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Let A be a unital separable amenable C^*-algebra and let C be a unital C^*-algebra with a certain infinite property. We show that two full monomorphisms $h_1, h_2 : A \to C$ are approximately unitarily equivalent if and only if $[h_1] = [h_2]$ in $KL(A, C)$. Let B be a nonunital but σ-unital C^*-algebra for which $M(B)/B$ has a certain infinite property. We prove that two full essential extensions are approximately unitarily equivalent if and only if they induce the same element in $KL(A, M(B)/B)$. The set of approximately unitarily equivalence classes of full essential extensions forms a group. If A satisfies the Universal Coefficient Theorem, the group can be identified with $KL(A, M(B)/B)$.

1. Introduction

The study of C^*-algebra extensions originated in the study of essentially normal operators on the infinite-dimensional separable Hilbert space. The original Brown–Douglas–Fillmore theory gives a classification of essential normal operators via certain Fredholm related indices (see [Brown et al. 1973b]). Later the theory was expanded to yield a classification of essential extensions of $C(X)$ by compact operators [Brown et al. 1973a; Brown 1984]. The study of C^*-algebra extensions developed into Kasparov's KK-theory and its applications can be found not only in operator theory and operator algebras but also in differential geometry and noncommutative geometry.

Let

$$0 \to B \to E \to A \to 0$$

be an essential extension of A by B. The extension is determined by a monomorphism $\tau : A \to M(B)/B$, the Busby invariant. When B is a σ-unital stable C^*-algebra, $KK^1(A, B)$ gives a complete classification of such essential extensions, up to stable unitary equivalence. However, $KK^1(A, B)$ gives little information, if any, about unitary equivalence classes of these mentioned extensions when $B \neq \mathcal{K}$ in

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general. There are examples in which $KK^1(A, B) = \{0\}$ but inequivalent nontrivial extensions exist [Lin 1998, Example 0.6]. There are also examples in which there are infinitely many inequivalent classes of trivial extensions [Lin 1995b, 7.4 and 7.5]. When B is not stable, $KK^1(A, B)$ certainly should not be used to understand unitary equivalence classes of the essential extensions mentioned above.

There are a number of results in classification of essential extensions (up to unitary equivalence or approximate unitary equivalence) when $B \neq \mathcal{K}$. Kirchberg's results [1996] on extensions in which B is a nonunital purely infinite simple C^*-algebra show that $KK^1(A, B)$ can be used to compute unitary equivalence classes of those extensions. If B is a nonunital but σ-unital simple C^*-algebra with continuous scale (see item (6) on page 391), then $M(B)/B$ is simple. A classification of essential extensions of a separable amenable C^*-algebra A by B (up to approximate unitary equivalence) was obtained in [Lin 2005b] (for some special cases in which $A = C(X)$, a classification up to unitary equivalence was obtained in [Lin 1995a; 1995b; 1997]. In this case, B may not be stable, so $KK^1(A, B)$ is not used as an invariant for essential extensions. Results about extensions of AF-algebras can be found in [Brown and Elliott 1982; Goodearl and Handelman 1982; Elliott and Handelman 1989].

Here we study full essential extensions. These are essential extensions $\tau : A \to M(B)/B$ such that $\tau(a)$ is a full element for each nonzero element $a \in A$. Since the Calkin algebra $M(\mathcal{K})/\mathcal{K}$ is simple, all essential extensions by $\mathcal{K}$ are full. If B is a nonunital but σ-unital purely infinite simple C^*-algebra, $M(B)/B$ is also simple. Therefore essential extensions by those C^*-algebras are all full. The homogeneous extensions of A by $C(X) \otimes \mathcal{K}$ studied by Pimsner, Popa and Voiculescu [Pimsner et al. 1979; 1980] are all full extensions. In these three cases, B is stable. There are nonstable, nonunital but σ-unital C^*-algebras that have continuous scale. In that case essential extensions by these C^*-algebras are also full. Furthermore, if A is a unital simple C^*-algebra and if the monomorphism $\tau : A \to M(B)/B$ is unital, then the essential extension induced by τ is always full for any nonunital C^*-algebra B.

With a technical condition on $M(B)/B$, we show that two full essential extensions are approximately unitarily equivalent if they induce the same element in $KL(A, M(B)/B)$ (see Theorem 2.5), provided that A is amenable and separable. When A is assumed to satisfy the (Approximate) Universal Coefficient Theorem, we show that there is a bijective correspondence between approximate unitary equivalence classes of essential and full extensions and $KL(A, M(B)/B)$. The advantage of studying these full extensions is that full extensions (in these cases) are "approximately absorbing". For stable B, we show that $KK^1(A, B)$ classifies the unitary equivalence classes of full essential extensions. In this case, full extensions are "purely large" in the sense of [Elliott and Kucerovsky 2001].

Section 2 describes the main results in this paper and introduces the technical conditions P1, P2 and P3. Section 3 shows that the corona algebras $M(B)/B$ of many stable C^*-algebras satisfy these conditions. In Section 4, we show that there are examples of nonstable, nonunital and σ-unital C^*-algebras B for which $M(B)/B$ has properties P1, P2 and P3. In Section 5 we give a few modified versions of some known results concerning amenable contractive completely positive linear maps. Section 6 discusses certain commutants in the ultrapower of corona algebras. In Section 7 we prove Theorem 2.5, mentioned above. In Section 8 we prove other main results described in Section 2.

Conventions and definitions. (1) Ideals are always closed and two-sided.

(2) Let A be a C^*-algebra and let $p, q \in A$ be two projections. We write $p \sim q$ if there exists $v \in A$ such that $v^*v = p$ and $vv^* = q$.

(3) Let A and B be C^*-algebras and let $L_1, L_2 : A \to B$ be linear maps. Take $\mathcal{F} \subset A$ and $\varepsilon > 0$. We write $L_1 \sim_\varepsilon L_2$ on $\mathcal{F}$ if

$$\|L_1(a) - L_2(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

(4) Let A and B be C^*-algebras. A contractive completely positive linear map $L : A \to B$ is said to *amenable* if for $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists an integer $n > 0$ and two contractive completely positive linear maps $\phi : A \to M_n$ and $\psi : M_n \to A$ such that

$$\psi \circ \phi \sim_\varepsilon L \quad \text{on } \mathcal{F}.$$

(5) A C^*-algebra A is said to be *amenable* (or nuclear) if id_A is amenable.

(6) Let B be a nonunital but σ-unital simple C^*-algebra. B is said to have continuous scale if there exists an approximate identity $\{e_n\}$ of B with $e_{n+1}e_n = e_n$ such that, for each nonzero element $b \in B$, there exists an integer $n > 0$ for which $e_{n+m} - e_n \lesssim b$ for all m; see [Lin 2004b].

Let $e \in B$ be a nonzero projection and $T_e(B)$ the set of all traces t on B for which $t(e) = 1$. Let B be a separable nonunital simple C^*-algebra with real rank 0, stable rank 1 and weakly unperforatated $K_0(B)$. If $\sup_n\{t(e_n)\}$ is a continuous function on $T_e(B)$, then B has continuous scale.

(7) Let $\{A_n\}$ be a sequence of C^*-algebras. Denote by $c_0(\{A_n\})$ the C^* direct sum of $\{A_n\}$ and by $l^\infty(\{A_n\})$ the C^*- product of $\{A_n\}$. Let $q_\infty(\{A_n\})$ be the quotient

$$q_\infty(\{A_n\}) = l^\infty(\{A_n\})/c_0(\{A_n\}).$$

When $A = A_n$ for all n, we write $c_0(A)$, $l^\infty(A)$ and $q_\infty(A)$ for simplicity.

(8) For each integer $n > 0$, define $f_n \in C_0((0, \infty))$ by

$$f_n(t) = \begin{cases} 1 & \text{if } t \geq 1/n; \\ \text{linear} & \text{if } 1/(n+1) \leq t < 1/n; \\ 0 & \text{if } 0 \leq t < 1/(n+1). \end{cases} \tag{1-1}$$

(9) An element a in a C^*-algebra A is said be *full* if the ideal generated by a is A itself. Let A and B be C^*-algebras and let $h : A \to B$ be a monomorphism. The monomorphism h is said to be *full* if $h(a)$ is full for every nonzero $a \in A$.

(10) Let $a \in A_+$ be a nonzero element, we write $\mathrm{Her}(a)$ for the hereditary C^*-subalgebra $\overline{aAa}$ generated by a.

2. Main results

Property P1. Let B be a unital C^*-algebra. We say that B has property P1 if for every full element $b \in B$ there exist $x, y \in B$ such that $xby = 1$. If b is positive, it is easy to see that $xby = 1$ implies that there is $z \in B$ such that $z^*bz = 1$.

It is obvious that an element b is full if and only if b^*b is full. It follows that B has property P1 if and only if for every full element $0 \leq b \leq 1$, there exists $x \in B$ such that $x^*bx = 1$.

Every unital purely infinite simple C^*-algebra has property P1.

Many other unital C^*-algebras have property P1. Let A be a unital C^*-algebra and $B = A \otimes \mathcal{K}$. In the next section we will show that $M(B)$ and $M(B)/B$ have property P1 for many such B. In Section 3 we will show that, for some nonstable (but σ-unital) C^*-algebra C, $M(C)$ and $M(C)/C$ can also have property P1.

Property P2. Let B be a unital C^*-algebra. We say that B has property P2 if 1 is proper infinite, that is, if there is a projection $p \neq 1$ and partial isometries $w_1, w_2 \in B$ such that $w_1^*w_1 = 1$, $w_1w_1^* = p$, $w_2^*w_2 = 1$ and $w_2w_2^* \leq 1 - p$.

In this case it is easy to see that, for each integer $n \geq 2$, there are mutually orthogonal and mutually equivalent projections $s_{11}, s_{22}, \ldots, s_{nn}$ such that $1_B \geq \sum_{i=1}^n s_{ii}$ and there exists an isometry $Z \in B$ such that $Z^*Z = 1_B$ and $ZZ^* = s_{11}$. Let $C = s_{11}Bs_{11}$. Then we may write $M_n(C) \subset B$.

It is clear that if B is stable then $M(B)$ and $M(B)/B$ have property P2.

Proposition 2.1. *Let B be a unital C^*-algebra having property P1. If B contains two mutually orthogonal full elements, B has property P2.*

Proof. Let $0 \leq a, b \leq 1$ be two mutually orthogonal full elements in B. Since B has property P1, there are $x, y \in B$ such that $x^*ax = 1$ and $y^*by = 1$. Set $v_1 = a^{1/2}x$ and $v_2 = b^{1/2}y$. Then $v_i^*v_i = 1$ and $s_{11} = v_1v_1^*$ and $s_{22} = v_2v_2^*$ are two projections. Thus B has property P2. □

Every purely infinite C^*-algebra (not necessary simple; see [Kirchberg and Rørdam 2000]) has properties P1 and P2.

Property P3. Let B be a unital C^*-algebra. We say that B has property P3 if, for any separable C^*-subalgebra $A \subset B$, there exists a sequence of sequences of elements $\{\{a_n^{(i)}\} : i = 1, 2, \dots\}$ in B satisfying these properties:

(a) $0 \le a_n^{(i)} \le 1$ for all i and n;

(b) $\lim_{n\to\infty} \|a_n^{(i)} c - c a_n^{(i)}\| = 0$ for all i and all $c \in A$;

(c) $\lim_{n\to\infty} \|a_n^{(i)} a_n^{(j)}\| = 0$ if $i \ne j$;

(d) $\{a_n^{(i)}\}$ is a full element in $l^\infty(B)$, for all i.

Even though property P3 looks more complicated than P1 and P2, it will be shown in Proposition 3.13 that $M(B)/B$ has property P3 for all $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra, for all B having continuous scale, and for many other nonunital σ-unital C^*-algebras B.

Proposition 2.2. *Let $B = C \otimes C_1$, where C_1 is a unital separable amenable purely infinite simple C^*-algebra. Then B has properties P1, P2 and P3.*

Let B be a nonunital but σ-unital C^*-algebra and let A be a unital separable amenable C^*-algebra. We study essential extensions of the following form:

$$0 \to B \to E \to A \to 0. \tag{2-1}$$

Using the Busby invariant, we study monomorphisms $\tau : A \to M(B)/B$. We will only consider the case in which the corona algebra $M(B)/B$ has properties P1, P2 and P3.

Definition 2.3. An essential extension $\tau : A \to M(B)/B$ is said to be *full* if τ is a full monomorphism. An extension τ is *weakly unital* if τ is unital monomorphism. If A is a unital simple C^*-algebra then every weakly unital essential extension is full. If $M(B)/B$ is simple, every essential extension is full.

Definition 2.4. Let A be a unital separable C^*-algebra and C a unital C^*-algebra. Let $h_1, h_2 : A \to C$ be homomorphisms. We say h_1 and h_2 are *approximately unitarily equivalent* if there exists a sequence of partial isometries $u_n \in C$ such that $u_n^* h_1(1_A) u_n = h_2(1_A)$, $u_n h_2(1_A) u_n^* = h_1(1_A)$ and

$$\lim_{n\to\infty} \|\operatorname{ad} u_n \circ h_1(a) - h_2(a)\| = 0 \quad \text{for all } a \in A.$$

Note that if both h_1 and h_2 are unital, the u_n can be chosen to be unitaries.

Let B be a nonunital but σ-unital C^*-algebra. Two essential extensions of A by B are said to be approximately unitarily equivalent if the corresponding Busby invariants $\tau_1, \tau_2 : A \to M(B)/B$ are approximately unitarily equivalent.

Recall that $\tau : A \to M(B)/B$ is trivial if there is a monomorphism $h : A \to M(B)$ such that $\pi \circ h = \tau$, where $\pi : M(B) \to M(B)/B$ is the quotient map. In the case that $B = C \otimes \mathcal{K}$, where C is a σ-unital C^*-algebra, the invariants τ_1 and τ_2 are *stably* unitarily equivalent if there exists a trivial extension $\tau_0 : A \to M(B)/B$ and a unitary $u \in M_2(M(B)/B)$ such that $\operatorname{ad} u \circ (\tau_1 \oplus \tau_0) = \tau_2 \oplus \tau_0$.

Let $\mathbf{Ext}(A, B)$ be the set of *stable* unitary equivalence classes of extensions of the form (2-1). When A is a separable amenable C^*-algebra, $\mathbf{Ext}(A, B)$ can be identified with $KK^1(A, B)$. When A satisfies the Universal Coefficient Theorem, $KK^1(A, B)$ is computable. However, as mentioned in the introduction, $KK^1(A, B)$ may not provide any useful information about unitary equivalence of extensions in general. In particular, when B is not stable, $KK^1(A, B)$ should not be used to describe unitary equivalence classes of essential extensions.

The first main result of this paper is the following:

Theorem 2.5. *Let A be a unital separable amenable C^*-algebra and let B be a nonunital but σ-unital C^*-algebra such that $M(B)/B$ has properties P1, P2 and P3. Two full monomorphisms $\tau_1, \tau_2 : A \to M(B)/B$ are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \quad \text{in } KL(A, M(B)/B).$$

We will describe $KL(A, C)$ in Definition 7.1. Theorem 2.5 is an easy corollary of the next theorem.

Theorem 2.6. *Let A be a unital separable amenable C^*-algebra and let B be a unital C^*-algebra having properties P1, P2 and P3. Two full monomorphisms $h_1, h_2 : A \to B$ are approximately unitarily equivalent, i.e., there exists a sequence of partial isometries $u_n \in B$ such that $u_n^* u_n = h_1(1_A)$, $u_n u_n^* = h_2(1_A)$ and*

$$\lim_{n\to\infty} \operatorname{ad} u_n \circ h_1(a) = h_2(a) \quad \text{for all } a \in A,$$

if and only if $[h_1] = [h_2]$ in $KL(A, B)$.

Corollary 2.7. *Let A be a unital separable amenable simple C^*-algebra and B a nonunital but σ-unital C^*-algebra such that $M(B)/B$ has properties P1, P2 and P3. Suppose that $\tau_1, \tau_2 : A \to M(B)/B$ are two weakly unital essential extensions. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \quad \text{in } KL(A, M(B)/B).$$

Definition 2.8. Let A be a unital separable amenable C^*-algebra and B a unital C^*-algebra having property P2. Fix a full monomorphism $j_o : A \to \mathcal{O}_2 \to B$. (Note that P2 implies such full monomorphisms do exist.) Let $h_1, h_2 : A \to B \otimes \mathcal{K}$ be homomorphisms. We write $h_1 \sim h_2$ if $h_1 \oplus j_o$ is approximately unitarily equivalent to $h_2 \oplus j_o$. Denote by $H(A, B)$ the set of $\sim$-equivalence classes of homomorphisms $A \to B \otimes \mathcal{K}$.

