# Pacific Journal of Mathematics


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Volume 229 No. 2 February 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 229, No. 2, 2007

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**We present a new and very efficient approach to study topology of algebraic curves in $\mathbb{C}^2$. It relies on using the Poincaré–Hopf formula, applied to a suitable Hamiltonian vector field, to estimate the number of double points and the Milnor numbers of singular points of the curve and on considering finite-dimensional spaces of curves with given asymptotics at infinity. We apply this method to classification of parametric lines with one self-intersection.**

## 1. Introduction

Everyone is familiar with plane algebraic curves; mathematicians encounter them in almost every day. Unfortunately there do not exist simple rules to determine the topology of a curve, whether given in implicit or parametric form.

Among papers devoted to the study of algebraic curves with fixed topology we highlight the work of S. Abhyankar and T. Moh [1973; 1975], who introduced the notion of approximate roots and proved the rectification of a smooth affine line. This approach was continued in [Sathaye and Stenerson 1994; A'Campo and Oka 1996; Nakazawa and Oka 1997].

Modern algebraic geometry tools, including the resolution of singularities, logarithmic Kodaira dimension, the Bogomolov–Miyaoka–Yau inequality, logarithmic deformations and zero-dimensional schemes, were used in [Miyanishi 2001; Wakabayashi 1978; Gurjar and Miyanishi 1996; Matsuoka and Sakai 1989; Zaidenberg and Orevkov 1996; Flenner and Zaidenberg 1996; Orevkov 2002; Yoshihara 1983; 1987; Greuel et al. 2000; Kleiman and Piene 1999].

Low-dimensional topologists have used some link theory invariants to study the topology of algebraic curves near singularities and near infinity; see [Eisenbud and Neumann 1985; Neumann 1989; Neumann and Norbury 2002; Rudolph 1982; 1983].

*MSC2000:* primary 14H50; secondary 32S05, 14H15, 14H20.
*Keywords:* affine algebraic curve, index of a vector field, Puiseux expansion.
Supported by Polish KBN Grants 2 P03A 010 22 and 1 P03A 015 29.

In this paper we propose a new approach to the problem. We consider parametric polynomial curves $\xi = (\varphi, \psi) : \mathbb{C} \to \mathbb{C}^2$ such that $\varphi(t) = t^p +$lower order terms and $\psi(t) = t^q +$l.o.t., as $t \to \infty$. For fixed $p, q$ such curves form a finite-dimensional space Curv.

Typical elements $\xi \in$ Curv represent curves $C = \xi(\mathbb{C})$ with transversal self-intersections. The number of these double points is calculated using the Poincaré–Hopf formula applied to the Hamiltonian vector field $X_F$, where $F(x, y)$ is the polynomial defining $C$; this number is quadratic in $p$ and $q$. Elements $\xi$ undergo degenerations along certain bifurcation surfaces in Curv; they acquire additional singularities or become degenerate at infinity. We prove the estimate $\mu \le n\nu$ for the Milnor number $\mu$ of a singularity in a curve given by $x = \tau^n$, $y = e_1\tau + e_2\tau^2 + \cdots$ in terms of the codimension $\nu$ of the singularity (Proposition 2.9). There is a similar estimate for the number of double points escaping to infinity and for the number of double points vanishing at the self-intersection point.

We propose in Conjecture 3.7 a probable estimate for the sum of external codimensions of singularities (e.g., $n + \nu - 2$ for a cusp) and of the degeneration at infinity, in terms of the dimension of the quotient space of Curv by the group of its natural automorphisms, induced by changes of variables $t$ and $(x, y)$.

Assuming this conjecture we classify rational curves with one place at infinity and one self-intersection point. This classification contains sixteen series and five special cases, and is given in Theorem 4.1. We believe that our list of curves is complete.

Analogous arguments appear in [Orevkov 2002]. The author introduces what he calls rough M-numbers, $\overline{M}_i$, corresponding to our external codimensions. He considers the $(3d-9)$-dimensional space of rational curves of degree $d$ (modulo Aut $\mathbb{CP}^2$) and proves the estimate $\sum \overline{M}_i \le 3d - 4$, which is not optimal. The same paper gives some estimates for the Milnor number, slightly weaker than $\mu \le n\nu$.

Unfortunately, it seems that our method cannot be applied to the classification of planar *projective* rational curves. Their description contains $\sim 3d$ parameters; with $\nu + n \le {\sim}3d$ one gets the estimate $\mu \le {\sim}\frac{9}{4}d^2$; since the number of double points is $\delta \sim \frac{1}{2}d^2$ and since $\mu \sim 2d$, one cannot deduce a bound for $d$. In the affine case we have $\nu + n \le {\sim}2d$ and therefore $\mu \le {\sim}d^2$.

Our estimate for the sum of external codimensions, Conjecture 3.7, holds true in all known examples. In particular, for any curve from Theorem 4.1 the sum of external codimensions of its singularities is as expected. We are working on this conjecture and we plan to devote a separate paper to it. In [Borodzik and Żołądek 2005] we studied also immersions of $\mathbb{C}^*$ into $\mathbb{C}^2$, among them algebraic annuli, confirming a conjecture analogous to Conjecture 3.7 in all found cases of embeddings of $\mathbb{C}^*$. (The problem of algebraic annuli was partly considered in [Neumann 1989; Kaliman 1996].)

In the next section we calculate the indices of the Hamiltonian vector field. In Section 3 we introduce the space Curv and describe its bifurcation set. In Section 4 we prove the main theorem about the classification of curves with given topological type.

## 2. The Poincaré–Hopf theorem

If $A \subset (\mathbb{C}^2, 0)$ is a germ of holomorphic curve defined by $G(x, y) = 0$, then the (complex) Hamiltonian vector field

$$X_G = G_y \partial_x - G_x \partial_y$$

is tangent to $A$. Below we shall regard $X_G$ as a real vector field in $\mathbb{R}^4$ (i.e. with real time). One can check that the real field $X_G$ is also Hamiltonian with $\operatorname{Re} G$ as the Hamilton function, but with respect to the symplectic structure given by $d \operatorname{Re} x \wedge d \operatorname{Re} y - d \operatorname{Im} x \wedge d \operatorname{Im} y$.

We set $Y := X_G|_A$. If 0 is an isolated singular point of $A$, then we consider the normalization $N : \tilde{A} \to A$; thus each topological component $\tilde{A}_j$, $j = 1, \dots k$ of $\tilde{A}$ (preimage of an analytic component $A_j$ of $A$) is a disc. The pull-back $\widetilde{Y} := N^* Y = (N_*)^{-1} Y \circ N$ of the vector field $Y$ is a vector field on the smooth manifold $\tilde{A}$ with isolated equilibrium points $p_j \in N^{-1}(0)$, $j = 1, \dots, k$. Thus one can define the indices $i_{p_j} \widetilde{Y}$.

**Lemma 2.1** [Milnor 1968; Lins Neto 1988]. *If $A$ is irreducible*, *then*

$$i_{p_1} \widetilde{Y} = \mu_0(G),$$

*where $\mu_0(G)$ is the Milnor number of the function $G$ at the point* 0.

*Proof.* The Milnor number $\mu_0(G)$ is the first Betti number of the manifold with boundary $A_z := B_\rho \cap \{G = z\}$, where $B_\rho$ is a ball of small radius $\rho$ around 0 and $z$ is a small noncritical value of $G$ (Milnor's Theorem). The vector field $X_G|_{A_z}$ does not vanish and its index at the boundary $\partial A_z$ equals $i_{p_1} \widetilde{Y}$. Consider the manifold $M = A_z / \partial A_z$ and a vector field $Z = f \cdot X_G$ on $M$, such that $f > 0$ on $A_z \setminus \partial A_z$ and $f = 0$ on $\partial A_z$. We have $i_{[\partial A_z]} Z = 2 - i_{p_1} \widetilde{Y}$; (if $dz/dt = z^\alpha$ on $S^1$, then $dz^{-1}/dt = z^{-2+\alpha}$). The Poincaré–Hopf theorem says that the Euler–Poincaré characteristic $\chi(M) = 2 - \mu_0(G)$ equals $\sum i_q Z$. □

**Definition 2.2.** We call the quantity

$$\delta_0 := \frac{1}{2} \sum_j i_{p_j} \widetilde{Y}$$

the *number of double points of $A$ hidden at* 0.

The next lemma justifies this definition. In [Serre 1959] an algebraic definition of the number of double points is given. In fact, the both definitions agree. In the literature $\delta_0$ is sometimes called the $\delta$-*invariant* of the singularity or the *virtual number of double points*.

**Lemma 2.3** [Milnor 1968]. *Let $A = A_1 \cup \dots \cup A_k = A_1 + \dots + A_k$, where the $A_j$ are the irreducible components of $A$, and let $\tilde{A}_j$ be the corresponding components of the normalization $\tilde{A}$. Then*

$$2\delta_0 = \sum_j \mu_0(A_j) + 2\sum_{i<j}(A_i \cdot A_j)_0,$$

*where $(A_i \cdot A_j)_0$ is the intersection number at $0$ of the components $A_i$ and $A_j$. In particular, if $0$ is a simple double point of $A$, then*

$$i_{p_1}\widetilde{Y} + i_{p_2}\widetilde{Y} = 2.$$

*Moreover, the Milnor number of the whole set $A$ equals*

$$\mu_0(A) = 2\delta_0 - k + 1. \tag{2-1}$$

*Proof.* Let $N_j : (\mathbb{C}, 0) \to (A_j, 0)$, $z \to (x(z), y(z))$ be a local parametrization (normalization) of $A_j$. Assume the coordinates $x$, $y$ are such that $A_j$ does not lie in the line $x = 0$. Then we get

$$\dot{z} = \left(\frac{dx}{dz}\right)^{-1} \frac{\partial G}{\partial y}\bigg|_{A_j} \quad \text{and} \quad i_{p_j}\widetilde{Y} = \operatorname{ord}_{z=0}\left(\frac{dx}{dz}\right)^{-1} \frac{\partial G}{\partial y}\bigg|_{A_j}.$$

If $G = G_1 \dots G_k$, where $G_j$ defines $A_j$, then

$$\begin{aligned}\operatorname{ord}_{z=0}&\left(\frac{dx}{dz}\right)^{-1} \frac{\partial G}{\partial y}\bigg|_{A_j} \\ &= \operatorname{ord}_{z=0}\left(\frac{\partial G_j}{\partial y}\Big/\frac{dz}{dx}\right)\bigg|_{A_j} + \sum_{i\neq j}\operatorname{ord}_{z=0} G_i|_{A_j} = \mu_0(A_j) + \sum_{i\neq j}(A_i \cdot A_j)_0.\end{aligned}$$

As to (2-1), the manifold $A_z = B_\rho \cap \{G = z\}$ is a surface of genus $g$ with $k$ holes and $\mu = 2g + k - 1$. After contracting the boundary circles to points and introducing the vector field $Z = f X_G$, we find that $\chi(M) = 2 - 2g = \sum_j (2 - i_{p_j}\widetilde{Y}) = 2k - 2\delta_0$. $\square$

**Remark 2.4.** Our proofs of Lemmas 2.1 and 2.3 are different (and hopefully simpler) than the ones given in [Lins Neto 1988; Milnor 1968].

Also our definition of the number $\delta_0$ is essentially topological. This number can be defined as the sum of indices of the Hamiltonian vector field $X_G$ along the loops $K_i = A_i \cap \partial B_\rho$ in the components $A_i$. Milnor [1968, Remark 10.9] asked whether

$\delta_0$ equals the minimal number of double points of a union $S = \bigcup D_i$ of immersed smooth (but maybe not holomorphic) discs $D_i$ in $B_\rho$ with $K_i$ as their boundaries, i.e., the gordian number of the link $K = \bigcup K_i$.

Milnor's conjecture was eventually proved by P. Kronheimer and T. Mrowka [1993]. The proof relies upon another theorem of theirs: *Let* $\Sigma$ *be a smooth* (*real*) *surface of genus* $g(\Sigma)$ *in a* $K3$ *complex surface* $X$ (*meaning that* $H^1(X) = 0$ $K_X$ *is trivial*), *and suppose* $\Sigma$ *represents the same* 2*-dimensional homology class as a smooth complex curve* $C \subset X$. *Then* $2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma = 2g(C) - 2$.

If one could find a vector field $X$ on $S$ such that $X|_K = X_G|_K$ with indices at the double points equal to the corresponding intersection indices, one would have another proof of the Milnor's conjecture. We looked for such a vector field, but without success.

Now we move to the application of the vector field $X_F$ to the curve $C = \{F = 0\}$, which is an image of $\mathbb{C}$ under the polynomial mapping $\xi = (\varphi, \psi)$, where

$$\varphi(t) = a_0 t^p + a_1 t^{p-1} + \cdots + a_p, \quad \psi(t) = b_0 t^q + b_1 t^{q-1} + \cdots b_q. \tag{2-2}$$

Using invertible changes of $x$, $y$ (tame transformations), we can assume that either

$$\varphi(t) = t, \quad \psi(t) \equiv 0,$$

or

$$1 < p < q, \quad q/p \notin \mathbb{Z}, \quad a_0 = b_0 = 1 \tag{2-3}$$

in (2-2). Moreover $\xi$ is a local embedding near typical points of $\mathbb{C}$.

