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BOUNDARY-CLUSTERED INTERFACES FOR THE ALLEN–CAHN EQUATION

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We consider the Allen–Cahn equation

$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$ and $\varepsilon > 0$ is a small parameter. We prove the existence of a radial solution u_ε having N interfaces $\{u_\varepsilon(r) = 0\} = \bigcup_{j=1}^N \{r = r_j^\varepsilon\}$, where $1 > r_1^\varepsilon > r_2^\varepsilon > \cdots > r_N^\varepsilon$ are such that $1 - r_1^\varepsilon \sim \varepsilon \log(1/\varepsilon)$ and $r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log(1/\varepsilon)$ for $j = 2, \ldots, N$. Moreover, the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is exactly N.

1. Introduction

The aim of this paper is to construct a family of *clustered* transitional layered solutions to the Allen–Cahn equation

$$\text{(1-1)} \qquad \varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator, $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$, $\varepsilon > 0$ is a small parameter, and $\nu(x)$ denotes the unit outer normal at $x \in \partial\Omega$.

Problem (1-1) and its parabolic counterpart have been a subject of extensive research for many years. In order to describe some known results, we define the Allen–Cahn functional (see [Allen and Cahn 1979]),

$$J_\varepsilon[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 - F(u) \right), \quad \text{where } F(u) = -\frac{1}{4}(1 - u^2)^2.$$

The set $\{x \in \Omega \mid u(x) = 0\}$ is called the interface of u. Let $\mathrm{Per}_\Omega(A)$ be the relative perimeter of the set $A \subset \Omega$. Using Γ-convergence techniques (see [Modica 1987]), Kohn and Sternberg [1989] obtained a general result stating that in a neighborhood

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of an isolated local minimizer of Per_Ω there exists a local minimizer to the functional J_ε. They further used this idea to show the existence of a stable solution for (1-1) in two-dimensional nonconvex domains, such as a dumb-bell. Since then, the existence of solutions with a single interface intersecting the boundary has been established and studied by many authors. See [Alikakos et al. 2000; Bronsard and Stoth 1996; Flores et al. 2001; Kowalczyk 2005; Padilla and Tonegawa 1998; Sternberg and Zumbrun 1998] and the references therein. However, the existence of multiple interfaces is only proved, in the one-dimensional case, for the Allen–Cahn equation (with inhomogeneous terms)

$$\varepsilon^2 u'' + a(x)(u-u^3) = 0, \quad -1 < x < 1, \quad u'(\pm 1) = 0 \tag{1-2}$$

(see [Nakashima 2003; Nakashima and Tanaka 2003]); and, in the higher-dimensional case, for the following nonlinear equation with bistable nonlinearity and inhomogeneous term:

$$\varepsilon^2 \Delta u + u\big(u - a(|x|)\big)(1-u) = 0 \quad \text{in } B_1(0), \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_1(0) \tag{1-3}$$

(see [Dancer and Yan 2003]). The result of this last paper states that if $a(r)$ has a critical point $r_0 \in (0,1)$ such that $a(r_0) = 1/2$, $a'(r_0) = 0$, $a''(r_0) < 0$, then there exists a clustered interior-layer solution to (1-3). All three papers use the properties of the inhomogeneous terms to construct multiple (interior) interfaces. (For the Allen–Cahn equation with inhomogeneity, $\Delta u + a(x)(u-u^3) = 0$ in $\mathbb{R}^2$, see [Rabinowitz and Stredulinsky 2003; 2004].)

Here, we continue our study, initiated in [Malchiodi et al. 2005], of clustered layered solutions for semilinear elliptic equations, and show that the *homogeneous* Allen–Cahn equation *itself* can generate multiple clustered interfaces near the boundary. In that paper we showed that the singularly-perturbed Neumann problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \\ u > 0 \ \text{ in } \Omega \quad \text{and} \quad \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1-4}$$

has a clustered layered solution near the boundary. (The existence of a one-layer solution to (1-4) near the boundary was first established in [Ambrosetti et al. 2003; 2004].) The purpose of this paper is to show that a similar phenomenon happens to the Allen–Cahn equation. In particular, we establish the existence of clustered interfaces — the so-called "phantom interfaces" — in higher dimensions. Moreover, we show that, for each fixed positive integer N, there exists a solution to (1-1) with Morse index N (in the space of radial functions).

Our main result is this:

Theorem 1.1. *Let N be a fixed positive integer. There exists $\varepsilon_N > 0$ such that, for all $\varepsilon < \varepsilon_N$, problem (1-1) admits a radially symmetric solution u_ε with the following properties*:

(1) *the set of interfaces $\{u_\varepsilon(r) = 0\}$ contains N spheres $\{r = r_j^\varepsilon\}$, $j = 1, \ldots, N$, with*

$$1 - r_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \qquad r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \quad j = 2, \ldots, N. \tag{1-5}$$

More precisely, we have $u_\varepsilon(r_j^\varepsilon + \varepsilon y) \to (-1)^j H(y)$, where $H(y)$ is the unique heteroclinic solution of

$$H'' + H - H^3 = 0, \qquad H(0) = 0, \qquad H(\pm\infty) = \pm 1. \tag{1-6}$$

(2) *The solution u_ε has the energy bound*

$$J_\varepsilon[u_\varepsilon] = \omega_{n-1} N \varepsilon I[H] + o(\varepsilon), \tag{1-7}$$

where

$$I[H] = \int_{\mathbb{R}} \bigl(\tfrac{1}{2}(H')^2 - F(H)\bigr),$$

and where ω_{n-1} denotes the volume of S^{n-1}.

(3) *The Morse index of u_ε in $H_r^1(\Omega)$ is exactly N, where $H_r^1(\Omega)$ denotes the space of radial functions in $H^1(\Omega)$.*

Remark 1.2. By a simple transformation, Theorem 1.1 readily extends to (1-3) with $a(r) \equiv 1/2$.

Our approach is similar to that of [Malchiodi et al. 2005], where a finite-dimensional reduction procedure combined with a variational approach is used. Such a method has been used successfully in many other papers, for example, [Ambrosetti et al. 2003; 2004; Dancer and Yan 1999; Gui and Wei 1999; 2000; Gui et al. 2000].

In the rest of section, we introduce some notation to be used later.

By the scaling $x = \varepsilon y$, problem (1-1) is reduced to the ODE

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + f(u) = 0 & \text{for } r \in (0,\ 1/\varepsilon), \\ u'(0) = u'(1/\varepsilon) = 0, \end{cases} \tag{1-8}$$

where $f(u) = u - u^3$. From now on, we will work with (1-8).

Let $H(y)$ be the unique solution to (1-6). Set

$$\Omega_\varepsilon = (1/\varepsilon) B_1(0) = B_{1/\varepsilon}(0), \qquad \text{and} \qquad I_\varepsilon = (0,\ 1/\varepsilon). \tag{1-9}$$

For $u \in C^2(\Omega_\varepsilon)$ and $u = u(r)$, we have

$$\Delta u = u'' + \frac{n-1}{r} u'. \tag{1-10}$$

For $k \in \mathbb{N}$, we denote by $H_r^k(\Omega_\varepsilon)$ the space of radial functions in $H^k(\Omega_\varepsilon)$. On $H_r^1(\Omega_\varepsilon)$, we define an inner product as follows:

$$(u, v)_\varepsilon = \int_0^{1/\varepsilon} (u'v' + 2uv) r^{n-1} dr. \tag{1-11}$$

Similarly, the inner product on $L_r^2(\Omega_\varepsilon)$ can be defined by

$$\langle u, v\rangle_\varepsilon = \int_0^{1/\varepsilon} (uv) r^{n-1} dr. \tag{1-12}$$

We also introduce a new energy functional that, up to a positive multiplicative constant, is equivalent to J_ε:

$$\mathscr{E}_\varepsilon[u] = \tfrac{1}{2}\int_0^{1/\varepsilon} |u'|^2 r^{n-1} - \int_0^{1/\varepsilon} F(u) r^{n-1} dr, \quad u \in H_r^1(\Omega_\varepsilon). \tag{1-13}$$

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants that are independent of ε, for ε sufficiently small. The notation $A_\varepsilon \gg B_\varepsilon$ means that $\lim_{\varepsilon\to 0} |B_\varepsilon|/|A_\varepsilon| = 0$, while $A_\varepsilon \ll B_\varepsilon$ means $(1/A_\varepsilon) \gg (1/B_\varepsilon)$.