Proposition 2.9. *Let A be a unital separable amenable C^*-algebra and B a unital C^*-algebra having property P2. Then $H(A, B)$ is a group with zero element $[j_o]$.*

Corollary 2.10. *Let A be a unital separable amenable C^*-algebra and B a unital C^*-algebra having properties P1, P2 and P3. Denote by $H_f(A, B)$ the set of approximate unitary equivalence classes of full monomorphisms from A to $B \otimes \mathcal{K}$. Then $H_f(A, B)$ is a group with zero element $[j_o]$.*

Definition 2.11. Let A be a unital separable amenable C^*-algebra and B a nonunital but σ-unital C^*-algebra. Denote by $\mathbf{Ext}_{ap}^f(A, B)$ the approximate unitary equivalence classes of full essential extensions. Denote by $\tau_o : A \to M(B)/B$ an essential extension that factors through $\mathcal{O}_2$. Note that $[\tau_o] = 0$ in $KL(A, M(B)/B)$. Suppose that $M(B)/B$ has properties P1, P2 and P3. By Corollary 2.7, τ_o is unique up to approximately unitary equivalence, Let $\tau_1, \tau_2 : A \to M(B)/B$ be full essential extensions. Since $M(B)/B$ has property P2, there are partial isometries $z_1, z_2 \in M(B)/B$ such that $z_1^* z_1 = 1_{M(B)/B}$, $z_1 z_1^* = \tau_1(1_A)$, $z_2^* z_2 = 1_{M(B)/B}$ and $z_2 z_2^* = \tau_2(1_A)$. Define $[\tau_1] + [\tau_2] = [\operatorname{ad} z_1 \circ \tau_1 \oplus \operatorname{ad} z_2 \circ \tau_2]$.

Note this is well defined, since $[\tau_o] = 0$ in $KL(A.M(B)/B)$ and $\operatorname{ad} z_1 \circ \tau \oplus \operatorname{ad} z_2 \circ \tau_o$ is approximately unitarily equivalent to τ by Corollary 2.7. With this addition $\mathbf{Ext}_{ap}^f(A, B)$ forms a semigroup.

By Corollary 2.10, we have:

Corollary 2.12. *Let A be a unital separable amenable C^*-algebra and B a nonunital but σ-unital C^*-algebra for which $M(B)/B$ has properties P1, P2 and P3. Then $\mathbf{Ext}_{ap}^f(A, B)$ is a group with zero element $[\tau_o]$, where $\tau_o : A \to M(B)/B$ is a full monomorphism that factors through $\mathcal{O}_2$.*

If, furthermore, A satisfies the so-called Approximate Universal Coefficient Theorem (see Definition 7.1 below), we can say more:

Theorem 2.13. *Let A be a unital separable amenable C^*-algebra satisfying the Approximate Universal Coefficient Theorem and let B be a nonunital but σ-unital C^*-algebra such that $M(B)/B$ has properties P1, P2 and P3. Then there is a bijection Γ from $\mathbf{Ext}_{ap}^f(A, B)$ onto $KL(A, M(B)/B)$.*

The Approximate Universal Coefficient Theorem will be briefly reviewed in Definitions Definition 7.1 and Definition 8.1. Note that, when B is not stable, $K_i(M(B)/B)$ is very different from $K_i(SB)$ [Lin 2005b, 1.7].

In the special case that $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra, we have:

Theorem 2.14. *Let A be a unital separable amenable C^*-algebra and set $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra such that $M(B)/B$ has the property P1.*

Two full essential extensions $\tau_1, \tau_2 : A \to M(B)/B$ are unitarily equivalent if and only if

$$[\tau_1] = [\tau_2] \quad \text{in } KK^1(A, B).$$

If $x \in KK^1(A, B)$, there is a full essential extension $\tau : A \to M(B)/B$ such that $[\tau] = x$.

Theorem 2.15. *Let A be a unital separable amenable C^*-algebra and set $B = C \otimes \mathcal{K}$, where C is a unital C^*-algebra for which the tracial state space $T(C)$ is nonempty. Suppose that there is $d > 0$ for which C satisfies the following*:

(1) *If $p, q \in B$ are two projections, the condition $t(p) > d + t(q)$ for all $t \in T(C)$ implies $q \sim p$ in B.*

(2) *If $b \in M_k(C)$ is such that $1 \geq b \geq 0$ and $\tau(b) > \alpha + d$ for all $\tau \in T(A)$, there is a projection $e \in \overline{bM_k(A)b}$ such that $\tau(e) > \alpha$ for all $\tau \in T(A)$.*

Then two full essential extensions $\tau_1, \tau_2 : A \to M(B)/B$ are unitarily equivalent if and only if

$$[\tau_1] = [\tau_2].$$

Remark 2.16. In the case that $B = \mathcal{K}$, Theorem 2.14 is the classical Brown–Douglas–Fillmore theorem, and $M(\mathcal{K})/\mathcal{K}$ is a purely infinite simple C^*-algebra. It has property P1 (as well as P2 and P3) and every essential extension is full. Let X be a compact metric space with finite dimension d. When $B = C(X) \otimes \mathcal{K}$, $M(B)/B$ has property P1 (Corollary 3.9). Theorems 2.14 and 2.15 deal with the extensions studied by Pimsner, Popa and Voiculescu [Pimsner et al. 1979; 1980]. The case where B is a nonunital purely infinite simple C^*-algebra was proved by Kirchberg.

Theorem 2.14 is closely related to a result of Elliott and Kucerovsky [2001]; see Remark 8.7 for discussion.

3. C^*-algebras have properties P1, P2 and P3

Let A be a unital C^*-algebra. Denote by $T(A)$ (or T if no confusion exits) the set of tracial states on A. If $t \in T(A)$, we extend t to a trace $(t \otimes Tr)$ on $A \otimes M_n$ by defining $t((a_{ij}) = \sum_{i=1}^{n} t(a_{ii})$. We further use t for the trace defined on a dense set of $A \otimes \mathcal{K}$. If $a \in A \otimes \mathcal{K}_+$, then $t(a)$ is well defined (although it could be infinity). Suppose that $h_n \in A \otimes \mathcal{K}_+$ is such that $h_n \nearrow h \in A^{**}$. Then $t(h) = \lim_{n\to\infty} t(h_n)$. These conventions will be used in this section.

Lemma 3.1. *Let A be a unital C^*-algebra and I a σ-unital ideal of A. If $a \in (A/I)_+$ is a full element, there exists a full element $b \in A_+$ such that $\pi(b) = a$, where $\pi : A \to A/I$ is the quotient map.*

Proof. This result is certainly known, but we prove it for completeness. Since $a \in (A/I)_+$ is full, there are $x_1, x_2, \ldots, x_m \in A/I$ such that

$$\sum_{i=1}^{m} x_i^* a x_i = 1.$$

Hence there are $c \in A_+$ and $y_1, y_2, \ldots, y_m \in A$ such that $\pi(c) = a$ and $1 - \sum_{i=1}^{m} y_i^* c y_i \in I$. Let e be a strictly positive element of I. Put $b = c + e$. Denote by J the ideal generated by b. Since $b \geq c$ and $b \geq e$, both c and e are in J. It follows that $I \subset J$. Since $\sum_{i=1}^{m} y_i^* c y_i \in J$, it follows that $1 \in J$. Thus $J = A$, and b is full. □

Corollary 3.2. *Let A be a unital C^*-algebra and I a σ-unital ideal of A. If A has property P1, then so does A/I.*

Lemma 3.3. *Let A be a unital C^*-algebra and let $B = A \otimes \mathcal{K}$. Suppose that $a \in M(B)$ is an element for which $b = \pi(a)$ is full in $M(B)/B$, where $\pi : M(B) \to M(B)/B$ is the quotient map. Then a is full in $M(B)$. If $M(B)/B$ has property P1, so does $M(B)$.*

Proof. There are $x_1, x_2, \ldots, x_m\ y_1, \ldots, y_m \in M(B)/B$ such that $\sum_{i=1}^{m} x_i b y_i = 1$. Then there are $w_1, w_2, \ldots, w_m, z_1, z_2, \ldots, z_m \in M(B)$ such that

$$1 - \sum_{i=1}^{m} w_i a z_i \in B.$$

Let $\{e_{ij}\}$ be a system of matrix units for $\mathcal{K}$. Put $E_n = \sum_{i=1}^{n} e_{ii}$. Then $\{E_n\}$ is an approximate identity consisting of projections. It follows that there exists $n > 0$ such that

$$\left\| \sum_{i=1}^{m} (1-E_n) w_i a z_i (1-E_n) - (1-E_n) \right\| < \tfrac{1}{2}.$$

Thus there exists $s \in (1-E_n)M(B)(1-E_n)$ such that

$$\sum_{i=1}^{m} s^*(1-E_n) w_i a z_i (1-E_n) s = 1 - E_n.$$

But there exists $V \in M(B)$ such that $V^*(1-E_n)V = 1$. Therefore a is full.

For the last statement, we take $m = 1$ in the argument above. □

Proposition 3.4. *Let B be a unital purely infinite simple C^*-algebra. Then $M(B \otimes \mathcal{K})$ and $M(B \otimes \mathcal{K})/A \otimes \mathcal{K}$ have property P1.*

Proof. It follows from [Zhang 1992] that $M(B \otimes \mathcal{K})/(B \otimes \mathcal{K})$ is purely infinite and simple. Therefore $M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ has property P1. Now Lemma 3.3 implies that $M(B \otimes \mathcal{K})$ has property P1. □

Theorem 3.5. *Let $B = A\otimes\mathcal{K}$, where A is a unital separable C^*-algebra for which $T(A)\neq\varnothing$. Let $d>0$. Suppose A satisfies the following*:

(1) *If $p, q\in B$ are two projections then $t(p)>d+t(q)$ for all $t\in T(A)$ implies $p\precsim q$ in B.*

(2) *If $1\geq b\geq 0$ in $M_k(A)$ such that $\tau(b)>\alpha+d$ for all $\tau\in T(A)$ (and some $\alpha>0$), then there is a projection $e\in\overline{bM_k(A)b}$ such that $\tau(e)>\alpha$ for all $\tau\in T(A)$.*

Then $M(B)$ and $M(B)/B$ have property P1.

Proof. Let $b\in M(B)$ be a full element. Without loss of generality, we may assume that $0\leq b\leq 1$. Let $\{e_{ij}\}$ be a system of matrix units for $\mathcal{K}$ and set $E_n=\sum_{k=1}^n e_{ii}$. Then E_nbE_n converges to b in the strict topology. Furthermore $b^{1/2}E_nb^{1/2}$ increasingly converges to b in the strict topology.

Since b is full, there are $x_1, x_2, \ldots, x_m\in M(B)$ such that

$$\sum_{k=1}^m x_i^*bx_i=1.$$

Let $\tau\in T(A)$ be a tracial state. We extend τ to B_+ and then to $M(B)_+$ in a usual way. Let T be the set of all (densely defined) traces on $M(B)_+$ whose restrictions to A are tracial states. With the usual weak *-topology, T is a compact convex set.

Because $b^{1/2}x_i^*x_ib^{1/2}\leq\|x_i\|^2b$, one has

$$\tau(x_i^*bx_i)=\tau\big(b^{1/2}x_i^*x_ib^{1/2}\big)\leq\|x_i\|^2\tau(b)$$

for all $\tau\in T(A)$ Therefore

$$\sum_{i=1}^m\tau(x_i^*bx_i)\leq\left(\sum_{i=1}^m\|x_i\|^2\right)\tau(b)$$

for all $\tau\in T(A)$. Since $\tau(1)=\infty$, it follows that $\tau(b)=\infty$. Because $b^{1/2}E_nb^{1/2}\nearrow b$ and because T is compact, by Dini's theorem, we have $\tau(b^{1/2}E_nb^{1/2})\to\infty$ uniformly on T. Since $\tau(E_nbE_n)=\tau(b^{1/2}E_nb^{1/2})$ for all $\tau\in T$, we have $\tau(E_nbE_n)\nearrow\infty$ uniformly on T. There is $n(1)\geq 1$ such that

$$\tau(E_{n(1)}bE_{n(1)})>1+2d\quad\text{for all } \tau\in T.$$

Let A_1 be the hereditary C^*-subalgebra of B generated by $E_{n(1)}bE_{n(1)}$. It follows from assumption (2) that there is a projection $p_1\in A_1$ such that $\tau(p_1)>1+d$ for all $\tau\in T$. Thus there is $v_1\in B$ such that $v_1^*v_1\leq p_1$ and $v_1v_1^*=E_1$. There are nonnegative continuous functions $f, g\in C_0((0, 2\|b\|]$ such that $gf=f$ and

$$\big\|f(E_{n(1)}bE_{n(1)})p_1f(E_{n(1)}bE_{n(1)})-p_1\big\|<\tfrac{1}{4}.$$

It follows by [Effros 1981, A8] that there is a projection

$$q_1 \in \overline{f(E_{n(1)}bE_{n(1)})Bf(E_{n(1)}bE_{n(1)})}$$

such that q_1 is unitarily equivalent to p_1. We conclude that $g(E_{n(1)}bE_{n(1)})q_1 = q_1$, since $gf = f$. By functional calculus, we see that there is $f_1 \in A_1$ such that

$$f_1 E_{n(1)} b E_{n(1)} f_1 = g.$$

Thus we obtain $z_1 \in E_{n(1)} B E_{n(1)}$ such that

$$z_1^* b z_1 = z_1^* E_{n(1)} b E_{n(1)} z_1 = E_1.$$

Note that $\tau\big((1-E_{n(1)})bE_{n(1)}\big) = \tau\big(bE_{n(1)}(1-E_{n(1)})\big) = 0$. It follows that

$$\tau\big((1-E_{n(1)})b(1-E_{n(1)})\big) = \tau\big((1-E_{n(1)})b\big).$$

Since $\tau(E_{n(1)}bE_{n(1)}) < \infty$ for all $\tau \in T$, we conclude that

$$\tau\big((1-E_{n(1)})b(1-E_{n(1)})\big) = \infty \quad \text{for all } \tau \in T.$$

By the earlier argument we obtain $n(2) > n(1)$ and $z_2 \in (E_{n(2)} - E_{n(1)})B(E_{n(2)} - E_{n(1)})$ such that

$$z_2^* b z_2 = z_2^*(E_{n(2)} - E_{n(1)})b(E_{n(2)} - E_{n(1)})z_2 = E_2 - E_1.$$

Continuing this fashion, we obtain a sequence $\{n(k)\}$ with $n(k+1) > n(k)$ and $z_k \in (E_{n(k+1)} - E_{n(k)})B(E_{n(k+1)} - E_{n(k)})$ such that

$$z_k^* b z_k^* = z_k^*(E_{n(k+1)} - E_{n(k)})b(E_{n(k+1)} - E_{n(k)})z_k = E_{k+1} - E_k,$$

for $k = 1, 2, \dots$. Hence $z = \sum_{k=1}^{\infty} z_k \in M(B)$, since the sum converges in the strict topology. Furthermore,

$$z^* b z = 1.$$

This shows that $M(B)$ has property P1. By Corollary 3.2, $M(B)/B$ also has property P1. □

Corollary 3.6. *Let A be a unital AF-algebra and take $B = A \otimes \mathcal{K}$. Then $M(B)$ and $M(B)/B$ have property P1.*

Proof. Clearly A satisfies (1) in Theorem 3.5 with any $d > 0$. To see that it satisfies (2), we let $1 \geq b \geq 0$ be an element in $M_n(A)$ such that $\tau(b) > \alpha + d$ for all $\tau \in T$. Set $C = \overline{bM_n(A)b}$ and let $\{e_n\}$ be an approximate identity for C consisting of projections. Then $\|e_n b e_n - b\| \to 0$ as $n \to \infty$. Since $0 \leq b \leq 1$, it follows that $\tau(e_n) > \alpha + d$ for some $n > 0$ and all $\tau \in T$. □

The proof of the corollary implies the following:

Corollary 3.7. *Let A be a unital separable C^*-algebra with real rank zero for which $T(A) \neq \varnothing$ and which satisfies* (1) *in Theorem 3.5. Then $M(B)$ and $M(B)/B$ have property P1, where $B = A \otimes \mathcal{K}$.*

Corollary 3.8. *Let $B = A \otimes \mathcal{K}$, where A is a unital simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Then both $M(B)$ and $M(B)/B$ have property P1.*

Corollary 3.9. *Let $A = C(X)$, where X is a compact Hausdorff space with finite covering dimension d. Then $M(A \otimes \mathcal{K})$ and $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ have property P1.*

Proof. Suppose $e, f \in A \otimes \mathcal{K}$ are projections. We may assume that $e, f \in M_n(C(X))$ for some integer $n > 0$. Suppose that $\tau(e) > \tau(f) + d + 1$ for all $t \in T(A)$. It follows that for each $x \in X$, the rank of $e(x)$ is greater than $d + 1$ + the rank of $f(x)$. It follows from [Husemoller 1966, 8.1.2 and 8.1.6] (see [Blackadar 1998, 6.10.3(d)]) that $f \lesssim e$. So Theorem 3.5(1) holds (for $(d+1)/2$).