The extension $\tilde{\xi} : \mathbb{CP}^1 \to \bar{C}$ is the normalization of the closure $\bar{C} = C \cup p_\infty \subset \mathbb{CP}^2$. As in [Żołądek 2003] we define a (real) vector field $Y$ on $\bar{C}$, or $\widetilde{Y}$ on $\mathbb{CP}^1$, by the formula

$$\widetilde{Y}(t) = f(t) \cdot \left(\xi^* X_F\right)(t), \quad t \in \mathbb{CP}^1 \setminus \infty.$$

Here $f(t) > 0$ is a smooth function tending to 0 as $t \to \infty$ in such a way that $\widetilde{Y}$ becomes smooth at $\infty$. That is, in the variable $\tau = 1/t$ the pull-back vector field $\xi^* X_F$ usually has a pole, $\xi^* X_F = \tau^{-\alpha}(c + \cdots)(d/d\tau)$ for $c \neq 0$. Then we put $f(t) = |\tau|^{2\alpha}$ near $\tau = 0$. We find that

$$i_\infty \widetilde{Y} = -\alpha. \tag{2-4}$$

**Lemma 2.5.** *If* $C$ *has only simple double points as singularities*, *their number equals* $1 - \frac{1}{2} i_\infty \widetilde{Y}$.

*Proof.* This follows from the Poincaré–Hopf formula and the equalities (2-4) and $\chi(\mathbb{CP}^1) = 2$. □

**Definition 2.6.** For a parametric line $C = \xi(\mathbb{C})$ of the form (2-2) the number

$$\delta := 1 - \tfrac{1}{2} i_\infty \widetilde{Y}$$

is called the *number of its (finite) double points*.

The same formula defines $\delta$ for a germ at $t=\infty$ of a curve $x=t^p+\text{l.o.t.}$, $y=t^q+\text{l.o.t.}$ Here $X_F$ and $\widetilde{Y}$ are germs of vector fields in $(\mathbb{CP}^2, (0:1:0))$ and $(\widehat{\mathbb{C}}, \infty)$ respectively (with $\widehat{\mathbb{C}}=\mathbb{C}\cup\infty$).

We calculate the number $i_\infty\widetilde{Y}$ in terms of the Puiseux expansion of the curve $C$ at infinity:

$$\begin{aligned} y &= x^{u_1/p_1}\big(d_1+\cdots+x^{-u_2/p_1p_2}(d_2+\cdots+x^{-u_r/p_1\cdots p_r}(d_r+\cdots)\cdots)\big) \\ &= (d_1x^{v_1/p}+\cdots)+(d_2x^{v_2/p}+\cdots)+\cdots+(d_rx^{v_r/p}+\cdots). \end{aligned} \tag{2-5}$$

Here $p_j>1$, $p=p_1\dots p_r$, $v_j=u_1p_2\dots p_r-u_2p_3\dots p_r-\dots-u_jp_{j+1}\dots p_r$ and $\gcd(u_j, p_j)=\gcd(v_j, p_j)=1$. The coefficients $d_j\neq 0$ and the dots denote power series in $x^{1/p_1\cdots p_j}$ in the $j$-th summand. Moreover, $v_1=q$ and $d_1=1$. The pairs $(p_1, u_1)$, $(p_2, -u_2)$, $\dots$, $(p_r, -u_r)$ are called the *characteristic pairs (at infinity)*.

We call the expansion (2-5) the curve's *topologically arranged Puiseux expansion*. The expansion of the form $y=c_0x^{q/p}+c_1x^{(q-1)/p}+\cdots$ is called the *standard Puiseux expansion*.

**Theorem 2.7.** *The number* $i_\infty\widetilde{Y}$ *equals*

$$2-\big((v_1-1)(p_1-1)p_2\dots p_r+(v_2-1)(p_2-1)p_3\dots p_r+\cdots+(v_r-1)(p_r-1)\big).$$

*In particular, the number of double points of $C$ equals*

$$\delta=\frac{1}{2}\sum(v_j-1)(p_j-1)p_{j+1}\dots p_r.$$

*Sketch of proof.* The formula (2-5) gives one branch $y=f_{\zeta^*}(x)$ of the multivalued solution to the equation $F(x, y)=0$. All branches of this solution take the form

$$y=f_\zeta(x)=\zeta_1\big(d_1x^{v_1/p}+\cdots+\zeta_2(d_2x^{v_2/p}+\cdots+\zeta_r(d_rx^{v_r/p}+\cdots)\dots)\big),$$

where $\zeta_1$ takes $p_1$values, $\zeta_2$ takes $p_2$ values, etc. We have $\zeta^*=(1, \dots, 1)$. The polynomial $F$ has the form $F=\prod_\zeta(y-f_\zeta(x))$ near infinity. Next, one rewrites the Hamiltonian differential equation on $C$ in the local variable $\tau=1/t$ and calculates the order of the pole at $\tau=0$ of the right-hand side. □

As a corollary of Theorem 2.7 we get:

**Proposition 2.8** [Milnor 1968]. *Let a germ $A\subset(\mathbb{C}^2, 0)$ be irreducible and have the standard Puiseux expansion*

$$y=c_1x^{v_1/n}+c_2x^{v_2/n}+\cdots,$$

(*with growing powers and nonzero coefficients*), *with $n$ is minimal* (*so the function $y=f(x)$ has $n$ branches*). *Set* $D_1=n$, $D_2=\gcd(n, v_1)$, $D_3=\gcd(n, v_1, v_2), \dots,$

$D_{r+1}=1$. *Then*

$$\begin{aligned}\mu_0(A) &= \sum (v_j-1)(D_j-D_{j+1})\\ &= (v_1-1)(D_1-1)+(v_2-v_1)(D_2-1)+\cdots+(v_r-v_{r-1})(D_r-1).\end{aligned}$$

The latter formula can be used in the estimation of the Milnor number in terms of the codimension of the singularity.

Fix the integer $n>0$ and consider the space $\mathscr{X}$ of germs in $(\mathbb{C}^2,0)$ of holomorphic curves of the form

$$x=\tau^n, \quad y=e_1\tau+e_2\tau^2+\cdots,$$

where the series defining $y$ represents an analytic function near $\tau=0$. We split $\mathscr{X}$ as a union $\bigcup_\mu \mathscr{X}(\mu)$ of disjoint subsets with constant Milnor number $\mu$. Each summand $\mathscr{X}(\mu)$ is a union $\mathscr{X}_i(\mu)$ of semialgebraic sets (equisingularity strata) defined by equations $e_j=0$ and inequalities $e_k\neq 0$. The number of equations is the *codimension* of the corresponding component: $\nu:=\operatorname{codim}\mathscr{X}_i(\mu)$. We allow also germs of the form $y=\psi(t^d)$, with $d$ dividing $n$; then $\mu=\nu=\infty$.

**Proposition 2.9.** *For any nonempty equisingularity stratum* $\mathscr{X}_i(\mu)$,

$$\mu\leq n\nu.$$

*Proof.* The proposition being true for $\mu=\nu=\infty$, we assume that $\mu<\infty$. Each stratum $\mathscr{X}_i(\mu)$ consists of curves with topologically arranged Puiseux expansion of the form

$$\begin{aligned}y &= x^{m_0}(d_0+\cdots)+x^{m_1/n_1}(d_1+\cdots)+\cdots+x^{m_r/n_1\dots n_r}(d_r+\cdots)\\ &= x^{m_0}(d_0+\cdots)+x^{u_1/n_1}\big(d_1+\cdots+x^{u_2/n_1n_2}(d_2+\cdots+x^{u_r/n_1\dots n_r}(d_r+\cdots)\dots)\big)\end{aligned}$$

with $n_1\dots n_r=n$, $\gcd(m_j,n_j)=1$, $n_j>1$ and *characteristic pairs* (at the singularity) $(n_1,u_1),\dots,(n_j,u_j)$. We have

$$\mu=\sum(v_j-1)(D_j-D_{j+1})=\left(\sum_{j=1}^r m_j(n_j-1)(n_{j+1}\dots n_r)^2\right)-(n-1);$$

(here $v_j=m_jn_{j+1}\dots n_r$, $D_j-D_{j+1}=(n_j-1)n_{j+1}\dots n_r$ and $\sum(D_j-D_{j+1})=n-1$).

The stratum $\mathscr{X}_i(\mu)$ is defined by:

- the vanishing of the coefficients $e_{jn_2\dots n_r}$ for $j<m_1$ and $j\neq 0 \pmod{n_1}$;
- the vanishing of the coefficients $e_{jn_3\dots n_r}$ for $j<m_2$ and $j\neq 0 \pmod{n_2}$;

  and so on up to
- the vanishing of the coefficients $e_j$ for $j<m_r$ and $j\neq 0 \pmod{n_r}$.

(The first term $d_0 x^{m_0}$, which may be absent, is inessential.) We thus conclude that *the codimension of the stratum* $\mathscr{X}_i(\mu)$ *equals*

$$\nu = \tilde{\nu}_{n_1}(m_1) + \cdots + \tilde{\nu}_{n_r}(m_r),$$

*where*

$$\tilde{\nu}_k(m) := m - 1 - \left[\frac{m-1}{k}\right], \tag{2-6}$$

*the brackets denoting the integer part.*

We extend the $\tilde{\nu}_k(x)$ to monotone piecewise linear functions on $\mathbb{R}$. Therefore we have the conditional extremum problem

$$\sup \left\{\mu(m_1, \ldots, m_r) : \tilde{\nu}_{n_1}(m_1) + \cdots + \tilde{\nu}_{n_r}(m_r) = \nu,\ m_j n_{j+1} \le m_{j+1} - 1\right\}.$$

The extremum of $\mu$ is achieved at the boundary, $m_j n_{j+1} = m_{j+1} - 1$. Indeed, we can assume that $m_j$ belongs to a closed interval where $\tilde{\nu}'_{n_i}(m_i) = 1$. Then, after fixing $m_j$, $j \ne i, r$, we have $m_r = -m_i + \text{const}$ and $\frac{\partial \mu}{\partial m_i} = (n_i - 1)(n_{i+1} \ldots n_r)^2 - (n_r - 1) > 0$. Therefore

$$\begin{aligned} m_{r-1} &= \frac{m_r}{n_r} - \frac{1}{n_r}, \\ m_{r-2} &= \frac{m_r}{n_r n_{r-1}} - \frac{1}{n_r n_{r-1}} - \frac{1}{n_{r-1}}, \\ &\vdots \\ m_1 &= \frac{m_r}{n_r \ldots n_2} - \frac{1}{n_r \ldots n_2} - \ldots - \frac{1}{n_2}. \end{aligned} \tag{2-7}$$

Note also that $\tilde{\nu}_k(x) \ge (x-1)(1-1/k)$. Thus

$$\nu \ge \hat{\nu}(m_1, \ldots, m_r) := (m_1 - 1)\Big(1 - \frac{1}{n_1}\Big) + \cdots + (m_r - 1)\Big(1 - \frac{1}{n_r}\Big).$$

When $r = 1$ and $r = 2$ standard calculations show that $n\hat{\nu} = \mu = (m_1 - 1)(n-1)$ and $n\hat{\nu} = \mu = (m_2 - 1)(n-1) - (n_1 - 1)n_2$, respectively. If $n \ge 3$ and (2-7) holds, we can write

$$n\hat{\nu} - \mu = n_r(An_r + B),$$

where

$$\begin{aligned} A = A(n_1, \ldots, n_{r-1}) &= (1 + n_{r-1} + n_{r-1}n_{r-2} + \cdots + n_{r-1} \ldots n_3)(n_1 - 1)n_2 \ldots n_{r-1} \\ &\quad + (1 + n_{r-1} + n_{r-1}n_{r-2} + \cdots + n_{r-1} \ldots n_4)(n_2 - 1)n_3 \ldots n_{r-1} + \cdots + (n_{r-2} - 1)n_{r-1} \\ &> 0 \end{aligned}$$

and $B = B(n_1, \ldots, n_{r-1})$ is a polynomial. The expression $(n\hat{\nu} - \mu)|_{n_r=1}$ is the same as the analogous expression depending on $r - 1$ variables $n_1, \ldots, n_{r-1}$. By induction we can assume that $n_r(An_r+B)|_{n_r=1}$ is nonnegative. Therefore

$$n_r(An_r+B)>0 \quad \text{for } n_r>1. \qquad \square$$

**Remark 2.10.** The codimension $\nu$ has also been computed by Orevkov [2002], but with slightly different notations. His rough M-number $\overline{M}$ (see Section 1) turns out to be equal to $\overline{M}=n-2+\sum\tilde{\nu}_{n_i}(m_i)=n-2+\nu$, when $n$ is the multiplicity of the singularity. In Section 3 we introduce the *external codimension* of the singularity, which is the same as $\overline{M}$ in the irreducible case in most situations. Orevkov [Orevkov 2002, Lemma 4.1] proves the inequality $\overline{M}-\mu/n>n-3$, which implies $\mu<n(\nu+1)$.

Note also that our estimate is optimal (in a sense); for one or two characteristic pairs it cannot be improved.

Later on we shall need the following variants of Proposition 2.9.

