## 2. Proofs

We start with some preliminary results from [Morales and Pacifico 2004] to be used in the proof of the theorems.

Let $X$ be a $C^r$ vector field on a compact boundaryless 3-manifold $r \geq 1$. Denote by $Cl(A)$ the closure of a set $A$. In the statement of the following two theorems we let $\Lambda$ be a compact invariant set of $X$ satisfying the following properties:

(1) $\Lambda$ is connected.

(2) $\Lambda$ is an attracting singular-hyperbolic set.

(3) The periodic orbits of $X$ contained in $\Lambda$ are dense in $\Lambda$.

(4) $\Lambda$ has a unique singularity $\sigma$.

Combining Lemma 2.1 and Theorem 2.8 in [Morales and Pacifico 2004] we obtain the following result.

**Theorem 8.** *$\Lambda$ is the union of transitive sets. More precisely, it is itself transitive or is the union of two homoclinic classes.*

It follows from [Morales et al. 1999] that the singularity $\sigma$ above has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

In particular, $\sigma$ has a two-dimensional *stable manifold* $W^s(\sigma)$ and a one-dimensional *unstable manifold* $W^u(\sigma)$, each one tangent at $\sigma$ to the eigenspaces associated to the eigenvalue sets $\{\lambda_2, \lambda_3\}$ and $\{\lambda_1\}$, respectively. It turns on also that $\sigma$ has a *strong stable manifold* $W^{ss}(\sigma)$ contained in $W^s(\sigma)$ and tangent at $\sigma$ to the set of eigenvalues $\{\lambda_2\}$. In particular, $W^{ss}(\sigma)$ divides $W^s(\sigma)$ in two connected components, denoted by $W^{s,+}$ and $W^{s,-}$.

**Theorem 9** [Morales and Pacifico 2004, Theorem 2.8]. *If $\Lambda$ is not transitive then for all $a \in W^u_X(\sigma) \setminus \{\sigma\}$ there is a periodic orbit $O$ of $X$ with a* ***positive*** *expanding eigenvalue and such that $a \in W^s_X(O)$.*

*Proof of Theorem 3.* Let $\Lambda$ be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity. Split $\Lambda$ into finitely many connected components. Such components are clearly attracting with dense periodic orbits and the nonsingular ones are hyperbolic hence transitive by the Spectral Theorem [Smale 1967]. On the other hand, the singular component satisfies properties (1)–(4) above, so it is the union of transitive sets, by Theorem 8. Then $\Lambda$, which is the union of its components, must be a finite union of transitive sets. □

As already noted, it has been conjectured that every singular-hyperbolic attractor is a homoclinic class. If this conjecture were true, we would be able to strengthen Theorem 3, obtaining a finite union of homoclinic classes instead of transitive sets.

**Remark 10.** The proof above implies that Theorem 3 holds for an arbitrary number of singularities as long as they belong to different connected components of $\Lambda$. It also implies that the union in Theorem 3 can be chosen to be disjoint or formed by homoclinic classes.

*Proof of Theorem 4.* Let $X$ a Kupka–Smale vector field in a compact 3-manifold and let $\Lambda$ be an attracting singular-hyperbolic set of $X$ with dense periodic orbits and a unique singularity $\sigma$. It suffices to prove that the component of $\Lambda$ containing $\sigma$ is transitive. Suppose, for a contradiction, that this is not so. Applying Theorem 9 to the (nontransitive) component containing $\sigma$, we would obtain $a \neq \sigma$ in the unstable manifold $W^u_X(\sigma)$ such that $\omega_X(a)$ is a periodic orbit $O$. But $\sigma$ has a one-dimensional unstable manifold $W^u_X(\sigma)$, so the vector field $X$ would exhibit a nontransversal intersection between $W^u(\sigma)$ and $W^s(O)$, a contradiction since $X$ is Kupka–Smale. □

*Proof of Theorem 6.* Take $X \in \chi^r(M)$ and let $\Lambda$ be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of $X$. If $z$ belongs to a hyperbolic periodic orbit of $X$, denote by $H_X(z)$ the homoclinic class of $X$ associated to $z$. Clearly one has $H_X(z) = H_X(z')$ whenever $z$, $z'$ belong to the same hyperbolic periodic orbit.

Assume that $\Lambda$ is *not* a disjoint union of transitive sets. Split $\Lambda$ into finitely many connected components as before. It suffices to prove that each such component $\Lambda_0$ satisfies

$$\#\mathcal{H}_X(\Lambda_0) < \infty.$$

The nonsingular ones are hyperbolic [Morales et al. 1999], so they satisfy this requirement. In addition, these components are also transitive.

Now consider the singular component $\Lambda_0$. It clearly contains the sole equilibrium $\sigma$ of $\Lambda$. As before, $\sigma$ has a one-dimensional unstable manifold $W^u_X(\sigma)$ [Morales et al. 1999]. Then $W^u_X(\sigma) \setminus \{\sigma\}$ consists of two regular orbits. Fix $a$, $a'$ in each orbit.