For (2), let $1 \geq b \geq 0$ be an element in $M_k(C(X))$ for which $\tau(b) > \alpha + (d+1)$. Let f_n be as in (1-1). For some large n, we have $\tau(f_n(b)) > \alpha + (d+1)$ for all $\tau \in T(A)$. Thus, for each $\xi \in X$, the rank of $f_n(b)(\xi)$ is at least $\alpha + (d+1)$. By [Blackadar et al. 1991, Lemma C], there is a projection $e \in \overline{bM_k(A)b}$ such that the rank of $e(\xi)$ is greater than α for all $\xi \in X$. It follows that $\tau(e) > \alpha$ for all $\tau \in T(A)$. □

To discuss property P2, we begin with an easy observation:

Proposition 3.10. *Let B be a unital C^*-algebra having property P2. For any integer $n > 0$, there are $s_{11}, s_{22}, \dots, s_{nn}$ such that $1_B \geq \sum_{i=1}^n s_{ii}$ and there exists an isometry $Z \in B$ such that $ZZ^* = e_{11}$. Moreover*:

(1) *If for some $n \geq 2$, $1_B = \sum_{i=1}^n s_{ii}$, then there exists a unital embedding from $\mathcal{O}_n$ to B.*

(2) *There is a unital embedding from $\mathcal{O}_\infty$ to B.*

(3) *There exists a full embedding $j : \mathcal{O}_2 \to B$.*

Conversely, if there is a unital embedding of $\mathcal{O}_\infty$ in B, then B has property P2. Furthermore, if B admits a full embedding from $\mathcal{O}_2$, then B has property P2.

Proposition 3.11. (1) *Let A be a unital C^*-algebra and $B = A \otimes \mathcal{K}$. Then $M(B)$ and $M(B)/B$ has property P2.*

(2) *Let A be a nonunital σ-unital simple C^*-algebra which has continuous scale. Then $M(A)/A$ has property P2*

(3) *Let A be a unital purely infinite simple C^*-algebra and $B = C_0(X, A)$, where X is a locally compact Hausdorff space. Then $M(B)$ and $M(B)/B$ have property P2.*

Proof. For (3), we note there is a unital embedding from $\mathcal{O}_\infty$ to A and the constant maps from X into A are in $C^b(X, A) = M(B)$. □

Now we turn to property P3. Every unital purely infinite simple C^*-algebra has property P3, by [Lin 2005b, 2.6]. Therefore, if B is a nonunital but σ-unital simple C^*-algebra with continuous scale, then $M(B)/B$ has property P3.

Proposition 3.12. *Let B be a unital C^*-algebra having property P1. Suppose that $0 \le a, b \le 1$, where $ab = a$ and a is full. Then there exists $x \in B$ with $\|x\| \le 1$ and*

$$x^*bx = 1.$$

Note that the proposition includes the case that a is a full projection.

Proof. There is $z \in B$ such that $z^*az = 1$. Then $a^{1/2}zz^*a^{1/2} = p$ must be a projection. Moreover, $p \in \mathrm{Her}(a)$. Therefore $pb = p$. Put $v = a^{1/2}z$. Then $v^*v = 1$ and $vv^* = p$. In particular, $\|v\| = 1$. Now

$$1 \ge \|b\| v^*v \ge v^*bv \ge v^*pv = 1.$$

We conclude that $v^*bv = 1$. □

Proposition 3.13. *Let A be a unital C^*-algebra and set $B = A \otimes \mathcal{K}$. Then $M(B)/B$ has property P3.*

Proof. Let $\pi : M(B) \to M(B)/B$ be the quotient map and D a separable C^*-algebra. Let $\{e_{i,j}\}$ be a system of matrix units for $\mathcal{K}$. Set $E_n = \sum_{i=1}^n e_{i,i}$. By [Pedersen 1979, 3.12.14] and the proof of [Lin 2001, 5.5.3], there is a sequence $\{e_n\} \subset \mathrm{Conv}\{E_k : k = 1, 2, \dots\}$ such that

$$e_{n+1}e_n = e_n \quad \text{and} \quad \|e_n a - a e_n\| \to 0 \quad \text{as } n \to \infty \tag{3-1}$$

for all $a \in D$.

Suppose that $e_n = \sum_{i=1}^{k(n)} \alpha_i E_i$, where the α_i are nonnegative scalars such that $\sum_{i=1}^{k(n)} \alpha_i = 1$. There exist $0 \le \beta_j \le 1$ such that $e_n = \sum_{i=1}^{k(n)} \beta_j e_{jj}$. Since, for each i,

$$\|e_m e_{ii} - e_{ii}\| \to 0 \quad \text{as } m \to \infty,$$

there is $N(n) > 0$ such that, for each $m > N(n)$, $e_m = \sum_{i=1}^{k(m)} \beta_i^{(m)} e_{ii}$ with $\beta_{k(n)+1} > \frac{1}{2}$. It follows that

$$\begin{aligned}(e_m - e_n)e_{k(n)+1,k(n)+1} &= \Bigl(\sum_{k(n)+1}^{k(m)} \beta_i^{(m)} e_{ii} + \sum_{i=1}^{k(n)} (\beta_i^{(m)} - \beta_i^{(n)}) e_{ii}\Bigr) e_{k(n)+1,k(n)+1} \\ &= \Bigl(\sum_{k(n)+1}^{k(m)} \beta_i^{(m)} e_{ii}\Bigr) e_{k(n)+1,k(n)+1} = \beta_i^{(m)} e_{k(n)+1,k(n)+1}.\end{aligned}$$

By passing to a subsequence if necessary we may as well assume that

$$(e_{n+1}-e_n)e_{k(n)+1,k(n)+1}=\lambda_n e_{k(n)+1,k(n)+1}$$

for some $\lambda_n > \frac{1}{2}$. Now let $F \subset \mathbb{N}$ be an infinite subset. Then

$$b_F=\sum_{n\in F}(e_{n+1}-e_n)\geq \frac{1}{2}\sum_{n\in F} e_{k(n)+1,k(n)+1}.$$

Thus b_F is a full positive element in $M(B)$. Suppose that $\{F_n\}$ is a sequence of infinite subsets of $\mathbb{N}$. By Proposition 3.12, the image $\pi\big(\{\sum_{j\in F_n} e_{k(j)+1,k(j)+1}\}\big)$ is full in $l^\infty(M(B)/B)$. So $\{\pi(b_{F_n})\}$ is full in $l^\infty(M(B)/B)$.

By (3-1), $\pi(b_F)$ commutes with $\pi(d)$ for each $d \in D$. Also by (3-1)

$$(e_{n+1}-e_n)(e_{m+1}-e_m)=0 \quad \text{if } |n-m|\geq 2.$$

It follows that $b_F b_{F'}=0$ if $|n-m|\geq 2$ for any $n\in F$ and any $m \in F'$. Note that one may write $b_F=\sum_{n\in S(F)}\lambda_n e_{n,n}$, where each $0<\lambda_n\leq 1$ is a positive number and $S(F)$ is an infinite subset of $\mathbb{N}$.

It is easy to find a family of (disjoint) infinite subsets $\{F_{i,j} : i, j = 1, 2, \dots\}$ of $\mathbb{N}$ such that $|n-m|\geq 2$ for any $n\in S_{i,j}$ and any $m\in S_{i',j'}$, if $i\neq i'$ or $j\neq j'$. Define $S_{i,j}=S(F_{i,j})$ as above. We note that $S_{i,j}\cap S_{i',j'}=\varnothing$ if $i\neq i'$ or $j\neq j'$. Write $b_{i,j}$ for $b_{F_{i,j}}$. It follows that $M(B)/B$ has property P3. □

4. Nonstable cases

In [Kirchberg and Rørdam 2000] the notion of purely infinite C^*-algebras was extended to nonsimple C^*-algebras. Let C_1 be a unital C^*-algebra and C_2 be a unital separable purely infinite simple C^*-algebra. Then $C_1\otimes C_2$ is purely infinite [Kirchberg and Rørdam 2000, 4.5]. Therefore, for any unital C^*-algebra C the C^*-algebra $B = C\otimes \mathcal{O}_\infty$ has properties P1 and P2 as well as P3.

Proof of Proposition 2.2. From the preceding paragraph, we know that B is purely infinite, and so has properties P1 and P2. Let A be a separable C^*-subalgebra of B. There is a separable C^*-subalgebra $C_0\subset C$ such that $A\subset C_0\otimes C_1$. It follows from [Kirchberg and Phillips 2000] that $C_1\otimes\mathcal{O}_\infty\cong C_1$. By [Rørdam 2002, 7.2.6] and [Kirchberg and Phillips 2000, 3.12], there is a sequence of unital monomorphisms $\phi_n : \mathcal{O}_\infty\to C_0\otimes C_1$ such that

$$\lim_{n\to\infty}\|\phi_n(x)a-a\phi_n(x)\|=0 \quad \text{for all } a\in C_0\otimes\mathscr{C}_1.$$

Let $\{e_k\}$ be a sequence of nonzero mutually orthogonal projections in $\mathcal{O}_\infty$. Define $a_n^{(i)}=\phi_n(e_i)$, $n, i = 1, 2, \dots$. One checks that $a_n^{(i)}$ satisfies the requirements for property P3. □

There are σ-unital but *nonstable* separable C^*-algebras B for which the corona C^*-algebra $M(B)/B$ has properties P1 and P2 as well as P3. For example, when B has continuous scale, $M(B)/B$ is a purely infinite simple C^*-algebra [Lin 2004b]. So in those cases $M(B)/B$ has all three properties. There are other nonstable separable C^*-algebras B for which B has properties P1, P2 and P3.

To make a point, we will present a very simple example of a nonstable σ-unital C^*-algebra B for which $M(B)/B$ is not simple but both $M(B)$ and $M(B)/B$ have properties P1, P2 and $M(B)/B$ has P3. Many such examples can be constructed.

Proposition 4.3 is not needed in Example 4.4 but will be used again later.

Lemma 4.1. *Let A be a unital C^*-algebra and let $0 \le a \le 1$ be an element in A. Suppose that there is $x \in A$ such that $x^*ax = 1$. Then there is $N > 0$, depending on $\|x\|$ but not on A or a, for which there is $y \in A$ with $\|y\| \le 1$ such that*

$$y^* f_N(a) y = 1.$$

In particular, $f_N(a)$ is full, where f_N is as defined in (1-1).

Proof. Let $q = a^{1/2}xx^*a^{1/2}$. Then q is a projection. There exists $k > 0$ depending on $\|x\|$ such that

$$\|f_k(t)t^{1/2} - t^{1/2}\| < \frac{1}{16\|x\|^2} \quad \text{for all } t \in [0, 1],$$

where f_k is as in (1-1). Then

$$\|f_k(a)q - q\| = \left\|(f_k(a)a^{1/2} - a^{1/2})x^*xa^{1/2}\right\| < \tfrac{1}{16}.$$

It follows from [Effros 1981, A8] that there is a projection $p \in \overline{f_k(a)Af_k(a)}$ such that

$$\|q - p\| < \tfrac{1}{2}.$$

Thus there exists $w \in A$ such that $w^*w = 1$ and $ww^* = p$. Choose $N = k + 1$. Then $f_N(a)q = q$, showing that

$$w^* f_N(a) w = 1. \qquad \square$$

Lemma 4.2. *Let A be a unital C^*-algebra and let $a \in A$ with $0 \le a \le 1$ be a full element. Suppose that there are $x_1, x_2, \dots, x_m \in A$ such that*

$$\sum_{i=1}^{m} x_i^* a x_i = 1.$$

Set $r = \sum_{i=1}^{m} \|x_i\|^2$. Suppose also that $1_{M_m(A)} \lesssim 1$. Then there exists an integer $N > 0$, depending on r but not on A on a, such that $f_N(a)$ is full. Moreover, there *are* $y_1, y_2, \ldots, y_m \in A$ *such that* $\sum_{i=1}^m \|y_i\|^2 \le 1$ *and*

$$\sum_{i=1}^m y_i^* f_N(a) y_i = 1.$$

Proof. Let

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a. \end{pmatrix}$$

Since $1_{M_m(A)} \lesssim 1$, one obtain $Y \in M_m(A)$ with $\|Y\| = 1$ and $Y^* \operatorname{diag}(1, 0, \ldots, 0) Y = 1_{M_m(A)}$. Note that $0 \le b \le 1$ and $XbX^* = \operatorname{diag}(1, 0, \ldots, 0)$. Thus

$$Y^* X b X^* Y = 1_{M_m(A)}.$$

We compute that $\|X^* Y\| \le r^{1/2}$. It follows from Lemma 4.1 that there exist $N > 0$ and $z \in M_m(A)$ with $\|z\| \le 1$ such that

$$z^* f_N(b) z = 1_{M_m(A)}.$$

So $Y z^* f_N(b) z Y^* = 1$. An easy computation shows that there are $y_1, y_2, \ldots, y_n \in A$ such that $\sum_{i=1}^m \|y_i\|^2 \le 1$ and

$$\sum_{i=1}^m y_i^* f_N(a) y_i = 1. \qquad \square$$

Proposition 4.3. *Let $\{A_n\}$ be a sequence of unital C^*-algebras having property P1. Then $l^\infty(\{A_n\})$ also has property P1.*

Proof. Let $a = \{a_n\}$ be a full element in $l^\infty(\{A_n\})$ such that $0 \le a \le 1$. (Note that full elements of $l^\infty(\{A_n\})$ cannot be in $c_0(\{A_n\})$.) By Lemma 4.2, there exists $N > 0$ for which $f_N(a)$ is full. For each n, there exists $x_n \in A_n$ such that $x_n^* f_N(a_n) x_n = 1$. Note that $f_{N+1}(a_n) f_N(a) = f_N(a)$. It follows from Proposition 3.12 that, for each n, there is $y_n \in A$ with $\|y_n\| \le 1$ such that

$$y_n^* f_{N+1}(a) y_n = 1.$$

Put $y = \{y_n\}$. Then $y \in l^\infty(\{A_n\})$. It is clear that there is $g \in C_0((0, 1])_+$ such that

$$\|g(a) a g(a) - f_{N+1}(a)\| < \tfrac{1}{4}.$$

Then

$$\|y^* g(a) a g(a) y - 1\| = \|y^* (g(a) a g(a) - f_N(a)) y\| \le \tfrac{1}{4}.$$

Hence there is $z \in l^\infty(\{A_n\})$ with $\|z\| < \frac{4}{3}$ such that

$$z^* y^* g(a) a g(a) y z = 1. \qquad \square$$

This proposition is not required in the following example. However it will be used in Lemma 6.5.

Example 4.4. Let A be a unital separable amenable purely infinite simple C^*-algebras. Denote by $B = c_0(A)$. Then $M(B) = l^\infty(A)$. Put $q_\infty(A) = l^\infty(A)/c_0(A)$. So $M(B)/B = q_\infty(A)$.

(1) $M(B)$ and $M(B)/B$ has properties P1 and P2.

(2) $M(B)/B$ has property P3.

Claim (1) is obvious (it also follows from Proposition 4.3). In fact, if $C = C_0((0, 1), A)$, then $M(C)$ and $M(C)/C$ also have properties P1 and P2. This can be proved rather easily.