**Proposition 2.11.** *Consider the space $\mathscr{X}'$ of germs in $(\mathbb{C}^2,0)$ of the form*

$$x=\tau^n,\qquad y=\tau^m(1+e_1\tau+e_2\tau^2+\cdots),$$

*where* $\gcd(m,n)=n'>1$. *For any equisingularity stratum in $\mathscr{X}'$ we have*

$$\mu\le\mu_{\min}+n'\nu',$$

*where $\nu'$ is the codimension in $\mathscr{X}'$ of the stratum and*

$$\mu_{\min}:=(m-1)(n-1)+(n'-1)$$

*is the minimal Milnor number.*

*Proof.* Set $m_0=m/n'$, $n_0=n/n'$. We have

$$\begin{aligned}\mu&=(m_0n'-1)(n_0-1)n'+\sum_{j\ge1}\big((v_j-m_0n'-1)+m_0n'\big)(n_j-1)n_{j+1}\dots n_r\\&=\big((m_0n'-1)(n_0-1)n'+n_0n'(n'-1)\big)+\sum(\tilde{v}_j-1)(n_j-1)n_{j+1}\dots n_r,\end{aligned}$$

where $\tilde{v}_j=v_j-m_0n'$. The expression $(m_0n'-1)(n_0-1)n'+n_0n'(n'-1)$ equals $\mu_{\min}$ and last sum is estimated by $n'\nu'$ like in Proposition 2.9. $\square$

**Proposition 2.12.** *Consider the space $\mathscr{X}_\infty$ of germs in $(\mathbb{CP}^2,(0:1:0))$ of the form*

$$x=\tau^p,\qquad y=\tau^q(1+e_1\tau^{-1}+e_2\tau^{-2}+\cdots),\qquad \tau\to\infty,$$

*where* $\gcd(p,q)=p'>1$. *For any equisingularity stratum in $\mathscr{X}_\infty$,*

$$2\delta\ge2\delta_{\max}-p'\nu_\infty,$$

*where $\delta$ is the number defined in Definition 2.6,*

$$\delta_{\max}=\tfrac12\big((p-1)(q-1)-(p'-1)\big)$$

*is the maximal number of double points and* $\nu_\infty$ *is the codimension in* $\mathscr{X}_\infty$ *of the stratum.*

We omit the proof.

Now we pass to the expression of the number of double points $\delta(A+B)$ (hidden at singularity) of a singular curve $A+B$ consisting of two local components $A$ and $B$. We need a formula for the intersection index $(A \cdot B)_0$ in terms of the Puiseux expansions of the germs $A$ and $B$.

Assume the following standard expansions:

$$\begin{aligned} A &: x = \iota^M, \quad y = d_1\iota + d_2\iota^2 + \cdots, \\ B &: x = \tau^N, \quad y = e_1\tau + e_2\tau^2 + \cdots \end{aligned} \tag{2-8}$$

Moreover, we assume that some initial parts of these expansions are the same. It is clear that the corresponding exponents $l/k$ (in $x^{l/k}$) are such that $k$ divides $M$ and $N$; thus $M = km$, $N = kn$. We have

$$\begin{aligned} A &: x = \iota^{km}, \quad y = f_1 x^{1/k} + \cdots + f_u x^{u/k} + d_{u_1 m'} x^{u_1/km_1} + \cdots \\ B &: x = \tau^{kn}, \quad y = f_1 x^{1/k} + \cdots + f_u x^{u/k} + e_{w_1 n'} x^{w_1/kn_1} + \cdots \end{aligned} \tag{2-9}$$

where $m = m_1 m'$, $n = n_1 n'$, $\gcd(u_1, m_1) = \gcd(w_1, n_1) = 1$ (note that $m_1, n_1$ may equal 1) and either

(i) $w_1/kn_1 < u_1/km_1$, or

(ii) $w_1/kn_1 = u_1/km_1$ but $d_{u_1 m'} \neq e_{w_1 n'}$ (here $m_1 = n_1$ and $u_1 = w_1$).

We assume that the part $\sum_{j=1}^{u} f_j x^{j/k}$ is the longest common part among analogous common parts for all Puiseux series for $A$ and $B$ (i.e., those parts that differ by changing branches of the roots of $x$). Assume also that $k$ is minimal (e.g., it is possible that $m$ and $n$ are not relatively prime).

We represent this common initial part $\sum_1^u f_j x^{j/k}$ in the topologically arranged form

$$(\tilde{f}_1 x^{l_1/k_1} + \cdots) + \cdots + (\tilde{f}_r x^{l_r/k_1 \ldots k_r} + \cdots), \tag{2-10}$$

with $k_1 \ldots k_r = k$, $\gcd(l_j, k_j) = 1$.

The number $(A \cdot B)_0 = (A, B)$ is the order at $\iota = 0$ of the function $G|_A$, where $G(x, y)$ is the function defining $B$. We represent $G$ as a product $\prod(y - \chi_\zeta(x))$ over $kn$ branches $\chi_\zeta$ of the multivalued function $y(x)$.

There are $n$ branches with the fixed common part $\sum_1^u f_j x^{j/k}$. In case (ii) it gives the contribution $n \cdot (w_1/kn_1) \cdot km = mn \cdot w_1/n_1$ into $(A \cdot B)$. In the case (i) these $n$ branches are divided into $n' = n/n_1$ groups, each containing $n_1$ branches and each with contribution $n_1 \cdot (w_1/kn_1) \cdot km$. Generally, the distinguished branches give $mnw_1/n_1$.

The contribution arising from the other branches is calculated from (2-10), as in Proposition 2.8. It equals

$$\sum_{j=1}^{r}(k_j-1)k_{j+1}\dots k_r n\frac{l_j}{k_1\dots k_j}\cdot km = mn\sum_{j=1}^{r} l_j(k_j-1)k_{j+1}^2\dots k_r^2.$$

**Proposition 2.13.** *Under the preceding assumptions we have*

$$(A\cdot B) = mn\left(\sum_{j=1}^{r} l_j\,(k_j-1)k_{j+1}^2\dots k_r^2+\frac{w_1}{n_1}\right).$$

We calculate the codimension of the corresponding singularity of $A+B$ in the space of pairs of germs of the form (2-8). The codimension of $A$ is $\nu(A)=\nu^{(A,B)}+\nu^{(A)}$ and that of $B$ is $\nu(B)=\nu^{(A,B)}+\nu^{(B)}$, where (with the notation $\tilde{\nu}_k(\cdot)$ of Equation (2-6))

$$\nu^{(A,B)}=\sum_{j=1}^{r}\tilde{\nu}_{k_j}(l_j),\qquad \nu^{(A)}=\sum_{i=1}^{r_A}\tilde{\nu}_{m_i}(u_i),\qquad \nu^{(B)}=\sum_{i=1}^{r_B}\tilde{\nu}_{n_i}(w_i),$$

according to the topologically arranged expansions $y=\sum(\tilde{f}_j x^{l_j/k_1\dots k_j}+\cdots)+\sum(\tilde{d}_i x^{u_i/km_1\dots m_i}+\cdots)$ and $y=\sum\tilde{f}_j x^{l_j/k_1\dots k_j}+\sum(\tilde{e}_i x^{w_i/kn_1\dots n_i}+\cdots)$ (see the proof of Proposition 2.9). The whole codimension of $A+B$ is $\nu(A+B)=\nu(A)+\nu(B)+\nu_{\tan}$, where $\nu_{\tan}:=u-\nu^{(A,B)}$ is the number of equalities among the nonzero coefficients before $x^{j/k}$'s (see (2-9)). We have proved the following:

**Lemma 2.14.** *The codimension of $A+B$ equals*

$$\nu(A+B)=\nu^{(A,B)}+\nu^{(A)}+\nu^{(B)}+u,$$

*where $u$ is the length of the common initial part of the Puiseux expansions for $A$ and $B$.*

Recall that by Definition 2.2 we have $\delta(A+B)=\delta(A)+\delta(B)+2(A\cdot B)$, where $\delta(A)$ and $\delta(B)$ are given via the Puiseux expansions of $A$ and $B$ (Proposition 2.8). The next result is proved in a similar way as Proposition 2.9.

**Lemma 2.15.**

$$(A\cdot B)\le M(\nu^{(B)}+u+1),\qquad (A\cdot B)\le N(\nu^{(A)}+u+1).$$

This lemma and Proposition 2.9 lead to the next result:

**Proposition 2.16.** *For the curves $A$ and $B$ given in (2-8) we have*

$$2\delta(A+B)\le(M+N)\cdot\big(\nu(A+B)+1\big),$$

*where $\nu(A+B)$ is given in Lemma 2.14.*

Note that when $M = N = 1$ this inequality becomes an equality.

We shall need also the following generalizations of Proposition 2.11. Consider the space of pairs of curves of the form

$$
\begin{aligned}
A &: x = \iota^{k_1k'm} = \iota^{\alpha}, \quad y = \iota^{l_1k'm}(d_0 + d_1\iota + \cdots) = \iota^{\gamma}(d_0 + \cdots), \\
B &: x = \tau^{k_1k'n} = \tau^{\beta}, \quad y = \tau^{l_1k'n}(e_0 + e_1\iota + \cdots) = \tau^{\delta}(e_0 + \cdots),
\end{aligned}
\tag{2-11}
$$

where $d_0e_0 \neq 0$, $\gcd(k_1, l_1) = 1$ and $k' = k_2 \ldots k_r$. Also set $\alpha' = \gcd(\alpha, \gamma) = k'm$ and $\beta' = \gcd(\beta, \delta) = k'n$.

The next result is rather easy and we omit its proof.

**Proposition 2.17.** *We have*

$$2\delta(A + B) \leq 2\delta_{\min}(A + B) + (\alpha' + \beta') \cdot \big(\nu'(A + B) + 1\big),$$

*where* $2\delta_{\min}(A+B) = (\alpha+\beta-1)(\gamma+\delta-1)+(\alpha'+\beta'-1)$ *is twice the minimal number of double points hidden at the singularity and* $\nu'(A+B)$ *is the codimension of the corresponding equisingularity stratum in the space of pairs* (2-11).

Consider the space of pairs of curves of the form

$$
\begin{aligned}
A &: x = \iota^{k\alpha'} = \iota^{\alpha}, \quad y = \iota^{l\alpha'}(d_0 + d_1\iota + \cdots) = \iota^{\gamma}(d_0 + \cdots), \\
B &: x = \tau^{p\beta'} = \tau^{\beta}, \quad y = \tau^{q\beta'}(e_0 + e_1\iota + \cdots) = \tau^{\delta}(e_0 + \cdots),
\end{aligned}
$$

where $d_0e_0 \neq 0$, $\kappa := kq - pl > 0$, $\gcd(k, l) = \gcd(p, q) = 1$.

**Proposition 2.18.** *We have*

$$2\delta(A + B) \leq 2\delta_{\min}(A + B) + \alpha' \cdot \nu'(A) + \beta' \cdot \nu'(B),$$

*where* $2\delta_{\min}(A+B) = (\alpha+\beta-1)(\gamma+\delta-1)+(\alpha'+\beta'-1)-\alpha'\beta'\kappa$ *and* $\nu'(A)$, $\nu'(B)$ *are the codimensions in the spaces* $A$ *and* $B$.

*Proof.* The contributions $2\delta(A)$ and $2\delta(B)$ equal $2\delta_{\min}(A)+\alpha'\nu'(A)$ and $2\delta_{\min}(B)+\beta'\nu'(B)$. The term $2(A \cdot B)$ equals

$$2(A \cdot B)_{\min} = 2 \cdot \min(\alpha\delta, \beta\gamma) = \alpha\delta + \beta\gamma - \alpha'\beta'\kappa. \qquad \square$$

## 3. The space of parametric lines

Recall that a *parametric line* is a plane curve

$$\xi : \mathbb{C} \to \mathbb{C}^2, \quad \xi(t) = (\varphi(t), \psi(t)),$$

where $\varphi$, $\psi$ are polynomials. In (2-2) it was assumed that $\xi$ is typically one-to-one; here we skip this restriction, but we require that $\xi$ not be constant.

By applying tame transformations of $\mathbb{C}^2$ we can assume that either

$$\varphi = t^p + a_1t^{p-1} + \cdots + a_p, \quad \psi \equiv 0, \tag{3-1}$$

or

$$\varphi = t^p + a_1 t^{p-1} + \cdots + a_p, \quad \psi = t^q + b_1 t^{q-1} + \cdots + b_q, \tag{3-2}$$

where $1 < p < q$, $q/p \notin \mathbb{Z}$ (see (2-3)) and

$$a_p = b_q = b_{q-p} = b_{q-2p} = \ldots = b_{q-[q/p]p} = 0. \tag{3-3}$$

We shall also use the notation $\xi = \xi_{a,b}$, $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_q)$.

**Definition 3.1.** Denote by $\overline{\mathrm{Curv}}_{p,q} = \overline{\mathrm{Curv}}$ the space of curves of the form (3-1) for $(p, q) = (p, 0)$, or (3-2). For $1 < p < q$ with $q/p \notin \mathbb{Z}$ we define the *space of parametric lines* $\mathrm{Curv} = \mathrm{Curv}_{p,q}$ as the space of curves (3-2) with the restriction (3-3).

Thus Curv is the quotient space of $\overline{\mathrm{Curv}}$ with respect to a natural action of the group $G_{p,q}$ of changes of coordinates $(\varphi, \psi) \to (\varphi + \alpha, \psi + P(\varphi))$, $\deg P \le [q/p]$ of changes stabilizing the form (3-2). We have $\dim \mathrm{Curv} = p + q - [q/p] - 2$.

**Lemma 3.2.** *The changes*

$$t \to \lambda^{-1} t + \mu, \quad x \to \lambda^p x, \quad y \to \lambda^q y$$

*induce an action of the group* $Aff(\mathbb{C}) = \mathbb{C}^* \rtimes \mathbb{C}$ (*semidirect product*) *of affine automorphisms of the complex line on* $\overline{\mathrm{Curv}}$. *The quotient space* $\overline{\mathrm{Curv}}/\mathbb{C}$ *can be identified with the space of curves of the form* (3-2) *such that* $a_1 = 0$, *and* $\mathrm{Curv}/\mathbb{C} = \overline{\mathrm{Curv}}/\mathbb{C} \rtimes G_{p,q}$ *is identified with the space of curves* (3-2) *satisfying* (3-3) *and* $a_1 = 0$.

