Pacific Journal of Mathematics

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Volume 229 No. 2 February 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 229, No. 2, 2007

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Let φ_t be a nonsingular flow on a 3-dimensional manifold M. Denote by $\pi_P : PX \to M$ the projectivized bundle of the quotient bundle of TM by the line bundle tangent to φ_t. The derivative of φ_t induces a flow ψ_t on PX, called the projective flow of φ_t. In this paper, we consider the dynamical properties of ψ_t restricted to $\pi_P^{-1}(\mathfrak{M})$ for a minimal set $\mathfrak{M}$ of φ_t, under the condition that the restriction of ψ_t to $\pi_P^{-1}(\mathfrak{M})$ has exactly two minimal sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$. If φ_t has no dominated splitting over $\mathfrak{M}$, we find two types of orbits of ψ_t in the domain between $\mathfrak{N}_1$ and $\mathfrak{N}_2$: one is "bounded below" and the other is "bounded above". As an application we prove that, if φ_t is further assumed to be almost periodic on the minimal set, there is a dense orbit in that domain.

1. Introduction

Let M be a closed oriented 3-dimensional manifold, and X a nonsingular C^1 vector field of M. We denote by TM the tangent bundle of M, and by φ_t the flow generated by X. Let $\pi_N : NX \to M$ denote the quotient bundle of TM by the 1-dimensional bundle tangent to X. We take the projectivized bundle $\pi_P : PX \to M$ of NX, that is,

$$PX = \bigcup_{z \in M} \big((\pi_N^{-1}(z) - 0)/v \sim kv\big)$$

for $v \in {\pi_N}^{-1}(z) - 0$, $k \in \mathbb{R} - 0$. The derivative $D\varphi_t$ of φ_t induces a *projective flow* ψ_t on PX (also called an *inductance flow*). Here, each fiber $\pi_P^{-1}(z)$ is oriented by the orientations of TM and X, and ψ_t preserves the orientation of fibers. Let $O(z, s)$ denote the orbit of ψ_t passing through $(z, s) \in PX$, and let $O_+(z, s)$ and $O_-(z, s)$ denote the positive and negative orbits of ψ_t passing through $(z, s) \in PX$.

In order to consider the dynamical properties of ψ_t, we will frequently use the renormalized linear Poincaré flow ν_t, defined as follows: Let $| \cdot |$ denote the norm

MSC2000: 37B05, 37D30.

Keywords: minimal set, projective flow, dominated splitting.

Partially supported by Grant-in-Aid for Scientific Research (No. 15540078), Japan Society for the Promotion of Science, Japan.

on each fiber of NX induced from a Riemannian metric of M. We take an orthonormal basis $\bigl(v_1(z), v_2(z)\bigr)$ of $\pi_N^{-1}(z)$ for any point z of M, with $\bigl(v_1(z), v_2(z)\bigr)$ not necessarily continuous with respect to z. The derivative $D\varphi_t|_{\pi_N^{-1}(z)}$ is represented by a matrix, and hence we can define its determinant $\det\bigl(D\varphi_t|_{\pi_N^{-1}(z)}\bigr)$. Since $\det\bigl(D\varphi_t|_{\pi_N^{-1}(z)}\bigr)$ is independent of the choice of the orthonormal basis, we can define a new flow ν_t on NX by

$$\nu_t(z, [v]_N) = \Bigl(\varphi_t(z),\ \bigl(\det(D\varphi_t|_{\pi_N^{-1}(z)})\bigr)^{-1/2}\,[(D\varphi_t)_z(v)]_N\Bigr),$$

where $[v]_N$ denotes the element of $\pi_N^{-1}(z)$ represented by the vector v of T_zM. The flow ν_t preserves the area form on each fiber, and the next diagram commutes:

$$\begin{tikzcd} NX \arrow[r] \arrow[d, "\nu_t"'] & PX \arrow[r, "\pi_P"] \arrow[d, "\psi_t"'] & M \arrow[d, "\varphi_t"'] \\ NX \arrow[r] & PX \arrow[r, "\pi_P"] & M \end{tikzcd}$$

Let $\mathfrak{M}$ be a minimal set of φ_t (that is, a closed φ_t-invariant set which is minimal with respect to the inclusion), and let $\widetilde{\mathfrak{M}}$ denote its lift ${\pi_P}^{-1}(\mathfrak{M})$. Since the restriction of ψ_t to $\widetilde{\mathfrak{M}}$ is a flow on a compact set, $\psi_t|_{\widetilde{\mathfrak{M}}}$ has a minimal set $\mathfrak{N}$. The set $\pi_P(\mathfrak{N})$ coincides with $\mathfrak{M}$ itself.

The number of minimal sets of $\psi_t|_{\widetilde{\mathfrak{M}}}$ is essential for the study of the dynamical structure of $\psi_t|_{\widetilde{\mathfrak{M}}}$. For example, if $\psi_t|_{\widetilde{\mathfrak{M}}}$ has more than two minimal sets, it was already proved in [Nakayama and Noda 2005] that $\varphi_t|_{\mathfrak{M}}$ is uniformly quasiconformal. In particular, if we further assume that φ_t is a C^∞ minimal flow on the 3-manifold M, then M is the 3-dimensional torus T^3 and φ_t is topologically equivalent to an irrational flow, by [Sullivan 1981; Brunella 1996; Ghys 1996] (see also [Matsumoto and Nakayama 1997]).