To see (2), let D be a separable C^*-subalgebra of $M(B)$. Suppose that $x^{(1)} = \{x_n^{(1)}\}$, $x^{(2)} = \{x_n^{(2)}\}, \ldots, x^{(k)} = \{x_n^{(k)}\}, \ldots$ is a dense sequence in the unit ball of D. Using the fact that $A \otimes \mathcal{O}_\infty \cong A$ [Kirchberg and Phillips 2000, Theorem 3.15], we obtain a sequence of homomorphisms $\phi_n : \mathcal{O}_\infty \to A$ such that

$$\lim_{n\to\infty} \|\phi_n(b)a - a\phi_n(b)\| = 0$$

for all $a \in A$ and $b \in \mathcal{O}_\infty$. Let $e_1 \in \mathcal{O}_\infty$ be a proper projection. There is an integer $n(1) > 0$ such that

$$\left\|\phi_{n(1)}(e_1)x_1^{(1)} - x_1^{(1)}\phi_{n(1)}(e_1)\right\| < \tfrac{1}{2}.$$

There is a projection $e_2 \in \mathcal{O}_\infty$ such that $e_1e_2 = e_2e_1 = 0$ and $1 > e_1 + e_2$. There is $n(2) > 0$ such that

$$\left\|\phi_{n(2)}(e_j)x_l^{(i)} - x_l^{(i)}\phi_{n(2)}(e_j)\right\| < \tfrac{1}{4} \quad \text{for } i, j, l = 1, 2.$$

Continuing in this fashion, we obtain a sequence of mutually orthogonal nonzero projections $\{e_m\} \subset \mathcal{O}_\infty$ and a subsequence $\{n(m)\}$ such that

$$\|\phi_{n(m)}(e_j)x_l^{(i)} - x_l^{(i)}\phi_{n(m)}(e_j)\| < 2^{-m} \quad \text{for } i, j, l = 1, 2, \ldots, m.$$

Put $p^{(j)} = \{\phi_{n(m)}(e_j)\} \in l^\infty(A)$, for $j = 1, 2, \ldots$.. Then $p_m^{(i)} p_m^{(j)} = 0$ if $i \neq j$. Moreover,

$$\left\|\pi(p^{(j)})\pi(\{x^{(i)}\}) - \pi(\{x^{(i)}\})\pi(p^{(j)})\right\| = 0.$$

This implies that

$$\pi(p^{(j)})\pi(d) = \pi(d)\pi(p^{(j)}).$$

Put $a_n^{(j)} = p^{(j)}$, $j = 1, 2, \ldots$. This shows that $M(B)/B$ has property P3.

It is clear, in fact, that $l^\infty(\{A_n\})/c_0(\{A_n\})$ has property P3 if each A_n is a unital purely infinite simple C^*-algebra.

5. Amenable contractive completely positive linear maps

Lemma 5.1 ([Akemann et al. 1986, 2.3]; see also [Lin 2001, 5.3.2]). *Let A be a separable C^*-algebra and $\psi : A \to \mathbb{C}$ a pure state. Denote by ψ also the extension of ψ to $\tilde{A}$ and put $L = \{a \in \tilde{A} : \psi(a^*a) = 0\}$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $z_1, z_2, z_3 \in \tilde{A}_+$ such that $\|z_i\| = 1$ and $z_i \notin L$ for $i = 1, 2, 3$, $z_{i+1}z_i = z_i$ for $i = 1, 2$, and*

$$\|z_i(\phi(a) - a)z_i\| < \tfrac{1}{2}\varepsilon \quad \text{for } i = 1, 2, 3 \text{ and all } a \in \mathcal{F}.$$

Moreover, if $\{e_n\}$ is an approximate identity for A, then, for some large N,

$$\|e_n z_i e_n(\phi(a) - a)e_n z_i e_n\| < \varepsilon \quad \text{and} \quad e_n z_i e_n \notin L$$

for all $a \in \mathcal{F}$ and all $n \geq N$.

Proof. To simplify notation, we assume that $\mathcal{F}$ is a subset of the unit ball of A. Let

$$N = \{a \in \tilde{A} : \phi(a) = 0\}.$$

Note that L is a closed left ideal. Let C be the hereditary C^*-subalgebra given by $L \cap L^*$. As in the proof of [Lin 2001, 5.3.2], we have $z_1, z_2, z_3, \in \tilde{A}$ with $\|z_i\| = 1$ ($i = 1, 2, 3$) such that $z_i \notin L$, $z_{i+1}y_i = z_i$, $i = 1, 2, 3$, and

$$\|z_i(\psi(a) - a)z_i\| < \tfrac{1}{2}\varepsilon, \quad i = 1, 2, 3.$$

Let $\{e_n\}$ be an approximate identity for A such that $e_n e_{n+1} = e_n$ for all n. Note that z_i has the form $\lambda_i 1_B + y_i'$, where $y_i' \in A$ and $\lambda_i \in \mathbb{C}$, $i = 1, 2$. Choose a large n such that

$$\|e_k a - a e_k\| < \tfrac{1}{4}\varepsilon, \quad \|e_k a - a\| < \tfrac{1}{4}\varepsilon \quad \text{and} \quad \|e_k z_i - z_i e_k\| < \tfrac{1}{4}\varepsilon$$

for all $a \in \mathcal{F} \cup \{z_1 a z_1, z_2 a z_2, z_3 a z_3 : a \in \mathcal{F}\}$ and for all $k \geq n$. Let $y_i = e_n z_i e_n$. Then, for $n \geq N$,

$$\|y_i(\psi(a) - a)y_i\| < \tfrac{1}{4}\varepsilon + \|e_n^2 z_i(\psi(a) - a)z_i e_n^2\| < \varepsilon \quad \text{for all } a \in \mathcal{F}. \qquad \square$$

The following is folklore.

Lemma 5.2. *Let A be a C^*-subalgebra of B and take $a \in A_+$. Denote by C the hereditary C^*-subalgebra of B generated by a. Then, for any approximate identity $\{e_n\}$ of A, the sequences $\|e_n b - b\|$ and $\|b e_n - b\|$ converge to 0 as $n \to \infty$, for all $b \in C$.*

Proof. There exists a sequence of positive functions $f_n \in C_0(sp(a))$ with $0 \leq f_n \leq 1$ such that $\{f_n(a)\}$ forms an approximate identity for C. Fix an element $b \in C$. For any $\varepsilon > 0$, there is f_k such that

$$\|f_k(a)b - b\| < \tfrac{1}{4}\varepsilon \quad \text{and} \quad \|b f_k(a) - b\| < \tfrac{1}{4}\varepsilon.$$

Choose an integer $N > 0$ such that

$$\|e_n f_k(a) - f_k(a)\| < \frac{\varepsilon}{4(\|b\|+1)} \quad \text{for all } n \geq N.$$

Then

$$\begin{aligned}\|e_n b - b\| &\leq \|e_n b - e_n f_k(a)b\| + \|e_n f_k(a)b - f_k(a)b\| + \|f_k(a)b - b\| \\ &< \tfrac{1}{4}\varepsilon + \|b\|\left(\frac{\varepsilon}{4(\|b\|+1)}\right) + \tfrac{1}{4}\varepsilon = \tfrac{3}{4}\varepsilon < \varepsilon.\end{aligned}$$

□

Lemma 5.3. *Let B be a unital C^*-algebra that has the property P1. Let A be a separable C^*-algebra and let I be an ideal of A. Suppose that $j : A \to B$ is an embedding such that $j(a)$ is a full element of B for all $a \notin I$. Then, for any pure state $\phi : A \to \mathbb{C}1_B \subset B$ which vanishes on I, any finite subset $\mathcal{F} \subset A$, and any $\varepsilon > 0$, there is a partial isometry $V \in B$ such that*

$$\|\phi(a) - V^* j(a)V\| < \varepsilon \quad \textit{for } a \in \mathcal{F}, \qquad V^*V = 1_B \quad \textit{and} \quad VV^* \in \mathrm{Her}(j(A)).$$

Proof. To simplify notation, we identify A with $j(A)$. Fix $0 < \varepsilon < \frac{1}{2}$. By Lemma 5.1, there are $z_1, z_2, z_3 \in \tilde{A}_+$ such that $\|z_i\| = 1$ and $z_i \notin L$ for $i = 1, 2, 3$, $z_{i+1}z_i = z_i$ for $i = 1, 2$, and

$$\|\phi(a)z_i^2 - z_i j(a)z_i\| < \tfrac{1}{4}\varepsilon \quad \text{for } i = 1, 2 \text{ and all } a \in \mathcal{F}.$$

Note that $L = \{a \in \tilde{A} : \psi(a^*a) = 0\}$. Therefore $I \subset L \cap L^* \subset L$. Let $\{e_n\}$ be an approximate identity for A such that $e_n e_{n+1} = e_n$, $n = 1, 2, \ldots$. Let N be an integer as in Lemma 5.1, such that

$$\left\|\phi(a)(e_n z_i e_n)^2 - e_n z_i e_n j(a) e_n z_i e_n\right\| < \tfrac{1}{2}\varepsilon, \quad \text{for } i = 1, 2, 3. \tag{5-1}$$

Put $y_1 = e_N z_1 e_N$. We may assume that $y_1 \notin L$. By the assumption, y_1 is full. Because B has property P1, there exists $x \in B$ such that $x^* y_1^2 x = 1_B$. Put $v_1 = y_1 x$. Then $v_1^* v_1 = 1_B$ and $v_1 v_1^* = p_1$ is a projection. Note that $p_1 \in \mathrm{Her}(y_1)$. There is a projection in $q_1 \in \mathrm{Her}(z_1^{1/2} e_N z_1^{1/2})$ such that q_1 is equivalent to p_1. Therefore there is a partial isometry $w_1 \in B$ such that $w_1^* q_1 w_1 = 1_B$ and $w_1 w_1^* = q_1$. Since $z_2^2 z_1 = z_1$, $z_2^2 q_1 = q_1$. By applying Lemma 5.2, one can choose a large integer $k > N$ such that, for all $n \geq k$,

$$\|e_n q_1 - q_1\| < \tfrac{1}{32}\varepsilon \quad \text{and} \quad \|e_n z_i - z_i e_n\| < \tfrac{1}{32}\varepsilon \qquad \text{for } i = 1, 2, 3. \tag{5-2}$$

Thus

$$\|(e_k z_2 e_k)^2 q_1 - q_1\| = \|e_k z_2 e_k^2 z_2 e_k q_1 - q_1\| < \tfrac{8}{32}\varepsilon = \tfrac{1}{4}\varepsilon.$$

Put $y_2 = e_k z_2 e_k$. Then one estimates

$$\|w_1^* y_2^2 w_1 - 1\| = \|w_1^* q_1 y_2^2 q_1 w_1 - w_1^* q_1 w_1\| < \tfrac{1}{2}\varepsilon.$$

Thus there is $s \in \mathrm{Her}(z_1^{1/2} e_N z_1^{1/2})_+ \subset B_+$ such that $\|s\| \le \dfrac{1}{1-\frac{1}{2}\varepsilon}$ and

$$s^{1/2} w_1^* y_2^2 w_1 s^{1/2} = 1.$$

Note that

$$\|w_1 s^{1/2}\| \le \sqrt{\frac{1}{1-\frac{1}{2}\varepsilon}} < \sqrt{\frac{2}{2-\frac{1}{2}}} = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}. \tag{5-3}$$

Define $V = y_2 w_1 s^{1/2}$. Note that

$$V^*V = 1_B \quad \text{and} \quad VV^* \in \mathrm{Her}(j(A)). \tag{5-4}$$

Put $y_3 = e_{k+1} z_3 e_{k+1}$. Then, by (5-2),

$$\|y_3 y_2 - y_2\| = \|e_{k+1}(z_3 e_k z_2 - z_2) e_k\| < \tfrac{1}{32}\varepsilon. \tag{5-5}$$

Furthermore, by (5-3) and (5-5),

$$\|y_3 V - V\| = \|y_3 y_2 w_1 s^{1/2} - y_2 w_1 s^{1/2}\| \le (\tfrac{1}{32}\varepsilon)\frac{2\sqrt{3}}{3} = \frac{\sqrt{3}}{48}\varepsilon. \tag{5-6}$$

We estimate, by applying (5-6), (5-4) and (5-1), that

$$\begin{aligned}\|\phi(a) - V^*aV\| = \|\phi(a)V^*V - V^*aV\| &\le \frac{\sqrt{3}}{48}\varepsilon + \left\|\phi(a)V^*y_2^2V - V^*y_2ay_2V\right\| \\ &\le \frac{\sqrt{3}}{48}\varepsilon + \|\phi(a)y_2^2 - y_2ay_2\| < \frac{\sqrt{3}}{48}\varepsilon + \tfrac{1}{2}\varepsilon < \varepsilon \quad \text{for all } a \in \mathcal{F}. \quad \square\end{aligned}$$

Remark 5.4. If A has a unit, the proof of Lemma 5.3 is almost identical to that of [Lin 2001, 5.3.2], which has its origin in [Akemann et al. 1986]. When A has no unit, the elements z_1, z_2, z_3 are not in A_+ but in $\tilde{A}_+$. By using an approximate identity $\{e_n\}$, one does have $\|y_3 y_2 - y_2\|$ small. However the norm of x could be large and it depends on the choice of z_i as well as N as in the proof above. By introducing q_1, we are able to control the norm of $w_1 s^{1/2}$.

Lemma 5.5. *Let B be a unital C^*-algebra having property P1 and let A be a separable C^*-algebra. Suppose there exists a sequence of homomorphism $\phi_n : A \to B$ such that $\{\phi_n(a) : n = 1, 2, \ldots\}$ is an orthogonal set in B for all $a \in A$. Let I be an ideal of A such that* $\ker \phi_n \subset I$ *and $\phi_n(a)$ is a full element in B for all $a \notin I$ and for all n. Then, for any state $\psi : A/I \to \mathbb{C}1_B \subset B$, any finite subset $\mathcal{F} \subset A$, and any $\varepsilon > 0$, there is a partial isometry $V \in B$ and an integer n such that $V^*V = 1B$,*

$$\left\|\psi \circ \pi(a) - V^*\Big(\sum_{k=1}^{n} \phi_k(a)\Big)V\right\| < \varepsilon \quad \textit{for } a \in \mathcal{F} \quad \textit{and} \quad VV^* \in \mathrm{Her}\sum_{i=1}^{n} \phi_i(A),$$

where $\pi : A \to A/I$ is the quotient map. If ψ is only assumed to be a nonzero positive linear functional with $\|\psi\| \le 1$, the conclusion still holds with the difference that V is merely a contraction.

Proof. By the Krein–Milman theorem, we have positive numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ with $\sum_{i=1}^m \alpha_i = 1$ and pure states $\psi_1, \psi_2, \dots, \psi_m$ of A/I such that

$$\left\| \psi \circ \pi(a) - \sum_{i=1}^m \alpha_i \psi_i(a) \right\| < \tfrac{1}{2}\varepsilon \quad \text{for } a \in \mathcal{F}.$$

Let $\pi_n : A \to A/\ker\phi_n$ and $\gamma_n : A/\ker\phi_n \to A/I$ be the quotient maps, for $n = 1, 2, \dots$. Note that $\psi_i \circ \gamma_n$ is a pure state of $A/\ker\phi_n$.

By Lemma 5.5, there are $V_i \in B$ such that $V_i^* V_i = 1_B$, $V_i V_i^* \in \mathrm{Her}\,\phi_i(A)$ and

$$\left\| \psi_i \circ \gamma_i(\phi_i(a)) - V_i^* \phi_i(a) V_i \right\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

(Note that $\psi_i \circ \gamma_i \circ \phi_i = \psi_i \circ \pi$.)