*The action of* $\mathbb{C}^*$ *on* $\overline{\mathrm{Curv}}$ (*and on* Curv, $\mathrm{Curv}/\mathbb{C}$) *results in the changes*

$$a_j \to \lambda^j a_j, \quad b_j \to \lambda^j b_j.$$

*The quotient space* $((\mathrm{Curv}/\mathbb{C}) \setminus 0)/\mathbb{C}^*$ *is the quasihomogeneous projective space*, *denoted* PCurv, *of dimension*

$$\sigma := p + q - [q/p] - 4.$$

(*Here the point* $0 \in \mathrm{Curv}/\mathbb{C}$ *corresponds to the quasihomogeneous curve* $x = t^p$, $y = t^q$).

Let $C = \xi(\mathbb{C}) \subset \mathbb{C}^2$ be the image of a $\xi \in \mathrm{Curv}$. The curve $C$ can have self-intersections (multiple points). The parametrization $\xi : \mathbb{C} \to C$ can be *primitive* or *nonprimitive*. In the primitive case the map $\xi$ is one-to-one above the set of simple points of $C$. In the nonprimitive case we have this result of Lüroth:

**Theorem 3.3.** *If the geometrical degree $d$ of $\xi : \mathbb{C} \to C$ is greater than one*, *there exists a polynomial $\omega(t)$ of degree $d$ such that $\xi(t) = \tilde{\xi} \circ \omega(t)$ for a polynomial map $\tilde{\xi} : \mathbb{C} \to \mathbb{C}^2$* (*of geometrical degree* 1). *In particular*, *$d$ divides $p$ and $q$*.

In the sequel we shall investigate the class $\mathrm{Curv}_{p,q}$ with $1 < p < q$ and $q/p \notin \mathbb{Z}$. The images $C = \xi(\mathbb{C})$ of maps $\xi \in \mathrm{Curv}_{p,q}$ have singularities, typically double points. However, there are reasons to regard them as not that singular.

**Definition 3.4.** An element $\xi \in \mathrm{Curv}$ is called *singular at* $t_0$ if $\xi'(t_0) = 0$. The curve $\xi$ is *singular* if it is singular at some point.

Thus nonprimitive curves are singular and primitive curves whose images have only double points are nonsingular.

The subset $\Sigma^{\sin} \subset \mathrm{Curv}$ of singular curves is an algebraic hypersurface, stratified by the Milnor number $\mu_{\xi(t_0)}(C_j)$ of the singular point of $C$. The typical singularity is the cusp singularity $\mathbf{A}_2$ with $\mu = 2$. We have the filtration $\Sigma^{\sin} = \Sigma_2^{\sin} \supset \Sigma_4^{\sin} \supset \ldots \supset \Sigma_\infty^{\sin}$. Here $\mu$ is always even and $\Sigma_\infty^{\sin}$ consists of nonprimitive curves; if $\gcd(p, q) = 1$ then $\Sigma_\infty^{\sin}$ is empty. Moreover, the hypersurface $\Sigma^{\sin}$ can have self-intersection points, which correspond to curves with several singular points.

Another degeneration corresponds to the situation when intersection of some local components $A$, $B$ of the curve ceases to be transversal. It occurs along an algebraic hypersurface $\Sigma^{\tan}$, which can be defined in an implicit form:

$$\frac{\varphi(s) - \varphi(t)}{s - t} = 0, \qquad \frac{\psi(s) - \psi(t)}{s - t} = 0, \qquad \frac{\varphi'(s)\psi'(t) - \varphi'(t)\psi'(s)}{s - t} = 0.$$

The nontransversality can be caused by tangency of two local smooth components or by singularity of one (or both) component.

One type of degeneration corresponds to intersection of three (or more) local branches. This occurs along a hypersurface $\Sigma^{\mathrm{tri}}$, which is also algebraic.

The last type of degeneration corresponds to some double point(s) escaping to infinity. It occurs along a hypersurface $\Sigma^{\inf}$. The corresponding bifurcation is governed by the standard Puiseux expansion of $C$ near infinity

$$y = x^{q/p} + c_1 y^{(q-1)/p} + \cdots .$$

This expansion is obtained by elimination of $t$ with the agreement that

$$x^{1/p} = t(1 + a_1 t^{-1} + \cdots + a_p t^{-p})^{1/p} = t + \frac{1}{p} a_1 t^{-1} + \cdots$$

as $t \to \infty$. Thus the coefficients $c_j = c_j(a, b)$ are uniquely defined. They are quasihomogeneous polynomials of degree $j$ with respect to the $\mathbb{C}^*$-action.

**Lemma 3.5.** *If* $\gcd(p, q) = 1$, *then* $\Sigma^{\inf} = \varnothing$. *If* $\gcd(p, q) = p' > 1$, *then*

$$\Sigma^{\inf} = \{c_1 = 0\},$$

*where $c_1 = b_1 - (q/p)a_1$. Any $\xi \in \mathrm{Curv} \setminus \Sigma^{\inf}$ defines a curve with total $\delta$-invariant equal to*

$$\delta_{\max} = \frac{1}{2}\big((p-1)(q-1) - (p'-1)\big).$$

For $\xi \in \mathrm{Curv}$ we define the *number of double points hidden at infinity* as

$$\delta_\infty := \delta_{\max} - \delta.$$

We define

$$\Sigma = \Sigma^{\sin} \cup \Sigma^{\tan} \cup \Sigma^{\mathrm{tri}} \cup \Sigma^{\inf}$$

as the *bifurcation set* of the space Curv. Note that there are intersections (and self-intersections) among these components.

The four types of bifurcations are local in nature. Therefore they are characterized by their codimension in the spaces of germs of analytic curves. We shall call this codimension as the external codimension.

**Definition 3.6.** For a cuspidal singularity $x = \tau^n$, $y = e_1\tau + \cdots$ we define the *external codimension* as

$$\operatorname{ext} \nu = (n-2) + \nu,$$

where $\nu$ is the codimension of the corresponding equisingularity stratum. Here the summand $n-2$ arises from the vanishing of $n-1$ terms in the expansion of $\dot{\varphi}(t)$ at the singular point, which may vary for $\xi = (\varphi, \psi) \in \mathrm{Curv}$.

For a self-intersection of two local branches $A$ and $B$ (as in (2-8)) we define the *external codimension* as

$$\operatorname{ext} \nu = (M-1) + (N-1) + \nu(A+B),$$

where $M-1$ and $N-1$ are the orders of $\dot{x}$ and $\nu(A+B)$ is the codimension of the corresponding equisingularity stratum (defined in Section 2).

Generally, for the intersection of $k$ local branches $A_1, \dots, A_k$ with the orders of $\dot{x}$ equal to $n^{(1)} - 1, \dots, n^{(k)} - 1$ and the codimension of the corresponding equisingularity stratum $\nu(A_1 + \cdots + A_k)$, the *external codimension* equals

$$\operatorname{ext} \nu = (n^{(1)} + \cdots + n^{(k)} - 2) + \nu(A_1 + \cdots + A_k).$$

Finally,

$$\operatorname{ext} \nu_\infty = \nu_\infty$$

is the *external codimension* of the degeneration at $t = \infty$; here $\nu_\infty$ is defined in Proposition 2.12.

Note that our definition of the external codimension of a singularity of $\xi \in \mathrm{Curv}_{p,q}$ is tied with the fixed choice of the coordinate system $(x, y)$ in the target complex plane. This is related to the fact that we always assume $p < q$.

(Here lies a difference with Orevkov's definition of the M-number of a cuspidal singularity (see Remark 2.10). In his formula, $\overline{M} = n - 2 + \nu$, the integer $n$ is the multiplicity of the singularity, that is, the minimum of the orders of $\dot{x}$ and $\dot{y}$. Therefore when $n - 1 = \operatorname{ord} \dot{x} \le \operatorname{ord} \dot{y}$, we have $\overline{M} = \operatorname{ext} \nu$; otherwise $\overline{M} < \operatorname{ext} \nu$.)

Now we can formulate the promised conjecture about codimensions.

**Conjecture 3.7.** *Any fixed collection of degenerations* (*singularities*) *in* $\Sigma^{\text{sin}}$, $\Sigma^{\text{tan}}$, $\Sigma^{\text{tri}}$ *and* $\Sigma^{\text{inf}}$ *in* $\mathrm{Curv}_{p,q}$, *where* $1 < p < q$ *and* $q/p \notin \mathbb{Z}$, *such that their external codimensions are finite*, *either does not occur or occurs along an algebraic subvariety of the space* $\mathrm{Curv} \setminus \Sigma^{\text{sin}}_{\infty}$ (*of primitive curves*) *whose codimension equals to the sum of external codimensions of the local degenerations.*

*In particular*, *when the curve is not equivalent to a quasihomogeneous curve*, *the sum of external codimensions does not exceed* $\sigma = \dim \mathrm{PCurv}$.

Conjecture 3.7 can be interpreted as the property of regularity of some sequences of essential Puiseux quantities defining the equisingularity strata of the singularities. Recall that a sequence $f_1, f_2, \ldots \in \mathbb{C}[X]$ of regular functions on a normal quasiprojective complex variety $X$ is *regular at* $x_0 \in X$ if any $f_j$ is not a zero divisor in the local ring $\mathbb{O}_{x_0}/(f_1, \ldots, f_{j-1})$ (see [Griffiths and Harris 1978]). In our situation $X = \mathrm{Curv} \setminus \Sigma^{\text{sin}}_{\infty}$ is the space of primitive curves.

**Examples 3.8.** (a) Let $p = 4, q = 6$. Thus $\sigma = p + q - 4 - [q/p] = 5$. One can calculate the first topologically essential Puiseux quantities at infinity $(f_1, \ldots, f_5) = (c_1, c_3, c_5, c_7, c_9)$. It turns out that the equalities $f_1 = \ldots = f_5 = 0$ hold only on the subspace $\Sigma^{\text{sin}}_{\infty}$ of nonprimitive curves; here $\dim P\Sigma^{\text{sin}}_{\infty} = 1$. Moreover, $PV_4 = \{f_1 = \ldots = f_4 = 0\}$ consists of two components: $P\Sigma^{\text{sin}}_{\infty}$ and a 1-dimensional variety $PV_4'$, such that $f_5|V_4' \not\equiv 0$. It follows that already $\sigma$ (not $\sigma + 1$) essential Puiseux coefficients form maximal (in a sense) regular sequence in $\mathrm{PCurv} \setminus P\Sigma^{\text{sin}}_{\infty}$.

(b) The curve $x = t^6 + 2\mu t^2$, $y = t^9 + 3\mu t^5 + \frac{3}{2}\mu^2 t$, $\mu \neq 0$ has the Puiseux expansion $y = x^{3/2} + \frac{1}{2}\mu^3 x^{-1/2} - \mu^4 x^{-7/6} + \cdots$. It suggests that the sequence $(f_1, \ldots, f_{11}) = (c_1, c_2, c_4, c_5, c_7, c_8, c_{10}, c_{11}, c_{13}, c_{14}, c_{16})$ of the first $\sigma + 1 = 11$ topologically essential Puiseux quantities is regular in $\mathrm{PCurv} \setminus P\Sigma^{\text{sin}}_{\infty}$. Here $\operatorname{codim} V_i = i$ for $i = 1, \ldots, 10$, and $\dim P\Sigma^{\text{sin}}_{\infty} = 3$.

(c) If $\gcd(p, q) = 1$, the quasihomogeneous curve $x = t^p$, $y = t^q$ has $\operatorname{ext} \nu_0 = p + q - [q/p] - 3 = \sigma + 1$. This explains the last statement in Conjecture 3.7.

Conjecture 3.7 may be regarded as an affine analogue of the following:

**Conjecture** [Orevkov 2002, Conjecture 2.3]. *If* $C \subset \mathbb{CP}^2$ *is a rational* (*not necessarily cuspidal*) *curve then* $\sum \overline{M}_i \le 3d - 9$ *where the sum is taken over all irreducible analytical branches of* $C$.

One could expect that in other natural spaces of parametric curves an analogue of the our conjecture and Orevkov's holds true. However this is not the case.

In the next section we shall consider curves of the form

(3-4)
$$\varphi(t)=t^{\alpha}(1-t)^{\beta}P(t),\quad \psi(t)=t^{\gamma}(1-t)^{\delta}Q(t),\quad P(0)P(1)Q(0)Q(1)\neq 0.$$

These curves form a semialgebraic space of dimension $\deg P+\deg Q-\epsilon$, where $\epsilon$ counts the changes $\psi\to\psi+\mathrm{const}\cdot\varphi^{j}$, which preserve the form (3-4). We still denote this space by Curv.

We use the notations

$$\begin{aligned}\alpha&=\alpha_1\alpha', & \gamma&=\gamma_1\alpha', & \alpha'&=\gcd(\alpha,\gamma),\\ \beta&=\beta_1\beta', & \delta&=\delta_1\beta', & \beta'&=\gcd(\beta,\delta),\end{aligned}\tag{3-5}$$

and

$$\kappa=\alpha_1\delta_1-\beta_1\gamma_1.\tag{3-6}$$

Note that $\alpha_1=\beta_1$ and $\gamma_1=\delta_1$ when $\kappa=0$.