On the other hand, if $\varphi_t|_{\mathfrak{M}}$ is uniformly quasiconformal, then $\psi_t|_{\widetilde{\mathfrak{M}}}$ does not have exactly two minimal sets, because $\varphi_t|_{\mathfrak{M}}$ is transversely conformal, again by [Sullivan 1981, p. 468], and a minimal set of $\psi_t|_{\widetilde{\mathfrak{M}}}$ is still a minimal set after the constant rotation along the fiber with respect to this transverse conformal structure. (This can also be proved in the following way: if $\varphi_t|_{\mathfrak{M}}$ is uniformly quasiconformal and the whole $\widetilde{\mathfrak{M}}$ is not a minimal set of $\psi_t|_{\widetilde{\mathfrak{M}}}$, we can show that, for any minimal set $\mathfrak{N}$ of $\psi_t|_{\widetilde{\mathfrak{M}}}$, the set $\mathfrak{N} \cap \pi_P^{-1}(z)$ does not contain an interval, for any $z \in \mathfrak{M}$, and the lengths of the connected components of $\pi_P^{-1}(z) \setminus \mathfrak{N}$ are bounded below for $z \in \mathfrak{M}$; therefore, $\mathfrak{N} \cap \pi_P^{-1}(z)$ consists of finitely many points, and furthermore $\pi_P|_{\mathfrak{N}}$ is a finite covering; we can also prove that an orbit disjoint from $\mathfrak{N}$ cannot approach $\mathfrak{N}$; therefore, $\psi_t|_{\widetilde{\mathfrak{M}}}$ does not have exactly two minimal sets.)

We will restrict our attention to the case when $\psi_t|_{\widetilde{\mathfrak{M}}}$ has exactly two minimal sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$. In this case, we have the following properties of $\mathfrak{N}_1$ and $\mathfrak{N}_2$ from

[Nakayama and Noda 2005], where $(S)_z$ denotes the intersection $S \cap \pi_P^{-1}(z)$ for a set S in PX and $z \in M$:

Proposition 1.1. (a) *Either $(\mathfrak{N}_1)_z$ or $(\mathfrak{N}_2)_z$ consists of a single point for any $z \in \mathfrak{M}$.*

(b) *There is a residual set in $\mathfrak{M}$ on which both $(\mathfrak{N}_1)_z$ and $(\mathfrak{N}_2)_z$ consist of a single point.*

(c) *For $i = 1, 2$, let $\xi_i^+(z)$ and $\xi_i^-(z)$ denote the maximum and minimum points of $(\mathfrak{N}_i)_z$ for $z \in \mathfrak{M}$, with respect to the orientation of the fiber $(PX)_z$. Here, these points are uniquely determined by (a). We observe that $\xi_i^+(z)$ and $\xi_i^-(z)$ may not be continuous in z, but are semicontinuous; this implies that, if $(\mathfrak{N}_i)_z$ is a single point, then z is a continuity point of ξ_i^+ and ξ_i^-. If we set*

$$K_i = \bigl\{(z, s) \in PX \bigm| \xi_i^-(z) \le s \le \xi_i^+(z)\bigr\},$$
$$E_1 = \bigl\{(z, s) \in PX \bigm| \xi_1^+(z) < s < \xi_2^-(z)\bigr\},$$
$$E_2 = \bigl\{(z, s) \in PX \bigm| \xi_2^+(z) < s < \xi_1^-(z)\bigr\},$$

then K_i is a closed invariant set in $\widetilde{\mathfrak{M}}$ and E_i is an open invariant set in $\widetilde{\mathfrak{M}}$.

(d) *By (c), there are two (continuous) sections $h_1 : \mathfrak{M} \to PX$ and $h_2 : \mathfrak{M} \to PX$ such that $h_1(\mathfrak{M})$ and $h_2(\mathfrak{M})$ are contained in E_1 and E_2, respectively. Thus, $h_1(\mathfrak{M})$ and $h_2(\mathfrak{M})$ separate $\mathfrak{N}_1$ and $\mathfrak{N}_2$ in PX. In particular, PX is a trivial bundle over $\mathfrak{M}$.*

Here we remark that [Nakayama and Noda 2005, Theorem 1 (2)] was proved under the stronger condition that φ_t is minimal on the whole 3-manifold and is not topologically equivalent to an irrational flow of T^3. However, its proof is also valid for the restriction of φ_t to the minimal set $\mathfrak{M}$, if $\varphi_t|_{\mathfrak{M}}$ is not uniformly quasiconformal. Now, as above, if $\psi_t|_{\widetilde{\mathfrak{M}}}$ has exactly two minimal sets, then $\varphi_t|_{\mathfrak{M}}$ is not uniformly quasiconformal.

The dynamics of $\psi_t|_{\widetilde{\mathfrak{M}}}$ are still very complicated, as indicated by an example of Johnson [1981] and our Example 1.4. Thus, we will apply the next result of Contreras [2002]:

Let $\pi : V \to B$ be a symplectic vector bundle over a compact set B. Let Ψ_t be a flow on V which is a bundle map that preserves the canonical symplectic structure on each fiber. The flow Ψ_t induces a flow Φ_t on B satisfying $\Phi_t \circ \pi = \pi \circ \Psi_t$. Contreras [2002] showed that, if Ψ_t is a weakly partially hyperbolic flow whose stable and unstable bundles have the same dimension, then the restricted flow $\Psi_t|_{\pi^{-1}(\Omega(\Phi_t))}$ is hyperbolic where $\Omega(\Phi_t)$ is a nonwandering set of Φ_t.

We will apply his result to the renormalized linear Poincaré flow ν_t and extend his theorem under the stronger condition that $\mathfrak{M}$ is a minimal set instead of a nonwandering set $\Omega(\Phi_t)$ and that $\psi_t|_{\widetilde{\mathfrak{M}}}$ has exactly two minimal sets (the precise definitions will be given in the next sections). In Section 2, we give relations between the dynamics of the renormalized linear Poincaré flow ν_t and the projective flow ψ_t. In Section 3, if $\psi_t|_{E_i}$ has a fiberwise-divergent orbit, we find two types of orbits of ψ_t in E_i, one that does not approach $\mathfrak{N}_1$ and the other that does not approach $\mathfrak{N}_2$. Here, the cross-ratio plays an important role. If all the orbits of ψ_t are fiberwise convergent, then φ_t has a dominated splitting over $\mathfrak{M}$, which will be proved in Section 4. Hence:

Theorem 1.2. *Let $\mathfrak{M}$ be a minimal set of φ_t. If $\psi_t|_{\widetilde{\mathfrak{M}}}$ has exactly two minimal sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$, then either*

(1) *φ_t has a dominated splitting over $\mathfrak{M}$; or*

(2) *for any $i = 1, 2$, there exist points (z, s) and (z', s') of E_i such that*

$$\overline{O(z,s)} \cap \mathfrak{N}_1 = \varnothing \qquad \text{and} \qquad \overline{O(z',s')} \cap \mathfrak{N}_2 = \varnothing;$$

in particular, $\psi_t|_{E_i}$ has an orbit which is not dense in E_i.