2. Some preliminary analysis

In this section we introduce a family of approximate solutions to (1-8) and derive some useful estimates.

Let H be the unique solution of (1-6). It is easy to see that

$$\begin{cases} H(y) - 1 = -A_0 e^{-\sqrt{2}|y|} + O\big(e^{-2\sqrt{2}|y|}\big) & \text{for } y > 1, \\ H(y) + 1 = A_0 e^{-\sqrt{2}|y|} + O\big(e^{-2\sqrt{2}|y|}\big) & \text{for } y < -1, \\ H'(y) = \sqrt{2} A_0 e^{-\sqrt{2}|y|} + O\big(e^{-2\sqrt{2}|y|}\big) & \text{for } |y| > 1, \end{cases} \tag{2-1}$$

where $A_0 > 0$ is a fixed constant.

We state the following well-known lemma on H. For a proof, see [Müller 1993, Lemma 4.1].

Lemma 2.1. *For the eigenvalue problem*

$$\phi'' + f'(H)\phi = \lambda\phi, \quad |\phi| \le 1, \quad \textit{in } \mathbb{R}, \tag{2-2}$$

there holds

$$\lambda_1 = 0, \qquad \phi_1 = cH', \qquad \lambda_2 < 0. \tag{2-3}$$

For $u \in H_r^2(\Omega_\varepsilon)$, we define the operator

$$\mathscr{S}_\varepsilon[u] := u_{rr} + \frac{n-1}{r} u_r + f(u). \tag{2-4}$$

We introduce the set

$$\Lambda = \left\{ \mathbf{t} = (t_1, \dots, t_N) \;\middle|\; \begin{array}{l} t_N > 1 - \varepsilon(\log(1/\varepsilon))^2, \quad 1 - t_1 \geq \eta\, \varepsilon \log(1/\varepsilon), \\ t_{j-1} - t_j > \eta\, \varepsilon \log(1/\varepsilon),\ j = 2, \dots, N \end{array} \right\}, \tag{2-5}$$

where $\eta \in (0,\ 1/8\sqrt{2})$ is a fixed number.

Let $\chi(s)$ be a cut-off function such that $\chi(s) = 1$ for $s \leq 1/4$ and $\chi(s) = 0$ for $s \geq 1/2$. For $t \in (3/4,\ 1)$, we define

$$\rho_\varepsilon(t) = H'\Big(\frac{1-t}{\varepsilon}\Big); \qquad \beta_\varepsilon(r) = \tfrac{1}{\sqrt{2}} e^{\sqrt{2}(r - 1/\varepsilon)}, \quad r \in [0,\ 1/\varepsilon], \tag{2-6}$$

and

$$\begin{aligned} H_t(r) &= H\Big(r - \frac{t}{\varepsilon}\Big), \\ H_{\varepsilon,t}(r) &= \Big(H\big(r - \frac{t}{\varepsilon}\big) - \rho_\varepsilon(t)\,\beta_\varepsilon(r)\Big)\big(1 - \chi(\varepsilon r)\big) - \chi(\varepsilon r). \end{aligned} \tag{2-7}$$

It is easy to see that, for $(1-t)/\varepsilon \gg 1$,

$$\rho_\varepsilon(t) = \sqrt{2} A_0 e^{-\sqrt{2}(1-t)/\varepsilon} + O\big(e^{-2\sqrt{2}(1-t)/\varepsilon}\big). \tag{2-8}$$

We first assume that N is *odd*. For $\mathbf{t} \in \Lambda$, we now define our approximate function:

$$H_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j}(r). \tag{2-9}$$

If N is even, we set

$$H_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon,t}(r) - 1 = \sum_{j=1}^{N+1} (-1)^j H_{\varepsilon,t_j}(r) \tag{2-10}$$

where we use the convention that $H_{\varepsilon,t_{N+1}} = 1$. So, without loss of generality, we can assume that N is *odd*.

Note that, for $r \leq 1/(2\varepsilon)$, there holds

$$|H_{\varepsilon,t}(r) - (-1)^N| + |H'_{\varepsilon,t}(r)| + |H''_{\varepsilon,t}(r)| \leq e^{-1/(C\varepsilon)}. \tag{2-11}$$

Observe also that, by construction, $H_{\varepsilon,\mathbf{t}}$ satisfies the Neumann boundary condition, namely $H'_{\varepsilon,\mathbf{t}}(0) = H'_{\varepsilon,\mathbf{t}}(1/\varepsilon) = 0$. Furthermore, $H_{\varepsilon,\mathbf{t}}$ depends smoothly on $\mathbf{t}$ as a map with values in $C^2\big([0,\ 1/\varepsilon]\big)$.

The next lemma shows that $H_{\varepsilon,\mathbf{t}}$ is a good approximate function to (1-8).

Lemma 2.2. *For ε sufficiently small and $\mathbf{t} \in \Lambda$, one has*

$$\big\|\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\big\|_{L^\infty} + \varepsilon^{n-1} \int_0^{1/\varepsilon} \big|\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\big| r^{n-1} dr \leq C\Big(\varepsilon + \sum_{j=1}^{N} \big(\rho_\varepsilon(t_j)\big)^2 + \sum_{i \neq j} e^{-\sqrt{2}|t_i - t_j|/\varepsilon}\Big). \tag{2-12}$$

Proof. Using (1-6) it is easy to see that

$$\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}] = \frac{n-1}{r} H'_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}}) - \sum_{l=1}^{N} (-1)^l f(H_{t_l}) - 2\sum_{l=1}^{N} (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r) + O\big(e^{-1/(C\varepsilon)}\big). \tag{2-13}$$

The first term in the right-hand side of (2-13) can be estimated as

$$\frac{1}{r} H'_{\varepsilon,\mathbf{t}} = \frac{1}{r}\sum_{j=1}^{N} (-1)^j \big(H'_{t_j} - \rho_\varepsilon(t_j)\beta'_\varepsilon(r)\big) + O\big(e^{-1/(C\varepsilon)}\big).$$

From the decay of H' and β_ε we deduce that

$$\Big\|\frac{1}{r} H'_{\varepsilon,\mathbf{t}}\Big\|_\infty + \varepsilon^{n-1}\int_0^{1/\varepsilon} \frac{1}{r}|H'_{\varepsilon,\mathbf{t}}| r^{n-1} dr \le C\varepsilon. \tag{2-14}$$

Next, we note that

$$\Big| f(H_{\varepsilon,\mathbf{t}}) - \sum_{l=1}^{N} f\big((-1)^l H_{t_l}\big) - 2\sum_{l=1}^{N} (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r)\Big| \le S_1 + S_2,$$

where

$$S_1 = \Big| f\Big(\sum_{j=1}^{N} (-1)^j H_{t_j}\Big) - \sum_{j=1}^{N} f\big((-1)^j H_{t_j}\big)\Big|,$$

$$S_2 = \Big| f\Big(\sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j}\Big) - f\Big(\sum_{j=1}^{N} (-1)^j H_{t_j}\Big) - 2\sum_{j=1}^{N} (-1)^j \rho_\varepsilon(t_j)\beta_\varepsilon(r)\Big|.$$

To estimate S_1 and S_2, we divide the domain $I_\varepsilon = (0, 1/\varepsilon)$ into the N intervals $I_{\varepsilon,1}, \dots, I_{\varepsilon,N}$ defined by

$$I_{\varepsilon,1} = \Big[\frac{t_1+t_2}{2\varepsilon}, \frac{1}{\varepsilon}\Big), \quad I_{\varepsilon,j} = \Big[\frac{t_j+t_{j+1}}{2\varepsilon}, \frac{t_j+t_{j-1}}{2\varepsilon}\Big), \; j = 2, \dots, N-1, \quad \text{and} \quad I_{\varepsilon,N} = \Big(0, \frac{t_N+t_{N-1}}{2\varepsilon}\Big). \tag{2-15}$$