The singularity $x=y=0$ is characterized by its external codimension $\operatorname{ext}\nu_0$, that means the codimension in the space of pairs of germs $A: x=\iota^{\alpha}$, $y=\iota^{\gamma}(1+\cdots)$, $B: x=\tau^{\beta}$, $y=\tau^{\delta}(1+\cdots)$. $\operatorname{ext}\nu_0$ is the same as $\nu'(A+B)$ from Proposition 2.17. Introducing $\operatorname{ext}\nu^{(A,B)}=\nu'^{(A,B)}$, $\operatorname{ext}\nu^{(A)}=\nu'^{(A)}$, $\operatorname{ext}\nu^{(B)}=\nu'^{(B)}$, $\operatorname{ext}u=u'$, we have $\operatorname{ext}\nu_0=\operatorname{ext}u+\operatorname{ext}\nu^{(A,B)}+\operatorname{ext}\nu^{(A)}+\operatorname{ext}\nu^{(B)}$. Moreover, $\operatorname{ext}\nu(A)=\operatorname{ext}\nu^{(A,B)}+\operatorname{ext}\nu^{(A)}$ and $\operatorname{ext}\nu(B)=\operatorname{ext}\nu^{(A,B)}+\operatorname{ext}\nu^{(B)}$ are the external codimensions of the components $A$ and $B$. Recall that $\operatorname{ext}u$ is the number of initial common terms in the Puiseux expansions of $A$ and $B$.

The next result shows that only when $\gamma<\alpha$ and $\beta<\delta$ is $\operatorname{ext}\nu_0$ bounded by the dimension of the space of curves (3-4) (modulo changes of $(x,y)$). In general, $\operatorname{ext}\nu_0$ can be greater than this dimension.

**Lemma 3.9.** *Assume that Conjecture 3.7 holds.*

(a)
$$\operatorname{ext}\nu_\infty+\sum_{i=0}^{N}\operatorname{ext}\nu_i\le\deg P+\deg Q+\left[\frac{\gamma-1}{\alpha}\right]+\left[\frac{\delta-1}{\beta}\right]-\left[\frac{q}{p}\right].$$

(b) *If* $\deg P=0$, $\deg Q=1$ *and* $\kappa\neq0$, *then* $\operatorname{ext}\nu^{(A)}+\operatorname{ext}\nu^{(B)}\le 2$ *and equality is possible only in three situations*:

- $\operatorname{ext}\nu(A)=\operatorname{ext}\nu(B)=1$, *when* $\alpha',\beta'>1$ *and* $\beta/\alpha=(\delta+1)/(\gamma+1)$;
- $\operatorname{ext}\nu(A)=2$ *and* $\operatorname{ext}\nu(B)=0$, *when* $\alpha'$ *is even and* $\beta/\alpha=(\delta-1)/(\gamma+3)$;
- $\operatorname{ext}\nu(A)=0$ *and* $\operatorname{ext}\nu(A)=2$, *when* $\beta'$ *is even and* $\beta/\alpha=(\delta+3)/(\gamma-1)$.

*Analogous statements hold when* $\deg P=1$, $\deg Q=0$.

(c) *Suppose* $\deg P = \deg Q = 1$ *and* $\kappa = 0$. *Then assertion* (a) *gives* $\operatorname{ext}\nu_0 \le 3$. *Assume also that* $\alpha_1 + \gamma_1 \le 5$ (*recall that* $\gcd(\alpha_1, \gamma_1) = 1$ *and* $\alpha_1 < \gamma_1 \le 2\alpha_1$, *since* $p < q < 2p$). *Then* $\operatorname{ext}\nu_0 = 3$ *may occur only when either* $\operatorname{ext}\nu(A) = 3$ *and* $\operatorname{ext} u = \operatorname{ext}\nu(B) = 0$, *or* $\operatorname{ext}\nu(B) = 3$ *and* $\operatorname{ext} u = \operatorname{ext}\nu(A) = 0$, *or* $\operatorname{ext} u = 3$, $\operatorname{ext}\nu(A) = \operatorname{ext}\nu(B) = 0$, $\alpha_1 = 2$, $\gamma_1 = 3$, *and* $\alpha' = \beta'$.

*Proof.* (a) From Conjecture 3.7 one obtains that the essential Puiseux quantities at $t = 0$, $t = 1$ and $t = t_j$ and the self-intersection quantities form regular sequences in the space $((\mathrm{Curv}_{p,q}/\mathbb{C}) \setminus 0)/\mathbb{C}^*$, normalized in such a way that $\xi(0) = \xi(1) = 0$. Next, we restrict the space of curves to the form

$$\varphi(t) = t^{\alpha}(1-t)^{\beta} P(t), \quad \psi(t) = t(1-t)Q(t). \tag{3-7}$$

Here we get the estimate $\sum$ ext. codim. $\le$ essential dim. $= \deg P + \deg Q - [q/p]$. The space of curves considered in Lemma 3.9 is a subspace of the space of curves of the form (3-7). It is defined by the vanishing of the first $\gamma - 1$ Puiseux quantities at $t = 0$ and of the first $\delta - 1$ Puiseux quantities at $t = t_0$. Among these quantities there are $\gamma - 1 - [(\gamma-1)/\alpha]$ and $\delta - 1 - [(\delta-1)/\beta]$ essential Puiseux quantities. Therefore the external codimension of the singularity (0, 0) in the space of curves (3-7) equals $\operatorname{ext}\nu_0 + \gamma - 1 - [(\gamma-1)/\alpha] + \delta - 1 - [(\delta-1)/\beta]$. Now the previous inequality gives the desired estimate for $\operatorname{ext}\nu_\infty + \sum \operatorname{ext}\nu_j$.

(b) We have $\varphi = t^{\alpha}(1-t)^{\beta}$, $\psi = t^{\gamma}(1-t)^{\delta}(1-bt)$. Here one calculates explicitly the successive Puiseux quantities, and in this way the statement of this point is obtained. We omit the details.

Part (c) is proved in the same way. □

**Examples 3.10.** (a) Consider curves of the form $\varphi = [t^{\alpha'}(1-t)^{\beta'}]^{\alpha_1}$, $\psi = [t^{\alpha'}(1-t)^{\beta'}]^{\gamma_1}(1-bt)$, $b \ne 0, 1$. It is clear that the vanishing of $c_1^{(A)}$ (if $\alpha' > 1$) means $b = 0$. Similarly, $c_1^{(B)} \sim b/(1-b)$. The first intersection quantity $c_0^{(A,B)}$ equals $\chi(1) - \chi(0)$, where $\chi(t) = \psi^{\alpha_1}/\varphi^{\gamma_1} = (1-bt)^{\alpha_1}$; thus $c_0^{(A,B)} = 0$ if and only if $1-b$ is a root of unity of degree $\alpha_1$. Here $c_1^{(A,B)} \ne 0$.

(b) Consider the case $\varphi = t^{\alpha}(1-t)^{\beta}$, $\psi = t^{\gamma}(1-t)^{\delta}(1 + b_1 t + b_2 t^2)$with $\alpha' > 2$, $\beta' > 1$. Calculations show that vanishing of $c_1^{(A)}$, $c_2^{(A)}$ and of $c_1^{(B)}$ leads to $b_1 = \tilde{\kappa}/\alpha$, $b_2 = \tilde{\kappa}(\tilde{\kappa}+\alpha)/2\alpha^2$ and $(2\beta - \alpha - \tilde{\kappa})(2\alpha + \tilde{\kappa}) = 0$. The possibility $2\alpha + \tilde{\kappa} = 0$ leads to $Q(1) = 0$, but the case $2\beta - \alpha - \tilde{\kappa} = 0$ is admissible.

If we additionally require that $c_2^{(B)} = 0$ (when $\beta' > 2$), we arrive at the condition $2\beta = (\tilde{\kappa} - \beta)(2\alpha + \tilde{\kappa})$. Eliminating $\alpha$ from the latter two equations we get the equation $\tilde{\kappa}^2 - 5\beta\tilde{\kappa} + 4\beta^2 - 2\beta = 0$ with the discriminant $\Delta = 9\beta^2 + 8\beta$. The requirement that $\Delta$ is a square, say $\rho^2$, leads to the Diophantine equation $(3\rho)^2 + 4^2 = (9\beta + 4)^2$ without solutions in positive integers. Therefore the maximal possible external codimension is 3.

These examples are in agreement with Lemma 3.9.

## 4. One-connected lines

In this section we consider curves $\xi \in \mathrm{Curv}_{p,q}$ (where $1 < p < q$ and $q/p \notin \mathbb{Z}$) that are primitive and have exactly one self-intersection. Simply connected curves were classified in [Abhyankar and Moh 1975; Suzuki 1974] (smooth case) and in [Zaidenberg and Lin 1983] (singular case).

The next theorem is the main result of this paper.

**Theorem 4.1.** *Any plane rational curve with one place at infinity and first Betti number equal to* 1, *obeying Conjecture 3.7, is equivalent* (*via an automorphism of* $\mathbb{C}^2$) *to one of the following* (*with* $\alpha\delta - \beta\gamma = 1$ *and* $\alpha\beta\gamma\delta \neq 0$):

(a) $x = t^2,\ \ y = t^{2l+1}(t^2-1)^k,\ k = 1, 2, \ldots,\ l = 0, 1, \ldots;$

(b) $x = t^3,\ \ y = t^{3k+1}(t-1),\ k = 1, 2, \ldots;$

(c) $x = t^4,\ \ y = t^{4k+1}(t-1),\ k = 1, 2, \ldots;$

(d) $x = t^4,\ \ y = t^{4k+2}(t-1),\ k = 0, 1, \ldots;$

(e) $x = t^6,\ \ y = t^{6k+2}(t-1),\ k = 1, 2, \ldots;$

(f) $x = t^6,\ \ y = t^{6k+3}(t-1),\ k = 0, 1, \ldots;$

(g) $x = t^\alpha(t-1)^{k\beta},\ y = t^\gamma(t-1)^{k\delta},\ k = 1, 2, \ldots,\ 2 < \alpha + k\beta < \gamma + k\delta;$

(h) $x = t^{2\alpha}(t-1)^{2\beta},\ y = t^{2\gamma}(t-1)^{2\delta},\ \alpha + \beta < \gamma + \delta;$

(i) $x = t^\alpha(t-1)^{k\beta-\alpha},\ y = t^\gamma(t-1)^{k\delta-\gamma},\ k = 1, 2, \ldots,\ 2 < k\beta < k\delta;$

(j) $x = t^2(t-1),\ y = x^k t(t-\frac{4}{3}),\ k = 1, 2, \ldots;$

(k) $x = t^3(t-1),\ y = x^k t(t-\frac{3}{2}),\ k = 1, 2, \ldots;$

(l) $x = \bigl(t(t-1)\bigr)^{2k},\ y = x^l\bigl(t(t-1)\bigr)^k(t-\frac{1}{2}),\ k = 1, 2, \ldots,\ l = 0, 1, \ldots;$

(m) $x = \bigl(t(t-1)\bigr)^{2k+1},\ y = x^l\bigl(t(t-1)\bigr)^k(t-\frac{1}{2}),\ k = 0, 1, \ldots,\ l = 1, 2, \ldots;$

(n) $x = \bigl(t^k(t-1)^{k+1}\bigr)^2,\ y = x^l t^k(t-1)^{k+1}(t-\frac{1}{2}),\ k = 1, 2, \ldots,\ l = 0, 1, \ldots;$

(o) $x = t^{2k+1}(t-1)^{2k+3},\ y = x^l t^k(t-1)^{k+1}(t-\frac{1}{2}),\ k = 0, 1, \ldots,\ l = 1, 2, \ldots;$

(p) $x = \bigl(t(t-1)\bigr)^3,\ y = x^k t(t-1)(t-\frac{1}{2}-\frac{1}{6}\sqrt{-3}),\ k = 1, 2, \ldots;$

(q) $x = t^3 - 3t,\ y = t^4 - 2t^2,$

(r) $x = t^3 - 3t,\ y = t^5 - 2\sqrt{-2}t^4 + 11\sqrt{-2}t^2 - \frac{37}{4}t;$

(s) $x = t^3 - 3t,\ y = t^5 + 10t^4 + 80t^2 - 205t;$

(t) $x = t^3 - 3t,\ y = t^5 - \frac{5}{2}t^4 + 5t^2 - 5t;$

(u) $x = t^3 - 3t,\ y = t^5 - \frac{7}{2}t^4 - 43t^2 + 11t.$

*All these curves are nonequivalent.*

***Commentary.*** We give here the singularities (degenerations) of the curves above.

(a) This curve has a cusp $\mathbf{A}_{2l}$ at $t_1 = 0$ with $n_1 = 2$ and $\nu_1 = l$ and a tangency of two smooth branches at $t = \pm 1$ of codimension $\nu_0 = k - 1$. Here $\operatorname{ext}\nu_0 + \operatorname{ext}\nu_1 = \sigma = k + l - 1$.

(b) The only singular point is at $t_1 = 0$, with one characteristic pair $(3, 3k + 1)$ and $\nu_1 = 2k$. Here $\operatorname{ext}\nu_1 = \sigma = 2k + 1$.

(c) The only singular point is at $t_1 = 0$, with one characteristic pair $(4, 4k + 1)$ and $\nu_1 = 3k$. Here $\operatorname{ext}\nu_1 = \sigma = 3k + 2$.

(d) The only singular point is at $t_1 = 0$, with two characteristic pairs $(2, 2k + 1)$, $(2, 1)$ and with $\nu_1 = 3k + 1$. Here $\operatorname{ext}\nu_1 = \sigma = 2k + 2$.

(e) The only singular point is at $t_1 = 0$, with two characteristic pairs $(3, 3k + 1)$, $(2, 1)$ with $\nu_1 = 5k + 1$. Here $\operatorname{ext}\nu_1 = \sigma = 5k + 5$.

(f) The only singular point is at $t_1 = 0$, with two characteristic pairs $(2, 2k + 1)$, $(3, 1)$ and with $\nu_1 = 5k + 2$. Here $\operatorname{ext}\nu_1 = \sigma = 5k + 6$.