To take advantage of finding two types of orbits, we apply Theorem 1.2 to almost-periodic flows (defined in Section 5):

Theorem 1.3. *If $\varphi_t|_{\mathfrak{M}}$ is an almost-periodic minimal flow and $\psi_t|_{\widetilde{\mathfrak{M}}}$ has exactly two minimal sets, then either*

(1) *φ_t has a dominated splitting over $\mathfrak{M}$; or*

(2) *both $\psi_t|_{E_1}$ and $\psi_t|_{E_2}$ have dense orbits.*

Finally, we give an example (communicated to the author by T. Noda) of an almost-periodic minimal flow on T^3 satisfying the conditions (2) of Theorems 1.2 and 1.3:

Example 1.4. Let $g : S^1 \to S^1$ be a C^ω diffeomorphism such that g is topologically conjugate to an irrational rotation R_α but is not C^1-conjugate to R_α; such an example was constructed by Arnold [Herman 1979]. Let β be an irrational number which is $\mathbb{Q}$-independent of α. We define a C^ω diffeomorphism $f : T^2 \to T^2$ by $f(x, y) = (g(x), y + \beta)$. Let φ_t denote its suspension flow on T^3. Since g is topologically conjugate to an irrational rotation, φ_t is an almost-periodic minimal flow. On the other hand, $\sup_{n\geq 0} |\log Dg^n(x)| = \infty$ for any $x \in S^1$, because g is not C^1-conjugate to a rotation [Herman 1979]. Therefore, the projective flow of φ_t has exactly two minimal sets, corresponding to the x-direction and the y-direction. Furthermore, by [Arroyo and Rodriguez Hertz 2003, Theorem B], we can show that φ_t has no dominated splitting. Thus, the conditions (2) of Theorems 1.2 and 1.3 hold. The key point of this example is that such an example can be constructed as a C^ω flow.

2. Relations between lengths and angles

Throughout this paper, we assume that $\mathfrak{M}$ is a minimal set of φ_t and $\psi_t|_{\widetilde{\mathfrak{M}}}$ has exactly two minimal sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$, where $\widetilde{\mathfrak{M}} = \pi_P^{-1}(\mathfrak{M})$. By (d) in Proposition 1.1, there exist two sections $h_1 : \mathfrak{M} \to PX$ and $h_2 : \mathfrak{M} \to PX$ separating $\mathfrak{N}_1$ and $\mathfrak{N}_2$. We choose a trivialization of $\widetilde{\mathfrak{M}} = \mathfrak{M} \times \mathbb{P}^1$ so that

$$h_1(\mathfrak{M}) = \mathfrak{M} \times \{0\} \qquad \text{and} \qquad h_2(\mathfrak{M}) = \mathfrak{M} \times \left\{\tfrac{\pi}{2}\right\},$$

where the coordinate of $\mathbb{P}^1$ is given by $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]/-\frac{\pi}{2} \sim \frac{\pi}{2}$. Let $p_2 : \widetilde{\mathfrak{M}} \to \mathbb{P}^1$ denote the projection to the second factor.

In this section, we prepare several properties of orbits of ν_t and ψ_t, which will be frequently used throughout this paper.

We only consider the properties of ψ_t restricted to

$$E_1 = \left\{(z, s) \in PX \;\middle|\; \xi_1^+(z) < s < \xi_2^-(z)\right\}.$$

Similar properties hold for $\psi_t|_{E_2}$. We define

$$\Delta_1 : E_1 \to \mathbb{R} \quad \text{by} \quad \Delta_1(z, s) = s - \xi_1^+(z) > 0,$$
$$\Delta_2 : E_1 \to \mathbb{R} \quad \text{by} \quad \Delta_2(z, s) = \xi_2^-(z) - s > 0.$$

Let $e_1(z)$ and $e_2(z)$ denote the unit vectors of $\pi_N^{-1}(z)$ representing $\xi_1^+(z)$ and $\xi_2^-(z)$, respectively, where $e_2(z)$ is assumed to be on the positive side of $e_1(z)$. Set

$$\Delta(z) = \xi_2^-(z) - \xi_1^+(z).$$

Lemma 2.1. *There is a real number $C > 0$ such that, for any $z \in \mathfrak{M}$,*

$$\frac{1}{C} \le |\nu_t(e_1(z))|\,|\nu_t(e_2(z))| \le C$$

Proof. Since $\xi_2^-(\varphi_t(z)) = \psi_t(\xi_2^-(z))$ and $\xi_1^+(\varphi_t(z)) = \psi_t(\xi_1^+(z))$, the angle between $\nu_t(e_1(z))$ and $\nu_t(e_2(z))$ is equal to $\Delta(\varphi_t(z))$. Now, ν_t preserves the area of the triangle defined by the vectors $e_1(z)$ and $e_2(z)$. Hence, the area

$$|\nu_t(e_1(z))|\,|\nu_t(e_2(z))| \;\sin\Delta(\varphi_t(z))$$

is invariant under $t \in \mathbb{R}$. Thus, we obtain

$$|\nu_t(e_1(z))|\,|\nu_t(e_2(z))| = \left|\frac{\sin\Delta(z)}{\sin\Delta(\varphi_t(z))}\right|.$$

