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We construct a small, hyperbolic 3-manifold M with the property that, for any integer $g \geq 2$, there are infinitely many separating slopes r in ∂M such that the 3-manifold $M(r)$ obtained by attaching a 2-handle to M along r contains an essential separating closed surface of genus g. The resulting manifolds $M(r)$ are still hyperbolic. This contrasts sharply with known finiteness results on Dehn filling and with the known finiteness result on handle addition for the cases $g = 0, 1$. Our 3-manifold M is the complement of a hyperbolic, small knot in a handlebody of genus 3.

1. Introduction

All manifolds in this paper are orientable and all surfaces F in 3-manifolds M are embedded and proper, unless otherwise specified. A surface $F \subset M$ is proper if $F \cap \partial M = \partial F$.

Let M be a compact 3-manifold. An incompressible, ∂-incompressible surface F in M is essential if it is not parallel to ∂M. A 3-manifold M is simple if M is irreducible, ∂-irreducible, anannular and atoroidal. In this paper, a compact 3-manifold M is said to be hyperbolic if M with its toroidal boundary components removed admits a complete hyperbolic structure with totally geodesic boundary. By Thurston's theorem, a Haken 3-manifold is hyperbolic if and only if it is simple. A knot K in M is hyperbolic if M_K, the complement of K in M, is hyperbolic. A 3-manifold M is *small* if M contains no essential closed surface. A knot K in M is *small* if M_K is small.

A *slope* r in ∂M is an isotopy class of unoriented essential simple closed curves in F. We denote by $M(r)$ the manifold obtained by attaching a 2-handle to M along a regular neighborhood of r in ∂M and then capping off the possible spherical component with a 3-ball. If r lies in a toroidal component of ∂M, this operation is known as Dehn filling.

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Essential surfaces are a basic tool in the study of 3-manifolds, and handle addition is a basic method to construct 3-manifolds. A central question connecting those two topics is the following:

Question 1. *Let M be a hyperbolic* 3*-manifold with nonempty boundary, containing no essential closed surface of genus g. How many slopes $r \subset \partial M$ are there such that $M(r)$ contains an essential closed surface of genus g?* (The question is asked only for hyperbolic 3-manifolds to avoid possibly infinitely many slopes produced by Dehn twists along essential discs or annuli. The mapping class group of a hyperbolic 3-manifold is finite.)

The main result of this paper shows that there can be many such slopes:

Theorem 1. *There is a small, hyperbolic knot K in a handlebody H of genus* 3 *such that, for any given integer $g \geq 2$, there are infinitely many separating slopes r in ∂H such that $H_K(r)$ contains an essential separating closed surface of genus g. Moreover the resulting manifolds $H_K(r)$ are still hyperbolic.*

Remarks. Let M be a hyperbolic 3-manifold with nonempty boundary.

(1) Suppose ∂M is a torus. W. Thurston's pioneer result [1982] asserts that there are at most finitely many slopes on ∂M such that $M(r)$ is not hyperbolic; hence the number of slopes in Question 1 is finite when $g = 0$ or 1. Sharp upper bounds for this number were given by Gordon and Luecke for $g = 0$, and by Gordon for $g = 1$; see the survey paper [Gordon 1997]. Hatcher [1982] proved that the number is finite for any g.

(2) Suppose ∂M has genus at least 2. Scharlemann and Wu [1993] have shown that if $g = 0$ or 1, there are only finitely many separating slopes r such that $M(r)$ contains an essential closed surface of genus g. Recently Lackenby [2002] generalized Thurston's finiteness result to handlebody attaching, proving that, for a hyperbolic 3-manifold M, there is a finite set C of exceptional curves on ∂M such that attaching a handlebody to M yields a hyperbolic-like manifold if none of those curves bounds a meridian disc of the handlebody.

(3) In [Qiu and Wang 2005] we proved Theorem 1 for g even.

Theorem 1 and the finiteness results just cited give a global view about the answer of Question 1.

Outline of the proof of Theorem 1 and organization of the paper. In Section 2 we first construct a knot K in the handlebody H of genus 3 for Theorem 1, then we construct infinitely many surfaces $S_{g,l}$ of genus g for each $g \geq 2$ such that (1) all those surfaces are disjoint from the given K, hence contained in H_K; and (2) for fixed g, all the $\partial S_{g,l}$ are connected and provide infinitely many slopes in ∂H as l varies. Those $\partial S_{g,l}$ will serve as the slopes r in Theorem 1. We denote

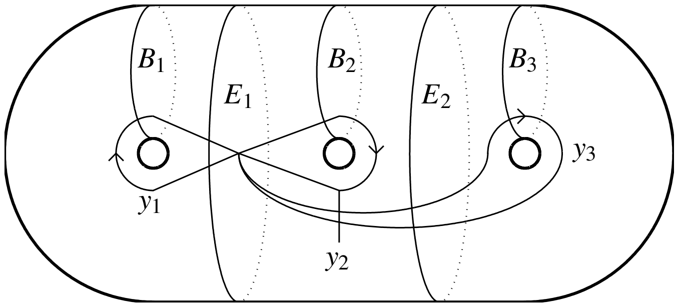

Figure 1

by $\hat{S}_{g,l} \subset H_K(\partial S_{g,l})$ the closed surface of genus g obtained by capping off the boundary of $S_{g,l}$ with a disk. We will prove in Section 3 that $\hat{S}_{g,l}$ is incompressible in $H_K(\partial S_{g,l})$. In Sections 4 and 5 we prove that the knot K is hyperbolic and small.

2. Construction of the knot K and the surfaces $S_{g,l}$ in H

Let H be a handlebody of genus 3. Suppose that B_1, B_2 and B_3 are basis disks of H, and E_1, E_2 are disks in H that separate H into three solid tori J_1, J_2 and J_3. See Figure 1.

Let c be a closed curve in ∂H as in Figure 2. The boundary of $E_1 \cup E_2$ separates c into 10 arcs $c_1, \ldots, c_{10}$, where $c_1, c_3, c_9 \subset J_1$ meet B_1 in two, one, one points respectively; $c_2, c_4, c_6, c_8, c_{10} \subset J_2$ meet B_2 in one, one, two, zero, one points respectively; $c_5, c_7 \subset J_3$ meet B_3 in one, three points respectively.

Let $u_1, \ldots, u_{2g}, v_1, \ldots, v_{2g}$ be $4g$ points located on ∂E_1 in the cyclic order u_1, u_3, $\ldots$, u_{2i-1}, $\ldots$, u_{2g-1}, u_{2g}, u_{2g-2}, $\ldots$, u_4, u_2, v_1, v_3, $\ldots$, v_{2i-1}, $\ldots v_{2g-1}$, v_{2g}, v_{2g-2}, $\ldots$, v_4, v_2 as in Figure 3. In view of the order of these points, C can be

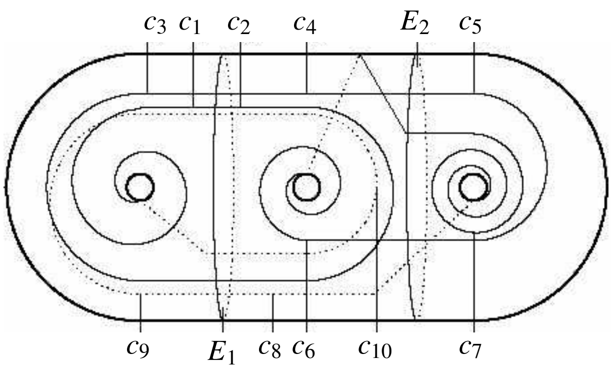

Figure 2

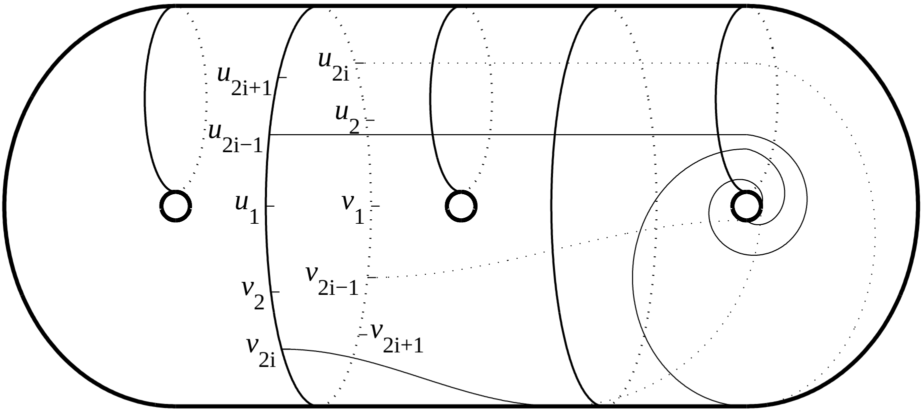

Figure 3

isotoped so that $\partial c_1 = \{u_1, v_1\}$, $\partial c_2 = \{u_1, v_2\}$, $\partial c_{10} = \{v_1, u_2\}$, $\partial c_3 = \{v_2, u_3\}$, $\partial c_9 = \{u_2, v_3\}$. Now suppose $u_{2i+1}v_{2i}$ and $v_{2i+1}u_{2i}$, for $1 \le i \le g-1$, are arcs in $\partial J_1 - \mathring{E}_1$ parallel to c_3 and c_9, and that $u_2v_1 = c_{10}$, $v_2u_1 = c_2$, and $u_{2i}v_{2i-1}$, $v_{2i}u_{2i-1}$, for $2 \le i \le g$, are parallel arcs in $\partial(J_2 \cup J_3) - \mathring{E}_1$, each of which intersects B_2 in one point and B_3 in l points (see Figure 3, where $l = 2$). Finally define $\alpha_1 = u_1v_1$, and let α_k be the union of $v_{k-1}u_k$, α_{k-1} and $u_{k-1}v_k$, for $k = 2, \dots, 2g$. Then $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_{2g}$ is an increasing sequence of arcs.