We choose $t_0 = 2 - t_1$ and $t_{N+1} = -t_N$, so that

$$I_{\varepsilon,j} = \Big[\frac{t_j+t_{j+1}}{2\varepsilon}, \frac{t_j+t_{j-1}}{2\varepsilon}\Big), \quad j = 1, \dots, N, \qquad I_\varepsilon = \bigcup_{j=1}^{N} I_{\varepsilon,j}. \tag{2-16}$$

For $r \in I_{\varepsilon,j}$ and $j < l$, we note that

$$H_{t_l}(r) = 1 + O\big(e^{-\sqrt{2}|r-t_j/\varepsilon|}\big),$$

while for $j > l$,

$$H_{t_l}(r) = -1 + O\big(e^{-\sqrt{2}|r-t_j/\varepsilon|}\big).$$

Since N is odd, we see that

$$\sum_{l\neq j}(-1)^l H_{t_l}=\sum_{l<j}(-1)^l(H_{t_l}+1)+\sum_{l>j}(-1)^l(H_{t_l}-1). \tag{2-17}$$

Thus, we can rewrite S_1 as:

$$\begin{aligned}S_1&=f\Big(\sum_{l<j}(-1)^l(H_{t_l}+1)+(-1)^j H_{t_j}+\sum_{l>j}(-1)^l(H_{t_l}-1)\Big)-(-1)^j f(H_{t_j})\\&\qquad\qquad-\sum_{l\neq j}(-1)^l f(H_{t_l})\\&=f'\big((-1)^j H_{t_j}\big)\Big(\sum_{l<j}(-1)^l(H_{t_l}+1)+\sum_{l>j}(-1)^l(H_{t_l}-1)\Big)-\sum_{l\neq j}(-1)^l f(H_{t_l})\\&\qquad\qquad+O\Big(\sum_{l<j}(H_{t_l}+1)^2+\sum_{l>j}(H_{t_l}-1)^2\Big)\end{aligned}$$

This quantity can also be written

$$\begin{aligned}S_1&=\big(f'\big((-1)^j H_{t_j}\big)-f'(1)\big)\Big(\sum_{l<j}(-1)^l(H_{t_l}+1)+\sum_{l>j}(-1)^l(H_{t_l}-1)\Big)\\&\qquad\qquad+O\Big(\sum_{l<j}(H_{t_l}+1)^2+\sum_{l>j}(H_{t_l}-1)^2\Big)\\&=O\big(\min\{H_{t_j}+1,\,H_{t_j}-1\}\big)\Big(\sum_{l<j}(H_{t_l}+1)+\sum_{l>j}(H_{t_l}-1)\Big)\\&\qquad\qquad+O\Big(\sum_{l<j}(H_{t_l}+1)^2+\sum_{l>j}(H_{t_l}-1)^2\Big).\end{aligned}$$

Then, with some elementary computations, one finds that

$$\|S_1\|_{L^\infty(I_{\varepsilon,j})}+\varepsilon^{n-1}\int_{I_{\varepsilon,j}}|S_1(r)|r^{n-1}dr\le C\sum_{i\neq j}e^{-\sqrt{2}|t_i-t_j|/\varepsilon}. \tag{2-18}$$

It remains to estimate S_2. For this, we note that, for $r\in I_{\varepsilon,j}$ and $j\ge 2$, we have

$$\rho_\varepsilon(t_j)\,\beta_\varepsilon(r)=O\big(e^{-\sqrt{2}(1-t_1)/\varepsilon}e^{\sqrt{2}(r-1/\varepsilon)}\big),$$

from which it follows that, for $j\ge 2$,

$$\|S_2\|_{L^\infty(I_{\varepsilon,j})}+\varepsilon^{n-1}\int_{I_{\varepsilon,j}}|S_2(r)|r^{n-1}dr=O\big(e^{-2\sqrt{2}(1-t_1)/\varepsilon}\big)=O\Big(\sum_{j=1}^N\big(\rho_\varepsilon(t_j)\big)^2\Big).$$

Therefore, we just need to consider the case when $r\in I_{\varepsilon,1}$. But, since $f'(\pm1)=-2$, we have

$$\begin{aligned}S_2&=f\Big(\sum_{l=1}^N(-1)^l H_{t_l}-\sum_{l=1}^N(-1)^l\rho_\varepsilon(t_l)\,\beta_\varepsilon(r)\Big)\\&\qquad\qquad-f\Big(\sum_{l=1}^N(-1)^l H_{t_l}\Big)-f'(-1)\sum_{l=1}^N(-1)^l\rho_\varepsilon(t_l)\,\beta_\varepsilon(r)\end{aligned}$$

$$= \Big(f'\Big(\sum_{l=1}^{N} (-1)^l H_{t_l} \Big) - f'(-1) \Big) \sum_{l=1}^{N} (-1)^l \rho_\varepsilon(t_l) \beta_\varepsilon(r) + O\Big(\sum_{l=1}^{N} \rho_\varepsilon(t_l)^2 \beta_\varepsilon(r)^2 \Big)$$

$$= O\Big(\sum_{l=1}^{N} e^{-\sqrt{2}|r - t_l/\varepsilon|} \Big) \Big(\sum_{l=1}^{N} \rho_\varepsilon(t_l) \beta_\varepsilon(r) \Big) + O\Big(\sum_{l=1}^{N} \rho_\varepsilon(t_l)^2 \beta_\varepsilon(r)^2 \Big).$$

Hence, we also get

$$\|S_2\|_{L^\infty(I_1)} + \varepsilon^{n-1} \int_{I_1} |S_2(r)| r^{n-1} dr \le C \rho_\varepsilon^2(t_1). \tag{2-19}$$

□

The proof of the next lemma is postponed to the appendix.

Lemma 2.3. *Let* $\mathbf{t} \in \Lambda$. *For* ε *sufficiently small, we have*

$$\begin{aligned} \mathscr{E}_\varepsilon\Big[\sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j} \Big] &= I[H] \sum_{i=1}^{N} \Big(\frac{t_i}{\varepsilon} \Big)^{n-1} - \Big(\frac{1}{\varepsilon} \Big)^{n-1} \big(\sqrt{2} A_0^2 + o(1) \big) e^{-2\sqrt{2}(1-t_1)/\varepsilon} \\ &\quad - \sum_{j=2}^{N} \Big(\frac{t_j}{\varepsilon} \Big)^{n-1} \big(\sqrt{2} A_0^2 + o(1) \big) e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon} + O(\varepsilon^{2-n}), \end{aligned} \tag{2-20}$$

where $A_0 > 0$ *is defined in* (2-1).

3. Lyapunov–Schmidt process: finite-dimensional reduction

In this section we outline the so-called Lyapunov–Schmidt reduction process. Since this can be proved along the same ideas of [Malchiodi et al. 2005, Sections 3], we skip some of the details.

Fix $\mathbf{t} \in \Lambda$. Integrating by parts, one can show that orthogonality to $\partial H_{\varepsilon,t_j}/\partial t_j$ in $H_r^1(\Omega_\varepsilon)$, $j = 1, \dots, N$, is equivalent to orthogonality in $L^2(\Omega_\varepsilon)$ to the functions

$$Z_{\varepsilon,t_j} = \Delta\Big(\frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big) - 2 \frac{\partial H_{\varepsilon,t_j}}{\partial t_j}, \qquad j = 1, \dots, N. \tag{3-1}$$

By elementary computations, differentiating (1-6) we obtain

$$\frac{\partial H_{\varepsilon,t_j}}{\partial t_j} = -\frac{1}{\varepsilon} H'\Big(r - \frac{t_j}{\varepsilon} \Big) + \frac{1}{\varepsilon} H''\Big(\frac{1 - t_j}{\varepsilon} \Big) \beta_\varepsilon(r) + O\big(e^{-1/(C\varepsilon)} \big), \tag{3-2}$$

$$\begin{aligned} Z_{\varepsilon,t_j} &= \big(f'(H_{t_j}) - f'(\pm 1) \big) \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} + \frac{n-1}{r} \Big(\frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big)' \\ &= -\frac{1}{\varepsilon} H'_{t_j} \big(f'(H_{t_j}) - f'(\pm 1) \big) + o(1/\varepsilon), \end{aligned} \tag{3-3}$$

where $O\big(e^{-1/(C\varepsilon)}\big)$ and $o(1/\varepsilon)$ are intended both in the C^1 and H_r^1 sense.