Since $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are disjoint closed sets, we can choose $C > 0$ (independent of $z \in \mathfrak{M}$) so that $1/C \le |\sin\Delta(z)/\sin\Delta(\varphi_t(z))| \le C$. ∎

Lemma 2.2. *For* $(z, s) \in E_1$, *we have*

$$\inf_{t\geq 0} \frac{|\nu_t(e_2(z))|}{|\nu_t(e_1(z))|} = 0$$

if and only if $\inf_{t\geq 0} \Delta_1(\psi_t(z, s)) = 0$. *In particular, if* $\inf_{t\geq 0} \Delta_1(\psi_t(z, s)) = 0$ *for some* $(z, s) \in E_1$, *then* $\inf_{t\geq 0} \Delta_1(\psi_t(z, s')) = 0$ *for any* $(z, s') \in E_1$. *Similarly,*

$$\inf_{t\geq 0} \frac{|\nu_t(e_1(z))|}{|\nu_t(e_2(z))|} = 0$$

if and only if $\inf_{t\geq 0} \Delta_2(\psi_t(z, s)) = 0$ *for* $(z, s) \in E_1$.

Proof. Let $v = k_1 e_1(z) + k_2 e_2(z)$ with $k_1, k_2 > 0$ denote the unit vector of $\pi_N^{-1}(z)$ representing (z, s) of E_1. We denote by ϑ_t the angle between $\nu_t(v)$ and $\nu_t(e_1(z))$. By the law of sines for the triangle defined by $\nu_t(k_1 e_1(z))$ and $\nu_t(k_2 e_2(z))$, we obtain

$$\frac{|\nu_t(k_1 e_1(z))|}{\sin\bigl(\Delta(\varphi_t(z)) - \vartheta_t\bigr)} = \frac{|\nu_t(k_2 e_2(z))|}{\sin \vartheta_t}.$$

Thus,

$$\frac{|\nu_t(e_2(z))|}{|\nu_t(e_1(z))|} = \frac{k_1}{k_2} \frac{\tan \vartheta_t}{\sin \Delta(\varphi_t(z)) - \cos \Delta(\varphi_t(z)) \tan \vartheta_t}.$$

Since $\sin \Delta(\varphi_t(z))$ is bounded below, $\inf_{t\geq 0} |\nu_t(e_2(z))|/|\nu_t(e_1(z))| = 0$ if and only if $\inf_{t\geq 0} \vartheta_t = 0$, which is equivalent to $\inf_{t\geq 0} \Delta_1(\psi_t(z, s)) = 0$. □

The next lemma is efficient for the study of the dynamical properties of $\psi_t|_{E_1}$.

Lemma 2.3. *For* $(z, s) \in E_1$, *we have* $\inf_{t\geq 0} \Delta_1(\psi_t(z, s)) = 0$ *if and only if*

$$\overline{O_+(z, s)} \cap \mathfrak{N}_1 \neq \varnothing.$$

Proof. It is enough to show that $\overline{O_+(z, s)} \cap \mathfrak{N}_1 = \varnothing$ if $\inf_{t\geq 0} \Delta_1(\psi_t(z, s))$ is greater than 0.

Let $C = \min\bigl\{\inf_{t\geq 0} \Delta_1(\psi_t(z, s)),\ \pi/2\bigr\}$. By (b) in Proposition 1.1, there is a point z_1 of $\mathfrak{M}$ such that $\mathfrak{N}_1 \cap \pi_P^{-1}(z_1)$ consists of a single point. Then, $\xi_1^{\pm}$ is continuous at z_1. We choose a neighborhood U_1 of z_1 so that $\sup_{x\in U_1} \xi_1^+(x) - \inf_{x\in U_1} \xi_1^-(x) < C$. For some small $\varepsilon > 0$, let W_1 be the open set

$$\Bigl\{(x, u) \in E_1 \Bigm| x \in U_1,\ \inf_{y\in U_1} \xi_1^-(y) - \varepsilon < u < \inf_{y\in U_1} \xi_1^-(y) + C\Bigr\}.$$

Since $\Delta_1(\psi_t(z, s)) \geq C$ for $t \geq 0$, the point $\psi_t(z, s)$ is not contained in W_1 for $t \geq 0$. On the other hand, $\pi_P^{-1}(U_1) \cap \mathfrak{N}_1$ is contained in W_1, because, if $x \in U_1$, then $\xi_1^+(x) \leq \sup_{y\in U_1} \xi_1^+(y) < \inf_{y\in U_1} \xi_1^-(y) + C$. Therefore, $\bigcup_{t\leq 0} \psi_t(W_1)$ is a neighborhood of $\mathfrak{N}_1$ disjoint from $O_+(z, s)$, which implies $\overline{O_+(z, s)} \cap \mathfrak{N}_1 = \varnothing$. □

Lemma 2.4. *If* (z, s) *is a point of* E_1 *satisfying*

$$\lim_{t\to+\infty} \Delta_1(\psi_t(z, s)) = 0$$

(*respectively*, $\lim_{t\to-\infty} \Delta_1(\psi_t(z, s)) = 0$), *then*

$$\lim_{t\to+\infty} |\nu_t(v)| = \infty$$

(*respectively*, $\lim_{t\to-\infty} |\nu_t(v)| = \infty$), *for any* $v \in \pi_N^{-1}(z) - \{0\}$ *with* $[v] \in \mathfrak{N}_1$, *where* $[v]$ *denotes the element of* PX *represented by* v.