(g) There are two (possible) singular points: at $t = 0$ (with one characteristic pair $(\alpha, \gamma)$) and at $t = 1$ (with two characteristic pairs $(\beta, \delta)$, $(k, 1)$). Here $\operatorname{ext}\nu_0 = \alpha + k\beta + \gamma + k\delta - 4 - [\gamma/\alpha] - [\delta/\beta] \leq \sigma$.

(h) There are two singular points at $t = 0$ and $t = 1$, both with two characteristic pairs $(\alpha, \gamma)$, $(2, 1)$ and $(\beta, \delta)$ and $(2, 1)$ respectively. Also at infinity there are two characteristic pairs $(\alpha + \beta, \gamma + \delta)$, $(2, -1)$ but $\nu_\infty = 0$. Here $\operatorname{ext}\nu_0 = 2\alpha + 2\beta + 2\gamma + 2\delta - 4 - [\gamma/\alpha] - [\delta/\beta] \leq \sigma$.

(i) There are two (possible) singular points at $t = 0$ and at $t = 1$, both with one characteristic pair. At infinity there are two characteristic pairs $(\alpha, \gamma)$, $(k, -1)$ but $\nu_\infty = 0$. Here $\operatorname{ext}\nu_0 = k\beta + k\delta - 4 - [\gamma/\alpha] - [(k\delta - \gamma)/(k\beta - \alpha)] \leq \sigma = k\beta + k\delta - 4 - [\delta/\beta]$.

(j) Besides the self-intersection of a smooth and a singular ($\mathbf{A}_{2k}$) branches at $x = y = 0$ with $\operatorname{ext}\nu_0 = 2k$ ($n_0 = 3$, $u = k - 1$, $\nu^{(A,B)} = k$, $\nu^{(A)} = \nu^{(B)} = 0$) there is a cusp $\mathbf{A}_2$ at $t_1 = \frac{2}{3}$. Here $\operatorname{ext}\nu_0 + \operatorname{ext}\nu_1 = \sigma = 2k + 1$.

(k) Besides the self-intersection of a smooth and a singular ($(t^3, t^{3k+1})$) branches at $x = y = 0$ with $\operatorname{ext}\nu_0 = 3k + 1$ ($n_0 = 4$, $u = k - 1$, $\nu^{(A,B)} = \nu^{(B)} = 0$, $\nu^{(A)} = 2k$) there is a cusp $\mathbf{A}_2$ at $t_1 = \frac{3}{4}$. Here $\operatorname{ext}\nu_0 + \operatorname{ext}\nu_1 = \sigma = 3k + 2$.

(l) The only singular point is $x = y = 0$ with $\operatorname{ext}\nu_0 = \sigma = 4kl + 6k - l - 3$ ($n_0 = 4k$, $u = (2l + 1)k$, $\nu^{(A,B)} = 2kl + k - l - 1$, $\nu^{(A)} = \nu^{(B)} = 0$).

(m) The only singular points is $x = y = 0$ with $\operatorname{ext}\nu_0 = \sigma = 4kl + 6k + l - 2$ ($n_0 = 4k + 2$, $u = l$, $\nu^{(A,B)} = 0$, $\nu^{(A)} = \nu^{(B)} = 2kl + k - 1$).

(n) The only singular point is $x=y=0$ with $\operatorname{ext}\nu_0=\sigma=4kl+6k+l$ ($n_0=4k+2$, $u=2l+1$, $\nu^{(A,B)}=l$, $\nu^{(A)}=(2l+1)(k-1)$, $\nu^{(B)}=(2l+1)k$).

(o) The only singular point is $x=y=0$ with $\operatorname{ext}\nu_0=4kl+6k+3l+1$ ($n_0=4k+4$, $u=l$, $\nu^{(A,B)}=0$, $\nu^{(A)}=2kl+k-1$, $\nu^{(B)}=2(k+1)l+k$). At infinity there are three characteristic pairs $(2,2l+1)$, $(k+1,-1)$, $(2,-1)$ and $\nu_\infty=1$. Here $\operatorname{ext}\nu_0+\operatorname{ext}\nu_\infty=\sigma$.

(p) The only singular points is $x=y=0$ with $\operatorname{ext}\nu_0=\sigma=4k+5$ ($n_0=6$, $u=3k+1$, $\nu^{(A,B)}=2k$, $\nu^{(A)}=\nu^{(B)}=0$). At infinity there are two characteristic pairs $(2,2k+1)$, $(3,-1)$ but $\nu_\infty=0$.

(q) The curve is a section of the swallowtail with two simple cusps $\mathbf{A}_2$ away from the transversal self-intersection. Here $\operatorname{ext}\nu_1+\operatorname{ext}\nu_2=\sigma=2$.

(r) The curve is nonsingular, but there is a tangency of third order $\mathbf{A}_7$ at the self-intersection. Here $\operatorname{ext}\nu_0=\sigma=3$.

(s) There is a cusp $\mathbf{A}_4$ and a tangency $\mathbf{A}_3$ at the self-intersection. Here $\operatorname{ext}\nu_0+\operatorname{ext}\nu_1=\sigma=3$.

(t) There is an ordinary cusp $\mathbf{A}_2$ and a cusp $\mathbf{A}_4$ away from the transversal self-intersection. Here $\operatorname{ext}\nu_1+\operatorname{ext}\nu_2=\sigma=3$.

(u) There is a cusp $\mathbf{A}_6$ away from the transversal self-intersection. Here $\operatorname{ext}\nu_1=\sigma=3$.

We see that there are at most three finite singular points, the self-intersection included. We have also checked that each of these curves is rectifiable by means of a birational automorphism of $\mathbb{CP}^2$.

All the cases above are discrete; the formulas do not contain moduli (parameters). This means that these curves represent isolated points in the space $\mathrm{PCurv}=\big((\mathrm{Curv}/\mathbb{C})\setminus 0\big)/\mathbb{C}^*$. On the other hand, the contractible curve corresponds to the point $0\in\mathrm{Curv}/\mathbb{C}$.

The remaining part of this section is devoted to the proof of Theorem 4.1. This proof relies on reductions to some special cases, which are treated separately. We begin with the special cases.

**Lemma 4.2.** *If $\varphi=t^p$ or if $\psi=t^q$, we have one of the cases* (a)–(f) *of Theorem 4.1.*

*Proof.* We shall assume $\varphi=t^p$.

<u>Case $p=2$</u>. We can assume $\psi=t\tilde\psi(t^2)$. The conditions $\xi(t')=\xi(t)$ for the double point read as $t'=-t$, $t\tilde\psi(t^2)=0$. The latter equation can have only two nonzero solutions $t_0,-t_0$. Assuming $t_0=1$ we get the case (a).

Case $p = 3$. The preimages $t_0$, $t_0'$ of the double point satisfy $t_0' = \zeta t_0$, $\zeta = e^{2\pi i/3}$; (by rescaling $t$ we can assume that $t_0 = -\zeta$ and $t_0' = \bar{t}_0$). The equation $\psi(\zeta t) - \psi(t) = 0$ can have $t = t_0$ and $t = 0$ as the solutions. Therefore

$$\psi(\zeta t) - \psi(t) = (\zeta^q - 1)(t - t_0)^\alpha t^\beta.$$

If $\alpha > 1$, then $\psi(\zeta t) - \psi(t)$ would be a polynomial with at least three nonzero consecutive monomials. But $\psi \circ \zeta - \psi$ does not contain the terms $t^{3l}$. It follows that $\psi$ contains only $t^q$ and $t^{q-1}$ with $q(q-1) \neq 0 \pmod 3$; i.e. the case (b) of Theorem 4.1. (Here $\psi(\zeta t) - \psi(t) = (\zeta^{-1} - 1)t^{3k+2} - (\zeta - 1)t^{3k+1}$ and $t_0 = -\zeta$, as we supposed. Also the equation $\psi(\zeta^{-1} t) - \psi(t) = 0$ has the unique nonzero root $t = -\zeta^{-1}$.)

Case $p = 4$. The polynomial $\psi$ must contain at least one odd power of $t$ (by the nonprimitivity). For the preimages $t_0$, $t_0'$, $t_0 \neq t_0'$ of the double point we have two excluding one another possibilities: $t_0' = it_0$, or $t_0' = -t_0$. In the second case the polynomial $\psi(-t) - \psi(t)$ is as in the case $p = 2$, with at least two terms; but then $\psi(it) - \psi(t)$ would contain at least two terms too. The same argument shows also that there is at most one odd power of $t$ in $\psi$. Consider the polynomial $\psi(it) - \psi(t)$. It contains as many monomials as $\psi$. Moreover, it is of the form $(i^q - 1)(t - t_0)^\alpha t^\beta$ where $\alpha = 1$ or $\alpha = 2$ (because there are no terms $t^{4l}$). If $q = 4k + 2$, there can be only two terms $t^{4k+2}$ and $t^{4k+1}$ (the case (c) of Theorem 4.1). The case $q = 4k + 1$ does not occur, because the term $t^{4k}$ is absent. If $q = 4k + 3$ and $\alpha = 2$, there would be two odd powers. So $q = 4k + 3$ and $\alpha = 1$, i.e. the case (d) of Theorem 4.1. Note that the case (d) with $k = 0$ corresponds to $x = t^3 - t^2$, $y = t^4$ (when we require $p < q$).

Case $p = 6$. Here we have three possibilities: $t_0' = e^{i\pi/3} t_0$, $t_0' = e^{2\pi i/3} t_0$, or $t_0' = -t_0$. As in the case $p = 4$ we show that the equations $\psi(e^{2i\pi/3} t) - \psi(t) = 0$ and $\psi(-t) - \psi(t) = 0$ (related to the last two of these possibilities) have only one solution $t = 0$. It follows that $\psi$ contains exactly one term of the type $t^{2l+1}$ and exactly one term of the type $t^{3m\pm 1}$. Thus $\psi(e^{i\pi/3} t) - \psi(t)$ has only two consecutive terms, with powers divisible by 2 and by 3 respectively. These are the cases (e) and (f) of Theorem 4.1. Note that case (f) with $k = 0$ corresponds to $x = t^4 - t^3$, $y = t^6$ (when we keep $p < q$).

Cases $p = 5$ and $p > 6$. There are at least four primitive roots of unity of order $p$, say $\zeta_1, \zeta_1^{-1}, \zeta_2, \zeta_2^{-1}$. Repeating the preceding analysis, we find that the equations $\psi(\zeta_1 t) - \psi(t) = 0$ and $\psi(\zeta_2 t) - \psi(t)$ have nonzero solutions corresponding to two different double points. $\square$

**Lemma 4.3.** *If* $\varphi = t^\alpha (t-1)^\beta$, $\psi = t^\gamma (t-1)^\delta$, $\alpha\beta\gamma\delta \neq 0$, *then we have one of the cases* (g)–(i) (*modulo the changes* $t \to 1 - t$, $x \to \mathrm{const} \cdot y$).

*Proof.* The equations for a double point different from (0, 0) are

$$\left(\frac{t'}{t}\right)^{\alpha}\left(\frac{t'-1}{t-1}\right)^{\beta}=1,\quad \left(\frac{t'}{t}\right)^{\gamma}\left(\frac{t'-1}{t-1}\right)^{\delta}=1.$$

We assume that this system does not admit solutions such that $tt'(t-1)(t'-1)\neq 0$. Let $\alpha=\alpha_1\alpha'$, $\beta=\beta_1\beta'$, $\gamma=\gamma_1\alpha'$, $\delta=\delta_1\beta'$ where $\alpha'=\gcd(\alpha,\gamma)$ and $\beta'=\gcd(\beta,\delta)$. We arrive at the equations

$$\left(\frac{t'}{t}\right)^{\alpha'\kappa}=1,\quad \left(\frac{t'-1}{t-1}\right)^{\beta'\kappa}=1,\quad \kappa:=\alpha_1\delta_1-\beta_1\gamma_1.$$

If $\kappa=0$, the parametrization is nonprimitive. We assume then that $\kappa>0$.

We find that

$$\frac{t'}{t}=\zeta:=e^{2\pi ij/\alpha'\kappa},\quad \frac{t'-1}{t-1}=\eta=e^{2\pi ik/\beta'\kappa},$$

where $j\in\mathbb{Z}/\alpha'\kappa\mathbb{Z}$ and $k\in\mathbb{Z}/\beta'\kappa\mathbb{Z}$ satisfy

$$\alpha_1 j+\beta_1 k=0 \pmod{\kappa}; \tag{4-1}$$

(the other equation $\gamma_1 j+\delta_1 k=0 \pmod{\kappa}$ follows automatically).

There are no additional double points if and only if we have only one of the three possibilities: 1. $\zeta=1$, or 2. $\eta=1$, or 3. $\zeta=\eta$.

The case 1 occurs when $\alpha'\kappa=1$ and the case 2 occurs when $\beta'\kappa=1$. We find case (g) of Theorem 4.1.

If $\alpha'\kappa=\beta'\kappa=2$, then one of the cases 1–3 necessarily occurs; if $\zeta\neq 1,\eta\neq 1$, then $\zeta=\eta=-1$. Here we have either case (h) or case (i) with $\kappa=2$.

If $\alpha'\geq 2,\beta'\geq 3$ (or $\alpha'\geq 3,\beta'\geq 2$), then one can find an additional double point. Indeed, there exists a solution $(j,k)$ to the equation (4-1) such that $j\neq 0$ and $k\neq 0$. If this solution is such that $\zeta=\eta$, we can replace $\eta$ by $\eta e^{2\pi im/\beta'}\neq 1,\zeta$ for some $m\in\mathbb{Z}/\beta'\mathbb{Z}$; (then $k\to k+m\kappa$).