Set $V = \sum_{i=1}^m \sqrt{\alpha_i}\, V_i \in B$. Since $\{\phi_n(a) : n = 1, 2, \dots\}$ is an orthogonal set for each $a \in A$, we compute that

$$V^*V = \sum_{i=1}^m 1_B = 1_B \quad \text{and} \quad VV^* = \sum_{i,j} \sqrt{\alpha_i \alpha_j} V_i V_j^* \in \mathrm{Her} \sum_{i=1}^m \phi_i(A).$$

Moreover

$$\begin{aligned}
&\left\| \psi \circ \pi(a) - V^* \Big(\sum_{i=1}^m \phi_i(a) \Big) V \right\| \\
&\qquad = \left\| \psi \circ \pi(a) - \sum_{i=1}^m \alpha_i V_i^* \phi_i(a) V_i \right\| \\
&\qquad \le \left\| \psi \circ \pi(a) - \sum_{i=1}^m \alpha_i \psi_i(a) \right\| + \sum_{i=1}^m \alpha_i \left\| \psi_i \circ \pi(a) - V_i^* \phi_i(a) V_i \right\| \\
&\qquad < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon \quad \text{for } a \in \mathcal{F}.
\end{aligned}$$

To check the last statement of the lemma, note that there is $0 < \lambda \le 1$ such that $\psi(a) = \lambda \cdot g(a)$ for some state g and for all $a \in A$. □

Lemma 5.6. *Let A be a separable C^*-algebra and let B be a unital C^*-algebra having properties P1 and P2. Let C be as described in the definition of Property P2, with $n = k$* (see Proposition 3.10). *Suppose that $\phi_n : A \to B$ is a sequence of homomorphisms such that $\{\phi_n(a) : n = 1, 2, \dots\}$ is an orthogonal set in B. Suppose that I is an ideal of A such that $I \supset \ker\phi_n$ and $\phi_n(a)$ is a full element for all $a \notin I$. Let $\psi : A/I \to M_k(\mathbb{C}) \subset M_k(C) \subset B$ be a contractive completely positive linear map. Then, for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a contraction $V \in B$*

and an integer $m > 0$ *such that*

$$\Big\|\psi(a) - V^*\Big(\sum_{i=1}^{m} \phi_i(a)\Big)V\Big\| < \varepsilon \quad \textit{for } a \in \mathcal{F} \quad \textit{and} \quad VV^* \in \operatorname{Her} \sum_{i=1}^{K} \phi_i(A),$$

where $\pi : A \to A/I$ *is the quotient map.*

Proof. Write $\psi(a) = \sum_{i=1}^{k} \psi_{ij}(a) \otimes s_{ij}$ for $a \in A$, where $\{s_{ij}\}$ is a system of matrix units for M_n and $\psi_{ij} : A \to \mathbb{C}$ is linear. Also assume that s_{ii} are as in Property P2 and Proposition 3.10, for $i = 1, 2, \ldots, k$. Define $\Phi : M_k(A) \to \mathbb{C} \subset C$ by $\Phi((a_{ij})_{k\times k}) = \sum_{i,j=1} \psi_{ij}(a_{ij})$, where $a_{ij} \in A$. Let Z be as in the definition of Property P2, so that $ZZ^* = s_{11}$. Put $J_n(a) = Z_k\phi_n(a)Z^*$ for all $a \in A$. Then J_n maps A into $C = s_{11}Bs_{11}$. Note that $\phi_n \otimes \mathrm{id} : M_k(A) \to M_k(C)$ is also full. Set $\mathcal{G} = \{(a_{ij}) : a_{ij} \in \mathcal{F} \cup \{0\}\}$. By applying Lemma 5.5, we see there is $W \in M_k(B)$ with $\|W\| \le 1$ such that

$$\Big\|\Phi(b) - W^*\Big(\sum_{k=1}^{m} J_k \otimes \mathrm{id}\, b\Big)W\Big\| < \frac{\varepsilon}{2n^2} \quad \text{for } b \in \mathcal{G}.$$

We may also assume that $W^*W \le \mathrm{diag}(1_C, 0, \ldots, 0)$. Choose a positive $d \in A$ such that $0 \le d \le 1$ and

$$\|da - a\| < \frac{\varepsilon}{2n^2} \quad \text{for all } a \in \mathcal{F}.$$

Let $v_i = 0, \ldots, 0, d, 0, \ldots, 0)$, where the nonzero entry is in i-th place. Let v_i' be the $n \times n$ matrix whose first row is v_i and whose remaining rows are zero. Put $r_i = \sum_{n=1}^{m} J_n \otimes \mathrm{id}(v_i')$. For any $a \in A$,

$$\Big\|r_i^*\Big(\sum_{n=1}^{m} J_n(a) \otimes \mathrm{id}(a \otimes s_{11})\Big)r_j - \sum_{n=1}^{m} J_n \otimes \mathrm{id}(a \otimes s_{ij})\Big\| < \frac{\varepsilon}{2n^2}.$$

Therefore

$$\Big\|\psi_{ij}(a) - W^*r_i^*\Big(\sum_{k=1}^{m} J_k \otimes \mathrm{id}(a \otimes s_{11})\Big)r_jW\Big\| < \frac{\varepsilon}{n^2} \quad \text{for all } a \in \mathcal{F}.$$

Put $V' = (v_1'W, v_2'W, \ldots, v_n'W)$. We view V' as an $n \times n$ matrix whose i-th column is the nonzero column $v_i'W$, for $i = 1, 2, \ldots, n$. Then

$$\Big\|\psi(a) - V'^* \sum_{k=1}^{m} J_k \otimes \mathrm{id}(a \otimes e_{11})V'\Big\| < \varepsilon \quad \text{for } a \in \mathcal{F}.$$

Define $V = Z^*V'$. We have

$$\Big\|\psi(a) - V^* \sum_{n=1}^{m} \phi_n(a)V\Big\| < \varepsilon \quad \text{for } a \in \mathcal{F}.$$

We also note that $VV^* \in \operatorname{Her} \sum_{n=1}^{m} \phi_n(A)$. □

Lemma 5.7. *Let A be a separable C^*-algebra and let B be a unital C^*-algebra having properties P1 and P2. Suppose that $\phi_n : A \to B$ is a sequence of homomorphisms such that the embedding $j_n : \phi_n(A) \to B$ is full, where $\{\phi_n(a) : n = 1, 2, \dots\}$ is an orthogonal set in B. Suppose that $\psi : A \to B$ is amenable and such that* $\ker\psi \supset \ker\phi_n$, $n = 1, 2, \dots$. *For any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a contraction $V \in B$ and an integer $K > 0$ such that*

$$\Big\|\psi(a) - V^*\Big(\sum_{i=1}^{K}\phi_i(a)\Big)V\Big\| < \varepsilon \quad \textit{for } a \in \mathcal{F} \quad \textit{and} \quad VV^* \in \operatorname{Her}\sum_{i=1}^{K}\phi_i(A).$$

Proof. Fix a finite subset $\mathcal{F}$ and $\varepsilon > 0$. Since ψ is amenable, we may as well assume that $\psi = \alpha \circ \beta$, where $\beta : A \to M_n = M_n(\mathbb{C} \cdot 1_C)$ and $\alpha : M_n \to B$ are contractive completely positive linear maps (however, n depends on $\mathcal{F}$ as well as ε). Write $M_n(C) \subset B$ as in the definition Property P2 (see also Proposition 3.10). Put $\mathcal{G} = \beta(\mathcal{F})$. It is convenient to assume that $\mathcal{F}$ lies in the unit ball of A so $\mathcal{G}$ lies the unit ball of $M_n(\mathbb{C} \cdot 1_C)$. Note that $\sigma : M_n \to M_n(\mathbb{C}) \subset B$ is full. There exists an integer $m > 0$ and a contraction $Z \in M_m(B)$ such that

$$\|\alpha(b) - Z^*(b \operatorname{Id}_m)Z\| < \tfrac{1}{4}\varepsilon \quad \text{for } b \in \mathcal{G},$$

where Id_m is the $m \times m$ identity matrix It follows from Lemma 5.6 that there is $N(1) > 1$ and a contraction $W_1 \in B$ such that

$$\Big\|\beta(a) - W_1^* \sum_{i=1}^{N(1)} \phi_i(a) W_1\Big\| < \frac{\varepsilon}{4m} \quad \text{for } a \in \mathcal{F}$$

as well as integers $N(k+1) > N(k)$ and a contractions $W_k \in B$ such that

$$\Big\|\beta(a) - W_{k+1}^* \sum_{i=N(k)+1}^{N(k+1)} \phi_i(a) W_{k+1}\Big\| < \frac{\varepsilon}{4m} \quad \text{for } a \in \mathcal{F} \text{ and } k = 1, 2, \dots.$$

Note that

$$\big\|\alpha \circ \beta(a) - Z^*(\beta(a) \operatorname{Id}_m)Z\big\| < \tfrac{1}{2}\varepsilon \quad \text{for } a \in \mathcal{F}.$$

It follows that

$$\Big\|\psi(a) - Z^*\Big(\operatorname{diag}\Big(W_1^* \sum_{i=1}^{m(1)} \phi_i(a) W_1, \cdots, W_m \sum_{i=N(m-1)+1}^{N(m)} \phi_i(a)\Big) W_m\Big)\Big) Z\Big\| < \tfrac{1}{2}\varepsilon$$

for all $a \in \mathcal{F}$. There exist $d_i \in \operatorname{Her}(\phi_i(A))_+$ with $0 \le d_i \le 1$ such that

$$\|d_i\phi_i(a) - \phi_i(a)\| < \frac{\varepsilon}{2m} \quad \text{and} \quad \|d_i\phi_i(a)d_i - \phi_i(a)\| < \frac{\varepsilon}{2m}$$

for all $a \in \mathcal{F}$. Note that $d_i d_j = 0$ if $i \neq j$, $i, j = 1, 2, \dots, m$. Now let Y be the $n \times n$ matrix whose first row is $(d_1, d_2, \dots, d_m)$ and the rest are zero. Put

$W = \mathrm{diag}(W_1, W_2, \dots, W_m)$ and $V = YWZ$. Then

$$\Big\|\mathrm{diag}\Big(W_1^* \sum_{i=1}^{m(1)} \phi_i(a) W_1, \dots, W_m \sum_{i=N(m-1)+1}^{N(m)} \phi_i(a) W_m\Big) - W^* Y^* \sum_{k=1}^{N(m)} \phi_k(a) Y W\Big\|$$

is less than $\frac{1}{2}\varepsilon$ for $a \in \mathcal{F}$. Moreover

$$\Big\|\psi(a) - V^* \sum_{k=1}^{N(m)} \phi_k(a) V\Big\| < \varepsilon \quad \text{for } a \in \mathcal{F} \quad \text{and} \quad VV^* \in \mathrm{Her} \sum_{k=1}^{N(m)} \phi_k(A). \quad \square$$

6. Commutants in the ultrapower of corona algebras

Definition 6.1. Recall that a family ω of subsets of $\mathbb{N}$ is an *ultrafilter* if

(i) $X_1, \dots, X_n \in \omega$ implies $\bigcap_{i=1}^n X_i \in \omega$,

(ii) $\varnothing \notin \omega$,

(iii) if $X \in \omega$ and $X \subset Y$, then $Y \in \omega$, and

(iv) if $X \subset \mathbb{N}$ then either X or $\mathbb{N} \setminus X$ is in ω.

An ultrafilter is said to be *free* if $\bigcap_{X \in \omega} X = \varnothing$. The set of free ultrafilters is identified with elements in $\beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone–Çech compactification of $\mathbb{N}$.

A sequence $\{x_n\}$ (in a normed space) is said to converge to x_0 along ω, written $\lim_\omega x_n = x_0$, if for any $\varepsilon > 0$ there exists $X \in \omega$ such that $\|x_n - x_0\| < \varepsilon$ for all $n \in X$.

Let $\{B_n\}$ be a sequence of C^*-algebras. Fix an ultrafilter ω. The ideal of $l^\infty(\{B_n\})$ consisting of those sequences $\{a_n\}$ in $l^\infty(\{B_n\})$ such that $\lim_\omega \|a_n\| = 0$ is denoted by $c_\omega(\{B_n\})$. Define

$$q_\omega(\{A_n\}) = l^\infty(\{B_n\})/c_\omega(\{B_n\}).$$

If $B_n = B, n = 1, 2, \dots,$ we write $c_\omega(B)$ for $c_\omega(\{B_n\})$ and $q_\omega(A)$ for $q_\omega(\{A_n\})$.

Lemma 6.2. *Let A be a C^*-algebra, I an ideal of A, and let $a \in A \setminus \{0\}$ be such that $0 \le a \le 1$. Suppose that $a \notin I$. Then there is $b \in C^*(a)$ with $0 \le b \le 1$ and $\|b\| = 1$ such that if $c \in C^*(b) \setminus J$, then $c \notin I$, where*

$$J = \{f(b) : f \in C_0(sp(b) \setminus \{0\}),\ f(1) = 0\}.$$

Proof. Let $\pi : A \to A/I$ be the quotient map. Then $\pi(a) \neq 0$. Suppose that $\xi \in sp(\pi(a)) \setminus \{0\}$. Note $sp(\pi(a)) \subset sp(a)$. Let $f \in C_0(sp(a) \setminus \{0\})$ such that $f(\xi) = 1$ and $0 < f(t) < 1$ for all other $t \in sp(a) \setminus \{0\}$. Set $b = f(a)$. Then, $\pi(b) \neq 0$ and $\|\pi(b)\| = 1$. If $c \notin J$, $c = g(b)$ for some $g \in C_0(sp(b) \setminus \{0\})$ such that $g(1) \neq 0$. Thus $c = g \circ f(a)$. Note that $g \circ f(\xi) \neq 0$. It follows that $\pi(c) = \pi(g \circ f(a)) \neq 0$. Therefore $c \notin I$. $\square$

Lemma 6.3. *Let B be a unital C^*-algebra and let $a \in B$ be an element with $0 \le a \le 1$. Suppose that there is $x \in B$ such that $x^*ax = 1$. Then there exists an element $b \in C^*(a)$ such that c is full for all $c \in C^*(b) \setminus J$, where*

$$J = \{f(b) : f \in C_0(sp(b) \setminus \{0\}),\ f(1) = 0\}.$$

Proof. Put $v = a^{1/2}x$. Then $v^*v = 1$ and $vv^* = q$ for some projection $q \in B$. Note that $q \in \mathrm{Her}(a^{1/2}xx^*a^{1/2}) \subset \mathrm{Her}(a)$. For any $0 < \varepsilon < \frac{1}{4}$, there is $N > 0$ such that

$$\|f_n(a)p - p\| < \tfrac{1}{2}\varepsilon \quad \text{for all } n \ge N,$$

with f_n is as in (1-1). It follows that

$$\|f_n(a)pf_n(a) - p\| < \varepsilon \quad \text{for all } n \ge N.$$

Hence there is a projection $q \in \mathrm{Her}(f_N(a))$ and a partial isometry $w \in B$ such that $w^*qw = 1$ and $ww^* = q$. Thus $f_{N+1}(a)q = q$. Put $b = f_{N+1}(a)$. For any function $g \in C_0((0, 1])$, if $g(1) \ne 0$, then $g(b)q = q$. It follows that $w^*g(b)w = 1$, so $g(b)$ is full and the lemma follows. □

Lemma 6.4. *Let A be a unital separable C^*-subalgebra of a unital C^*-algebra B which has properties P1 and P3. Suppose that every nonzero element in A is full in B. Then there exists a sequence of sequences of positive elements $\{a_n^{(i)}\}$, $i = 1, 2, \ldots$ with $0 \le a_n^{(i)} \le 1$ satisfying the following:*

(1) $\lim_{n\to\infty} \|a_n^{(i)}a - aa_n^{(i)}\| = 0$ *for all $a \in A$ and $i = 1, 2, \ldots$.*

(2) $\lim_{n\to\infty} \|a_n^{(i)}a_n^{(j)}\| = 0$ *if $i \ne j$.*

(3) $\Pi(\{a_n^{(i)}\})\Pi \circ J(a)$ *is full in $q_\omega(A)$ for any free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, where $J : B \to l^\infty(B)$ is defined by $J(b) = (b, b, \ldots, b, \ldots)$ for $b \in B$ and $\Pi : l^\infty(B) \to q_\omega(B)$ is the quotient map.*

Proof. For each nonzero element $0 \le a \le 1$ in A, define

$$r(a) = \inf\{\|x\| : x^*ax = 1\}.$$

Let $b_1, b_2, \ldots, b_n, \ldots$ be a dense sequence of the unit ball of A. We may assume that $\{b_n\}$ contains a subsequence of positive elements which is dense in the positive part of the unit ball. For each $0 \le b_k \le 1$ in the sequence, from the assumption, there is $x_k \in B$ such that $x_k^*b_kx_k = 1$ and $\|x_k\| \le \frac{4}{3}r(b_k)$. Let D be the separable C^*-subalgebra generated by A and $\{x_k\}$.

We claim that, for each nonzero $a \in A$ with $0 \le a \le 1$ there is $x \in D$ such that $x^*ax = 1$. There is $z \in B$ such that $z^*az = 1$ and $\|z\| < \frac{4}{3}r(a)$. There is b_k with $0 \le b_k \le 1$ for which

$$\|a - b_k\| < \frac{1}{8\left(\frac{4}{3}r(a) + 1\right)^2}.$$

Then

$$\|z^*b_kz-1\|\le\|z^*(b_k-a)z\|\le\tfrac{1}{8}.$$

We obtain $y\in D$ with $\|y\|<\frac{8}{7}$ such that

$$y^*z^*b_kzy=1.$$

It follows that $r(b_k)\le\frac{8}{7}r(a)$. Hence there is $x_k\in D$ with $\|x_k\|\le\frac{4}{3}\cdot\frac{8}{7}r(a)$ such that $x_k^*b_kx_k=1$. It follows that

$$\|x_k^*ax_k-1\|\le\|x_k^*(a-b_k)x_k\|<\big(\tfrac{1}{8}(\tfrac{4}{3}r(a)+1)^2\big)\big(\tfrac{4}{3}\cdot\tfrac{8}{7}r(a)\big)^2<\tfrac{8}{49}<1.$$

Thus there is $d\in D$ such that

$$d^*x_kax_kd=1.$$

This proves the claim.

Now since B has property P3 and D is separable, there exists a sequence of sequences of nonzero elements $\{a_n^{(i)}\}$ in B with $0\le a_n^{(i)}\le 1$ such that

(i) $\lim_{n\to\infty}\|a_n^{(i)}d-da_n^{(i)}\|=0$ for all $d\in D$,

(ii) $\lim_{n\to\infty}\|a_n^{(i)}a_n^{(j)}\|=0$ if $i\neq j$, and

(iii) for each i, $\{a_n^{(i)}\}$ is full in $l^\infty(A)$.