Proof. Suppose that $\lim_{t\to+\infty} \Delta_1(\psi_t(z, s)) = 0$. By the same proof as that of Lemma 2.2, $|\nu_t(e_2(z))|/|\nu_t(e_1(z))|$ converges to 0 as $t \to +\infty$. By Lemma 2.1, $|\nu_t(e_1(z))|\,|\nu_t(e_2(z))|$ is bounded below. Thus, $|\nu_t(e_1(z))|$ tends toward ∞ as $t \to +\infty$, because

$$|\nu_t(e_1(z))|^2 = \frac{|\nu_t(e_1(z))|}{|\nu_t(e_2(z))|}\,|\nu_t(e_1(z))|\,|\nu_t(e_2(z))|.$$

Again by Lemma 2.1, we obtain $\lim_{t\to+\infty} |\nu_t(e_2(z))| = 0$, and hence, for any $v \in \pi_N^{-1}(z) - \{0\}$ with $[v] \in \mathfrak{N}_1$, $\lim_{t\to+\infty} |\nu_t(v)| = \infty$.

The other case (that is, when $\lim_{t\to-\infty} \Delta_1(\psi_t(z, s)) = 0$) can be shown in the same way. □

3. Fiberwise divergent orbits

An orbit $O(z, s)$ of ψ_t in E_i $(i = 1, 2)$ is called *fiberwise convergent* if either

$$\lim_{t\to+\infty} \Delta_1(\psi_t(z, s)) = 0 \qquad \text{or} \qquad \lim_{t\to+\infty} \Delta_2(\psi_t(z, s)) = 0$$

and either

$$\lim_{t\to-\infty} \Delta_1(\psi_t(z, s)) = 0 \qquad \text{or} \qquad \lim_{t\to-\infty} \Delta_2(\psi_t(z, s)) = 0.$$

If not, we call it *fiberwise divergent*.

Lemma 3.1. *If there is a point* (z_1, s_1) *of* E_i $(i = 1, 2)$ *whose orbit is fiberwise divergent*, *then there exist points* (z_2, s_2) *and* (z_3, s_3) *of* E_i *such that*

$$\overline{O(z_2, s_2)} \cap \mathfrak{N}_1 = \varnothing \qquad \textit{and} \qquad \overline{O(z_3, s_3)} \cap \mathfrak{N}_2 = \varnothing.$$

Proof. Let $O(z_1, s_1)$ be an orbit of ψ_t in E_1 which is fiberwise divergent. We assume first that neither $\lim_{t\to+\infty} \Delta_1(\psi_t(z_1, s_1)) = 0$ nor $\lim_{t\to+\infty} \Delta_2(\psi_t(z_1, s_1)) = 0$. Since $\omega(z_1, s_1)$ contains a minimal set, $\omega(z_1, s_1)$ contains $\mathfrak{N}_1$ or $\mathfrak{N}_2$. If $\omega(z_1, s_1)$ contains $\mathfrak{N}_1$, then $\inf_{t\geq 0} \Delta_1(\psi_t(z_1, s_1)) = 0$ by Lemma 2.3. Then, there exist points (z_2, s_2) and (z_3, s_3) of E_1 such that $\overline{O(z_2, s_2)} \cap \mathfrak{N}_1 = \varnothing$ and $\overline{O(z_3, s_3)} \cap \mathfrak{N}_2 = \varnothing$ by Lemma 3.3 at the end of this section. The other cases in Lemma 3.1 can be proved similarly. □

In order to later prove Lemma 3.3, we introduce the cross-ratio of straight lines in the plane: Let s be an element of $\mathbb{P}^1$. We denote by l the straight line in the plane, passing through the origin and representing s. If $\rho(s)$ (in $\mathbb{R}\cup\{\infty\}$) denotes the slope of l, then $(1, \rho(s)) \in \mathbb{R}^2$ is the intersection of l and the line $\{(1, y) \mid y \in \mathbb{R}\}$. We give the coordinate of $\mathbb{P}^1$ by $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]/\sim$, and thus $\rho(s) = \tan s$.

Let s_1, s_2, s_3 and s_4 be elements of $\mathbb{P}^1$. The cross-ratio (s_1, s_2, s_3, s_4) is

$$(s_1, s_2, s_3, s_4) = \frac{\rho(s_1)-\rho(s_3)}{\rho(s_3)-\rho(s_2)} \Big/ \frac{\rho(s_1)-\rho(s_4)}{\rho(s_4)-\rho(s_2)}.$$

It is invariant under the action of $PSL(2;\mathbb{R})$. In particular, for (z, s) and (z, s') in E_1, the cross-ratio $(\xi_1^+(\varphi_t(z)),\ \xi_2^-(\varphi_t(z)),\ p_2\psi_t(z, s),\ p_2\psi_t(z, s'))$ is invariant under $t \in \mathbb{R}$. Now, we define $R_1, R_2 : E_1 \to \mathbb{R}$ by

$$R_1(z, s) = \frac{\rho(s)-\rho(\xi_1^+(z))}{\rho(\xi_2^-(z))-\rho(\xi_1^+(z))} \qquad \text{and} \qquad R_2(z, s) = \frac{\rho(\xi_2^-(z))-\rho(s)}{\rho(\xi_2^-(z))-\rho(\xi_1^+(z))}.$$

We have

$$\begin{aligned}\bigl(\xi_1^+(\varphi_t(z)),\ \xi_2^-(\varphi_t(z)),\ p_2\psi_t(z, s),\ p_2\psi_t(z, s')\bigr)&\\ = \frac{R_1(\psi_t(z, s))}{1-R_1(\psi_t(z, s))} \Big/ \frac{R_1(\psi_t(z, s'))}{1-R_1(\psi_t(z, s'))}&.\end{aligned}$$

Lemma 3.2. *If there exists a sequence $\{(z_n, s_n)\}_{n=1,2,\dots}$ in E_1 such that*

$$R_1(\psi_t(z_n, s_n)) \geq R_1(z_n, s_n)$$

for $-n \leq t \leq n$, then there is a z_∞ in $\mathfrak{M}$ such that $\overline{O(z_\infty, 0)} \cap \mathfrak{N}_1 = \varnothing$.