We consider first the following linear problem: Given $h \in L^\infty(\Omega_\varepsilon)$, find a function ϕ satisfying

$$\begin{cases} L_\varepsilon[\phi] := \phi'' + \dfrac{n-1}{r}\phi' + f'(H_{\varepsilon,\mathbf{t}})\phi \;=\; h + \displaystyle\sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}; \\ \phi'(0) = \phi'(1/\varepsilon) = 0 \qquad \text{and} \qquad \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \quad j = 1, \dots, N, \end{cases} \tag{3-4}$$

for some constants c_j, $j = 1, \dots, N$. For this, define the norm

$$\|\phi\|_* = \sup_{r \in (0,\, 1/\varepsilon)} |\phi(r)|. \tag{3-5}$$

Assuming a solution to (3-4) exists, we have an estimate on ϕ:

Proposition 3.1. *Let ϕ satisfy (3-4). For ε sufficiently small, we have*

$$\|\phi\|_* \le C\|h\|_*, \tag{3-6}$$

where C is a positive constant independent of ε and $\mathbf{t} \in \Lambda$.

Proof. The argument is similar in spirit of that of [Malchiodi et al. 2005, Proposition 3.1]. For the sake of completeness, we include a proof here.

Arguing by contradiction, assume that

$$\|\phi\|_* = 1, \qquad \|h\|_* = o(1). \tag{3-7}$$

We multiply (3-4) by $\partial H_{\varepsilon,t_j}/\partial t_j$ and integrate over Ω_ε to obtain

$$\sum_{i=1}^{N} c_i \Big\langle Z_{\varepsilon,t_i}, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big\rangle_\varepsilon = -\Big\langle h, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big\rangle_\varepsilon + \Big\langle \Delta\phi + f'(H_{\varepsilon,\mathbf{t}})\phi, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big\rangle_\varepsilon. \tag{3-8}$$

From the exponential decay of H', one finds

$$\Big\langle h, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big\rangle_\varepsilon = \int_0^{1/\varepsilon} h \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} r^{n-1} dr = O\big(\|h\|_* \varepsilon^{-n}\big).$$

Moreover, integrating by parts and using (3-2) and (3-3), we deduce

$$\Big\langle \Delta\phi + f'(H_{\varepsilon,\mathbf{t}})\phi, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big\rangle_\varepsilon = \Big\langle Z_{\varepsilon,t_j} + f'(H_{\varepsilon,\mathbf{t}})\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}, \phi \Big\rangle_\varepsilon = o\big(\varepsilon^{-n}\|\phi\|_*\big).$$

From (3-2) and (3-3), we also see that

$$\Big\langle Z_{\varepsilon,t_i}, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \Big\rangle_\varepsilon = -\varepsilon^{-n-1}\Big(t_i^{n-1}\delta_{ij} \int_{\mathbb{R}} f'(H)(H')^2 + o(1)\Big), \tag{3-9}$$

where δ_{ij} denotes the Kronecker symbol. Note that, using the equation $H''' + f'(H)\,H' = 0$, we find

$$\int_{\mathbb{R}} f'(H)(H')^2 = \int_{\mathbb{R}} (H'')^2 > 0.$$

This shows that the left-hand side of the equation (3-8) is diagonally dominant in the indices i and j, and hence, by (3-7), we have

$$c_i = O\big(\varepsilon\|h\|_*\big) + o\big(\varepsilon\|\phi\|_*\big) = o(\varepsilon), \qquad i = 1, \dots, N. \tag{3-10}$$

Also, since we are assuming that $\|h\|_* = o(1)$ and since $\|Z_{\varepsilon,t_j}\|_* = O\,(1/\varepsilon)$, there holds

$$\Big\| h + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j} \Big\|_* = o(1). \tag{3-11}$$

Thus, (3-4) yields

$$\begin{cases} \phi'' + \dfrac{n-1}{r}\phi' + f'(\pm 1) + \big(f'(H_{\varepsilon,\mathbf{t}}) - f'(\pm 1)\big)\phi \;=\; o(1), \\ \phi'(0) = \phi'(1/\varepsilon) = 0 \qquad \text{and} \qquad \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \quad j = 1, \dots, N, \end{cases} \tag{3-12}$$

where $o(1)$ is in the sense of $L^\infty(0,\ 1/\varepsilon)$.

We show that (3-12) is incompatible with our assumption that $\|\phi\|_* = 1$. First, we claim that

$$|\phi| \to 0 \quad \text{on} \quad y \in \bigcup_{j=1}^{N} \Big(\frac{t_j}{\varepsilon} - R,\ \frac{t_j}{\varepsilon} + R\Big), \quad \text{as } \varepsilon \to 0, \tag{3-13}$$

where R is any fixed positive constant.

Indeed, assuming the contrary, there exist $\delta_0 > 0$, $j \in \{1, \dots, N\}$, and sequences $\varepsilon_k, \phi_k, y_k \in \big(t_j/\varepsilon - R,\ t_j/\varepsilon + R\big)$ such that ϕ_k satisfies (3-4) and

$$|\phi_k(y_k)| \geq \delta_0. \tag{3-14}$$

Let $\tilde{\phi}_k = \phi_k(y - t_j/\varepsilon_k)$. Then, using (3-12) and $\|\phi\|_* = 1$, as $\varepsilon_k \to 0$, $\tilde{\phi}_k$ converges weakly in $H^2_{\text{loc}}(\mathbb{R})$ and strongly in $C^1_{\text{loc}}(\mathbb{R})$ to a bounded function ϕ_0 which satisfies

$$\phi_0'' + f'(H)\phi_0 = 0 \quad \text{in } \mathbb{R}, \quad |\phi_0| \leq C.$$

By Lemma 2.1, we have $\phi_0 = cH'$ for some c. Since $\tilde{\phi}_k \perp Z_{\varepsilon,t_j}$, we conclude that

$$\int_{\mathbb{R}} \phi_0 f'(H)(H')^2(y) = 0,$$

which yields $c = 0$. Hence $\phi_0 = 0$ and $\tilde{\phi}_k \to 0$ in $B_{2R}(0)$. This contradicts (3-14), so (3-13) holds true.

Given $\delta > 0$, the decay of $f'(H) - f'(\pm 1)$ together with (3-13) (with R sufficiently large) imply that

$$\big\| \big(f'(H_{\varepsilon,\mathbf{t}}) - f'(\pm 1)\big)\phi \big\|_* \leq \delta + \tfrac{1}{2}\|\phi\|_*. \tag{3-15}$$

Using (3-12) and the Maximum Principle, one finds

$$\|\phi\|_* \le \big\|(f'(H_{\varepsilon,\mathbf{t}}) - f'(\pm 1))\phi\big\|_* + \sum_{j=1}^{N} |c_j|\, \|Z_{\varepsilon,t_j}\|_* + \|h\|_* \le 2\delta + \tfrac{1}{2}\|\phi\|_*,$$

and hence

$$\|\phi\|_* \le 4\delta < 1,$$

if we choose $\delta < 1/4$. This contradicts (3-7). □

Next, we consider the following nonlinear problem: Find a function ϕ such that for some constants c_j, $j = 1, \ldots, N$, the equation

$$\begin{cases} \Delta(H_{\varepsilon,\mathbf{t}} + \phi) + f(H_{\varepsilon,\mathbf{t}} + \phi) = \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j} & \text{in } \Omega_\varepsilon, \\ \phi'(0) = \phi'(\frac{1}{\varepsilon}) = 0 \qquad \text{and} \qquad \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, & j = 1, \ldots, N. \end{cases} \tag{3-16}$$

holds true.

The proof of the next result follows the same lines of [Malchiodi et al. 2005, Proposition 4.2].