Let $\alpha \subset \partial H$ be an arc which meets ∂S exactly at its two endpoints for a proper separating surface $S \subset H$. The surface resulting from tubing S along α in H, denoted by $S(\alpha)$, is obtained by first attaching a 2-dimensional 1-handle $N(\alpha) \subset \partial H$ to S, then making the surface $S \cup N(\alpha)$ to be proper, that is, pushing its interior into the interior of H. The image of $N(\alpha)$ after the pushing is still denoted by $N(\alpha)$. In fact, $S \cup N(\alpha)$ is a once punctured torus. Since S is orientable and separating, $S(\alpha)$ is still separating and orientable.

Since α_1 meets E_1 exactly in its two endpoints, we do tubing on E_1 along α_1 to get a proper surface $E_1(\alpha_1)$. Now α_2 meets $E_1(\alpha_1)$ exactly in its two endpoints. We do tubing on $E_1(\alpha_1)$ along α_2 to get $E_1(\alpha_1, \alpha_2) = E_1(\alpha_1)(\alpha_2)$, where the tube $N(\alpha_2)$ is thinner and closer to ∂H so that it goes over the tube $N(\alpha_1)$. Hence $E_1(\alpha_1, \alpha_2)$ is a properly embedded surface (indeed, a one-punctured torus). By the same argument, we do tubing along $\alpha_3, \dots, \alpha_{2g}$ to get a proper embedded surface $E_1(\alpha_1, \dots, \alpha_{2g})$ in H, denoted by $S_{g,l}$. This surface is orientable and separating.

Since $S_{g,l}$ is obtained from the disc E_1 by attaching $2g$ 1-handles to E_1 such that the ends of any two handles are alternating, $S_{g,l}$ is a once punctured orientable surface of genus g. We summarize the facts just discussed:

Lemma 2.1. *$S_{g,l}$ is a once punctured surface of genus g and is separating in H.*

Now let K be a knot in $\mathring{H}$ obtained by first pushing c_6 into $\mathring{H}$ deeply and then pushing $C - c_6$ into $\mathring{H}$ so that it stays between $N(\alpha_3)$ and $N(\alpha_4)$. The following fact is clear:

Lemma 2.2. *K is disjoint from $S_{g,l}$ for all g, l.*

3. Proof of Theorem 1 assuming that K is hyperbolic and small

We denote by $\hat{S}_{g,l} \subset H_K(\partial S_{g,l}) \subset H(\partial S_{g,l})$ the surface obtained by capping off the boundary of $S_{g,l}$ with a disk. Then $\hat{S}_{g,l}$ is a closed surface of genus g.

From the definition of $S_{g,l}$ for a given genus g, the boundary $\partial S_{g,l}$ provides infinitely many boundary slopes as l varies from 1 to infinity. Then Theorem 1 follows from the next two propositions (apart from the last assertion, which follows directly from [Scharlemann and Wu 1993]).

Proposition 3.0. *$K \subset H$ is a hyperbolic, small knot.*

Proposition 3.1. *$\hat{S}_{g,l}$ is incompressible in $H_K(\partial S_{g,l})$.*

We postpone the proof of the first of these results and prove the second here. Recall that a surface F in a 3-manifold is *compressible* if either F is a 2-sphere that bounds a 3-ball, or there is an essential simple closed curve in F that bounds a disk in M; otherwise, F is *incompressible*. Hence Proposition 3.1 is a consequence of the following result:

Proposition 3.2. *$\hat{S}_{g,l}$ is incompressible in $H(\partial S_{g,l})$.*

We choose the center of E_1 as the common base point for the fundamental groups of H and of all surfaces $S_{g,l}$.

Now $\pi_1(S_{g,l})$ is a free group of rank $2n$ generated by $(x_1, \dots, x_{2n})$, where x_i is the generator given by the centerline of the tube $N(\alpha_i)$; and $\pi_1(H)$ is a free group of rank three generated by curves y_1, y_2, y_3 corresponding to B_1, B_2, B_3, as in Figure 1. Let $i : S_{g,l} \to H$ be the inclusion. One can read $i_*(x_i)$ directly as words in y_1, y_2, y_3:

$$\begin{aligned} i_*(x_1) &= y_1^2, \\ i_*(x_2) &= y_2 y_1^2 y_2, \\ i_*(x_3) &= y_1 y_2 y_1^2 y_2 y_1, \\ i_*(x_4) &= y_2 y_3^l y_1 y_2 y_1^2 y_2 y_1 y_2 y_3^l, \end{aligned}$$

and in general, for $2 \le i \le g$,

$$\begin{aligned} i_*(x_{2i-1}) &= y_1 (y_2 y_3^l y_1)^{i-2} y_2 y_1^2 y_2 (y_1 y_2 y_3^l)^{i-2} y_1, \\ i_*(x_{2i}) &= (y_2 y_3^l y_1)^{i-1} y_2 y_1^2 y_2 (y_1 y_2 y_3^l)^{i-1}. \end{aligned}$$

Lemma 3.3. *$S_{g,l}$ is incompressible in H.*

The proof is the same as that in [Qiu 2000].

Now $S_{g,l}$ separates H into two components P_1 and P_2 with $\partial P_1 = T_1 \cup S_{g,l}$ and $\partial P_2 = T_2 \cup S_{g,l}$, where $T_1 \cup T_2 = \partial H$ and $\partial T_1 = \partial T_2 = \partial S_{g,l}$.

Lemma 3.4. *T_1 and T_2 are incompressible in H.*

Proof. We have $H_1(H) = \mathbb{Z} + \mathbb{Z} + \mathbb{Z}$, with the three generators y_1, y_2 and y_3. By the preceding argument, $i_*(H_1(S_{g,l}))$ is a subgroup of $H_1(H)$ generated by $2y_1$, $2y_2$ and $2ly_3$. Thus $H_1(H)/i_*(H_1(S_{g,l})) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2l}$ is a finite group.

Suppose T_1 or T_2 is compressible. Then it bounds a compressing disk D_1 in H. Since $\partial D \cap \partial S_{g,l} = \varnothing$ and $S_{g,l}$ is incompressible in H, by a standard argument in 3-manifold topology, we may assume that $D_1 \cap S_{g,l} = \varnothing$. Since H is a handlebody, we may also assume that D_1 is nonseparating in H. Thus there are two properly embedded disks D_2 and D_3 in H such that $\{D_1, D_2, D_3\}$ is a set of basis disks of H. Let z_1, z_2, z_3 be generators of $\pi_1(H)$ corresponding to D_1, D_2, D_3. Since $S_{g,l}$ misses D_1, we have $i_*(\pi_1(S_{g,l})) \subset G$, where G is a subgroup of $\pi_1(H)$ generated by z_2 and z_3. Then $H_1(H)/i_*(H_1(S_{g,l}))$ is an infinite group, a contradiction. $\square$

Proof of Proposition 3.2. Since H is a handlebody and $S_{g,l}$ is incompressible in H, P_1 and P_2 are handlebodies. By Lemmas 3.3, 3.4 and the Handle Addition Lemma [Jaco 1984], $\hat{S}_{g,l}$ is incompressible in $P_i(\partial S_{g,l})$ for $i = 1, 2$. Since $H(\partial S_{g,l}) = P_1(\partial S_{g,l}) \cup_{\hat{S}_{g,l}} P_2(\partial S_{g,l})$, the surface $\hat{S}_{g,l}$ is incompressible in $H(\partial S_{g,l})$. $\square$

4. H_k is irreducible, ∂-irreducible and anannular

By construction, K is cut by $E_1 \cup E_2$ into ten arcs $a_1, \dots, a_{10}$, where a_i arises from pushing c_i into $\mathring{H}$. Now let $N(K) = K \times D$ be a regular neighborhood of K in H, where the product structure has been adjusted so that $\bigcup_{i=1}^{10} \partial a_i \times D$ is contained in $E_1 \cup E_2$. Let $H_K = H - \mathring{N}(K)$ and $F_i = E_i - \mathring{N}(K)$; also set $M_i = H_K \cap J_i$, for $i = 1, 2, 3$, and $T = \partial(K \times D)$. Then $F_1 \cup F_2$ separates T into ten annuli $A_1, \dots, A_{10}$ such that $A_i = a_i \times \partial D$.

K and C bound a nonembedded annulus A_*, which is cut by $E_1 \cup E_2$ into ten disk $D_{1*}, \dots, D_{10*}$ in H. Note that $D_* = \bigcup_{i \neq 6} D_{i*}$ is still a disk. Let $D_i = D_{i*} \cap H_K$ for $i \neq 6$. Then D_i is a proper disk in some M_l and $\bigcup_{i \neq 6} D_i$ is still a disk; see Lemma 4.1. Now we number the ∂A_i such that $\partial_1 A_i = \partial_2 A_{i-1}$ and $\partial_2 A_i = \partial_1 A_{i+1}$. For $i \neq 6$, let $W_i = \overline{\partial N(D_i \cup A_i) - \partial M_l}$. Then W_i is a proper separating disk in M_l. Each W_i intersects $F_1 \cup F_2$ in two arcs l_i and l_{i+1}. Note that $W = \bigcup_{i \neq 6} W_i$ is a disk. Thus ∂W is a union of two arcs in ∂H and $l_6 \cup l_7$; see Figure 4. Since c_3, c_9 are parallel in $\partial J_1 - \mathring{E}_1$, there are two arcs parallel to c_3 in $\partial J_1 - \mathring{E}_1$, say l', l'', and two arcs in F_1, say l^1, l^2, such that $l' \cup l'' \cup l^1 \cup l^2$ bounds a disk W' that separates M_1 into two handlebodies of genus two H^1, H^2 with $A_1 \subset H^1$ and $A_3, A_9 \subset H^2$. We denote by μ the meridian slope on T and by τ the longitude slope on T.