On the other hand, if there exists a sequence $\{(z'_n, s'_n)\}_{n=1,2,\dots}$ in E_1 such that

$$R_2(\psi_t(z'_n, s'_n)) \geq R_2(z'_n, s'_n)$$

for $-n \leq t \leq n$, then there is a z'_∞ in $\mathfrak{M}$ such that $\overline{O(z'_\infty, 0)} \cap \mathfrak{N}_2 = \varnothing$.

Proof. Let $a_t = R_1(\psi_t(z_n, s_n))$ and $b_t = R_1(\psi_t(z_n, 0))$, where $(z_n, 0)$ is contained in the section $h_1(\mathfrak{M})$ by the choice of projective structure in Section 2. Since $h_1(\mathfrak{M})$ is contained in E_1, we have $0 < a_t < 1$ and $0 < b_t < 1$. By the invariance of the cross-ratios along orbits, we have

$$\frac{a_t}{1-a_t} \Big/ \frac{b_t}{1-b_t} = \frac{a_0}{1-a_0} \Big/ \frac{b_0}{1-b_0}.$$

Since $a_t \geq a_0$ for $t \in [-n, n]$, we obtain $b_t \geq b_0$. Hence,

$$\frac{\rho(p_2\psi_t(z_n, 0))-\rho(\xi_1^+(\varphi_t(z_n)))}{\rho(\xi_2^-(\varphi_t(z_n)))-\rho(\xi_1^+(\varphi_t(z_n)))} \geq \frac{\rho(0)-\rho(\xi_1^+(z_n))}{\rho(\xi_2^-(z_n))-\rho(\xi_1^+(z_n))}.$$

Now, $\rho(\xi_2^-) - \rho(\xi_1^+)$ is bounded above, because $(\mathfrak{N}_1 \cup \mathfrak{N}_2) \cap h_2(\mathfrak{M}) = \varnothing$. Since $\mathfrak{N}_1$ and $h_1(\mathfrak{M})$ are disjoint closed sets, $\rho(0) - \rho(\xi_1^+(z_n))$ is bounded below. Thus, there exists $C_2 > 0$ such that $\Delta_1(\psi_t(z_n, 0)) > C_2$ for any n and $t \in [-n, n]$.

Let $(z_\infty, 0)$ be an accumulating point of $\{(z_n, 0)\}$. Let x be a point of $\mathfrak{M}$ on which $(\mathfrak{N}_1)_x$ is a single point. There is a neighborhood W of $(x, \xi_1^+(x))$ such that $\Delta_1(y, w) < C_2$ whenever $(y, w) \in W \cap E_1$.

We claim that $O(z_\infty, 0) \cap W = \varnothing$. Suppose there is a t_1 such that $\psi_{t_1}(z_\infty, 0) \in W$. If n is a sufficiently large number satisfying $-n \le t_1 \le n$, then $\Delta_1(\psi_{t_1}(z_n, 0)) > C_2$, and hence $\psi_{t_1}(z_n, 0)$ is not contained in W. However, this contradicts the continuity of ψ_{t_1}. Hence, $O(z_\infty, 0)$ is disjoint from W.

The invariant set $\bigcup_{t\in\mathbb{R}} \psi_t(W)$ is a neighborhood of $\mathfrak{N}_1$ disjoint from $O(z_\infty, 0)$. Therefore, $\overline{O(z_\infty, 0)} \cap \mathfrak{N}_1 = \varnothing$.

The second part of the lemma can be proved in the same way as above, for $R_2 = 1 - R_1$. □

Lemma 3.3. *If* (z_1, s_1) *is a point of* E_1 *satisfying* $\inf_{t\ge 0} \Delta_1(\psi_t(z_1, s_1)) = 0$ *but* $\Delta_1(\psi_t(z_1, s_1))$ *does not converge to* 0 *as* $t \to \infty$, *then there exist points* (z_2, s_2) *and* (z_3, s_3) *of* E_1 *such that*

$$\overline{O(z_2, s_2)} \cap \mathfrak{N}_1 = \varnothing \qquad \textit{and} \qquad \overline{O(z_3, s_3)} \cap \mathfrak{N}_2 = \varnothing.$$

Proof. Since $R_1(\psi_t(z_1, s_1))$ does not converge to 0 as $t \to \infty$, there are $C_1 > 0$ and $\{t_n\}_{n=1,2,\dots}$ $(t_n \ge 0)$ such that $\lim_{n\to\infty} t_n = \infty$ and $R_1(\psi_{t_n}(z_1, s_1)) > C_1$. Set

$$\begin{aligned} K &= \big\{(z, s) \in E_1 \;\big|\; R_1(z, s) \ge C_1\big\}, \\ W_n &= E_1 \setminus \big\{\psi_t(z, s) \;\big|\; (z, s) \in K,\ -n \le t \le n\big\}. \end{aligned}$$

Let $C_2 = C_1 \min_{z\in\mathfrak{M}}\{\rho(\xi_2^-(z)) - \rho(\xi_1^+(z))\}$. Since the set

$$\big\{(z, s, s', t) \in \mathfrak{M} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{R} \;\big|\; |\rho(s) - \rho(s')| \ge C_2,\ -n \le t \le n\big\}$$

is compact, the number

$$\varepsilon_n = \inf\left\{ \frac{|\rho(p_2\psi_t(z, s)) - \rho(p_2\psi_t(z, s'))|}{\rho(\xi_2^-(\varphi_t(z))) - \rho(\xi_1^+(\varphi_t(z)))} \;\middle|\; (z, s), (z, s') \in \bar{E}_1,\ -n \le t \le n,\ |\rho(s) - \rho(s')| \ge C_2 \right\}$$

is positive.