Proposition 3.2. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, there exists a unique* $\phi = \phi_{\varepsilon,\mathbf{t}}$ *such that* (3-16) *holds. Moreover,* $t \mapsto \phi_{\varepsilon,\mathbf{t}}$ *is of class* C^1 *as a map into* $H^1_r(\Omega_\varepsilon)$, *and we have*

$$\|\phi_{\varepsilon,\mathbf{t}}\|_* \le C\Big(\varepsilon + \sum_{j=1}^{n} e^{-(3/2)\sqrt{2}(1-t_j)/\varepsilon} + \sum_{i\neq j} e^{-(3/4)\sqrt{2}|t_i - t_j|/\varepsilon}\Big). \tag{3-17}$$

4. Energy computation for reduced energy functional

We expand the quantity

$$\mathcal{M}_\varepsilon(\mathbf{t}) := \varepsilon^{n-1}\mathscr{E}_\varepsilon\big[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}\big] : \Lambda \to \mathbb{R} \tag{4-1}$$

in ε and $\mathbf{t}$, where $\phi_{\varepsilon,\mathbf{t}}$ is given by Proposition 3.2. Up to negligible error terms, the same expansion of Lemma 2.3 holds true.

Lemma 4.1. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, we have*

$$\begin{aligned} \mathcal{M}_\varepsilon(\mathbf{t}) &= \varepsilon^{n-1}\mathscr{E}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] \\ &= I[H]\sum_{j=1}^{N} t_j^{n-1} - \big(\sqrt{2}A_0^2 + o(1)\big)e^{-2\sqrt{2}(1-t_1)/\varepsilon} \\ &\qquad - \big(\sqrt{2}A_0^2 + o(1)\big)\sum_{j=2}^{N} t_j^{n-1} e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon} + O(\varepsilon). \end{aligned} \tag{4-2}$$

Proof. It is sufficient to show that

$$\mathcal{M}_\varepsilon(\mathbf{t}) = \varepsilon^{n-1}\mathscr{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] + o\Big(\sum_{j=1}^{N} e^{-2\sqrt{2}(1-t_j)/\varepsilon} + \sum_{i\neq j} e^{-\sqrt{2}|t_i-t_j|/\varepsilon}\Big) + O(\varepsilon),$$

and to apply Lemma 2.3. In order to do this, we write

$$\varepsilon^{1-n}\mathcal{M}_\varepsilon = \mathscr{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] + K_1 + K_2 - K_3,$$

where

$$K_1 = \int_0^{1/\varepsilon}\big(H'_{\varepsilon,\mathbf{t}}\phi'_{\varepsilon,\mathbf{t}} - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big)r^{n-1}dr;$$

$$K_2 = \tfrac{1}{2}\int_0^{1/\varepsilon}\big(|\phi'_{\varepsilon,\mathbf{t}}|^2 - f'(H_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\big)r^{n-1}dr;$$

$$K_3 = \int_0^{1/\varepsilon}\big(F(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - F(H_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \tfrac{1}{2}f'(H_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\big)r^{n-1}dr.$$

Integrating by parts, using Lemma 2.2 and Proposition 3.1, we find

$$\text{(4-3)}\quad |K_1| = \Big|\int_0^{1/\varepsilon}\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}}r^{n-1}dr\Big| \leq C\|\phi_{\varepsilon,\mathbf{t}}\|_*\int_0^{1/\varepsilon}\big|\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\big|r^{n-1}dr$$
$$\leq C\varepsilon^{1-n}\Big(\varepsilon^2 + \sum_{j=1}^{N}\big(\rho_\varepsilon(t_j)\big)^{2+3/2} + \sum_{i\neq j} e^{-(7/4)\sqrt{2}|t_i-t_j|/\varepsilon}\Big).$$

To estimate K_2, we note that $\phi_{\varepsilon,\mathbf{t}}$ satisfies

$$\text{(4-4)}\qquad \Delta\phi_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) + \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}.$$

Multiplying (4-4) by $\phi_{\varepsilon,\mathbf{t}}r^{n-1}$ and integrating over I_ε, we obtain

$$\text{(4-5)}\quad \int_{I_\varepsilon}\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}}r^{n-1}dr = \int_{I_\varepsilon}\big(|\phi'_{\varepsilon,\mathbf{t}}|^2 - f'(H_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\big)r^{n-1}dr$$
$$+ \int_{I_\varepsilon}\big(f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big)\phi_{\varepsilon,\mathbf{t}}r^{n-1}dr.$$

Hence, we find

$$2K_2 = -\int_{I_\varepsilon}\big(f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big)\phi_{\varepsilon,\mathbf{t}}r^{n-1}dr$$
$$+ \int_{I_\varepsilon}\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}}r^{n-1}dr.$$

From Taylor's formula, we get

$$\big|f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big| \leq C|\phi_{\varepsilon,\mathbf{t}}|^2,$$

so we deduce

$$|K_2| \le C \int_{I_\varepsilon} |\phi_{\varepsilon,\mathbf{t}}|^3 r^{n-1} dr \,+\, C\|\phi_{\varepsilon,\mathbf{t}}\|_* \int_{I_\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}] r^{n-1} dr.$$

From the exponential decay of $H(\pm y) - (\pm 1)$ one finds that $\phi_{\varepsilon,\mathbf{t}}(r)$ satisfies

$$\phi''_{\varepsilon,\mathbf{t}} + \frac{n-1}{r}\phi'_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) = O\Big(\sum_{j=1}^{N} e^{-\sqrt{2}|r - t_j/\varepsilon|}\Big),$$

$$\phi'_{\varepsilon,\mathbf{t}}(0) = \phi'_{\varepsilon,\mathbf{t}}(1/\varepsilon) = 0.$$

From (4-4) and a comparison principle, we obtain

$$|\phi_{\varepsilon,\mathbf{t}}(r)| \le C \sum_{j=1}^{N} e^{-(\sqrt{2}/\tilde{C})|r - t_j/\varepsilon|} \tag{4-6}$$

for some $\tilde{C} < 1$.

Using Proposition 3.2 and (4-6), we get

$$|K_2| \le C\varepsilon^{1-n}\Big(\varepsilon^2 + \sum_{j=1}^{N}\big(\rho_\varepsilon(t_j)\big)^3 + \sum_{i \ne j} e^{-2\sqrt{2}|t_i - t_j|/\varepsilon}\Big). \tag{4-7}$$

From the Hölder continuity of f', we deduce

$$\big|F(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - F(H_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \tfrac{1}{2} f'(H_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\big| \le C|\phi_{\varepsilon,\mathbf{t}}|^3,$$

so, again, it follows that

$$|K_3| \le C\varepsilon^{1-n}\Big(\varepsilon^2 + \sum_{j=1}^{N}\big(\rho_\varepsilon(t_j)\big)^3 + \sum_{i \ne j} e^{-2\sqrt{2}|t_i - t_j|/\varepsilon}\Big). \tag{4-8}$$

Combining with (2-20) of Lemma 2.2, we obtain the conclusion. □

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Fix $\mathbf{t} \in \overline{\Lambda}$ and let $\phi_{\varepsilon,\mathbf{t}}$ be given by Proposition 3.2. Let also $\mathcal{M}_\varepsilon(\mathbf{t})$ denote the reduced energy functional defined by (4-1).

Proposition 5.1. *For ε small, the following maximization problem*

$$\sup\big\{\mathcal{M}_\varepsilon(\mathbf{t}) \mid \mathbf{t} \in \Lambda\big\} \tag{5-1}$$

has a solution $\mathbf{t}^\varepsilon$ in the interior of Λ.

Proof. Since $\mathcal{M}_\varepsilon(\mathbf{t})$ is continuous in $\mathbf{t}$, it achieves a maximum in $\bar{\Lambda}$. Let $\mathbf{t}^\varepsilon$ be a maximum point. We claim that $\mathbf{t}^\varepsilon \in \Lambda$.

We argue by contradiction and assume that $\mathbf{t}^\varepsilon \in \partial\Lambda$. From the definition of Λ, there are three possibilities: either $1 - t_1 = \eta\varepsilon\log(1/\varepsilon)$, or there exists $j \ge 2$ such that $t_{j-1} - t_j = \eta\varepsilon\log(1/\varepsilon)$, or, finally, $t_N = 1 - \varepsilon\big(\log(1/\varepsilon)\big)^2$.