We list some elementary facts about K and a_i:

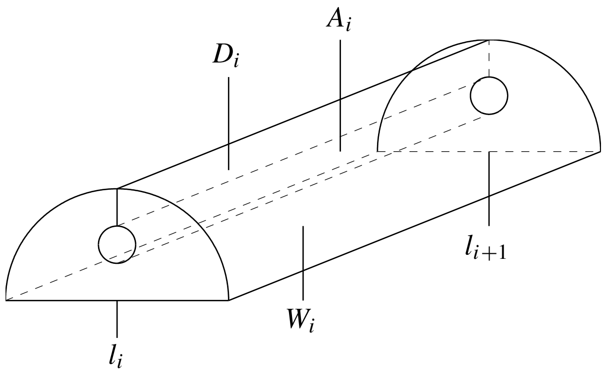

Figure 4

Lemma 4.0. (1) *$K \neq 1$ in $\pi_1(H)$.*

(2) *Suppose $a_i \subset J_m$, where $i \neq 4, 8$. Let $b_i \subset E_1 \cup E_2$ be a given arc with $\partial b_i = \partial a_i$ and let $B \subset J_m$ be a nonseparating proper disk. Then $a_i \cup b_i$ intersects ∂B in at least one point for all i, in at least three points when $i = 7$, and in at least two points when $i = 1, 6$.*

(3) *There is no relative homotopy on $(J_m, E_1 \cup E_2)$ sending a_i to $E_1 \cup E_2$.*

Recall that a 3-manifold M is *irreducible* if it contains no essential 2-spheres. M is *∂-irreducible* if ∂M is incompressible. M is *atoroidal* if it contains no essential tori. M is *anannular* if it contains no essential annuli.

Lemma 4.1. *H_K is irreducible.*

Proof. Suppose that H_K is reducible, so there is an essential 2-sphere S in H_K. Since H is irreducible, S bounds a 3-ball B^3 in H and $K \subset B^3$, which contradicts Lemma 4.0(1). □

Recall that F is *∂-compressible* if there is an essential arc a in F which, together with an arc b in ∂M, bounds a disk D in M such that $D \cap F = a$; otherwise, F is ∂-incompressible.

Lemma 4.2. *$F_1 \cup F_2$ is incompressible and ∂-incompressible in H_K.*

Proof. Suppose first that $F_1 \cup F_2$ is compressible in H_K. Then there is a disk B in M such that $B \cap (F_1 \cup F_2) = \partial B$ and ∂B is an essential circle on $F_1 \cup F_2$. Without loss of generality, we assume that $\partial B \subset F_1$ and $B \subset M_2$. Denote by B' the disk bounded by ∂B in E_1. Then $B \cup B'$ is a 2-sphere $S \subset J_2$, and it follows easily from Lemma 4.1 that S bounds a 3-ball B^3 in J_2. Since ∂B is essential in F_1, B' contains at least one component of ∂a_i. Since S is separating and a_i is connected, we must have $(a_i, \partial a_i) \subset (B^3, B')$, which provides a relative homotopy on (J_2, E_1) sending a_i to E_1. This contradicts Lemma 4.0(2).

Now suppose $F_1 \cup F_2$ is ∂-compressible in H. There is an essential arc a in $F_1 \cup F_2$ which, with an arc b in ∂H_k, bounds a disk B in H_K with $B \cap (F_1 \cup F_2) = a$. Without loss of generality, we assume that $a \subset F_2$ and $B \subset M_2$. There are two cases:

$b \subset T$. Then b is a proper arc in one of A_4, A_6, and A_8, say A_6. If b is not essential in A_6, then a and an arc b' in ∂A_6 form an essential circle in F_2 bounding a disc in M_2. This contradicts the incompressibility of F_2 we just proved. If b is essential in A_6, the disk B provides a relative homotopy on (J_2, E_2) sending a_6 to E_2, which contradicts Lemma 4.0(2).

$b \subset \partial H$. If B is nonseparating in J_2, then b_6 can be chosen so that $a_6 \cup b_6$ intersects ∂B in at most one point, where b_6 is an arc in E_2 connecting the endpoints of a_6; this contradicts Lemma 4.0(2). If B is separating in J_2, then B separates J_2 into a 3-ball B^3 and a solid torus J. We denote by D_1, D_2 the two components of $E_2 - a$. Since a is essential in F_2, each of $\mathring{D}_1$ and $\mathring{D}_2$ contains at least one endpoint of a_4, a_6 and a_8.

Suppose that $D_1 \subset B^3$ and $D_2 \cup E_1 \subset J$. By construction, $\partial_1 a_4, \partial_1 a_8 \subset E_1$, $\partial_2 a_4, \partial_2 a_8 \subset E_2$, and $\partial a_6 \subset E_2$. Since a_4, a_6 and a_8 are disjoint from B, we have $a_4, a_8 \subset J$ and $a_6 \subset B^3$. This contradicts Lemma 4.0(2).

Suppose that $D_1 \subset J$ and $D_2 \cup E_1 \subset B^3$. Then $a_2, a_{10} \subset B^3$. This contradicts Lemma 4.0(2). □

Lemma 4.3. *H_K is ∂-irreducible.*

Proof. Suppose H_K is ∂-reducible. Let B be a compressing disk of ∂H_K. If $\partial B \subset T$, then H_K contains an essential 2-sphere, which contradicts Lemma 4.1. Below we assume that $\partial B \subset \partial H$. Since $F_1 \cup F_2$ is incompressible and ∂-incompressible in H_K (Lemma 4.2), by a standard cut and paste argument, we may assume that $B \cap (F_1 \cup F_2) = \varnothing$. We assume that $B \subset M_2$. (The other cases are similar.) Then B misses b_6. If B is nonseparating in J_2, by Lemma 4.0(2), B intersects a_6, a contradiction. If B is separating, then B separates a 3-ball B^3 from J_2. Since ∂B is essential in ∂H_K, there are two cases: Either B^3 contains only one of E_1 and E_2, say E_1, in which case $a_8 \cap B \neq \varnothing$, a contradiction; or B^3 contains both E_1 and E_2, in which case there is a relative homotopy on (J_2, E_2) sending a_6 to E_2, in contradiction with Lemma 4.0(2). □

Lemma 4.4. *M is anannular.*

Proof. Suppose H_K contains an essential annulus A. We can choose A so that $\left|A \cap (F_1 \cup F_2)\right|$ is minimal among all essential annuli in H_K. This condition, together with Lemma 4.2 and the proof of Lemma 4.3, implies that each component of $A \cap (F_1 \cup F_2)$ is essential in both A and $F_1 \cup F_2$. There are three cases:

Case 1: $\partial A \subset T$. Here A is separating in H_k; otherwise, H contains either a nonseparating 2-sphere or a nonseparating torus. Hence the union of A and an annulus A' on T makes a separating torus T', cutting off a manifold with boundary $T \cup T'$. Since M is irreducible, T' is incompressible, so by Lemma 5.5 T' is parallel to T, which implies that A is inessential. (The arguments in Section 5 are independent of those in Section 4.)

Case 2: $\partial_1 A \subset T$ and $\partial_2 A \subset \partial H$. By Lemma 4.3, both ∂H and T are incompressible in H_K. Clearly H_K is not homeomorphic to $T \times I$. Since Dehn fillings along μ and ∂A_1 both compress ∂H, by an important theorem in Dehn filling, $\Delta(\partial_1 A, \mu) \leq 1$. See [Culler et al. 1987, 2.4.3].

We first suppose that $\partial_1 A$ is the meridian slope μ. Then $\partial_1 A$ is disjoint from $F_1 \cup F_2$. We claim that A is disjoint from $F_1 \cup F_2$.

Suppose, to the contrary, that $A \cap (F_1 \cup F_2) \neq \varnothing$. Since $F_1 \cup F_2$ is incompressible and ∂-incompressible in H_K (Lemma 4.2), by a standard cut and paste argument, we may assume that $\partial_2 A \cap (F_1 \cup F_2) = \varnothing$. Now each component of $A \cap (F_1 \cup F_2)$ is an essential simple closed curve in A. Let a be an outermost circle in $A \cap (F_1 \cup F_2)$. Then a and $\partial_1 A$ bound an annulus A^* in A such that $\mathring{A}^*$ is disjoint from $F_1 \cup F_2$. We may assume that $a \subset F_1$ and $\partial_1 A \subset A_i$ for some i. Let B^* be the disk bounded by a on E_1 and let D be the meridian disk of $N(K)$ bounded by $\partial_1 A$. Since a is essential on F_1, B^* contains at least one component of ∂F_1. In H, $B^* \cup A^* \cup D$ is a separating 2-sphere S^2 that bounds a 3-ball B^3. For $j \neq i$, if $\partial_1 a_j \subset B^*$, then $\partial_2 a_j \subset B^*$ and $a_j \subset B^3$. This possibility is ruled out by Lemma 4.0(2). Note also that $\partial_1 a_i \subset B^*$ and that $\partial_2 a_i$ is not contained in B^*. Now let A' be the annulus bounded by a and $\partial_1 a_i \times \partial D = \partial_1 A_i$ in F_1. Then $A^* \cup A'$ is isotopic to an annulus disjoint from $F_1 \cup F_2$. By the preceding argument, $A^* \cup A'$ is inessential. Thus we can properly isotope A by pushing the annulus A^* to the other side of F_1 to reduce $|A \cap (F_1 \cup F_2)|$, contradicting our choice of A at the beginning of the proof.