We denote by V_n the set $\{(z, s) \in E_1 \mid R_1(z, s) < \varepsilon_n\}$. We claim that V_n is contained in W_n. Suppose that a point (z, s) of E_1 is not contained in W_n. Then, there is a $t_0 \in [-n, n]$ such that $\psi_{t_0}(z, s)$ is contained in K. Hence,

$$\rho(p_2\psi_{t_0}(z, s)) - \rho(\xi_1^+(\varphi_{t_0}(z))) \ge C_1\big(\rho(\xi_2^-(\varphi_{t_0}(z))) - \rho(\xi_1^+(\varphi_{t_0}(z)))\big) \ge C_2.$$

The following inequality holds:

$$R_1(z,s)=\frac{\rho\big(p_2\psi_{-t_0}(\varphi_{t_0}(z),\,p_2\psi_{t_0}(z,s))\big)-\rho\big(p_2\psi_{-t_0}(\varphi_{t_0}(z),\,\xi_1^+(\varphi_{t_0}(z)))\big)}{\rho\big(\xi_2^-(\varphi_{-t_0}(\varphi_{t_0}(z)))\big)-\rho\big(\xi_1^+(\varphi_{-t_0}(\varphi_{t_0}(z)))\big)}\geq\varepsilon_n.$$

Thus, (z, s) is not a point of V_n. This implies that $V_n \subset W_n$.

By the assumption that $\inf_{t\geq 0}\Delta_1(\psi_t(z_1, s_1)) = 0$, we have

$$\inf_{t\geq 0} R_1(\psi_t(z_1, s_1)) = 0.$$

We choose an increasing sequence $\{v_n\}_{n=1,2,\ldots}$ in $\mathbb{R}$ so that $\psi_{v_{2n-1}}(z_1, s_1) \in K$ and $\psi_{v_{2n}}(z_1, s_1) \in V_n$. Let u_n be the time between v_{2n-1} and v_{2n+1} when R_1 is minimum at $\psi_{u_n}(z, s)$, that is,

$$R_1(\psi_{u_n}(z_1, s_1)) = \min\big\{R_1(\psi_t(z_1, s_1))\;\big|\; v_{2n-1} \leq t \leq v_{2n+1}\big\}.$$

Then, $R_1(\psi_{u_n}(z_1, s_1))$ is smaller than ε_n, because $R_1(\psi_{v_{2n}}(z_1, s_1)) < \varepsilon_n$. Hence, $\psi_{u_n}(z_1, s_1)$ is contained in V_n and thus in W_n. Since we have $\psi_{u_n}(z_1, s_1) \in W_n$ and $\psi_{v_{2n-1}}(z_1, s_1) \in K$, the time difference $u_n - v_{2n-1}$ is greater than n. Furthermore, we obtain $v_{2n+1} - u_n > n$, because $\psi_{u_n}(z_1, s_1) \in W_n$ and $\psi_{v_{2n+1}}(z_1, s_1) \in K$.

By the above construction, if $-n \leq t \leq n$, the following inequalities hold:

$$v_{2n-1} < u_n - n \leq t + u_n \leq u_n + n < v_{2n+1}, \qquad R_1(\psi_{t+u_n}(z_1, s_1)) \geq R_1(\psi_{u_n}(z_1, s_1)).$$

Therefore, $\{\psi_{u_n}(z_1, s_1)\}_{n=1,2,\ldots}$ satisfies the first condition of Lemma 3.2, and hence there is a point z_2 of $\mathfrak{M}$ such that $\overline{O(z_2, 0)} \cap \mathfrak{N}_1 = \varnothing$.

Let w_n be the time between v_{2n} and v_{2n+2} when R_2 is minimum, that is,

$$R_2(\psi_{w_n}(z_1, s_1)) = \min\big\{R_2(\psi_t(z_1, s_1))\;\big|\; v_{2n} \leq t \leq v_{2n+2}\big\}.$$

Since $R_1 = 1 - R_2$, $\psi_{w_n}(z_1, s_1)$ is contained in K. Hence, $w_n - v_{2n} \geq n$ and $v_{2n+2} - w_n \geq n$. Thus, $R_2(\psi_{t+w_n}(z_1, s_1)) \geq R_2(\psi_{w_n}(z_1, s_1))$ for $-n \leq t \leq n$. Therefore, $\{\psi_{w_n}(z_1, s_1)\}_{n=1,2,\ldots}$ satisfies the second condition of Lemma 3.2. As a consequence, there exists a point z_3 of $\mathfrak{M}$ such that $\overline{O(z_3, 0)} \cap \mathfrak{N}_2 = \varnothing$. □

4. Fiberwise-convergent orbits

Let Λ be a closed φ_t-invariant set in M. A C^1 flow φ_t admits a dominated splitting over Λ if there is a continuous splitting of $\pi_N^{-1}(\Lambda)$ into a direct sum of 1-dimensional bundles S and U, invariant under $\tilde{D}\varphi_t$, such that there are constants $C > 0$ and $\lambda \in (0, 1)$ satisfying

$$\frac{\|\tilde{D}\varphi_t|_{S(z)}\|}{\|\tilde{D}\varphi_t|_{U(z)}\|} \leq C\lambda^t$$

for any z and $t \geq 0$, where $\tilde{D}\varphi_t$ is the map on $NX = TM/T\varphi_t$ induced from $D\varphi_t$, while $\|\cdot\|$ denotes the operator norm from some continuous Riemannian metric. The above inequality is equivalent to

$$\frac{\|\nu_t|_{S(z)}\|}{\|\nu_t|_{U(z)}\|} \le C\lambda^t.$$

We consider the case when Λ is a minimal set. For example, if Λ is a hyperbolic closed orbit, then φ_t admits a dominated splitting over Λ. On the other hand, a minimal C^2 flow on a closed 3-manifold does not admit a dominated splitting over the whole 3-manifold, as was proved by Arroyo and Rodriguez Hertz [2003].