In the first case, we have

$$
\begin{aligned}
I[H]t_1^{n-1} - \big(\sqrt{2}A_0^2 + o(1)\big)e^{-2\sqrt{2}(1-t_1)/\varepsilon}& \\
&= I[H]\Big(1 - \eta\varepsilon \log\frac{1}{\varepsilon}\Big)^{n-1} - \sqrt{2}A_0^2 e^{-2\eta\sqrt{2}\log(1/\varepsilon)} + o\big(\varepsilon^{2\sqrt{2}\eta}\big) \\
&\le I[H] - A_0^2\varepsilon^{2\sqrt{2}\eta}.
\end{aligned}
$$

Since $\eta < 1/8\sqrt{2}$, we obtain

$$
\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \le N I[H] - A_0^2\varepsilon^{2\sqrt{2}\eta}. \tag{5-2}
$$

In the second case, there holds

$$
\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \le I[H]\sum_{j=1}^{N} t_j^{n-1} - \big(\sqrt{2}A_0^2 + o(1)\big)\varepsilon^{\sqrt{2}\eta} t_j^{n-1} \le N I[H] - A_0^2\varepsilon^{\sqrt{2}\eta}. \tag{5-3}
$$

In the latter case, we have $t_N = 1 - \varepsilon\big(\log(1/\varepsilon)\big)^2$, and therefore

$$
\begin{aligned}
\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) &\le I[H](N - 1 + t_N^{n-1}) + O(\varepsilon) \\
&\le I[H]\big(N - (n-1)\varepsilon(\log(1/\varepsilon))^2\big) + O(\varepsilon).
\end{aligned} \tag{5-4}
$$

On the other hand, choosing $t_j = 1 - (j/\sqrt{2})\varepsilon\log(1/\varepsilon)$, $j = 1, \dots, N$, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{N} t_j^{n-1} = 1 - \frac{N(N+1)(n-1)}{2\sqrt{2}}\varepsilon\log(1/\varepsilon) + O\big(\varepsilon^2(\log(1/\varepsilon))^2\big); \\
&e^{-2\sqrt{2}(1-t_1)/\varepsilon} = \varepsilon^2; \qquad e^{-\sqrt{2}|t_{j-1}-t_j|/\varepsilon} = \varepsilon,
\end{aligned} \tag{5-5}
$$

and we find

$$
\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \ge N I[H] - \frac{N(N+1)(n-1)^2}{2\sqrt{2}}\varepsilon\log(1/\varepsilon) + O(\varepsilon),
$$

which contradicts either (5-2) or (5-3) or (5-4). This completes the proof of Proposition 5.1. □

Remark 5.2. The above argument also shows that

$$
1 - t_1^\varepsilon \sim \varepsilon\log(1/\varepsilon), \qquad t_{j-1}^\varepsilon - t_j^\varepsilon \sim \varepsilon\log(1/\varepsilon). \tag{5-6}
$$

Finally, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 3.2, there exists ε_N such that, for $\varepsilon < \varepsilon_N$, we have a C^1 map $\mathbf{t} \mapsto \phi_{\varepsilon,\mathbf{t}}$ from $\overline{\Lambda}$ into $C^2(I_\varepsilon)$ such that

$$
\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}, \tag{5-7}
$$

for some constants $\{c_j\} \subseteq \mathbb{R}$, which are also of class C^1 in $\mathbf{t}$.

By Proposition 5.1, there exists $\mathbf{t}^\varepsilon \in \Lambda$ that achieves the maximum of $\mathcal{K}_\varepsilon : \mathbf{t} \mapsto \mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}]$. Let

$$u_\varepsilon = \sum_{i=1}^{N} (-1)^i H_{\varepsilon,t_i^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon} = H_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}.$$

Then we have

$$\partial_{t_i}\Big|_{\mathbf{t}=\mathbf{t}^\varepsilon} \mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) = 0, \qquad i = 1, \dots, N,$$

and hence

$$\int_{I_\varepsilon} \Big(\nabla u_\varepsilon \nabla \partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) + u_\varepsilon \partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(u_\varepsilon) \partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) \Big)\Big|_{\mathbf{t}=\mathbf{t}^\varepsilon} r^{n-1} dr = 0.$$

Therefore, by (5-7), we find

$$\sum_{j=1}^{N} c_j \int_{I_\varepsilon} \big(Z_{\varepsilon,t_j} \partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) \big) r^{n-1} dr = 0. \tag{5-8}$$

Differentiating the equation $\langle \phi, Z_{\varepsilon,t_j} \rangle_\varepsilon = 0$ with respect to t_j, we get

$$\langle \partial_{t_i} \phi, Z_{\varepsilon,t_j} \rangle_\varepsilon = -\langle \phi, \partial_{t_i} Z_{\varepsilon,t_j} \rangle_\varepsilon = O(\|\phi\|_*) \varepsilon^{-n-1}.$$

Using (3-3), we see that (5-8) is diagonally dominant in the coefficients $\{c_i\}$, which implies that $c_j = 0$ for $j = 1, \dots, N$. Hence

$$u_\varepsilon = H_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}$$

is a solution of (1-1).

By our construction, one can easily check that $\varepsilon^{n-1}\mathcal{E}_\varepsilon(u_\varepsilon) \to N I[H]$ as $\varepsilon \to 0$, and that u_ε has only N zeroes $s_1^\varepsilon/\varepsilon, \dots, s_N^\varepsilon/\varepsilon$. By the structure of u_ε, we see that (up to a permutation) $s_i^\varepsilon - t_i^\varepsilon = o(1)$. This proves (1) and (2) of Theorem 1.1.

It remains to prove (3). First we note that u'_ε satisfies

$$\Delta u'_\varepsilon + f'(u_\varepsilon) u'_\varepsilon = \frac{n-1}{r^2} u'_\varepsilon. \tag{5-9}$$

By our construction, at each interval $(s_j^\varepsilon/\varepsilon,\ s_{j-1}^\varepsilon/\varepsilon)$, for $j = 2, \dots, N$, there exists a point $\tilde{s}_{j-1}^\varepsilon/\varepsilon \in (s_j/\varepsilon,\ s_{j-1}/\varepsilon)$ such that $u'_\varepsilon(\tilde{s}_{j-1}^\varepsilon/\varepsilon) = 0$. Now, we set

$$\varphi_1(r) = \begin{cases} u'_\varepsilon(r) & \text{for } r \in \big(\tilde{s}_1^\varepsilon/\varepsilon,\ 1\big), \\ 0 & \text{otherwise}; \end{cases}$$

$$\varphi_j(r) = \begin{cases} u'_\varepsilon(r) & \text{for } r \in \big(\tilde{s}_j^\varepsilon/\varepsilon,\ \tilde{s}_{j-1}^\varepsilon/\varepsilon\big), \\ 0 & \text{otherwise}, \end{cases} \qquad j = 2, \dots, N-1;$$

$$\varphi_N(r) = \begin{cases} u'_\varepsilon(r), & \text{for } r \in \big(1/(2\varepsilon),\ \tilde{s}_{N-1}^\varepsilon/\varepsilon\big), \\ 2\varepsilon(r - 1/(4\varepsilon)) u'_\varepsilon(r), & \text{for } 1/(4\varepsilon) \le r \le 1/(2\varepsilon), \\ 0, & \text{for } r < 1/(4\varepsilon) \text{ or } r \ge \tilde{s}_{N-1}^\varepsilon/\varepsilon. \end{cases}$$

Next, we define a quadratic functional

$$\mathbf{Q}[\phi] = \int_{I_\varepsilon} \big(|\nabla\phi|^2 - f'(u_\varepsilon)\phi^2\big) r^{n-1} dr. \tag{5-10}$$

It is easy to check that

$$\int_{I_\varepsilon} \varphi_i \varphi_j r^{n-1} dr = 0 \qquad \text{for } i \neq j. \tag{5-11}$$

Using equation (5-9), we obtain

$$\mathbf{Q}[\varphi_i] = -\int_{I_\varepsilon} \varphi_i^2 r^{n-3} dr < 0, \qquad i = 1, \dots, N-1. \tag{5-12}$$

When $i = N$, we have

$$\mathbf{Q}[\varphi_N] = -\int_{I_\varepsilon} \varphi_N^2 r^{n-3} dr + O\big(e^{-1/(C\varepsilon)}\big) < 0. \tag{5-13}$$

From (5-12) and (5-13), the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at least N.