We may assume that A is contained in M_2. Let D be the meridian disk of $N(K)$ bounded by $\partial_1 A$ and set $B = A \cup_{\partial_1 A} D$. Then B is a proper disk in J_2, meeting K in exactly one point; hence B is a meridian disk of J_2. Let b_6 be an arc on E_2 connecting the two endpoints of c_6. Then $c_6 \cup b_6$ would be a closed curve of winding number 2 in the solid torus J_2 intersecting B at most once, which is absurd.

Next we suppose that $\Delta(\partial_1 A, \mu) = 1$. Then A is cut by $(F_1 \cup F_2)$ into ten squares S_i, $i = 1, \ldots, 10$, each of which has two opposite sides in $F_1 \cup F_2$, the other two sides being the longitude arc a_i in A_i and $a_i^* \subset \partial H$. Let b_2^* be the arc connecting the two endpoints of a_2^* in E_1 and let b_6^* be the arc connecting the two endpoints of a_6^* in E_2. The two simple closed curves $b_2^* \cup a_2^*$ and $b_6^* \cup a_6^*$ on ∂J_2 are disjoint. But in $\pi_1(J_2)$, we have $b_2^* \cup a_2^* = y_2$ and $b_6^* \cup a_6^* = y_2^2$, a contradiction.

Case 3: $\partial A \subset \partial H$. Suppose first that $A \cap (F_1 \cup F_2) = \varnothing$. Then A is contained in one of M_1, M_2 and M_3. Since A is essential and H_K is ∂-irreducible, A is disjoint from

D_i for $i \neq 6$. Since each component of $\partial H \cap J_1 - c_1 \cup c_3$ and $\partial H \cap J_3 - c_5 \cup c_7$ is a disc, $A \subset M_2$. Since A is disjoint from c_2, c_4, c_8, c_{10}, each component of ∂A intersects B_2 in only one point in J_2 (see Figure 2). Thus A is isotopic to each component of $\partial J_2 - \partial A$ in J_2. This means that A is not essential in M_2, a contradiction.

Now suppose that $A \cap (F_1 \cup F_2) \neq \varnothing$. There are two subcases:

Case 3a: Each component of $A \cap (F_1 \cup F_2)$ is an essential circle. Let a be an outermost component of $A \cap (F_1 \cup F_2)$. That means that $\partial_1 A$, together with a, bounds an annulus A^* in A such that $A^* \cap (F_1 \cup F_2) = a$. Then $A^* \subset M_i$. We denote by B^* the disk bounded by a in $E_1 \cup E_2$. Let $D^* = A^* \cup B^*$. Then D^* is a disk. Let D be the disk obtained from D^* by pushing B^* slightly into J_l. Then D is a properly embedding disk in J_l such that D intersects each a_i in at most two points. Furthermore, if D intersects a_i in two points for some i, the two endpoints of a_i lie in B^*. Thus, in this case, the algebraic intersection number of a_i and D is 0. By Lemma 4.0, A^* is separating in J_l.

Suppose that A^* is contained in one of J_1 and J_3, say J_1. Then $\partial_1 A$ is parallel to ∂E_1. We denote by A' the annulus bounded by $\partial_1 A$ and a in ∂J_1. Since a is essential in F_1, B_* contains at least one endpoint of a_1, a_3, a_9. Furthermore, $\partial_1 a_i \subset B^*$ if and only if $\partial_2 a_i \subset B^*$. Now if $\partial_1 a_j \subset A'$ for some j, then $\partial_2 a_j \subset A'$. This means that a_j is disjoint from B_1 as in Figure 1, a contradiction. Thus for each i, j, we have $\partial_j a_i \subset B^*$, which means that a is parallel to ∂E_1 in F_1. Now ∂D_i, for $i = 1, 3, 9$, intersects each component of ∂A^* in two points, which means that D_i intersects A^* in two arcs each of which has its two endpoints in distinct components of ∂A^*. (Otherwise, since $\partial_1 A$ is isotopic to ∂E_1, we would have $a_i \cup b_i = 1$ in $\pi_1(J_1)$, where b_i is an arc in ∂E_1 connecting the two endpoints of a_i, a contradiction.) Thus we can push $\partial_1 A$ into M_2 to reduce $|A \cap (F_1 \cup F_2)|$, contradicting our assumption on A.

Suppose instead that $A^* \subset M_2$. Without loss of generality, we assume that $a \subset F_1$. We denote by A' the annulus bounded by ∂E_1 and a in E_1. Then A' and B^* lie on distinct sides of $J_2 - A^*$. If $\partial_1 A$ is isotopic to ∂E_2, then $a_6 \cup b_6 = 1$ in $\pi_1(J_2)$ where b_6 is an arc in E_2 connecting the two endpoints of a_6, a contradiction. If $\partial_1 A$ bounds a disk D in ∂J_2 such that $E_1, E_2 \subset D$, then $a_4 \cup a_8 \cup b^1 \cup b^2 = 1$ in $\pi_1(J_2)$, where b^i is an arc in E_i connecting the endpoints of a_{4i} and a_8, a contradiction. Now $\partial_1 A$ is isotopic to ∂E_1. Then D_4 intersects A^* in an arc. By the preceding argument, we can push $\partial_1 A$ into M_1 to reduce $|A \cap (F_1 \cup F_2)|$.

Case 3b: Each component of $A \cap (F_1 \cup F_2)$ is an essential arc. Then $F_1 \cup F_2$ cuts A into proper squares S_i in $M_l \subset J_l$, each S_i having two opposite sides in $F_1 \cup F_2$ and the remaining two sides in ∂H. If $S_i \subset J_l$ for $l = 2$ or 3, then S_i is a separating disc in J_l. Otherwise, say S_i is a nonseparating disc in J_2. By the same reason as that at the end of the proof of Lemma 4.3, the fact that $S \cap (F_1 \cup F_2)$ consists of

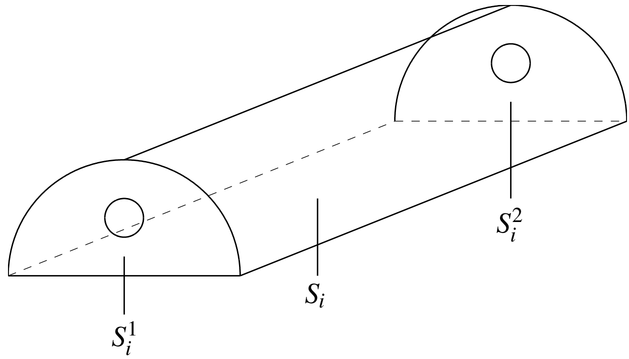

Figure 5

two proper arcs in $E_1 \cup E_2$ implies that b_6 can be chosen so as to intersect ∂S_i in at most two points; furthermore, if b_6 intersects ∂S_i in two points then $S_i \cap F_1 = \varnothing$ and $S_i \cap b_2 = \varnothing$, where b_i is an arc in $E_1 \cup E_2$ connecting the two endpoints of a_i. This means that S_i meets a_2 or a_6 by Lemma 4.0(1), a contradiction. Now each S_i cuts off a 3-ball B_i^3 from J_l for $l = 2$ or 3 as in Figure 5. Let S_i^1 and S_i^2 be the two disks of $B_i^3 \cap (E_1 \cup E_2)$ and $S_i \subset J_l$ where $l = 2$ or 3. By Lemma 4.0(2), we have:

(i) $\partial_1 a_j \subset S_i^1$ if and only if $\partial_2 a_j \subset S_i^2$.

(ii) If a_j is contained in B_i^3, then a_l is not contained in B_i^3.

This means that for each i, there is only one boundary component of $F_1 \cup F_2$ lying in each of S_i^1 and S_i^2. Thus if S_i lies in M_1 for some i, then S_i is also separating in J_1. Otherwise, say S_i is nonseparating in J_1. By (i) and (ii), the three circles $a_1 \cup b_1$, $a_3 \cup b_3$, $a_9 \cup b_9$ intersect S_i in two points, a contradiction. It follows that S_i is also as in Figure 5 and A cuts off a solid torus P from H. Thus D_{i*} can be chosen to be disjoint from A even if $i = 6$. This means that K and a component of ∂A bound an annulus, which has been ruled out in Case 2. □

5. H_K contains no closed essential surface

Suppose H_K contains essential closed surfaces. Let W, W' and W_i be the disks defined in Section 4. Denote by $X(F)$ the union of the components of $F \cap M_1$ isotopic to $\partial H \cap M_1$. We define the complexity on the essential closed surfaces F in H_K by the quadruple

$$C(F) = \big(|F \cap W|,\ |F \cap F_2|,\ |(F \cap M_1 - X(F)) \cap W'|,\ |F \cap F_1|\big).$$

We rank complexities in lexicographic order. Suppose F minimizes $C(F)$. By a standard argument in 3-manifold topology, we derive the following facts:

Lemma 5.0. (1) *Each component of $F \cap (F_1 \cup F_2)$ is an essential circle in both F and $F_1 \cup F_2$.*

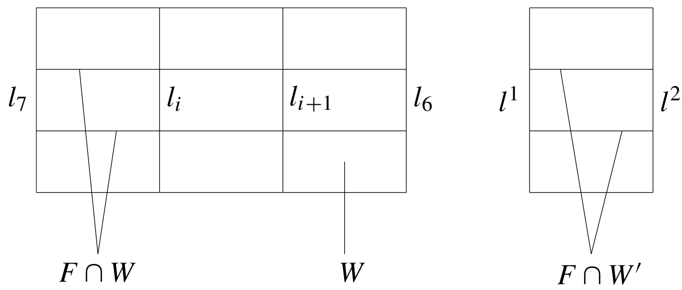

Figure 6

(2) *Each component of* $F \cap W$ *is an arc in* W *one of whose endpoints lies in* l_6 *and the other in* l_7. *Similarly each component of* $F \cap W'$ *is an arc in* W' *one of whose endpoints lies in* l^1 *and the other in* l^2. *Hence* $|F \cap l_i| = |F \cap l_j|$ *for all* i, j *and* $|F \cap l^1| = |F \cap l^2|$ *as in Figure 6.*

(3) *Each component of* $F \cap (F_1 \cup F_2)$ *isotopic to* ∂A_i *is disjoint from* $W \cup W'$.