Thus (1) and (2) follow. To see (3), let $a\in A$. From the claim, there is $d\in D$ such that

$$d^*ad=1.$$

Put $a_i=\{a_n^{(i)}\}$. Then, by Proposition 4.3, there is $z\in l^\infty(A)$ such that $z^*a_iz=1$. Note that (i) implies that

$$\Pi(a_i)\Pi\circ J(b)=\Pi\circ J(b)\Pi(a_i)\quad\text{for all } b\in D.$$

Put $g=\Pi\circ J(d)\Pi(z)$. Then

$$\begin{aligned}g^*\Pi(a_i)\Pi\circ J(a)g&=\Pi(z^*)\Pi\circ J(d^*)\Pi(a_i)\Pi\circ J(a)\Pi\circ J(d)\Pi(z)\\&=\Pi(z^*)\Pi(a_i)\Pi\circ J(d^*)\pi\circ J(a)\Pi\circ J(d)\Pi(z)\\&=\pi(z^*)\Pi(a_i)\Pi(z)=1.\end{aligned}$$

□

Lemma 6.5. *Let A be a unital separable amenable C^*-algebra and B a unital C^*-algebra having properties P1, P2 and P3. Let $\omega\in\beta\mathbb{N}\setminus\mathbb{N}$ be a free ultrafilter. Suppose that $\tau:A\to B$ is a full unital embedding. Let $\tau_\infty:A\to l^\infty(B)$ be defined by $\tau_\infty(a)=(\tau(a),\tau(a),\dots)$ and let $\psi=\Pi\circ\tau_\infty$, where $\Pi:l^\infty(B)\to q_\omega(B)$. Then there is a unital C^*-subalgebra $C\cong\mathcal{O}_\infty$ in the commutant of $\psi(A)$ in $q_\omega(B)$.*

Proof. Let $\{a_n^{(i)}\}$ be the sequence of sequences of elements given by Lemma 6.4. Put $a_i = \{a_n^{(i)}\}$, $i = 1, 2, \ldots$.

Applying Lemma 6.3, and introducing and using D as in the proof of Lemma 6.4, we may assume that each a_i has the property that $sp(a_i) \subset [0, 1]$ and $f(\Pi(a_i))$ is full for all $0 \le f \le 1$ in $C_0((0, 1])$ for which $f(1) \ne 0$.

Let $X = (0, 1]$ and fix i. Define $\phi'_j, L' : C_0(X) \otimes A \to q_\omega(B)$ by $\phi'_i(f \otimes a) = f(\Pi(a_i))\psi(a)$ and $L'(f \otimes a) = f(1)\psi(a)$ for $a \in A$. By Lemma 6.4(3), ϕ'_i is full. Let $\{\mathcal{F}_j\}$ be an increasing sequence of finite subsets of A for which $\bigcup_{n=1}^\infty \mathcal{F}_j$ is dense in A and let $\{g_n\}$ be a dense sequence of $C_0((0, 1])$.

Let $\{a_{i(k)}\}_{k=1}^\infty$ be a subsequence of $\{a_i\}$. It follows from Lemma 5.7 that there exists $s_n \in B$ such that

$$\Big\| s_n^*\Big(\sum_{k=1}^{m(n)} g_j(a_{i(k)})\psi(a)\Big)s_n - g_j(1)\psi(a)\Big\| < \frac{1}{2^n} \quad \text{for } a \in \mathcal{F}_n \text{ and } j = 1, 2, \ldots, n.$$

Moreover, $s_n s_n^* \in \mathrm{Her}(\sum_{k=1}^{m(n)}(a_{i(k)})\psi(A))$. Suppose that $s_n = \Pi((s_{n,1}, s_{n,2}, \ldots))$, $n = 1, 2, \ldots$. We may assume that

$$\Big\| s_{n,k(n)}^*\Big(\sum_{k=1}^{m(n)} g_j(a_{k(n)}^{i(k)})\tau(a)\Big)s_{n,k(n)} - g_j(1)\tau(a)\Big\| < \frac{1}{2^n} \quad \text{for } n = 1, 2, \ldots.$$

Now put $t_n = s_{n,k(n)}$, $t' = (t_1, t_2, \ldots)$ and $t = \Pi(t')$. Define $\Phi : C_0(X) \otimes A \to l^\infty(B)$ by

$$\Phi(f \otimes a) = \Big\{\sum_{k=1}^{m(n)} f(a_{k(n)}^{(i(k)})\tau(a)\Big\} \quad \text{for all } f \in C_0(X) \text{ and } a \in A.$$

It follows that

$$t^*\Pi \circ \Phi(f \otimes a)t = f(1)\psi(a) \quad \text{for all } f \in C_0(X) \text{ and } a \in A.$$

Put $b(\{i(k)\}) = \Pi(\{a_{k(n)}^{i(k)}\})$. Note that $0 \le b(\{i(k)\}) \le 1$. We have (with $\iota(t) = t$ for all $t \in (0, 1]$)

$$t^*b(\{i(k)\})t = \iota(1) = 1_{q_\omega(B)}.$$

Put

$$w(\{i(k)\}) = b(\{i(k)\})^{1/2}t \quad \text{and} \quad q = b(\{i(k)\})^{1/2}tt^*b(\{(i(k)\})^{1/2}.$$

Since $b(\{i(k)\}) \in \psi(A)'$ and $\iota(1) = 1$, we have

$$\text{(6-1)} \qquad t^*b(\{i(k)\})^{1/2}\psi(a)b(\{i(k)\})^{1/2}t = t^*b(\{i(k)\})\psi(a)t = \iota(1)\psi(a)$$
$$= \psi(a) \quad \text{for all } a \in A.$$

It follows from [Rørdam 2002, 6.36] that $w(\{i(k)\}) = b(\{i(k)\})^{1/2}t \in \psi(A)'$. If $\{i(k)\}$ and $\{i(k)'\}$ are disjoint infinite subsets of $\mathbb{N}$, the corresponding projections q and q' are orthogonal. Thus there is a sequence of isometries $v_k \in \psi(A)'$ such that $v_k^*v_k = 1_{q_\omega(B)}$ and $1 \geq \sum_{k=1}^n v_kv_k^*$, $n = 1, 2, \ldots$. Thus $\psi(A)'$ admits a unital embedding of $\mathcal{O}_\infty$, □

7. Full extensions

Definition 7.1. Let $\mathbf{Ext}(A, B)$ be the set of *stable* unitary equivalence classes of extensions of the form (2-1). When A is amenable, it is known (by work of Arveson, Choi and Effros) that $\mathbf{Ext}(A, B)$ is a group. Moreover, the group can be identified with $KK^1(A, B)$. Let $\mathcal{T}(A, B)$ be the set of all *stable* unitary equivalence classes of approximately trivial extensions. It is known that $\mathcal{T}(A, B)$ is a subgroup of $KK^1(A, B)$ (see [Lin 2005a]). Following Rørdam, one defines $KL^1(A, B) = KK^1(A, B)/\mathcal{T}(A, B)$.

Let G_1, G_2, G_3 be abelian groups. A group extension $0 \to G_1 \to G_3 \to G_2 \to 0$ is said to be *pure* if every finitely generated subgroup of G_2 lifts. Denote by $\mathrm{Pext}(G_2, G_1)$ the set of all pure extensions and by $E(G_2, G_1)$ the quotient $\mathrm{ext}_{\mathbb{Z}}(G_2, G_1)/\mathrm{Pext}(G_2, G_1)$.

If A satisfies the Approximate Universal Coefficient Theorem, or AUCT (see [Lin 2005a]), one has the following short exact sequence:

$$0 \to E(K_i(A), K_i(B)) \to KL^1(A, B) \to \mathrm{Hom}(K_i(A), K_{i-1}(B)) \to 0. \tag{7-1}$$

So $KL^1(A, B)$ may be computable in theory. It should be noted every separable amenable C^*-algebra which satisfies the Universal Coefficient Theorem (UCT) satisfies the AUCT. Rosenberg and Schochet [1987] have shown that every separable C^*-algebras in the so-called *bootstrap class* satisfies the UCT (therefore the AUCT). We also use the notation $KL(A, B) = KL^1(A, SB)$.

As mentioned in the introduction, two stably unitarily equivalent extensions are in general not unitarily equivalent and trivial extensions are not unitarily equivalent. Furthermore, an essential extension which is zero in $KK^1(A, B)$ may not be trivial (or approximately trivial). We will use $KL^1(A, M(B)/B)$ to give a classification of full essential extensions up to approximately unitary equivalence.

Proposition 7.2. *Let D be a unital C^*-algebra for which there is a unital embedding from $\mathcal{O}_2$ to D. Let $h_1, h_2 : \mathcal{O}_2 \to D$ be two full homomorphisms. Suppose that $h_1(1_{\mathcal{O}_2}) \sim h_2(1_{\mathcal{O}_2})$. Then there is a sequence of partial isometries v_n such that*

$$v_n^*v_n = h_2(1_{\mathcal{O}_2}),\ v_nv_n^* = h_1(1_{\mathcal{O}_2}) \quad \text{and} \quad \lim_{n\to\infty} \|v_n^*h_1(a)v_n - h_2(a)\| = 0$$

for all $a \in \mathcal{O}_2$.

Proof. This is the combination of Theorem 6.5 and Lemma 7.2 in [Lin 2007]. □

Lemma 7.3. *Let A be a unital separable C^*-algebra and let B and C be unital C^*-algebras such that $B \otimes \mathcal{O}_2$ is a unital C^*-subalgebra of C and C has property P1.*

Suppose that $h_1, h_2 : A \to B \otimes \mathbb{C}\cdot 1 \subset B \otimes \mathcal{O}_2$ are two unital full monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent in C.

Proof. By [Rørdam 1994] we have $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$. Let $p_n = 1_B \otimes q_n \otimes 1_{\mathcal{O}_2}$, where $\{q_n\}$ is a sequence of mutually orthogonal nonzero projections in $\mathcal{O}_2$. Note that $p_n \sim 1_{B\otimes\mathcal{O}_2\otimes\mathcal{O}_2}$, $n = 1, 2, \ldots$. Define $\phi_i(a) = p_i h_1(a)$ and $\psi_i(a) = p_i h_2(a)$ for all $a \in A$. Also define $\Phi_n(a) = \big(1-\sum_{i=1}^n p_i\big)h_1(a)$ and $\Psi_n(a) = \big(1-\sum_{i=1}^n p_i\big)h_2(a)$ for all $a \in A$. Then, for each n, $h_1 = \sum_{i=1}^n \phi_i \oplus \Phi_n$ and $h_2 = \sum_{i=1}^n \psi_i \oplus \Psi_n$. Note that ϕ_i, Φ_n, ψ_i and Ψ_n are all full. Now we work in $B \otimes \mathcal{O}_2 \otimes 1$. There are partial isometries $v_{i,j} \in \mathcal{O}_2$ such that $v_{i,j}^* v_{i,j} = p_j$ and $v_{i,j} v_{i,j}^* = p_i$ for $i, j = 1, 2, \ldots, n$ and

$$v_{n+1,j}^* v_{n+1,j} = p_j,\ \ v_{n+1,j} v_{n+1,j}^* = 1 - \sum_{i=1}^n p_i \quad \text{for } j = 1, 2, \ldots, n.$$

Put $w_{i,j} = 1 \otimes v_{i,j} \otimes 1$. Then we also have

$$w_{i,1}^* \phi_1 w_{i,1} = \phi_i \quad \text{for } i = 1, 2, \ldots, n \quad \text{and} \quad w_{n+1,1}^* \phi_1 w_{n+1,1} = \Phi_n.$$

Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots$ be an increasing sequence of finite subsets of A such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A. It follows from [Lin 2001, Lemma 5.4.2] that, for each n, there are isometries $u_n, v_n \in B \otimes \mathcal{O}_2 \otimes 1$ such that

$$\big\|u_n^* h_1(a) u_n - h_2(a)\big\| < 1/n \quad \text{and} \quad \big\|v_n^* h_2(a) v_n - h_1(a)\big\| < 1/n \quad \text{for } a \in \mathcal{F}_n.$$

Note that the relative commutant of $B \otimes \mathcal{O}_2 \otimes 1$ contains a unital C^*-subalgebra $1_B \otimes 1_{\mathcal{O}_2} \otimes \mathcal{O}_2$ which is isomorphic to $\mathcal{O}_2$. It follows from [Kirchberg and Phillips 2000, 1.10] that h_1 and h_2 are approximately unitarily equivalent. □

Lemma 7.4. *Let A be a unital separable nuclear C^*-algebra, let B_1 and B_2 be two unital C^*-algebra and let C be another unital C^*-algebra. Suppose that $j_i : B_i \otimes \mathcal{O}_2 \to C$ are two full monomorphisms so that $j_1(1) \sim j_2(1)$ and $h_i : A \to B_i$ are two full unital monomorphisms. Then there is a sequence of partial isometries $v_n \in C$ such that $v_n^* v_n = j_1(1)$, $v_n v_n^* = j_2(1)$ and*

$$\lim_{n\to\infty} \big\|v_n^*(j_2 \circ h_2(a)) v_n - j_1 \circ h_1(a)\big\| = 0 \quad \textit{for all } a \in A.$$

Proof. To simplify notation, we may assume that $j_1(1) = j_2(1)$. Therefore we may assume that both j_1 and j_2 are unital. Define $J_i : B_i \otimes \mathcal{O}_2 \to l^\infty(C)$ by $J_i(b) = (j_i(b), j_i(b), \ldots)$ for $b \in B_i \otimes \mathcal{O}_2$ and $H_i = J_i \circ h_i$, respectively, $i = 1, 2$. Note that these maps are full in $l^\infty(C)$. Since there is a unital $\mathcal{O}_2$ embedding to $l^\infty(C)$, by Proposition 7.2, we obtain unitaries $u_n \in C$ such that

$$\lim_{n\to\infty} \big\|u_n^* J_2(1 \otimes b) u_n - J_1(1 \otimes b)\big\| = 0 \quad \text{for all } b \in \mathcal{O}_2.$$

Denote $U = \{u_n\}$ in $l^\infty(C)$. Let ω be a free ultrafilter on $\mathbb{N}$ and $\pi : l^\infty(C) \to q_\omega(C)$ be the quotient map. Let D be the C^*-subalgebra generated by $\pi \circ J_1(B_1 \otimes \mathbb{C}\cdot 1_{\mathcal{O}_2})$

and $\pi \circ \operatorname{ad} U \circ J_2(B_2 \otimes \mathbb{C} \cdot 1_{\mathcal{O}_2})$. It follows that D', the commutant of D, contains $J_1(1_{B_1} \otimes \mathcal{O}_2)$ which is isomorphic to $\mathcal{O}_2$. Therefore we may write $D \subset D \otimes \mathcal{O}_2$. Now $\pi \circ H_1$ and $\pi \circ \operatorname{ad} W \circ H_2$ are two full unital monomorphisms from A into $D \subset D \otimes \mathcal{O}_2$. It follows from Lemma 7.3 that $\pi \circ H_1$ and $\pi \circ \operatorname{ad} W \circ H_2$ are approximately unitarily equivalent. It follows from [Rørdam 2002, Lemma 6.2.5] that $j_1 \circ h_1$ and $j_2 \circ h_2$ are approximately unitarily equivalent. □

Theorem 7.5. *Let A be a unital separable amenable C^*-algebra and let B be a unital C^*-algebra which has properties P1, P2 and P3. Let $j_o : A \to \mathcal{O}_2 \to B$ be a full embedding of A into B which factors through $\mathcal{O}_2$. Suppose that $\tau : A \to B$ is a full monomorphism. Then there is a sequence of partial isometries $V_n \in M_2(B)$ such that $V_n^* V = 1_B \oplus j_o(1_A)$, $V_n V_n^* = 1_B$ and*

$$\lim_{n\to\infty} \left\| V_n(\tau \oplus j_o)(a)V_n^* - \tau(a) \right\| = 0 \quad \textit{for all } a \in A.$$