Finally, we also show that the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at most N. In fact, we define

$$z_j^\varepsilon(r) = H'_{\varepsilon, t_j^\varepsilon} \chi\Big(\frac{\varepsilon r - t_j^\varepsilon}{\varepsilon\sqrt{|\log(1/\varepsilon)|}}\Big), \qquad j = 1, \dots, N, \tag{5-14}$$

and consider the following minimization problem

$$\mu_j^\varepsilon = \inf_{\substack{\phi \in H^1(I_{\varepsilon,j}) \\ \int_{I_{\varepsilon,j}} \phi z_j^\varepsilon r^{n-1} dr = 0}} \frac{\displaystyle\int_{I_{\varepsilon,j}} \big(|\nabla\phi|^2 - f'(u_\varepsilon)\phi^2\big) r^{n-1} dr}{\displaystyle\int_{I_{\varepsilon,j}} \phi^2 r^{n-1} dr}. \tag{5-15}$$

Assume that $\mu_j^\varepsilon \le 0$. By standard regularity theory, μ_j^ε is attained by a function ϕ_j^ε which satisfies

$$\begin{aligned} &\Delta\phi_j^\varepsilon + f'(u_\varepsilon)\phi_j^\varepsilon = -\mu_j^\varepsilon \phi_j^\varepsilon + c_j^\varepsilon z_j^\varepsilon, \\ &(\phi_j^\varepsilon)'|_{\partial I_{\varepsilon,j}} = 0 \qquad \text{and} \qquad \int_{I_{\varepsilon,j}} \phi_j^\varepsilon z_j r^{n-1} dr = 0, \end{aligned} \tag{5-16}$$

where c_j^ε is a constant.

First, we notice that $c_j^\varepsilon = o(\|\phi_j^\varepsilon\|_*)$, which follows by reasoning as for (3-10) of Proposition 3.1. Then, from Lemma 2.1 we deduce that $\mu_j^\varepsilon \to 0$; moreover, the same argument leading to Proposition 3.1 shows that $\phi_j^\varepsilon = 0$.

Thus, $\mu_j^\varepsilon > 0$. Let $\phi = \phi(r)$ be such that $\int_{I_\varepsilon} \phi z_j^\varepsilon r^{n-1} = 0$, $j = 1, \dots, N$, which is equivalent to $\int_{I_{\varepsilon,j}} \phi z_j^\varepsilon r^{n-1} = 0$. This then implies that

$$\int_{I_{\varepsilon,j}} \big(|\nabla\phi|^2 - f'(u_\varepsilon)\phi^2\big) r^{n-1} dr \ge \mu_j^\varepsilon \int_{I_{\varepsilon,j}} |\phi|^2 r^{n-1} dr, \qquad j = 1, \dots, N, \tag{5-17}$$

and hence

$$\begin{aligned}\int_{I_\varepsilon} \big(|\nabla\phi|^2 - f'(u_\varepsilon)\phi^2\big) r^{n-1} dr &= \sum_{j=1}^{N} \int_{I_{\varepsilon,j}} \big(|\nabla\phi|^2 - f'(u_\varepsilon)\phi^2\big) r^{n-1} dr \\ &\ge \min_{j=1,\dots,N} \mu_j^\varepsilon \int_{I_\varepsilon} |\phi|^2 r^{n-1} dr.\end{aligned} \tag{5-18}$$

This yields

$$\lambda_{N+1} = \sup_{v_1,\dots,v_N} \inf_{\substack{\int_{I_\varepsilon} \phi v_j r^{n-1} = 0 \\ j=1,\dots,N}} \frac{\int_{I_\varepsilon} \big(|\nabla u|^2 - f'(u_\varepsilon)\phi^2\big) r^{n-1}}{\int_{I_\varepsilon} \phi^2 r^{n-1}} \ge \min_{j=1,\dots,N} \mu_j^\varepsilon > 0, \tag{5-19}$$

and hence the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at most N.

Combining the upper and lower bound for the Morse index, we see that the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is exactly N. This proves (3) of Theorem 1.1. □

Appendix

In this appendix we expand the quantity $\mathscr{E}_\varepsilon\big[\sum_{j=1}^N (-1)^j H_{\varepsilon,t_j}\big]$ as a function of ε and $\mathbf{t}$. Several facts will be used repeatedly:

$$\begin{aligned}
&H(y) = 1 - A_0 e^{-\sqrt{2}|y|} + O\big(e^{-2\sqrt{2}|y|}\big), \text{ for } y > 1;\\
&H(y) = -1 + A_0 e^{-\sqrt{2}|y|} + O\big(e^{-2\sqrt{2}|y|}\big), \text{ for } y < -1;\\
&H'(y) = \sqrt{2} A_0 e^{-\sqrt{2}|y|} + O\big(e^{-2\sqrt{2}|y|}\big), \text{ for } |y| > 1;\\
&\rho_\varepsilon(t_1) = \sqrt{2}\big(A_0 + o(1)\big) e^{-\sqrt{2}(1-t_1)/\varepsilon};\\
&\rho_\varepsilon(t_j) = o\big(\rho_\varepsilon(t_1)\big) \text{ for } j \ge 2.
\end{aligned}$$

From a Taylor expansion we find

$$\mathscr{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] = I_1 + I_2 + I_3 + O\big(\varepsilon^{1-n} \rho_\varepsilon^3(t_1)\big),$$

where

$$I_1 = \mathscr{E}_\varepsilon\Big[\sum_{j=1}^N (-1)^j H_{t_j}\Big],$$

$$I_2 = -\Big(\sum_{l=1}^K (-1)^l \rho_\varepsilon(t_l)\Big)\int_{I_\varepsilon}\Big(\Big(\sum_{j=1}^N (-1)^j H_{t_j}\Big)'\beta_\varepsilon' - f\Big(\sum_{j=1}^N (-1)^j H_{t_j}\Big)\beta_\varepsilon\Big) r^{n-1}dr,$$

$$I_3 = \tfrac{1}{2}\Big(\sum_{l=1}^N (-1)^l \rho_\varepsilon(t_l)\Big)^2\int_{I_\varepsilon}\Big(|\beta_\varepsilon'|^2 - f'\Big(\sum_{j=1}^N (-1)^j H_{t_j}\Big)\beta_\varepsilon^2\Big) r^{n-1}.$$

Recalling that $f'(\pm 1) = -2$, the term I_3 can be estimated by

$$\begin{aligned} I_3 &= \tfrac{1}{2}\Big(\sum_{j=1}^N (-1)^j \rho_\varepsilon(t_j)\Big)^2\int_{I_\varepsilon}\Big(2 - f'\Big(\sum_{j=1}^N (-1)^j H_{t_j}\Big)\Big)\beta_\varepsilon^2 r^{n-1}dr + o(\varepsilon^{1-n}\rho_\varepsilon^2(t_1)) \\ &= \big(\rho_\varepsilon(t_1)\big)^2\int_{I_\varepsilon}\beta_\varepsilon^2 r^{n-1}dr + o\big(\varepsilon^{1-n}\rho_\varepsilon^2(t_1)\big) = \tfrac{1}{2\sqrt{2}}\varepsilon^{1-n}\big(\rho_\varepsilon(t_1)\big)^2 + o\big(\varepsilon^{1-n}\rho_\varepsilon^2(t_1)\big) \\ &= \frac{A_0^2 + o(1)}{\sqrt{2}}\varepsilon^{1-n}e^{-2\sqrt{2}(1-t_1)/\varepsilon}. \end{aligned}$$