For two surfaces P_1 and P_2 in a 3-manifold, a pattern of $P_1 \cap P_2$ is a set of disjoint arcs and circles representing isotopy classes of $P_1 \cap P_2$. For each isotopy class s, we denote by $\nu(s)$ the number of components of $P_1 \cap P_2$ in the isotopy class s.

The proof of the next lemma is similar to that of [Qiu and Wang 2004, Lemma 4.3].

Lemma 5.1. *Each component of* $F \cap M_3$ *is isotopic to one of* $\partial H \cap M_3$, A_5 *and* A_7.

Proof. The four arcs l_5, l_6, l_7, l_8 separate F_2 into four annuli A^5, A^6, A^7, A^8 and a disk D. By the minimality of $|F \cap W|$, the pattern of $F \cap A^j$ is as in Figure 7, left, and the pattern of $F \cap D$ is as in Figure 7, right. Since $|F \cap l_i|$ is a constant, $\nu(d_5) = 0$. If $\nu(d_i) \neq 0$ for $1 \leq i \leq 4$, then $F \cap F_2$ contains $\min(\nu(d_1), \ldots, \nu(d_4))$

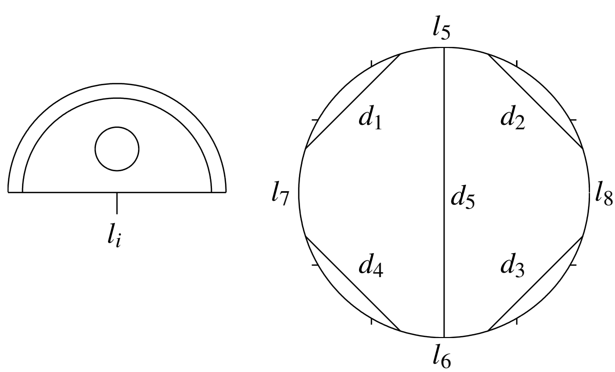

Figure 7

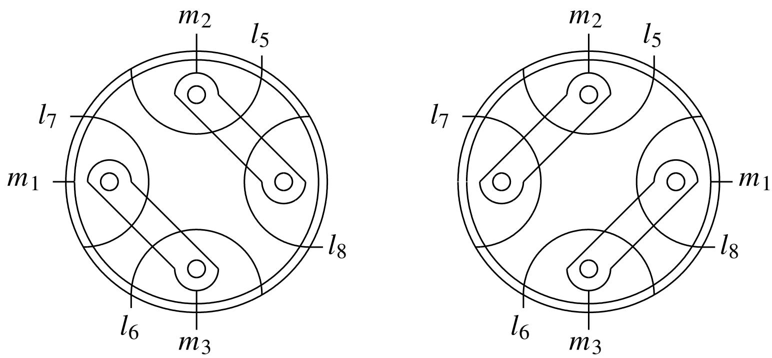

Figure 8

components parallel to a disk on ∂E_2. Now if $\nu(d_1)=0$, then $\nu(d_3)=0$. Similarly, if $\nu(d_2)=0$, then $\nu(d_4)=0$. Thus according to the order of l_5, l_6, l_7, l_8 in F_2, the pattern of $F \cap F_2$ is as in one of the diagrams in Figure 8, with $\nu(m_2)=\nu(m_3)$. Note that W_5 and W_7 separate M_3 into three solid tori J^1, J^2, J^3. Without loss of generality, we assume that $A_5 \subset J^1$, $A_7 \subset J^2$. Let $S = F \cap M_3$ and S' be a component of S.

Now we claim that if one of component of $\partial S'$ is isotopic to ∂E_2, then S' is isotopic to $\partial H \cap M_3$.

Let $\partial_1 S$ be the outermost component of ∂S isotopic to ∂E_2. Now $\partial_1 S$ intersects l_i as in Figure 8. Without loss of generality, we assume that $\partial_1 S \subset \partial S'$. We denote by e_i the arc $\partial_1 S \cap A^i$. Now let $S_l = S' \cap J^l$, then S_l is an incompressible surface in J^l. Note that $\partial S_1 = e_5 \cup e_6 \cup (S \cap W_5)$ bounds a disk in J^1 parallel to a disk on ∂M_3. Similarly S_2 is a disk in J^2 parallel to a disk on ∂M_3 bounded by $e_7 \cup e_8 \cup (S \cap W_7)$. ∂S_3 also has one component which is trivial in ∂M_3, as in Figure 9, left. Hence one component of S_3 is a disk in J^3 parallel to ∂J^3, Thus $S' = S_1 \cup_{S\cap W_5} S_3 \cup_{S \cap W_7} S_2$ is isotopic to $M_3 \cap \partial H$.

Now we claim that $\nu(m_2)=\nu(m_3)=0$ in both parts of Figure 8.

Let $S_0 = S - X'$, where X' is a subset of S each of whose components is isotopic to $\partial H \cap M_3$. Then no component of ∂S_0 is isotopic to ∂E_2. Let $P_3 = S_0 \cap J^3$. If

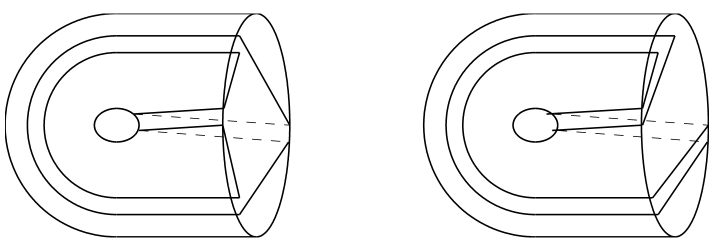

Figure 9

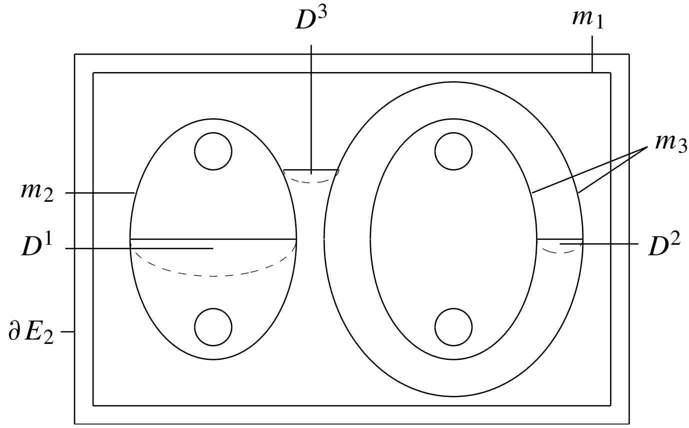

Figure 10

$\nu(m_2) \neq 0$, then P_3 is incompressible in J^3 and ∂P_3 contains $2\nu(m_2) = 2\nu(m_3)$ components c, as in Figure 9, right. Since a_7 intersects a basis disk B_3 of J_3 in three points and a_5 intersects B_3 in one point, c does not bound a disk in J^3. Since J^3 is a solid torus, each component of P_3 is a ∂-compressible annulus. Let D^* be a ∂-compressing disk of an outermost component of P_3. This disk can be isotoped so that $D^* \cap \partial J^3 \subset E_2 \cap J^3$. Then, back in J_3, D^* is isotopic to one of D^1, D^2, D^3 as in Figure 10. In the case of D^1 or D^2, one can push F along the disc to reduce $|F \cap W|$; in the case of D^3, one can push F along the disc to reduce $|F \cap F_2|$, without increasing $|F \cap W|$. Either way, the minimality of $C(F)$ is contradicted.

Now let P be a component of $S = F \cap M_3$. If one component of ∂P is isotopic to ∂E_2, then P is isotopic to $M_3 \cap \partial H$. If not, each component of ∂P is isotopic to one component of $\partial A_5 \cup \partial A_7$. By the minimality of $C(F)$, P is contained in J^1 or J^2. It is easy to see that P is isotopic to one of A_5 and A_7. □

Now we consider $S = F \cap M_1$. Note that W_1 and W' separate M_1 into two solid tori J^1, J^2 and a handlebody of genus two H' such that $A_1 \subset J^1$ and $A_3, A_9 \subset H'$; moreover l_1, l_2, l^1, l^2 separate F_1 into two annuli and two planar surfaces with three boundary components and a disk D such that $\partial J^2 \cap F_1 = D$. See Figure 11. Let k_1 be a component of $F \cap W_1$, k_2 a component of $F \cap W'$, and k'_i, for $i = 1, 2$, an arc in D connecting the two endpoints of k_i. Let $\alpha = k_1 \cup k'_1$ and $\beta = k_2 \cup k'_2$. Note that k'_1 and k'_2 can be chosen so that β intersects α in one point. Furthermore, by construction, α intersects a basis disk of J^2 in two points and β intersects a basis disk of J^2 in one point. Now we fix the orientations of α and β so that $\alpha = y^2$ and $\beta = y$, where y is a generator of $\pi_1(J^2)$. Then $\alpha\beta^{-2}$ is an essential circle in ∂J^2 and null homotopic in J^2.

The next lemma follows immediately from the proof of Lemma 5.1.