If $\kappa>1$ and $\alpha'=1,\beta'\geq 2$ (or $\alpha'\geq 2,\beta'=1$), then also there is a new double point. Indeed, if $\beta_1\neq 0 \pmod{\kappa}$, we can (if necessary) replace $\eta=\zeta$ by $\eta e^{2\pi im/\beta'\kappa}\neq 1,\zeta$ (see (4-1)). Otherwise we get $\alpha_1\neq 0 \pmod{\kappa}$ (because $\gcd(\alpha_1,\gamma_1)=1$ and $\kappa=\alpha_1\delta_1-\beta_1\gamma_1$), $j\neq 0 \pmod{\kappa}$ and $k\neq 0 \pmod{\kappa}$. So we have freedom in replacing $\eta$ by $\eta e^{2\pi m/\beta'}\neq 1,\zeta$.

Let $\alpha'=\beta'=1$ and $\kappa>1$. Then Equation (4-1) admits only solutions $j=k \pmod{\kappa}$ (for $j,k\neq 0 \pmod{\kappa}$); that is, the equation $\alpha j+\beta k=0 \pmod{\kappa}$ should take the form $\alpha(j-k)=0 \pmod{\kappa}$ with $\alpha$ relatively prime to $\kappa$. We can rewrite this case as follows: $\beta=m\kappa-\alpha$, $\delta=n\kappa-\gamma$, where $\alpha n-\gamma m=1$ and $\gcd(\alpha,\kappa)=\gcd(\gamma,\kappa)=1$, $\kappa>1$. So we get case (i) of Theorem 4.1 with $\kappa=k$, $m=\beta$ and $n=\delta$.

Since there are no other possibilities, the lemma is complete. □

(Note that, when $\kappa = 1$, the invertible rational change $(x, y) \to (x_1, y_1)$ gives the curve $x_1 = t^{\alpha'}$, $y_1 = (t-1)^{\beta'}$ with the number of double points given by $2\delta_{\max} = (\alpha'-1)(\beta'-1) - (\gcd(\alpha', \beta') - 1)$. One can check that $2\delta_{\max} = 0$ if and only if either $\alpha' = 1$, or $\beta' = 1$, which is case (g), or $\alpha' = \beta' = 2$, which is case (h).)

**Lemma 4.4.** *In cases* (j) *and* (k) *of Theorem 4.1 the curve has only one self-intersection point $x = y = 0$* (*where a smooth and singular local curves meet*) *and a simple cusp $\mathbf{A}_2$.*

*Proof.* Consider case (j). The change $x_1 = x$, $y_1 = y/x^k = t(t - \frac{4}{3})$ gives a curve without self-intersection corresponding to $t = 0$ and $t = 1$. But $\dot{x}_1 = 3t^2 - 2t$ and $\dot{y}_1 = 2t - \frac{4}{3}$ vanish at $t_1 = \frac{2}{3}$ (the cusp point). This means that the curve $(x_1, y_2)(\mathbb{C})$ is contractible.

In case (k) the curve $x_1 = x(t)$, $y_1 = y/x^k = t(t - \frac{3}{2})$ also has simple cusp at $t_1 = \frac{3}{4}$ and no other singularities. □

**Lemma 4.5.** *Suppose a curve $\varphi = (t^{\alpha'}(t-1)^{\beta'})^{\alpha_1} = t^{\alpha}(t-1)^{\beta}$, $\psi = (t^{\alpha'}(t-1)^{\beta'})^{\gamma_1}(t-b) = t^{\gamma}(t-1)^{\delta}(t-b)$, where $b \neq 0, 1$, $\gcd(\alpha_1, \gamma_1) = 1$ and $\gamma_1 < \alpha_1$, has only one self-intersection point. Then we have one of the following possibilities*:

- $\alpha_1 = 2$, $\gamma_1 = 1$, $b = \frac{1}{2}$ *and* $\alpha' = \beta'$;
- $\alpha_1 = 2\gamma_1 + 1$, $b = \frac{1}{2}$ *and* $\alpha' = \beta' = 1$;
- $\alpha_1 = 2$, $\gamma_1 = 1$, $b = \frac{1}{2}$ *and* $|\alpha' - \beta'| = 1$;
- $\alpha_1 = 3$, $\gamma_1 = 1$, $b = \frac{1}{2} + \frac{1}{6}\sqrt{-3}$ *and* $\alpha' = \beta' = 1$.

*This gives the cases* (l), (m), (n) *and* (p) *of Theorem 4.1.*

*Proof.* The condition for a double point implies that

$$(t')^{\alpha'}(t'-1)^{\beta'} = \eta t^{\alpha'}(t-1)^{\beta'}, \quad t' - b = \zeta(t-b), \quad \zeta = \eta^{-\gamma_1} \tag{4-2}$$

for some roots $\zeta, \eta$ of unity of order $\alpha_1$ (if $\gcd(\alpha', \beta') > 1$, then (4-2) can be replaced by several analogous systems). We can assume that $\zeta \neq 1$. This system can have only $(t, t') = (b, b)$, $(0, 1)$ and $(1, 0)$ as solutions, but it can have no solutions.

If $(b, b)$ is a solution, then $\eta = 1$ (we reject this possibility). If $(0, 1)$ (respectively $(1, 0)$) is a solution, then $b = 1/(1-\zeta)$ (respectively $b = \zeta/(\zeta - 1)$). If the both are solutions, then $\zeta = -1$ and $b = \frac{1}{2}$.

Suppose that (4-2) does not have solutions. We use the variables $\tau = t - b$, $\tau' = t' - b$ and the transformation $\sigma(\tau) = \zeta\tau$. Then (4-2) takes the form $\phi \circ \sigma(\tau) = \eta\phi(\tau)$, where

$$\phi(\tau) = (\tau + b)^{\alpha'}(\tau - 1 + b)^{\beta'}.$$

This equation does not have solutions if and only if $\phi \circ \sigma \equiv \eta\phi + d$, $d = \mathrm{const} \neq 0$. Differentiating it we find that $\phi' \circ \sigma \equiv \lambda\phi'$, $\lambda = \eta/\zeta$, i.e. $\phi'$ is a semiinvariant with the weight $\lambda$. It follows that $\phi'(\tau) = \tau^{\varepsilon}\chi(t^{\theta})$ and $\phi(\tau) = \tau^{\varepsilon+1}\tilde{\chi}(\tau^{\theta}) + e$, where $\theta$ is the order of the root $\zeta$, $\varepsilon$ is an integer, $\chi$, $\tilde{\chi}$ are some polynomials and $e \neq 0$. It is easy to see that it should be $\alpha' = \beta' = 1$ and $b = \frac{1}{2}$. Therefore the case $b = \frac{1}{2}$ must be treated separately.

Assume that $b = \frac{1}{2}$ and $\alpha' = \beta'$. Then we get the curve

$$(\varphi, \psi^2) = (z^{\alpha}, z^{2\gamma+1} + \tfrac{1}{4}z^{2\gamma}),$$

for $z = t(t-1)$. If $\alpha = 2\gamma + 1$ or $\alpha = 2\gamma$, then the latter curve does not have self-intersections (since we have assumed $\gamma < \alpha$, the possibilities that $2\gamma + 1$ or $2\gamma$ are multiples of $\alpha$ are excluded). Otherwise, we have the equations $(\zeta\tau)^2 - \frac{1}{4} = \eta'(\tau^2 - \frac{1}{4})$, $\eta'$ being a root of 1 of order $\alpha$ such that $\zeta = (\eta')^{-\gamma}$. There are two self-intersection points $(\tau, \tau') = (\pm\tau_0, \pm\zeta\tau_0)$, with $\tau_0^2 \neq 0, \frac{1}{4}$ given by $\big((\eta')^{2\gamma+1} - 1\big)\tau_0^2 = \frac{1}{4}(\eta')^{2\gamma}(\eta' - 1)$.

If $b = \frac{1}{2}$ and $\alpha' < \beta'$ (the case $\beta' < \alpha'$ is reduced to the previous one), then (4-2) must have a solution (for any $\eta$). So any $\zeta = -1$ (i.e. $\alpha_1 = 2$, $\gamma_1 = 1$) and we get the equation

$$t^{\alpha'}(t-1)^{\alpha'}\big((-1)^{\alpha'+\beta'}t^{\beta'-\alpha'} - \eta(t-1)^{\beta'-\alpha'}\big) = 0$$

We see that it should be $\beta' - \alpha' = 1$.

Let $b \neq \frac{1}{2}$. We can assume that $(0, 1)$ is the only solution to (4-2) (there is at least one). Then $t' = \zeta t + 1$, $t' - 1 = \zeta t$. It should be $\alpha' = \beta' = 1$ and we get the equation

$$(\zeta^2 - \eta)t + \zeta + \eta = 0.$$

Thus either $\eta = \zeta^2$, or $\eta = -\zeta$ (here $1 = \eta^{\alpha_1} = (-1)^{\alpha_1}$). If $(1, 0)$ is the only solution to (4-2), we also get the same possibilities.

If $\alpha_1$ is odd, we have only the case $\eta = \zeta^2$, and so $\eta^{2\gamma_1+1} = 1$, since $\zeta = \eta^{-\gamma_1}$. Since for any $\zeta$ either $b = 1/(1-\zeta)$ or $b = \zeta/(\zeta - 1)$, it should be $\alpha_1 = 3$, $\gamma_1 = 1$. Putting $\zeta = e^{2\pi i/3}$ we get $b = \frac{1}{2} + \frac{1}{6}\sqrt{-3}$.

If $\alpha_1$ even, an analogous argument shows that $\alpha_1 = 2$ and $b = \frac{1}{2}$, a contradiction. □

**Lemma 4.6.** *The curves from the item* (o) *of Theorem 4.1 have only one self-intersection point.*

*Proof.* Let $y_1 = y/x^l = t^{k-1}(t-1)^k(t - \frac{1}{2})$. Then $x/y_1^2 = t(t-1)/(t - \frac{1}{2})^{-2}$. We get $t' = 1 - t$ when $\xi(t') = \xi(t)$. Finally, $x(t')/x(t) = \big((t-1)/t\big)^2 = 1$ if and only if $t = t' = \frac{1}{2}$.

Note that the curve (o) for $l = 0$ is obtained from the curve (n) for $l = 0$ by applying the transformation $x_1 = y^2 - \frac{1}{4}x$ (and vice versa). Similarly, the curve (m) for $l = 0$ is obtained from the curve (l) for $l = 0$ via the same transformation (and vice versa). □

**Lemma 4.7.** *If $p = 3$, $q = 4$ and $\varphi \neq t^p$, we have one of the cases* (q), (d) *with* $k = 0$, (l) *with* $k = 1$, $l = 0$, *or* (g) *with* $k = 0$. *These cases correspond to four possible degenerations of the* $\delta_{\max} - 1 = 2$ *double points* (*of curves from* $\mathrm{Curv}_{3,4}$): *two cusps* $\mathbf{A}_1 + 2\mathbf{A}_2$, *or a cusp of order four* $\mathbf{A}_1 + \mathbf{A}_4$, *or a tangency of order two* $\mathbf{A}_5$, *or a cusp at the self-intersection point* $\mathbf{D}_5$.

*Proof.* We first transform the double point condition. We can assume that

$$\varphi = t^3 - 3t, \quad \psi = t^4 - 2at^2 - 4bt;$$

note that the change $t \to -t$, $x \to -x$ allows one to make the change $b \to -b$. We put $t + t' = u$, $tt' = v$ and get $(u^2 - v) - 3 = 0$, $(u^3 - 2uv) - 2au - 4b = 0$, that is,

$$f(u) := u^3 + (2a - 6)u + 4b = 0. \tag{4-3}$$

If there is a cusp, which can occur at $t = 1$ or at $t = -1$ (since $\varphi' = 3(t^2 - 1)$), then $t' = t = 1$ (or $t' = t = -1$ respectively) represents a "degenerate" double point. In this case $u = 2$ (respectively $u = -2$) is a solution to (4-3). If $u = -1$ is a solution to (4-3), we have $t = 1$, $t' = -2$; if, additionally, $u = 2$ is another solution to (4-3), then the cusp lies at the self-intersection point; this may cause that the "order of tangency" grows. All other solutions correspond to intersection points of local smooth branches.

Suppose we have two cusps. Then $f$ should take the form $f = (u - u_0)(u^2 - 4) = u^3 - u_0u^2 - 4u + 4u_0$. Therefore $u_0 = 0$, $a = 1$ and $b = 0$, which is case (q) of Theorem 4.1.

Suppose we have the cusp of multiplicity four $\mathbf{A}_4$. Then $f = (u - u_0)(u - 2)^2 = u^3 - (4 + u_0)u^2 + (4 + 4u_0)u - 4u_0$, and hence $u_0 = -4$ and $y = t^4 + 6t^2 - 16t$. But further calculations show that $y - 4x + 1 = (t - 1)^4$ and $x + 2 = (t - 1)^3 + 3(t - 1)^2$ — that is, case (d) with $k = 0$.

Suppose two local components have a tangency of order two. Then $f = (u - u_0)^3$. One finds $u_0 = 0$ (which is different from $\pm 2$), $b = 0$ and $a = 3$, $x = (t^2 - 3)t$, $y + 9 = (t^2 - 3)^2$. After the change $t \to (t + \sqrt{3})/2\sqrt{3}$ one gets case (l) of Theorem 4.1 with $k = 1$, $l = 0$.