Proof. Let $J : B \to l^\infty(B)$ be defined by $J(c) = (c, c, \dots)$ for $c \in B$. Define $\tau_\infty = J \circ \tau$ and $J_o = J \circ j_o$. Let ω be a free ultrafilter on $\mathbb{N}$ and $\pi : l^\infty(B) \to q_\omega(B)$ be the quotient map. It follows from Lemma 6.5 that $\pi \circ \tau_\infty(A)'$ contains a unital C^*-subalgebra which is isomorphic to $\mathcal{O}_\infty$. Denote this C^*-subalgebra by $\mathcal{O}_\infty$. Let $q \in \mathcal{O}_\infty$ be a nonzero projection such that $[q] = 0$ in $K_0(\mathcal{O}_\infty)$. There is a C^*-subalgebra C of $\mathcal{O}_\infty$ for which $1_C = q$ and $C \cong \mathcal{O}_2$. Put $\tau_0(a) = q\pi \circ \tau_\infty(a)$. So we may view τ_0 is a unital full homomorphism from A into $\tau_0(A) \otimes \mathcal{O}_2$. Since $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ by [Rørdam 1994], it follows from Lemma 7.4 that $\tau_0 \oplus \pi \circ J_o$ and τ_0 are approximately unitarily equivalent. Thus $\pi \circ \tau_\infty$ and $\pi \circ \tau_\infty \oplus \pi \circ J_o$ are approximately unitarily equivalent. It follows from [Rørdam 2002, 6.2.5] that τ and $\tau \oplus j_o$ are approximately unitarily equivalent. □

Proof of Theorem 2.6. Since A is separable, there is a unital embedding $j : A \to \mathcal{O}_2$, by [Kirchberg and Phillips 2000, 2.8]. Since B has property P2, there is a full monomorphism $\sigma : \mathcal{O}_2 \to B$. Define $\bar{j} = \sigma \circ j$. Note $\bar{j}$ is full. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. It follows from [Lin 2005a, Theorem 3.9] that there is an integer n and a unitary $v \in M_{n+1}(B)$ such that

$$\left\| v^* \operatorname{diag}\big(h_1(a), \bar{j}(a), \bar{j}(a), \dots, \bar{j}(a)\big)v - \operatorname{diag}\big(h_2(a), \bar{j}(a), \bar{j}(a), \dots, \bar{j}(a)\big) \right\| < \tfrac{1}{4}\varepsilon$$

for all $a \in \mathcal{F}$. On the other hand, by Proposition 7.2, there is an isometry $u \in M_n(\pi \circ \sigma(\mathcal{O}_2))$ with $uu^* = 1_{\pi\circ\sigma(O_2)}$ such that

$$\left\| u^* \bar{j}(a)u - \operatorname{diag}\big(\bar{j}(a), \bar{j}(a), \dots, \bar{j}(a)\big) \right\| < \tfrac{1}{4}\varepsilon$$

for $a \in \mathcal{F}$. Thus, we obtain an isometry $w \in M_2(B)$ with $ww^* = 1_B$ such that

$$\left\| w^* \operatorname{diag}(h_1(a), \bar{j}(a))w - \operatorname{diag}(h_2, \bar{j}(a)) \right\| < \tfrac{1}{2}\varepsilon \quad \text{for all } a \in \mathcal{F}.$$

By applying Theorem 7.5, we obtain a partial isometry $z \in B$ such that $z^*h_1(1_A)z = h_2(1_A)$, $zh_2(1_A)z^* = h_1(1_A)$ and

$$\|z^*h_1(a)z - h_2(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}. \qquad \square$$

Remark 7.6. If both h_1 and h_2 are unital, it is clear that z can be chosen to be unitary. If one of them is unital and the other is not, z can never be unitary. Suppose that both are not unital. Since B has properties P1, P2 and P3, we obtain full $\mathcal{O}_2$ embeddings into $h_1(1_A)Bh_1(1_A)$ and $h_2(1_A)Bh_2(1_A)$. Therefore there is a projection $e \le h_1(1_A)$ such that $h_1(1_A)$ is equivalent to $h_1(1_A) - e$ and e is a full projection. So there is a partial isometry $v \in B$ such that $v^*v = h_1(1_A)$ and $vv^* = h_1(1_A) - e$. Thus $1 - \operatorname{ad} v^* \circ h_1(1_A)$ is full. Similarly, there is a partial isometry $w \in B$ with $w^*w = h_2(1_A)$ such that $1 - \operatorname{ad} w^* \circ h_2(1_A)$ is full. Now apply Theorem 2.6 to the case that $A = \mathbb{C}$. we know that $1 - \operatorname{ad} v^* \circ h_1(1_A)$ and $1 - \operatorname{ad} w^* \circ h_2(1_A)$ are equivalent. This implies that we can choose z to be unitary in the proof of Theorem 2.6 in the case that both h_1 and h_2 are not unital.

Corollary 7.7. *Theorem 2.6 also holds for the case that $B = q_\infty(\{C_n\})$, where each C_n is a unital purely infinite simple C^*-algebras.*

Proof. It is clear that B has properties P1 and P2. From the proof of Theorem 2.6 above, we only need an absorbing Theorem 7.5 for this B. Let $\tau : A \to B$ be a full monomorphism and $j_0 : A \to \mathcal{O}_2 \to B$ be a full embedding of A into B which factors through $\mathcal{O}_2$. So we may write $j_0 = \Phi \circ j$, where $j : A \to \mathcal{O}_2$ is a monomorphism and $\Phi : \mathcal{O}_2 \to B$ is a full homomorphism. Let $L : A \to l^\infty(\{C_n\}$ be a contractive completely positive linear map for which $\pi \circ L = \tau$, where $\pi : l^\infty(\{C_n\}) \to q_\infty(\{C_n\})$ is the quotient map. Write $L = \{L_n\}$, where $L_n : A \to C_n$ is a contractive completely positive linear map. Let $\phi_n : \mathcal{O}_2 \to C$ such that $\pi \circ \{\psi_n\} = \Phi$. Denote by D_n the separable unital purely infinite simple C^*-algebra containing $L_n(A)$ and $\psi_n(\mathcal{O}_2)$. Then $q_\infty(\{D_n\}) \subset B$ and $\tau : A \to q_\infty(\{C_n\})$ and $j_0 : A \to \mathcal{O}_2 \to q_\infty(\{C_n\})$. Thus one can apply [Lin 2004a, 7.5]. $\square$

Proof of Proposition 2.9. Let $h_1 : A \to B \otimes \mathcal{K}$ be a homomorphism. It follows from [Lin 2004a, 4.5] that there is a sequence of asymptotically multiplicative contractive completely positive linear maps $\{\phi_n\}$ from A to $B \otimes \mathcal{K}$ and a sequence of unitaries $u_n \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\lim_{n\to\infty} \big\|(h \oplus \phi_n)(a) - \operatorname{ad} u_n \circ j(a)\big\| = 0 \quad \text{for all } a \in A.$$

Since B has property P2, it is easy to see that we may assume that ϕ_n maps A into B and u_n are unitaries in B. It follows from 6.5 in [Lin 2007] that, for each k, there exists a sequence of unitaries $v_n(k) \in M_2(B)$ such that

$$\lim_{n\to\infty} \big\|v_n(k)^*(\phi_n(a) \oplus j_o(a))v_n(k) - (\phi_{n+k}(a) \oplus j_o(a))\big\| = 0 \quad \text{for all } a \in A.$$

It follows from [Lin 2004a, 4.7] that there exists a homomorphism $h_1 : A \to M_2(B)$ and a sequence of unitaries $w_n \in M_2(B)$ such that

$$\lim_{n\to\infty} \left\| \operatorname{ad} w_n \circ h_1(a) - (\phi_n(a) \oplus j_o(a)) \right\| = 0 \quad \text{for all } a \in A.$$

By applying the fact that B has property P2 and applying Proposition 7.2, we obtain a sequence of isometries $z_n \in M_3(B)$ with $z_n z_n^* = j_o(1_A)$ such that

$$\lim_{n\to\infty} \left\| (h \oplus h_1 \oplus j_o)(a) - z_n^* j_o(a) z_n \right\| = 0 \quad \text{for all } a \in A.$$

Hence $[h_1] = -[h]$ in $H(A, B)$. □

Proof of Corollary 2.10. Combine Proposition 2.9 and Theorem 7.5. □

8. Classification of full extensions

Definition 8.1. Let C_n be a commutative C^*-algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Put $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$. One has the following six-term exact sequence (see [Schochet 1984]):

$$\begin{array}{ccccc} K_0(A) & \to & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \to & K_1(A) \\ \uparrow\mathbf{k} & & & & \downarrow\mathbf{k} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A)\,. \end{array}$$

In [Dadarlat and Loring 1996], $K_i(A, \mathbb{Z}/n\mathbb{Z})$ is identified with $KK^i(\mathbb{I}_n, A)$ for $i = 0, 1$. As in that paper, we use the notation

$$\underline{K}(A) = \bigoplus_{\substack{i=0,1\\ n\in\mathbb{Z}_+}} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect to direct sum decomposition and the so-called Bockstein operations; see [Dadarlat and Loring 1996]. It follows from these authors' definition that if $x \in KK(A, B)$, the Kasparov product $KK^i(\mathbb{I}_n, A) \times x$ gives an element in $KK^i(\mathbb{I}_n, B)$, which we identify with $\operatorname{Hom}(K_i(A, \mathbb{Z}/n\mathbb{Z}), K_0(B, \mathbb{Z}/n\mathbb{Z}))$. Thus one obtains a map $\Gamma : KK(A, B) \to \operatorname{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$. It was shown in the same paper that if A is in $\mathcal{N}$ then, for any σ-unital C^*-algebra B, the map Γ is surjective and $\ker \Gamma = \operatorname{Pext}(K_*(A), K_*(B))$. In particular,

$$\Gamma : KL(A, B) \to \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

is an isomorphism. It is shown in [Lin 2005a] that if A satisfies the AUCT, then Γ is also an isomorphism from $KL(A, B)$ onto $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.

Lemma 8.2. *Let B be a unital C^*-algebra which admits a full $\mathcal{O}_2$ embedding and let G_i be a countable subgroup of $K_i(B)$ $(i = 0, 1)$. There exists a unital separable*

C^*-algebra $B_0 \subset B$ which has a full $\mathcal{O}_2$ embedding such that $K_i(B_0) \supset G_i$ and $j_{*i} = \mathrm{id}_{K_i(B_0)}$, where $j : B_0 \to B$ is the embedding.

Proof. Let $p_1, \dots, p_n, \dots$ be projections and $u_1, u_2, \dots, u_n, \dots$ be unitaries in $\bigcup_{k=1}^{\infty} M_k(B)$ such that $\{p_n\}$ and $\{u_n\}$ generates of G_0 and G_1, respectively. There is a countable set S such that

$$p_n, u_n \in \bigcup_{n=1}^{\infty} \{(a_{ij})_{n\times n} \in M_n(B) : a_{ij} \in S\}.$$

Let $j_o : \mathcal{O}_2 \to B$ be a full embedding. Let $p = j(1_{\mathcal{O}_2})$ and $x_1, x_2, \dots, x_m \in B$ such that $\sum_{i=1}^{m} x_i^* p x_i = 1$. Let B_1 be the unital separable C^*-subalgebra generated by S, $\{x_1, x_2, \dots, x_m\}$ and $j(\mathcal{O}_2)$. Then B_1 has a full $\mathcal{O}_2$ embedding and $p_n, u_n \in \bigcup_{k=1}^{\infty} M_k(B_1)$ for all n. Note that $K_i(B_1)$ is countable. The embedding $j_1 : B_1 \to B$ gives homomorphisms $(j_1)_{*i} : K_i(B_1) \to K_i(B)$. Let $F_{1,i}$ be the subgroup of $K_0(B_1)$ generated by $\{p_n\}$ and $\{u_n\}$, respectively. It is clear that $(j_1)_{*i}$ is injective on $F_{1,i}$, $i = 0, 1$. In particular, the image of $(j_1)_{*i}$ contains G_i, $i = 0, 1$. Let $N'_{1,i} = \ker(j_1)_{*i}$ and let $N_{1,i}$ be the set of all projections (if $i = 0$), or unitaries (if $i = 1$) in $\bigcup_{k=1}^{\infty} M_k(B_1)$ which have images in $N'_{1,i}$. Let $\{p_{1,n}\}$ be a dense subset of projections in $\bigcup_{k=1}^{\infty} M_k(B_1)$. There are countable pairs of projections $\{e_n, e'_n\}$ in $\{p_{1,n}\}$ such that $[e_n] = [e'_n]$ in $K_0(B)$. There are $w_n \in \bigcup_{k=1}^{\infty} M_k(B)$ such that $w_n^* w_n = e_n \oplus 1_{k(n)}$ and $w_n w_n^* = e'_n \oplus 1_{k(n)}$.

Let $\{u_{1,n}\}$ be a dense subset of unitaries in $\bigcup_{k=1}^{\infty} M_k(B_1)$. For each $u_{1,n}$, there are unitaries $z_{1,n,k} \in \bigcup_{j=1}^{\infty} M_j(B)$, $k = 1, 2, \dots, m(n)$ such that

$$\|z_{1,n,1} - 1\| < \tfrac{1}{2}, \|z_{1,n,m(n)} - u_{1,n}\| < \tfrac{1}{2} \quad \text{and} \quad \|z_{1,n,k} - z_{1,n,k+1}\| < \tfrac{1}{2},$$

$k = 1, 2, \dots, m(n)$, $n = 1, 2, \dots$. Let B_2 be a separable unital C^*-algebra containing B_1 such that $\bigcup_{k=1}^{\infty} M_k(B_2)$ contains all $\{w_{1,n}\}$ and $\{z_{1,n,k}\}$. Note that there is a full embedding of $\mathcal{O}_2$ to B_2. Note also that if $p, q \in \bigcup_{k=1}^{\infty} M_k(B_1)$ are projections such that $[p] - [q] \in N_{1,0}$ then $[p] - [q] = 0$ in $K_0(B_2)$. Similarly, if $u \in B_1$ and $[u] \in N_{1,1}$, then $[u] = 0$ in B_2. Suppose that B_l has been constructed. Let $j_l : B_l \to B$ be the embedding. Let $N_{l,i} = \ker(j_l)_{*i}$, for $i = 0, 1$. As before, we obtain a unital separable C^*-algebra $B_{l+1} \supset B_l$ such that every pair of projections $p, q \in \bigcup_{k=1}^{\infty} M_k(B_l)$ with $[p] - [q] \in N_{l,0}$ has the property that $[p] = [q]$ in $K_0(B_{l+1})$, and every unitary $u \in B_l$ with $[u] \in N_{l,1}$ has the property that $[u] = 0$ in $K_1(B_{l+1})$. Let B_0 be the closure of $\bigcup_{l=1}^{\infty} B_l$. Note that B_0 admits a full $\mathcal{O}_2$ embedding, say $j : B_0 \to B$, and that B_0 is separable.