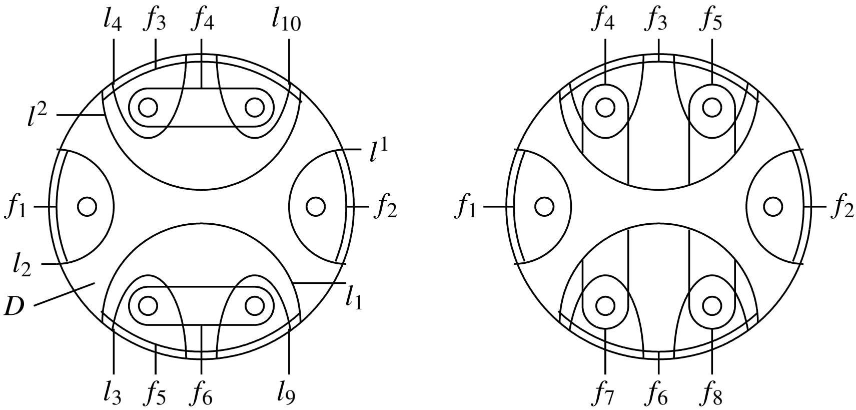

Figure 11

Lemma 5.2. *Let P be a component of $S = F \cap M_1$. If one component of ∂P is isotopic to ∂E_1. Then P is isotopic to $M_1 \cap \partial H$.*

By the construction and Lemma 5.0, the pattern of $\partial S \cap (F_1 \cap (J^1 \cup H'))$ is as in one of the diagrams in Figure 11, and moreover

(1) in Figure 11, left, we have $\nu(f_1) = \nu(f_2)$, $\nu(f_3) = \nu(f_5)$, $\nu(f_4) = \nu(f_6)$ and $\nu(f_3) + \nu(f_4) = \nu(f_1)$;

(2) in Figure 11, right, we have $\nu(f_1) = \nu(f_2) = \nu(f_3) + \nu(f_4)$, $\nu(f_3) = \nu(f_6)$ and $\nu(f_4) = \nu(f_5) = \nu(f_7) = \nu(f_8) \neq 0$.

Lemma 5.3. *If the pattern of $S \cap (F_1 \cap (J^1 \cup H'))$ is as in Figure 11, left, the pattern of $S \cap F_1$ is as in Figure 12 with $\nu(n_2) = \nu(n_3) = \nu(n_4)$.*

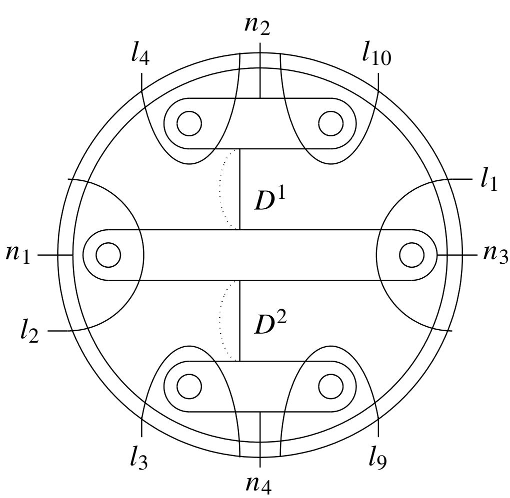

Figure 12

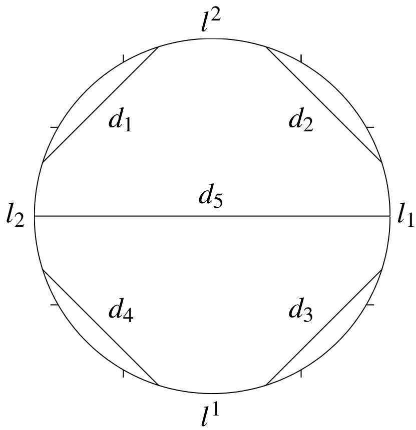

Figure 13

Proof. If $\nu(f_3) = 0$, the pattern of $S \cap F_1$ is as in Figure 12 with $\nu(n_2) = \nu(n_3) = \nu(n_4)$ and $\nu(n_1) = 0$.

Suppose instead that $\nu(f_3) \neq 0$. Since $\nu(f_3) = \nu(f_5) \leq \nu(f_1) = \nu(f_2)$, the pattern of $S \cap D$ is as in Figure 13, where $\nu(d_1) = \nu(d_3)$ and $\nu(d_2) = \nu(d_4)$. If $\nu(d_1)$, $\nu(d_2) \neq 0$, then $S \cap F_1$ contains $\min(\nu(d_1), \nu(d_2))$ components isotopic to ∂E_1. Thus if $\nu(d_1) = \nu(d_2)$, then $S \cap F_1$ is as in Figure 12 with $\nu(n_2) = \nu(n_3) = \nu(n_4)$. Now without loss of generality, we assume that $\nu(d_1) < \nu(d_2)$. Let $k = \nu(d_2) - \nu(d_1)$. By Lemmas 5.0(2) and 5.2, $\partial(S \cap J^2)$ contains $n = \gcd(k, k + \nu(d_5))$ components c isotopic to $\alpha^p \beta^q$, where $|p| = (k + \nu(d_5))/n$ and $|q| = k/n$. Since $y + \nu(d_5) \geq y$, c is not null homotopic in J^2. Moreover, c intersects both d_2 and d_4; if $\nu(d_5) \neq 0$, then c also intersects d_5. Thus these curves separate ∂J^2 into m annuli $A^1, \ldots, A^m$ such that, for each j, there is an arc in $D \cap A^j$ connecting the two boundary components of A^j. Since J^2 is a solid torus, each component of $(S - X(F)) \cap J^2$ is an annulus. Let D^* be a ∂-compressing disk of $(S - X(F)) \cap J^2$. Then D^* can be moved so that $D^* \cap \partial J^2 = D^* \cap D = a$. Thus there are three possibilities:

1. The two endpoints of a lie in one of d_2, d_4, d_5. Then D^* is one of D^1, D^3 as in Figure 14, left. In each case, one can push F along the disc to reduce $|F \cap W|$, a contradiction.

2. One endpoint of a lies in $d_2 \cup d_4$ and the other lies in d_5. Then D^* is D^2 as in Figure 14, left. This case is similar to the previous case.

3. One endpoint of a lies in d_2 and the other lies in d_4. In this case, $\nu(d_5) = 0$. By Lemma 5.0(2), we have $\nu(f_4) = \nu(f_6) = 0$ in Figure 11, left. Now the pattern of $S \cap F_1$ is as in Figure 14, right, and D^* is also as in the same figure. By doing a surgery on F along D^*, we obtain a surface F' isotopic to F such that $|F' \cap W| = |F \cap W|$, $|F' \cap F_2| = |F \cap F_2|$ and $\left|(F' \cap M_1 - X(F')) \cap W'\right| < \left|(F \cap M_1 - X(F)) \cap W'\right|$ (by Lemma 5.2), contradicting minimality. $\square$

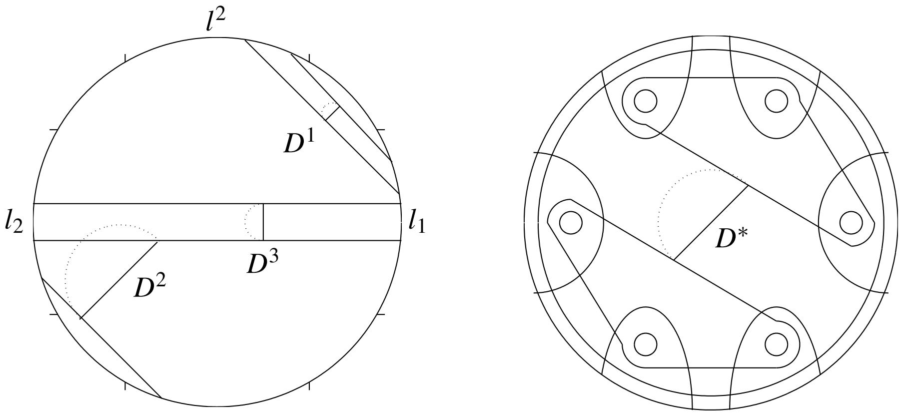

Figure 14

Lemma 5.4. *If the pattern of $S \cap (F_1 \cap (J_1 \cup H'))$ is as in Figure 11, right, then the pattern of $S \cap F_1$ is as in Figure 15.*

Proof. We have $\nu(f_1) = \nu(f_2) = \nu(f_3) + \nu(f_4) = \nu(f_6) + \nu(f_7)$. Thus the pattern of $S \cap D$ is as in Figure 16, where $\nu(d_1) = \nu(d_3)$, $\nu(d_2) = \nu(d_4)$, and $\nu(d_5) = 2\nu(f_5)$. Therefore $\nu(d_5) \neq 0$. Referring to Figure 11, right, we distinguish two cases: $\nu(f_3) = \nu(f_6) = 0$ and $\nu(f_3) = \nu(f_6) = 0$.