Observe that $\Lambda_0$ cannot be transitive, for otherwise $\Lambda$ would be a disjoint union of transitive sets, contrary to the hypothesis. Then Theorem 9 applied to $\Lambda_0$ implies that $\omega_X(a) = O_0$ and $\omega_X(a') = O_0'$, where $O_0$ and $O_0'$ are periodic orbits with positive eigenvalues of $X$.

Now assume for a contradiction that $\#\mathcal{H}_X(\Lambda_0) = \infty$. There is an infinite sequence of periodic orbits $O_n \subset \Lambda_0$ and a infinite sequence $z_n \in O_n$ such that

$$H_X(z_n) \neq H_X(z_m) \quad \text{for } n \neq m. \tag{1}$$

Define $A = Cl\big(\bigcup_n H_X(z_n)\big)$. If $\sigma \notin A$ then $A$ is hyperbolic, yielding $\mathcal{H}_X(A) < \infty$, which contradicts (1). We conclude that $\sigma \in A$, that is,

$$\sigma \in Cl\big(\bigcup_n H_X(z_n)\big).$$

Thus there is a sequence $x_n \in H_X(z_n)$ such that $x_n \to \sigma$. The Birkhoff–Smale Theorem [Palis and Takens 1993] implies that $x_n$ is an accumulation point of periodic orbits homoclinically related to $O_n$. Hence, we can assume that $x_n = z_n$ without loss of generality.

Since $O_n$ is periodic and $z_n$ lies in $O_n$, we have $z_n \notin W^s_X(\sigma)$. So, $O_n$ accumulates on either $a_0$ or $a'_0$. We shall assume the first case since the proof for the second case is analogous.

Because the expanding eigenvalue of $O$ is positive, $O$ divides its own unstable manifold $W^u_X(O)$ into two connected components $W^{u,+}$, $W^{u,-}$ labeled according to the following rule: Let $W^{s,+}$, $W^{s,-}$ be the two connected components of $W^s(\sigma)\setminus W^{ss}(\sigma)$. If $I^{\pm}$ is an interval with boundary point $a$ and pointing to the side of $W^{s,\pm}$, then the positive orbit of $I^{\pm}$ accumulates on $W^{u,\pm}$, by the Inclination Lemma [Palis and de Melo 1982]. For details we refer the reader to [Morales and Pacifico 2004, Definition 2.9, p. 335]. The main property of $W^{u,\pm}$ is that if $z$ lies in a periodic orbit and is sufficiently close to some point in $W^{u,\pm}$ then

$$H_X(z) = Cl(W^{u,\pm}). \tag{2}$$

This property is described in [Morales and Pacifico 2004, Proposition 2.13, p. 336].

Now we obtain the desired contradiction. Since $\omega_X(a_0) = O$ and $O_n$ accumulates at $a_0$ we can find $z'_n \in O_n$ passing close to $O$ as indicated in Figure 3. In particular, by the Inclination Lemma [Palis and de Melo 1982], we can assume that $z'_n$ converges to a point in either $W^{u,+}$ or $W^{u,-}$. Again we assume the first case since the second one is analogous.

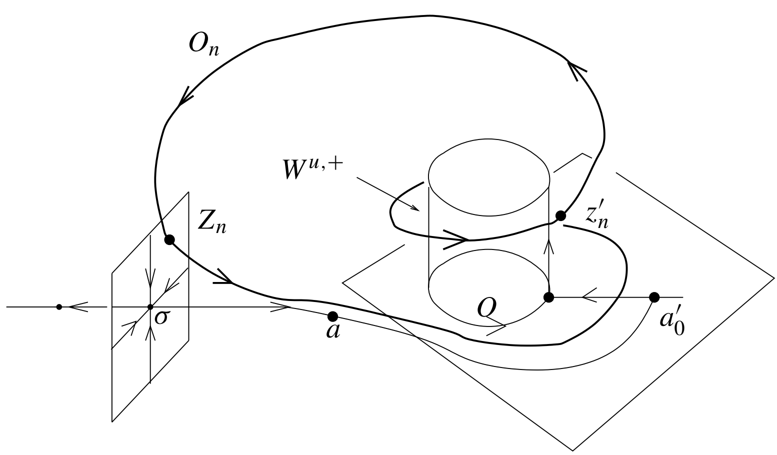


**Figure 3**

Then, (2) implies

$$H_X(z'_n) = Cl(W^{u,+}) \quad \text{for all } n.$$

But $z'_n$ is in the orbit of $z_n$ so

$$H_X(z_n) = H_X(z'_n).$$

Then, $H_X(z_n) = Cl(W^{u,+})$ and so

$$H_X(z_n) = H_X(z_m), \quad \text{for all } n, m.$$

However, this is a contradiction by (1). □

By Remark 10, the set $\Lambda$ in Theorem 6 is also a finite union of homoclinic classes.

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