We claim that j_{*i} is injective. Suppose that $p, q \in M_k(B_0)$ is a pair of projections for which $[p] - [q] \in \ker j_{*0}$ and $[p] - [q] \neq 0$ in B_0. Without loss of generality, we may assume that $p, q \in M_k(B_l)$ for some large integer l. Then $[p] - [q]$ must be in the $\ker(j_l)_{*0}$. By the construction, $[p] - [q] = 0$ in $K_0(B_{l+1})$. This would imply that $[p]-[q]=0$ in $K_0(B_0)$. Thus j_{*0} is injective. An exactly same argument shows that j_{*1} is also injective. The lemma follows. □

Lemma 8.3. *Let B be a unital C^*-algebra which admits a full $\mathcal{O}_2$ embedding. Suppose that $G_i \subset K_i(B)$ and $F_i(k) \subset K_i(B, \mathbb{Z}/k\mathbb{Z})$ are countable subgroups such that the image of $F_i(k)$ in $K_{i-1}(B)$ is contained in G_{i-1}, for $i=0,1$ and $k=2,3,\dots$. Then there exists a separable unital C^*-algebra $C \subset B$ admitting a full $\mathcal{O}_2$ embedding and such that $K_i(C) \supset G_i$, $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset F_i(k)$ and the embedding $j : C \to B$ induces an injective map*

$$j_{*i} : K_i(C) \to K_i(B)$$

and an injective map

$$j_* : K_i(C, \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}) \quad \textit{for } k = 2, 3, \dots.$$

Proof. By Lemma 8.2, there is a separable unital C^*-algebra C_1 admitting a full $\mathcal{O}_2$ embedding $j : C_1 \to B$ and such that $K_0(C_1) \supset G_0$, $K_1(C_1) \supset G_1$ and j induces an identity map on $K_0(C_1)$ and $K_1(C_1)$. Fix k, and let $\{x \in K_i(C_1) : kx = 0\} = \{g_1^{(i)}, g_2^{(i)}, \dots, \}$. Suppose that $\{s_1^{(i)}, s_2^{(i)}, \dots, \}$ is a subset of $K_{i-1}(B, \mathbb{Z}/k\mathbb{Z})$ such that the map from $K_{i-1}(B, \mathbb{Z}/k\mathbb{Z})$ to $K_i(B)$ maps $s_j^{(i)}$ to $g_j^{(i)}$. For each $z^{(i)} \in K_{i-1}(C_1, \mathbb{Z}/k\mathbb{Z})$, there is $s_j^{(i)}$ such that

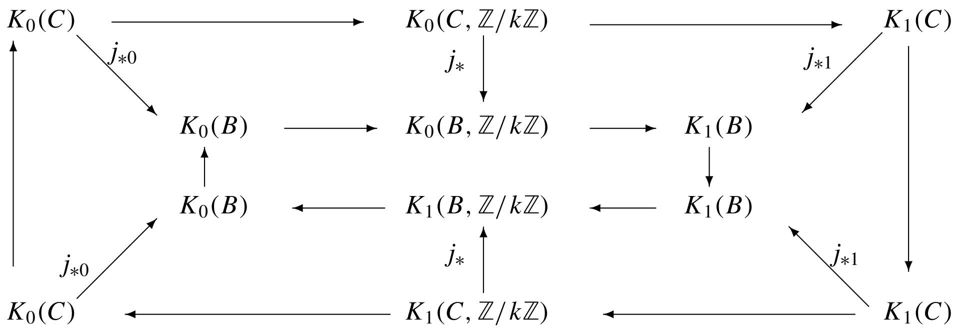
$$z^{(i)} - s_j^{(i)} \in K_i(B)/kK_i(B).$$

Since $K_i(C_1)$ is countable, the set of all possible $z^{(i)} - s_j^{(i)}$ is countable. Thus one obtains a countable subgroup G_i' that contains $K_i(C_1)$ and for which $G_i'/kK_i(B)$ contains the countable set just mentioned, as well as $F_i(k) \cap (K_i(B)/kK_i(B))$ for each k. Since countably many countable sets is still countable, we obtain a countable subgroup $G_i^{(2)} \subset K_i(B)$ such that $G_i^{(2)}$ contains G_i' and $kK_i(B) \cap G_i^{(2)} = kG_i^{(2)}$, $k = 1, 2, \dots$, and $i = 0, 1$. Note also $F_i(k) \cap (K_i(B)/kK_i(B)) \subset G_i^{(2)}/kK_i(B)$. By applying Lemma 8.2, we obtain a separable unital C^*-algebra $C_2 \supset C_1$ such that $K_i(C_2) \supset G_i^{(2)}$ and an embedding from C_2 to B gives an injective map on $K_i(C_2)$, $i = 0, 1$. Repeating what we have done above, we obtain an increasing sequence of countable subgroups $G_i^{(n)} \subset K_i(B)$ such that $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)}$ for all k and $i = 0, 1$ and an increasing sequence of separable C^*-subalgebra s C_n such that $K_i(C_n) \supset G_i^{(n)}$ and embeddings from C_n into B giving injective maps on $K_i(C_n)$, $i = 0, 1$, and $n = 1, 2, \dots$. Moreover $F_i^{(k)} \cap (K_i(B)/kK_i(B)) \subset K_i(C_n)/kK_i(B)$. Let C denote the closure of $\bigcup_n C_n$ and $j : C \to B$ be the embedding. Then C is a separable unital C^*-algebra and j_{*i} is an injective map, $i = 0, 1$. Since $C \supset C_1$ and C_1 is unital, C admits a full $\mathcal{O}_2$ embedding. We claim that $K_i(C) \cap kK_i(B) = kK_i(C)$, $k = 1, 2, \dots$, and $i = 0, 1$. Note that $K_i(C) = \bigcup_n G_i^{(n)}$. Since $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)} \subset kK_i(C)$, we see that $K_i(C) \cap kK_i(B) = kK_i(C)$,

$i = 0, 1$. Thus $K_i(C)/kK_i(C) = K_i(C)/kK_i(B)$. Since

$$K_i(C)/kK_i(B) \supset F_i^{(k)} \cap (K_i(B)/kK_0(B)),$$

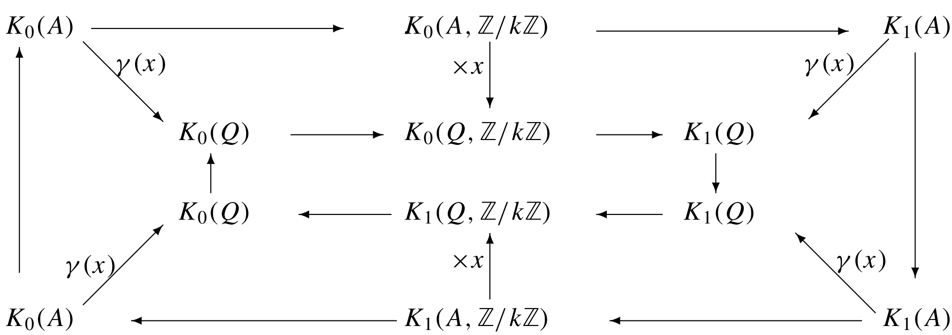
we conclude also that $K_i(C, \mathbb{Z}/k\mathbb{Z})$ contains $F_i(k)$. Since j_{*i} is injective, j induces an injective map from $K_i(C)/kK_i(C)$ into $K_i(B)/kK_i(B)$ for all integers $k \geq 1$. Using this and the fact that $j_{*i} : K_i(C) \to K_i(B)$ is injective, by chasing around the commutative diagram



one sees that j induces an injective map from $K_i(C, \mathbb{Z}/k\mathbb{Z})$ to $K_i(B, \mathbb{Z}/k\mathbb{Z})$. $\square$

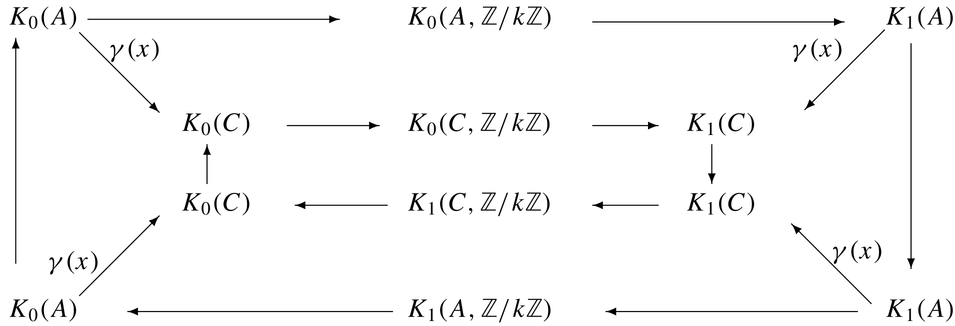
Corollary 8.4. *Without assuming that B has a full $\mathcal{O}_2$ embedding, both Lemma 8.3 and Lemma 8.2 hold if we do not require that C (or B_0) has a full $\mathcal{O}_2$ embedding.*

Proof of Theorem 2.13. By Theorem 2.5, it suffices to show that, for each $x \in KL(A, M(B)/B)$, there is a full monomorphism $h : A \to M(B)/B$ such that $[h] = x$. Put $Q = M(B)/B$. Since A satisfies the AUCT, we may view x as an element in $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q))$. Note that $K_i(A)$ is a countable abelian group ($i = 0, 1$). Let $G_0^{(i)} = \gamma(x)(K_i(A))$, $i = 0, 1$, where $\gamma : \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q)) \to \mathrm{Hom}(K_*(A), K_*(Q))$ is the surjective map. Then $G_0^{(i)}$ is a countable subgroup of $K_i(Q)$, $i = 0, 1$. Consider the commutative diagram

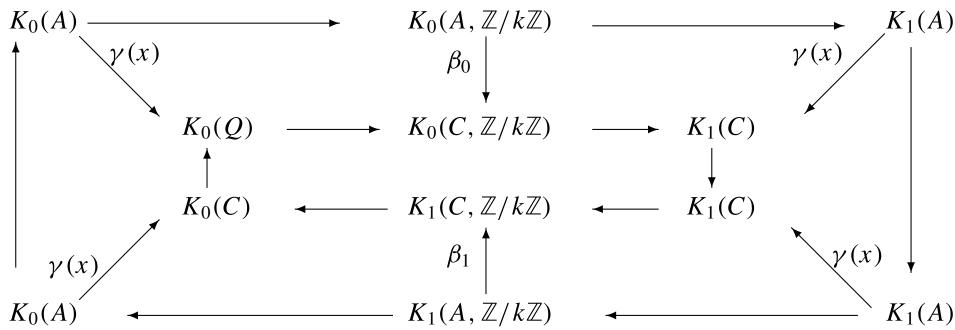


It follows from Lemma 8.3 that there is a unital C^*-algebra $C \subset Q$ which has a full $\mathcal{O}_2$ embedding such that $K_i(C) \subset G_0^{(i)}$, $K_i(C) \cap kK_i(Q) = kK_i(C)$, $k = 1, 2, \ldots,$ and $i = 0, 1$, and the embedding $j : C \to Q$ induces injective maps on $K_i(C)$

as well as on $K_i(C, \mathbb{Z}/k\mathbb{Z})$ for all k and $i = 0, 1$. Moreover $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset (\times x)(K_i(A, \mathbb{Z}/k\mathbb{Z}))$ for $k = 1, 2, \ldots$ and $i = 0, 1$. We have the commutative diagram



We will add two more maps on the above diagram. From the fact that the image of $K_i(A, \mathbb{Z}/k\mathbb{Z})$ under $\times x$ is contained in $K_i(C, \mathbb{Z}/k\mathbb{Z})$, ($k = 2, 3, \ldots, i = 0, 1$), we obtain two maps $\beta_i : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_i(C, \mathbb{Z}/k\mathbb{Z})$, $k = 2, 3, \ldots, i = 0, 1$ such that $j_* \circ \beta_i = \times x$ and obtain the commutative diagram



Consider the commutative diagram

$$\begin{array}{ccccccccc}
\to & K_i(A, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(A, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(A, \mathbb{Z}/m\mathbb{Z}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & K_i(Q, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(Q, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(Q, \mathbb{Z}/m\mathbb{Z}) & \to
\end{array}$$

Since $j_* \circ \beta_i = \times x$ and all vertical maps in the diagram

$$\begin{array}{ccccccccc}
\to & K_i(C, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(C, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(C, \mathbb{Z}/m\mathbb{Z}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & K_i(Q, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(Q, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(Q, \mathbb{Z}/m\mathbb{Z}) & \to
\end{array}$$

are injective, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
\to & K_i(A, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(A, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(A, \mathbb{Z}/m\mathbb{Z}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & K_i(C, \mathbb{Z}/mn\mathbb{Z}) & \to & K_i(C, \mathbb{Z}/n\mathbb{Z}) & \to & K_{i-1}(C, \mathbb{Z}/m\mathbb{Z}) & \to
\end{array}$$

Thus we obtain $y \in KL(A, C)$ such that $y \times [j] = x$. Since A satisfies the AUCT, one checks that $KL(A, C) = KL(A \otimes \mathcal{O}_\infty, C)$. This also follows from the fact that the unital embedding from $A \to A \otimes \mathcal{O}_\infty$ gives a KK-equivalence; see [Pimsner 1997]. It follows from [Lin 2007, 6.6, 6.7] that there exists a homomorphism $\phi : A \otimes \mathcal{O}_\infty \to C \otimes \mathcal{K}$ such that $[\phi] = y$. Define $\psi = \phi|_{A\otimes 1}$. By the same result of Pimsner, one obtains that $[\psi] = y$. Since A is unital, we may assume that the image of ψ is in $M_m(C)$ for some integer $m \geq 1$. Since C admits a full $\mathcal{O}_2$ embedding, C has property P2. Thus 1_m is equivalent to a projection in C. Thus we may further assume that ψ maps A into C. Put $h_1 = j \circ \psi$. To obtain a full monomorphism, we use the embedding $\iota : A \to \mathcal{O}_2$ given by [Kirchberg and Phillips 2000, Theorem 2.8]. Since $M(B)/B$ has property P2, we obtain a full monomorphism $\psi : \mathcal{O}_2 \to M(B)/B$. Let $e = \psi(1_{\mathcal{O}_2})$. There is a partial isometry $w \in M_2(M(B)/B)$ such that $w^*w = 1_{M(B)/B}$ and $ww^* = 1 \oplus e$. Define $h = w^*(h_1 \oplus \psi \circ \iota)w$. One checks that $[h] = [h_1] = x$ and h is a full monomorphism. □

Corollary 8.5. *Let A be a unital separable amenable C^*-algebra satisfying the AUCT. Let B be a unital C^*-algebra which has property P2. Then, for each $x \in KL(A, B)$, there is a full monomorphism $h : A \to B$ such that $[h] = x$.*

Proof. In the proof above, we may replace $M(B)/B$ by B. □

Proof of Theorem 2.14. For the first part of the theorem, it suffices to show that every essential and full extension is absorbing. Let τ be a such extension. Following Elliott and Kucerovsky, we will show that τ is purely large. Denote $E = \tau^{-1}(A)$. Choose $c \in E \setminus C$. Then, by Lemma 3.3, c is a full element. Since $M(C)$ has property P1, there exists $x \in M(B)$ such that $x^*cc^*x = 1$. Therefore there exists a projection $p \leq cc^*$ for which there is $v \in M(B)$ such that $v^*v = 1$ and $vv^* = p$. Note $\overline{cBc^*} = \overline{cM(B)c^*} \cap B$. Thus $pBp \subset \overline{cBc^*}$. Now $v^*pBpv = B$, so pBp is stable and pBp is full. Thus τ is purely large, hence absorbing. The last part of the theorem follows from the next corollary. □

Corollary 8.6. *Let A be a separable unital amenable C^*-algebra and C a unital C^*-algebra, and set $B = C \otimes \mathcal{K}$. Then* **Ext**$(A, B)$ *is the same set as unitary equivalence classes of essential and full extensions of A by B.*

Proof. It suffices to show that given any element $x \in \mathbf{Ext}(A, B)$, there exists an essential and full extension $\tau : A \to M(B)/B$ so that $[\tau] = x$. There exists an essential extension $\tau_1 : A \to M(B)/B$ such that $[\tau_1] = x$. Take a monomorphism $j : A \to \mathcal{O}_2$ (see [Kirchberg and Phillips 2000]). Let $h : \mathcal{O}_2 \to M(\mathcal{K})$ be a monomorphism (given by a faithful representation of $\mathcal{O}_2$ on a separable Hilbert space). Let $\phi : M(\mathcal{K}) \to M(B)$ be the standard unital embedding and $\pi : M(B) \to M(B)/B$ be the quotient map. Then $\tau_2 = \pi \circ \phi \circ h \circ j$ gives a full essential trivial extension. It follows that $\tau = \tau_1 \oplus \tau_2$ is an full essential extension. Since $[\tau_2] = 0$, $[\tau] = [\tau_1] = x$. □

Remark 8.7. Let B be a nonstable, nonunital but σ-unital C^*-algebra. Suppose that $M(B)/B$ has properties P1, P2 and P3, and suppose that $\tau : A \to M(B)/B$ is an essential and full extension. One should not expect that such extension is purely large in general. Let $0 \to B \to E \to A \to 0$ be an essential and full extension corresponding to τ. Recall that the extension is purely large if cBc^* contains a C^*-subalgebra which is stable and cBc^* is full in B; see [Elliott and Kucerovsky 2001]. Given any element $c \in E \setminus B$, $\pi(c)$ is full in $M(B)/B$. But, in general, c need not be full in $M(B)$, nor does cBc^* need to be full in B. Examples are easily seen in the case that $B = c_0(C)$, where C is a unital purely infinite simple C^*-algebra. Suppose that $0 \to c_0(C) \to E \to A \to 0$ is a full extension and $c' \in E \setminus c_0(C)$. Write $c' = \{c'_n\} \in l^\infty(C)$. Define $c_n = c'_n$ if $n \geq N > 1$ and $c_n = 0$ if $n \leq N$. Put $c = \{c_n\}$. Then $c \in E \setminus c_0(C)$. However, it is clear that $cc_0(C)c^*$ is not full in $c_0(C)$. By Theorem 7.5, the full extension τ is approximately absorbing in the sense of Theorem 7.5 but not purely large. It should be also noted that, even if c^*Bc is full for all $c \in E \setminus B$, the full extension may not be purely large. Let B be a nonstable, nonunital but σ-unital simple C^*-algebra with continuous scale (see [Lin 2004b] for more examples). Then B may be stably finite. No hereditary C^*-subalgebra of B contains a stable C^*-subalgebra. So none of the essential extensions of a unital separable amenable C^*-algebra A by B could be possibly purely large in the sense of [Elliott and Kucerovsky 2001]; nevertheless, all of these extensions are approximately absorbing in the sense of Theorem 7.5 (and many of them are actually absorbing: for example, when $A = C(X)$).

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