If $\nu(f_3) = \nu(f_6) = 0$, we have $\nu(d_5) = \nu(d_1) + \nu(d_2)$. There are three subcases:

Suppose first that $\nu(d_1) = \nu(d_2)$. Since $\nu(d_5) \neq 0$, $\partial(S \cap J^2)$ contains $\nu(d_1)$ trivial components in ∂J^2 bounding some disks in S as in Figure 9, left, and $\nu(d_5)$ components isotopic to β. Since β intersects a basis disk of J^2 in one point, each nontrivial component of $S \cap J^2$, say A^*, is an annulus parallel to each component

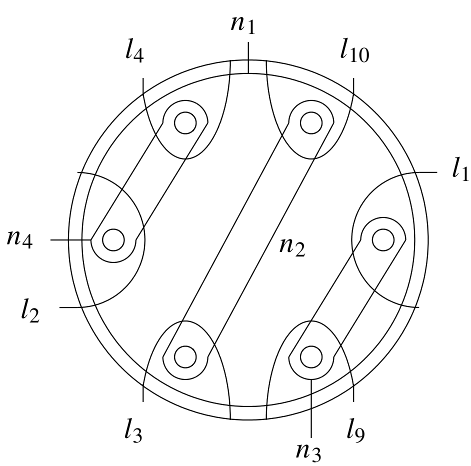

Figure 15

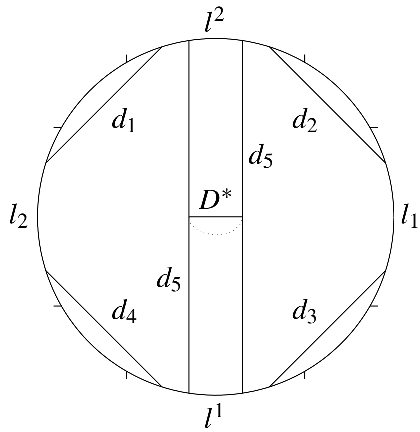

Figure 16

of $\partial J^2 - \partial A^*$. Thus there is a ∂-compressing disk D^* of $S \cap J^2$ as in Figure 16. By doing a surgery on F along D^*, we can obtain a surface F' isotopic to F such that $|F' \cap W| = |F \cap W|$, $|F' \cap F_2| = |F \cap F_2|$, and $\left|(F' \cap M_1 - X(F')) \cap W'\right| < \left|(F \cap M_1 - X(F)) \cap W'\right|$, a contradiction.

Suppose instead that $\nu(d_1) < \nu(d_2)$. Set $k = |\nu(d_2) - \nu(d_1|$ and $n = \gcd(k, k + \nu(d_5)$. Then $\partial(S \cap J^2)$ contains $\nu(d_1)$ trivial components and n components c isotopic to $\alpha^p\beta^q$, where $|q| = (k + \nu(d_5))/n$ and $|p| = k/n$. By construction, $p > 0$ if and only if $q > 0$. (See Figure 2.) That means that c is not null homotopic in J^2. By the proof of Lemma 5.3, we can obtain a surface F' isotopic to F such that $C(F') < C(F)$, a contradiction.

Finally, suppose that $\nu(d_1) > \nu(d_2)$, and define k as in the previous case. By the preceding argument, $\partial(S \cap J^2)$ contains $\nu(d_2)$ trivial components and n components c isotopic to $\alpha^p\beta^q$, where $|q| = (k + \nu(d_5))/n$ and $|p| = k/n$. If c is not null-homotopic in J_2, then by the preceding argument, we can obtain a surface F' isotopic to F so that $C(F') < C(F)$, a contradiction. Assume that $q = -2p$. Then $\nu(d_5) = \nu(d_1) - \nu(d_2)$. Since $\nu(d_5) = \nu(d_1) + \nu(d_2)$, $\nu(d_2) = 0$ and $\nu(d_5) = \nu(d_1)$. Thus $F_1 \cap F$ is as in Figure 15 with $\nu(n_2) = \nu(n_3) = \nu(n_4)$ and $\nu(n_1) = 0$. This completes the analysis when $\nu(f_3) = \nu(f_6) = 0$.

If $\nu(f_3) = \nu(f_6) \neq 0$ in Figure 11, right, there are two subcases:

Suppose first that $\nu(d_1) \le \nu(d_2)$. Then $S \cap F_1$ contains $\min(\nu(d_1), \nu(f_3))$ components isotopic to ∂E_1. If $\nu(d_1) \ge \nu(f_3)$, we can obtain, by the same argument as in the preceding case, a surface F' isotopic to F such that $C(F') < C(F)$, a contradiction. Assume that $\nu(d_1) < \nu(f_3)$, then $S \cap F_1$ contains $\nu(d_1)$ components isotopic to ∂E_1. Now $2\nu(f_1) = \nu(d_1) + \nu(d_2)$. By assumption, $\nu(f_1) = \nu(f_3) + \nu(f_4)$. Thus $\nu(d_1) < \nu(d_2)$. Then, by the proof of Lemma 5.3, $\partial(S \cap J^2)$ contains $\gcd(k, k + \nu(d_5))$ components each of which is isotopic to $\alpha^p\beta^q$, where

$|q| = (k + \nu(d_5))/n$ and $|p| = k/n$ (here again we have set $k = |\nu(d_2) - \nu(d_1|$ and $n = \gcd(k, k + \nu(d_5))$. If $q \neq -2p$, then by the proof of Lemma 5.3, there is in H_K an essential closed surface F' isotopic to F such that $C(F') < C(F)$, a contradiction. Since $y = \nu(d_2) - \nu(d_1) = 2(\nu(f_1) - \nu(d_1)) > 2(\nu(f_1) - \nu(f_3)) = 2\nu(f_5)$, we conclude that $\nu(d_5) = 2\nu(f_5)$. Thus $q \neq -2p$.

If, on the other hand, $\nu(d_1) > \nu(d_2)$, then $S \cap F_1$ contains $\min(\nu(d_2), \nu(f_3))$ components isotopic to ∂E_1. If $\nu(d_2) \geq \nu(f_3)$, then by the same argument as before the pattern of $F \cap F_1$ is as in Figure 15, with $\nu(n_1) = \nu(f_3)$ and $\nu(n_2) = \nu(n_3) = \nu(n_4)$. But then we see that it is impossible to have $\nu(d_1) < \nu(f_3)$. □

Lemma 5.5. *H_K contains no closed essential surface.*

Proof. Suppose, to the contrary, that H_K contains an essential closed surface F such that the complexity $C(F)$ is minimal among all surfaces isotopic to F. By Lemma 5.1, the pattern of $F \cap F_2$ is as in one of the diagrams of Figure 8. Furthermore, $\nu(m_2) = \nu(m_3) = 0$ for any case. By Lemmas 5.3 and 5.4, the pattern of $F \cap F_1$ is as in one of Figures 12 and 15. Furthermore, $\nu(n_2) = \nu(n_3) = \nu(n_4)$ for any case. By Lemma 5.0, $\nu(n_1) + \nu(n_2) = \nu(m_1)$.

In M_2, the pattern of $F \cap F_1$ can be labeled as in one of the diagrams on the top row of Figure 17, and the pattern of $F \cap F_2$ can be labeled as in Figure 17, bottom.

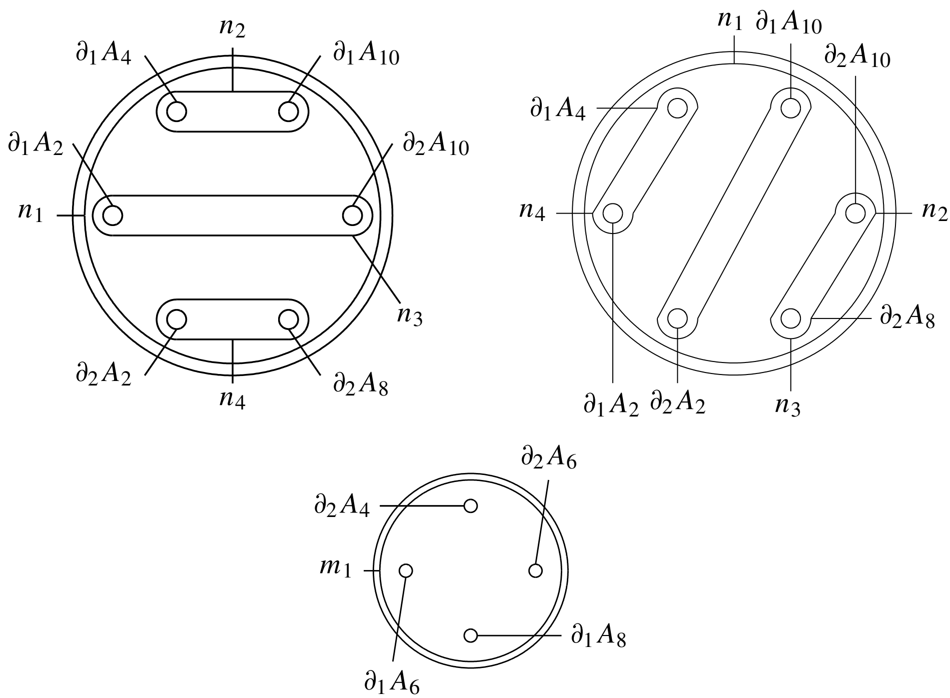

Figure 17

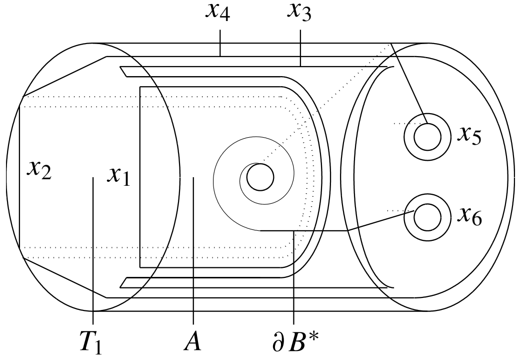

Figure 18

Note that W_2, W_4, W_8, W_{10} separate M_2 into four solid tori J^1, J^2, J^4, J^5 and a handlebody of genus two H' such that $A_{2i} \subset J^i$ for $i = 1, 2, 4, 5$ and $A_6 \subset H'$. Let $S = F \cap H'$.