Suppose the tangency occurs and that there is a cusp, which we may assume to occur for $t = t' = 1$. Then we have $f = (u - u_0)^2(u - 2) = u^3 - (2 + 2u_0)u^2 + (u_0^2 + 4u_0)u - 2u_0^2$. Thus $u_0 = -1$, $a = \frac{3}{2}$ and $b = -\frac{1}{2}$. Here $y = t^4 - 3t^2 + 2t$ and one checks that $x + 2 = (t - 1)^2(t + 2)$ and $y + 2x + 4 = (t - 1)^2(t + 2)^2$ — a subcase of (g). □

**Lemma 4.8.** *If $p = 3$, $q = 5$ and $\varphi \neq t^p$, we have cases* (r)–(u), (g) *or* (j).

*Proof.* We have $\varphi = t^3 - 3t$, $\psi = t^5 - at^4 - bt^2 - ct$, with possible symmetry $a \to -a$, $b \to -b$ (induced by $t \to -t$, $x \to -x$, $y \to -y$). The analogue of equation (4-3) is $f(u)=u^4-au^3-3u^2+(b+6a)u+(c-9)=0$. Further analysis goes as in the proof of Lemma 4.7. □

Now we turn to the bounds. This is the most complicated (and boring) part of the proof of Theorem 4.1. So we do not provide all the details and we present only main steps of the estimates.

Assume that $\xi\in\mathrm{Curv}_{p,q}$ (where $p<q$ and $q/p\notin\mathbb{Z}$) is such that $\xi(t_0^{(A)})=\xi(t_0^{(B)})$ (where $t_0^{(A)}\neq t_0^{(B)}$) is an intersection of local branches $A$ and $B$ of the curve and that there are singular points $t_j$, $j=1,\dots N$. Each of these singular points is characterized by the number $\delta_{t_j}=\delta_{\xi(t_j)}(C)$ of its double points, i.e. the number of double points vanishing at $\xi(t_j)$ (or the $\delta$-invariants). Moreover, $\delta_\infty=\delta_\infty(C)$ double points has escaped to infinity. We have

$$\delta_\infty+\sum\delta_{t_j}=\delta_{\max}. \tag{4-4}$$

The irreducible singularities are known; they have their external codimensions $\mathrm{ext}\,\nu_j=n_j+\nu_j-2$ and the bounds $2\delta_{t_j}\leq n_j\nu_j$, $j=1,\dots,N$. The degeneration at infinity has the external codimension $\mathrm{ext}\,\nu_\infty=\nu_\infty$ and the bound $2\delta_\infty\leq n_\infty\nu_\infty$, where $n_\infty=0$ if $p'=\gcd(p,q)=1$ and $=p'$ otherwise.

Near the self-intersection we have the expansions

$$\begin{aligned} x&=x_0+(t-t_0^{(A)})^{n_0^{(A)}}(a_0^{(A)}+\cdots), & y&=y_0+b_1^{(A)}(t-t_0^{(A)})+\cdots,\\ x&=x_0+(t-t_0^{(B)})^{n_0^{(B)}}(a_0^{(B)}+\cdots), & y&=y_0+b_1^{(B)}(t-t_0^{(B)})+\cdots, \end{aligned}$$

with $n_0^{(A)},n_0^{(B)}\geq 1$. The external codimension equals $\mathrm{ext}\,\nu_0=n_0^{(A)}+n_0^{(B)}-2+\nu_0$, where $\nu_0=\nu(A+B)$ is defined in Lemma 2.14. We have the bound $2\delta_0\leq n_0(\nu_0+1)$, where $n_0=n_0^{(A)}+n_0^{(B)}$ (see Proposition 2.16).

Assuming Conjecture 3.7 we have the bound

$$\sum_{j=0}^{N} ext\nu_j+\mathrm{ext}\,\nu_\infty\leq\sigma=p+q-4-[q/p].$$

By the lemmas above we can assume that

$$\begin{cases} 2\leq n_0\leq p;\\ 2\leq n_j\leq p-1,\ \ \nu_j\geq 1 \text{ for } j\geq 1;\\ \sum_0^N n_j\leq p+N+1;\\ p\geq 3;\\ [q/p]\geq 2 \text{ if } p=3. \end{cases} \tag{4-5}$$

***The extremum problem.*** Define

$$\Xi := n_0(1+\nu_0) + \sum_1^N n_i\nu_i + n_\infty\nu_\infty,$$

where we assume the condition

$$\sum_0^N (n_i+\nu_i-2)+\nu_\infty \le \sigma$$

and also the restrictions (4-5) are imposed. The following lemma is rather easy.

**Lemma 4.9.** (a) *Suppose $n_\infty = p' \ge n_i$, for $i=0,\dots,N$. Then $\Xi$ is maximal when $N=0$, $n_0=2$, $\nu_0=0$ and $\nu_\infty=\sigma$ is maximal; thus*

$$\Xi = 2+p'\sigma, \quad p'>2.$$

(b) *If $n := \max\{n_0,\dots,n_N\} > n_\infty$ and at least two of the $n_i$'s are equal $n$, then $\Xi$ is maximal when $\nu_\infty=0$, $N=1$, $n_0=n_1=n$ and $\nu_0+\nu_1$ is maximal; here*

$$\Xi = n(\sigma+5-2n), \quad n\le \min\{p-1,(p+3)/2\}$$

(see (4-5)).

(c) *If $n := \max\{n_0,\dots,n_N\} > n_\infty$, then $\Xi$ is maximal when $\nu_\infty=0$, only one of the $n_i$'s equals $n$ (if $n>2$) with $\nu_i$ maximal, and all other $n_j=2$ with $\nu_j=1$. Moreover,*

$$\Xi = n(\sigma+2-n-N)+2N$$

*when $n_0\ne n$.*

***Estimates for $\Delta := 2\delta_{\max} - \Xi$.*** We focus attention on some exemplary calculations.

Recall that $2\delta_{\max} = (p-1)(q-1)-(p'-1) = pq-p-q-p'+2$ and $\sigma = p+q-4-[q/p] \le p+q-5$.

In case (a) of Lemma 4.9 we have

$$\Delta \ge (p')^2(p_1q_1-p_1-q_1)-p'(p_1+q_1-4) > p'\,[2(p_1-3/2)(q_1-3/2)-1/2]>0.$$

In case (b) the quadratic function $\Xi=\Xi(n)$ takes its maximal value at $n_* := \frac14(\sigma+5)$ and $\Xi|_{n=n_*} = \frac18(\sigma+5) \le \frac18(p+q)$. Therefore

$$\Delta \ge \left(p-\tfrac98\right)\left(q-\tfrac{17}{8}\right)+\left(p-p'-\tfrac{25}{64}\right)>0.$$

In case (c) of Lemma 4.9 similar (but more involved) calculations lead to the following result:

**Lemma 4.10.** *To complete the proof of Theorem 4.1 it is enough to consider curves of the form*

$$\varphi(t) = t^{\alpha}(1-t)^{\beta}P(t), \quad \psi(t) = t^{\gamma}(1-t)^{\delta}Q(t);$$

*that is*, (3-4) *with* $\deg P = 0, 1$, $\deg P + \deg Q > 0$, *with at most two singular points*: *self-intersection* $(0,0)$ *and a simple cusp* $\xi(t_1)$ $(n_1 = 2,\ \nu_1 = 1)$, *with* $\nu_\infty = 0, 1$, $\nu_1 + \nu_\infty \le 1$ *and satisfying* (4-5). *Moreover*, *when* $\deg P = 1$ *one can assume* $[q/p] = 1$ *and only one singular point* $(0,0)$.

Recall the notations introduced in (3-5) and (3-6). The external codimensions $\operatorname{ext}\nu_0$, $\operatorname{ext}\nu_1$ (for $t_1$) and $\operatorname{ext}\nu_\infty$ are defined as in Section 3. We recall from Lemma 3.9 the bound

(4-6) $\operatorname{ext}\nu_0 + \operatorname{ext}\nu_1 + \operatorname{ext}\nu_\infty \le \deg P + \deg Q + [\gamma - 1/(\alpha)] + [(\delta-1)/\beta] - [q/p]$.

***The case $\kappa = 0$.*** We have

$$\varphi = [t^{\alpha'}(1-t)^{\beta'}]^{\alpha_1}P(t), \quad \psi = \left(t^{\alpha'}(1-t)^{\beta'}\right)^{\gamma_1}Q(t)$$

and we estimate the quantity $\Xi = 2\delta_0 + 2\delta_{t_1} + 2\delta_\infty$.

By Proposition 2.17, twice the number of double points at $(0,0)$ is estimated by

(4-7) $\quad 2\delta_0 \le (\alpha+\beta-1)(\gamma+\delta-1) + (\alpha'+\beta'-1) + (\alpha'+\beta')(1+\operatorname{ext}\nu_0).$

If the system of equations defining a corresponding stratum in Curv consists only of equations for vanishing of Puiseux quantities (not self-intersection quantities), that is, if $\operatorname{ext} u = \operatorname{ext}\nu^{(A,B)} = 0$, we have

(4-8)

$2\delta_0 \le (\alpha+\beta-1)(\gamma+\delta-1) + (\alpha'+\beta'-1) - \alpha'\beta'|\kappa| + \alpha'\cdot\operatorname{ext}\nu(A) + \beta'\cdot\operatorname{ext}\nu(B),$

following from Proposition 2.18.

<u>Case $\deg P = 1$</u>. The subcase $\gamma_1 < \alpha_1$, after calculations, leads to $\Delta = 2\delta_{\max} - \Xi > 0$ (a contradiction). When $\gamma_1 = \alpha_1$ a change $y \to y + \text{const}\cdot x$ leads to another case: either $\kappa \ne 0$ or $\kappa = 0$ and $\gamma_1/\alpha_1 > 1$.

If $\gamma_1/\alpha_1 > 1$ and $\deg Q = 0$, we use Lemma 4.5. If $\gamma_1/\alpha_1 > 1$ and $\deg Q \ge 1$ then one proves that $\Delta > 0$; here the point (c) of Lemma 3.9 is essentially used.

<u>Case $\deg P = 0$</u>. If $\gamma_1 < \alpha_1$, then one shows that $\Delta > 0$. The subcase $\gamma_1/\alpha_1 = 1$ is destroyed by the change $y \to y + \text{const}\cdot x$.

When $\gamma_1/\alpha_1 > 1$ we can make the change $y \to y_1 = y/x^{[\gamma_1/\alpha_1]}$ which leads us to another cases (with lower degrees). These are the cases (l) and (m) and (p) of Theorem 4.1.

Note however that it may happen (and happens) that $\deg y_1$ divides $\deg x$. Then one must apply additional elementary transformation $x \to x_1 = x + R(y_1)$; (maybe several elementary transformations should be applied). Anyway, this should lead to one of the cases from Theorem 4.1 (see also analogous conclusion in the end of the next point). Analysis of the list in Theorem 4.1 shows the only possible such changes are the following: (l) for $l = 1 \to$ (m) for $l = 0$, (m) for $l = 0 \to$ (l) for $l = 0$, (n) for $l = 0 \to$ (o) for $l = 0$ (see also the proof of Lemma 4.6).

***The case $|\kappa| > 0$.*** Here the intersection index of two local components $A$ and $B$ at $(0, 0)$ is fixed and equal $\min(\alpha\delta, \beta\gamma) = \frac{1}{2}(\alpha\delta + \beta\gamma - \alpha'\beta'|\kappa|)$. The two local branches, at $t = 0$ and at $t = 1$, are characterized by their external codimensions $\operatorname{ext}\nu(A)$ and $\operatorname{ext}\nu(B)$ in the spaces of germs of curves with fixed leading terms. The singularity at $t_1$ has external codimension $\nu_1 = 0, 1$.

By (4-6) we have the general estimate $\operatorname{ext}\nu(A) + \operatorname{ext}\nu(B) + \nu_1 + \nu_\infty \le \deg P + \deg Q + [(\gamma - 1)/\alpha] + [(\delta - 1)/\beta] - [q/p]$. The numbers of double points at $x = y = 0$ (i.e. $\delta_0$) and at $t_1 \ne 0, 1$ (i.e. $\delta_{t_1}$) and at $\infty$ (i.e. $\delta_\infty$) are estimated in (4-8), by $2\delta_{t_1} \le 2\nu_1$ and by $2\delta_\infty \le p'\nu_\infty$ respectively.

If $\alpha' = \beta' = 1$, then estimates leave only three possibilities the inequality $\Delta \le 0$. These are the cases (j), (k) and (o) of Theorem 4.1.

If $\max(\alpha', \beta') \ge 2$ and $\deg P = 1$, then one shows that always $\Delta > 0$. Here the point (b) of Lemma 3.9 is essentially used.

If $\max(\alpha', \beta') \ge 2$ and $\deg P = 0$, the estimates lead either to the cases considered in Lemma 4.3, or to the case (o) of Theorem 4.1, or to the following situation.

We have $\alpha_1 < \gamma_1$ and $\beta_1 < \delta_1$ and the change $(x, y) \to (x, y/x) = (x, y_1)$ leads to a curve of analogous form, but with lower degrees (possibly after interchanging $x$ and $y$). We use induction and obtain cases (j), (k) and (o) of Theorem 4.1.

As in the case with $\kappa = 0$ and $\deg P = 0$ (see page 335) it can (and does) happen that $\deg y_1$ divides $\deg x$. Then an additional change should be done in order to remove the divisibility of the degrees. Inverse to such change should start from one of the cases from Theorem 4.1 and should lead to a product of powers of $t$ and of $t - 1$. It is not difficult to check that the only possible reductions are (o) for $l = 0$ and (n) for $l = 0$. No new cases are obtained in this way.

***Conclusion.*** Theorem 4.1 follows from Lemmas 4.2–4.10 and the subsequent case analysis. □

**Acknowledgements**

We thank Stepan Orevkov for his comments and for drawing our attention to the Bogomolov–Miyaoka–Yau inequality. We thank also Mariusz Koras for his interest in our work.

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Received July 18, 2005.

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