Next we estimate the integral in I_2. We have

$$\begin{aligned} &\int_{I_\varepsilon}\Big(\sum_{j=1}^N (-1)^j H_{t_j}'\beta_\varepsilon' - f\Big(\sum_{j=1}^N (-1)^j H_{t_j}\Big)\beta_\varepsilon\Big) r^{n-1}dr \\ &\qquad = \int_{I_\varepsilon}\Big(\sqrt{2}\sum_{j=1}^N (-1)^j H_{t_j}' - f\Big(\sum_{j=1}^N (-1)^j H_{t_j}\Big)\Big)\beta_\varepsilon r^{n-1}dr \\ &\qquad = \int_{I_{\varepsilon,1}}\big(-\sqrt{2}H_{t_1}' - f(-H_{t_1})\big)\beta_\varepsilon r^{n-1}dr + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big) \\ &\qquad = -\tfrac{1}{\sqrt{2}}e^{-\sqrt{2}(1-t_1)/\varepsilon}\int_{\mathbb{R}}\big(\sqrt{2}H' - f(H)\big)e^{\sqrt{2}y}dy\big(t_1/\varepsilon\big)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big) \\ &\qquad = -A_0 e^{-\sqrt{2}(1-t_1)/\varepsilon}\big(t_1/\varepsilon\big)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big), \end{aligned}$$

since

$$\int_{\mathbb{R}}\big(\sqrt{2}H' - f(H)\big)e^{\sqrt{2}y}dy = \big(H'e^{\sqrt{2}y}\big)\Big|_{-\infty}^{+\infty} = \sqrt{2}A_0.$$

Thus,

$$I_2 = -\big(\sqrt{2}A_0^2 + o(1)\big)e^{-2\sqrt{2}(1-t_1)/\varepsilon}\big(t_1/\varepsilon\big)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big) + O(\varepsilon^{2-n}),$$

which implies that

$$I_2 + I_3 = -\frac{A_0^2 + o(1)}{\sqrt{2}}e^{-2\sqrt{2}(1-t_1)/\varepsilon}\big(t_1/\varepsilon\big)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big) + O(\varepsilon^{2-n}), \tag{5-20}$$

since $t_1 = 1 + O\big(\varepsilon(\log(1/\varepsilon))^2\big)$.

It remains to consider I_1. For this purpose, we decompose it as

$$I_1 = \sum_{j=1}^{N} E_{\varepsilon,j},$$

where

$$\begin{aligned}
E_{\varepsilon,j} &= \int_{I_{\varepsilon,j}} \Big(\tfrac{1}{2}\Big|\sum_{l=1}^{N}(-1)^l H'_{t_l}\Big|^2 - F\Big(\sum_{l=1}^{N}(-1)^l H_{t_l}\Big)\Big) r^{n-1}\,dr \\
&= \int_{I_{\varepsilon,j}} \Big(\tfrac{1}{2}\Big|H'_{t_j} + \sum_{l\neq j}(-1)^{j+l} H'_{t_l}\Big|^2 - F\Big(H_j + \sum_{l\neq j}(-1)^{j+l} H_{t_l}\Big)\Big) r^{n-1}\,dr \\
&= I_4 + I_5 + I_6 + o\Big(\varepsilon^{1-n}\sum_{i\neq j} e^{-\sqrt{2}|t_i-t_j|/\varepsilon}\Big),
\end{aligned}$$

with

$$\begin{aligned}
I_4 &= \int_{I_{\varepsilon,j}} \big(\tfrac{1}{2}|H'_{t_j}|^2 - F(H_{t_j})\big) r^{n-1}\,dr, \\
I_5 &= \int_{I_{\varepsilon,j}} \Big(H'_{t_j}\sum_{l\neq j}(-1)^{l+j} H'_{t_l} - f(H_{t_j})\sum_{l\neq j}(-1)^{l+j} H_{t_l}\Big) r^{n-1}\,dr, \\
I_6 &= \frac{1}{2}\int_{I_{\varepsilon,j}} \Big|\sum_{l\neq j}(-1)^{j+l} H_{t_l}\Big|^2 \big(2 - f'((-1)^j H_{t_j})\big) r^{n-1}\,dr.
\end{aligned}$$

Using the fact that $|H'|^2 = 2F(H)$, for I_4 we find

$$\begin{aligned}
I_4 &= \int_{I_{\varepsilon,j}} |H'_{t_j}|^2 r^{n-1}\,dr \\
&= \int_R |H'|^2 dy\,(t_j/\varepsilon)^{n-1} - \frac{A_0^2 + o(1)}{\sqrt{2}}\Big(e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon} + e^{-\sqrt{2}|t_j-t_{j+1}|/\varepsilon}\Big)(t_j/\varepsilon)^{n-1} \\
&\qquad + O(\varepsilon^{2-n}).
\end{aligned}$$

For $j \geq 2$, I_5 can be estimated (by recalling the exponential-decay property of $H(y) \pm 1$) as

$$\begin{aligned}
I_5 &= (t_j/\varepsilon)^{n-1} H'_{t_j}\sum_{l\neq j}(-1)^{l+j} H_{t_l}\Big|_{\partial I_{\varepsilon,j}} + O(\varepsilon^{2-n}) \\
&= -\big(A_0^2 + o(1)\big)\sqrt{2}\big(e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon} + e^{-\sqrt{2}|t_j-t_{j+1}|/\varepsilon}\big)(t_j/\varepsilon)^{n-1} + O(\varepsilon^{2-n}).
\end{aligned}$$

For $j = 1$, we have

$$\begin{aligned}
I_5 &= (t_1/\varepsilon)^{n-1} H'_{t_j}\sum_{l>1}(-1)^{l+1} H_{t_l}\Big|_{\partial I_{\varepsilon,1}} + O(\varepsilon^{2-n}) \\
&= -\big(A_0^2 + o(1)\big)\sqrt{2}\,e^{-\sqrt{2}|t_1-t_2|/\varepsilon}(t_j/\varepsilon)^{n-1} + O(\varepsilon^{2-n}).
\end{aligned}$$

I_6 can be estimated similarly: for $j \geq 2$, we have

$$I_6 = 2\int_{I_{\varepsilon,j}} \Big|\sum_{l\neq j}(-1)^{j+l}H_{t_l}\Big|^2 r^{n-1}dr$$
$$= \frac{A_0^2+o(1)}{\sqrt{2}}\big(e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon}+e^{-\sqrt{2}|t_j-t_{j+1}|/\varepsilon}\big)(t_j/\varepsilon)^{n-1}+O(\varepsilon^{2-n}),$$

while for $j = 1$,

$$I_6 = 2\int_{I_{\varepsilon,1}} \Big|\sum_{l>1}(-1)^{l+1}H_{t_l}\Big|^2 r^{n-1}dr = \frac{A_0^2+o(1)}{\sqrt{2}}e^{-\sqrt{2}|t_1-t_2|/\varepsilon}(t_1/\varepsilon)^{n-1}+O(\varepsilon^{2-n}).$$

Combining the estimates of I_4, I_5, and I_6, we obtain

$$I_1 = I[H]\sum_{j=1}^{N}(t_j/\varepsilon)^{n-1} - \sqrt{2}\big(A_0^2+o(1)\big)\sum_{j=2}^{N}e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon}(t_j/\varepsilon)^{n-1}$$
$$- \frac{A_0^2+o(1)}{\sqrt{2}}e^{-2\sqrt{2}(1-t_1)/\varepsilon}+O(\varepsilon^{2-n})$$
$$= I[H]\sum_{j=1}^{N}(t_j/\varepsilon)^{n-1} - \sqrt{2}\big(A_0^2+o(1)\big)\sum_{j=2}^{N}e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon}(t_j/\varepsilon)^{n-1}$$
$$- \frac{A_0^2+o(1)}{\sqrt{2}}e^{-2\sqrt{2}(1-t_1)/\varepsilon}(t_1/\varepsilon)^{n-1}+O(\varepsilon^{2-n}).$$

Adding this to the estimate in (5-20), we obtain the asymptotic expansion (2-20) of $\mathscr{E}_\varepsilon\big[\sum_{j=1}^{N}(-1)^j H_{\varepsilon,t_j}\big]$.

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ANDREA MALCHIODI
SECTOR OF FUNCTIONAL ANALYSIS AND APPLICATIONS, SISSA
VIA BEIRUT 2-4
34014 TRIESTE
ITALY
malchiod@sissa.it

WEI-MING NI
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN 55455
UNITED STATES
ni@math.umn.edu
www.math.umn.edu

JUNCHENG WEI
DEPARTMENT OF MATHEMATICS
THE CHINESE UNIVERSITY OF HONG KONG
SHATIN
HONG KONG
wei@math.cuhk.edu.hk
www.math.cuhk.edu.hk