Now we claim that $\nu(n_2) = \nu(n_3) = \nu(n_4) = 0$. There are two cases:

Case 1. The pattern of $F \cap F_1$ is as in Figure 17, top left. Now each component of ∂S is contained in one of the eight families $x_1, \dots, x_8$ as in Figures 18 and 19, where the boundary components of ∂S contained in $\bigcup_{i=1}^4 x_i$ are produced by cutting along the arcs in $F \cap (W_2 \cup W_4 \cup W_8 \cup W_{10})$ whose endpoints lie in $m_1 \cup n_1$ and the components of ∂S contained in $x_7 \cup x_8$ are produced by cutting along the arcs whose endpoints lie in $n_2 \cup n_3 \cup n_4 \cup m_1$, and each component in $x_5 \cup x_6$ is isotopic to one component of ∂A_6. Each component lying in $x_3 \cup x_4$ is trivial in $\partial H'$. By observation, there are two disks D^1 and D^2 in $\partial H'$ such that $\partial D^i = b_i \cup b'_i$, where $b_i \subset F_1$ and $b'_i \subset S$ as in Figure 19. Back to M_2, D^1 and D^2 are as in Figure 12. Thus by doing surgeries on F along D^1 and D^2, we can obtain a surface F' isotopic to F such that $|F' \cap W| = |F \cap W|$, $|F' \cap F_2| = |F \cap F_2|$ and $\left|(F' \cap M_1 - X(F')) \cap W'\right| < \left|(F \cap M_1 - X(F)) \cap W'\right|$, contradicting minimality.

Case 2. The pattern of $F \cap F_1$ is as in Figure 17, top right. This is similar to Case 1.

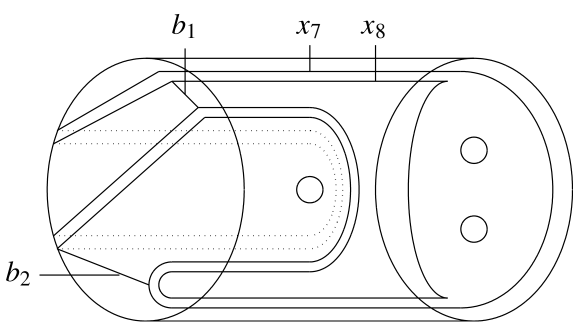

Figure 19

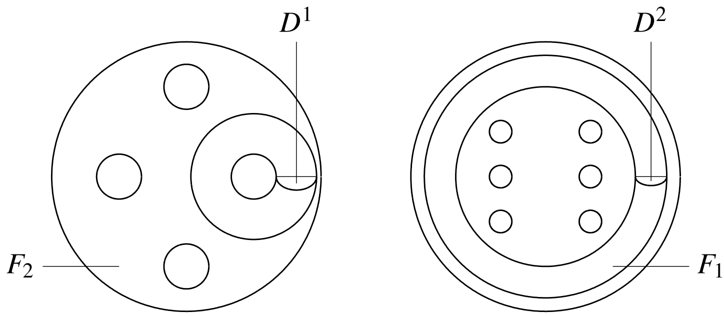

Figure 20

Now $\nu(n_2) = \nu(n_3) = \nu(n_4) = 0$ and ∂S is as in Figure 18. By construction, there is a disk $B^* = H' \cap D_{6*}$ in H' such that ∂B^* intersects each component in $x_1 \cup x_2 \cup x_5 \cup x_6$ in only one point as in Figure 18. Thus $S \cap B^*$ offers a ∂-compressing disk D^* of S such that D^* is disjoint from $\mathring{A}_6$. We denote by A the annulus bounded by an outermost component of x_1, say e_1, and an outermost component of x_2, say e_2, in $\partial H'$, and T_1 the punctured torus bounded by an outermost component of x_1 and an outermost component of x_2 in $\partial H'$as in Figure 18. Now if $\partial D^* \cap \partial H' = a \subset A$, then $e_1 \cup e_2$ bounds an annulus in S parallel to A. This means that one component of $F \cap M_2$ is parallel to $\partial H \cap M_2$.

Let $X_0(F)$ be a union of components in $F \cap M_2$ parallel to $\partial H \cap M_2$ or A_6, and set $S = (F \cap M_2 - X_0(F)) \cap H'$. Then $(F \cap M_2 - X_0(F)) \cap H' \cap B^*$ offers a ∂-compressing disk, also denoted by D^*, of S such that $\partial D^* \cap \partial H' = a$.

We claim each component of S is isotopic to one component of ∂A_6. There are five possibilities:

(1) *The two endpoints of a lies in $x_5(x_6)$.* Then D^* can be moved to be D^1 as in Figure 20(a), Thus by doing a surgery on F along D^1, we can obtain a surface F' isotopic to F such that $|F' \cap W| = |F \cap W|$, $|F' \cap F_2| < |F \cap F_2|$, a contradiction.

(2) *The two endpoints of a lies in $x_1(x_2)$.* Then D^* can be moved to be D^2 as in Figure 20(b), contradicting the minimality of $|F \cap W|$.

(3) *One endpoint of a lies in x_5 and the other lies in x_6.* Since ∂B^* intersects $\bigcup_{i=1}^{6} x_i$ in the order x_6, x_3, x_1, x_2, x_4, x_5, there is by the argument in (1) an outermost component of $S \cap B^*$ in B^*, say b, which, together with an arc b^* in $\partial H'$, bounds an outermost disk D such that $\partial_1 b$ is contained in x_5, $\partial_2 b$ is contained in x_6 and b^* intersects A_6 in an arc. Since S is incompressible, by the standard argument, the component of S containing b is parallel to A_6, a contradiction.

(4) *One endpoint of a lies in x_1 and the other lies in x_2.* Then $\partial_1 a \subset c_1$ and $\partial_2 a \subset c_2$, where c_1 is a component of x_1 and c_2 is a component of x_2. We denote again by A the annulus bounded by c_1, c_2 in $\partial H'$ and by T_1 the punctured torus bounded by

c_1, c_2 in $\partial H'$. Note that a is disjoint from $\mathring{A}_6$ and $A_6 \subset T_1$. Hence $a \subset A$. By the preceding argument, the component of $F \cap M_2$ consisting of c_1 and c_2 is parallel to $\partial H \cap M_2$. By the definition of S, this is impossible.

(5) *One endpoint of a lies in $x_1 \cup x_2$ and the other lies in $x_5 \cup x_6$.* Since S is incompressible, each component c of $x_3 \cup x_4$ bounds a disk D_c in S parallel to a disk D_c^* on $\partial H'$; see Figure 18. Let $S^* = S - \cup_{c \in x_3 \cup x_4} D_c$. Note that ∂B^* intersects $\bigcup_{i=1}^6 x_i$ in the order x_6, x_3, x_1, x_2, x_4, x_5. Hence each component of $S \cap B^*$ is an arc b such that $\partial_1 b \subset x_1 \cup x_2$ and $\partial_2 b \subset x_5 \cup x_6$. Otherwise there would be an outermost component b^* of $S^* \cap B^*$ in B^* such that ∂b^* is as in one of the above four cases, a contradiction.

Each component of $S \cap B^*$ is an arc b such that $\partial_1 b \subset x_1 \cup x_2$ and $\partial_2 \subset x_5 \cup x_6$. Set $H^* = H' - B^* \times (0, 1)$ and $S^{**} = S^* - B^* \times (0, 1)$, where $B^* \times I$ is a regular neighborhood of B^* in H'. Then H^* is a solid torus. Since each component of $x_1 \cup x_2 \cup x_5 \cup x_6$ intersects ∂B^* in one point, each component h of ∂S^{**} is obtained by doing a band sum of one component h_1 of $x_5 \cup x_6$ and one component h_2 of $x_1 \cup x_2$ along a component of $S^* \cap B^*$. Since $h_1 = 1 \in \pi_1(H)$, we have $h_2 \neq 1 \in \pi_1(H)$, so $h \neq 1 \in \pi_1(H^*)$. Recall the disk B_2 in H defined in Section 2. The intersection $B_2 \cap H'$ is a planar surface P such that one component of ∂P, say $\partial_1 P$, is disjoint from A_6, and the other components of ∂P lie in $\mathring{A}_6$. Furthermore, $\partial_1 P$ intersects each component in $x_1 \cup x_2$ in one point. Hence $P - B^* \times (0, 1)$ is a properly embedded disk in H^* intersecting each component of ∂S^{**} in one point. This means that each component of S^{**} is an annulus A parallel to each component of $\partial H^* - \partial A$.

Suppose that D is a ∂-compressing disk of A in H^* such that the arc $\alpha = D \cap \partial H^*$ lies on the annulus A^* on ∂H^* which contains the disk $A_6 - B^* \times (0, 1)$. Then D is disjoint from $x_3 \cup x_4$. Since the disk $D^* = B^* \times \{0, 1\} \cup (A_6 - B^* \times (0, 1))$ intersects ∂A^* in two arcs, D can be moved to have the arc α lying on $A^* - D^*$. Furthermore, since each component h of ∂S^{**} is obtained by doing a band sum of one component h_1 of $x_5 \cup x_6$ and one component h_2 of $x_1 \cup x_2$, we may assume that $\partial \alpha \subset x_1 \cup x_2$. Hence D is also a ∂-compressing disk of S^* in H'. By the preceding argument, this is impossible.

Also by the preceding argument, if one component of $F \cap (F_1 \cup F_2)$ is parallel to ∂E_1 or ∂E_2 then it is parallel to ∂H. Suppose that each component of $F \cap (F_1 \cup F_2)$ is isotopic to one component of ∂A_i. By the minimality of $C(F)$, F is disjoint from W_i for $i \neq 6$ and F is also disjoint from $\overline{\partial N(B^* \cup A_6) - \partial H'}$ in H'. Thus each component of $F \cap M_j$ is an annulus parallel to A_i for some i. That means that F is isotopic to T, a contradiction. □

Proof of Proposition 3.0. The proposition follows immediately from Lemmas 4.1, 4.3, 4.4 and 5.5 and [Scharlemann and Wu 1993, Theorem 1]. □

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