In this section, we will consider the case when all the orbits of ψ_t contained in E_1 are fiberwise convergent. The orbit of ψ_t passing through $(z, s) \in E_1$ is then classified into four types:

$$\begin{array}{llll}
\text{(I)} & \lim_{t\to+\infty} \Delta_1(\psi_t(z,s)) = 0 & \text{and} & \lim_{t\to-\infty} \Delta_2(\psi_t(z,s)) = 0; \\
\text{(II)} & \lim_{t\to+\infty} \Delta_2(\psi_t(z,s)) = 0 & \text{and} & \lim_{t\to-\infty} \Delta_1(\psi_t(z,s)) = 0; \\
\text{(III)} & \lim_{t\to+\infty} \Delta_1(\psi_t(z,s)) = 0 & \text{and} & \lim_{t\to-\infty} \Delta_1(\psi_t(z,s)) = 0; \\
\text{(IV)} & \lim_{t\to+\infty} \Delta_2(\psi_t(z,s)) = 0 & \text{and} & \lim_{t\to-\infty} \Delta_2(\psi_t(z,s)) = 0.
\end{array}$$

To prove Theorem 1.2, we will investigate the case when there is no orbit of type IV. The following lemma of Contreras and Iturriaga [1999] plays an important role; we change their hypothesis to suit our purpose, but the proof is the same.

Lemma 4.1 [Contreras and Iturriaga 1999, Lemma 3.3]. *If* $\sup_{t\in\mathbb{R}} |\nu_t(v)| = \infty$ *for any* $v \neq 0$ *with* $[v] \in \mathfrak{N}_1$, *then there is a* $C_1 > 0$ *such that*

$$|\nu_t(e_1(z))| \le C_1\big(1 + |\nu_s(e_1(z))|\big)$$

for any z, s *and* t *with* $0 \le t \le s$.

Lemma 4.2. *If all the orbits of* ψ_t *in* E_1 *are of type* I, II *or* III, *then all the orbits are of type* I *or all the orbits are of type* II.

Proof. It is enough to show that, if all the orbits of ψ_t are of type I, II or III and there is an orbit of type I or III, then all the orbits are of type I.

By the assumption that there is no orbit of type IV, we have

$$\lim_{t\to+\infty} \Delta_1(\psi_t(z,s)) = 0 \qquad \text{or} \qquad \lim_{t\to-\infty} \Delta_1(\psi_t(z,s)) = 0$$

for any $(z, s) \in E_1$. By Lemma 2.4, $\sup_{t\in\mathbb{R}} |\nu_t(v)| = \infty$ for any $v \neq 0$ with $[v] \in \mathfrak{N}_1$. Thus, by Lemma 4.1, there is a $C_1 > 0$ such that $|\nu_t(e_1(z))| \le C_1\big(1 + |\nu_s(e_1(z))|\big)$ for any z, s and t with $0 \le t \le s$. If (z_1, s_1) is a point of type I or III, then $\lim_{t\to+\infty} \Delta_1(\psi_t(z_1, s_1)) = 0$, and hence $\lim_{t\to+\infty} |\nu_t(e_1(z_1))| = \infty$ by Lemma 2.4.

Let z_2 be an arbitrary point of $\mathfrak{M}$. Since $\mathfrak{M}$ is a minimal set, there is a sequence $\{t_n\}_{n=1,2,\ldots}$ of positive numbers such that $\lim_{n\to+\infty} t_n = \infty$ and $\varphi_{t_n}(z_1) \to z_2$.

Taking a subsequence of $\{t_n\}$, we can assume that $\nu_{t_n}(e_1(z_1))/|\nu_{t_n}(e_1(z_1))|$ converges to some unit vector v_2. Then $[v_2]$ is contained in $(\mathfrak{N}_1)_{z_2}$. Since we have $\lim_{t\to+\infty}|\nu_t(e_1(z_1))| = \infty$, the constant $C_2 = \inf_{t\geq 0}|\nu_t(e_1(z_1))|$ is positive. For $0 \leq t \leq t_n$, we have

$$\left|\nu_{-t}\left(\frac{\nu_{t_n}(e_1(z_1))}{|\nu_{t_n}(e_1(z_1))|}\right)\right| = \frac{|\nu_{t_n-t}(e_1(z_1))|}{|\nu_{t_n}(e_1(z_1))|} \leq \frac{C_1\bigl(1+|\nu_{t_n}(e_1(z_1))|\bigr)}{|\nu_{t_n}(e_1(z_1))|} \leq C_1\left(1+\frac{1}{C_2}\right).$$

Therefore, $|\nu_{-t}(v_2)|$ is bounded above for $t \geq 0$. By Lemma 2.4, for $(z_2, s) \in E_1$, $\Delta_1(\psi_t(z_2, s))$ does not converge to 0 as $t \to -\infty$. Since all the orbits are of type I, II or III, the orbit passing through (z_2, s) is of type I. □

Proof of Theorem 1.2. We prove that, if φ_t has no dominated splitting over $\mathfrak{M}$, there exist points (z, s) and (z', s') of E_1 with $\overline{O(z,s)} \cap \mathfrak{N}_1 = \varnothing$ and $\overline{O(z',s')} \cap \mathfrak{N}_2 = \varnothing$. The proof for $\psi_t|_{E_2}$ is the same. By Lemma 3.1, we can further assume that all the orbits of ψ_t in E_1 are fiberwise convergent.

If there is no orbit of type IV, then, by Lemma 4.2, all the orbits are of type I or all the orbits are of type II. But then φ_t has a dominated splitting over $\mathfrak{M}$ by a standard argument (for example, $\varphi_t|_{\mathfrak{M}}$ is weakly partially hyperbolic [Contreras 2002]). Therefore, there exists an orbit of type IV. We can show in the same way that there exists an orbit of type III. Thus, there exist points (z, s) and (z', s') of E_1 such that $\overline{O(z,s)} \cap \mathfrak{N}_1 = \varnothing$ and $\overline{O(z',s')} \cap \mathfrak{N}_2 = \varnothing$. □

5. Almost-periodic minimal flows

We prove Theorem 1.3. A subset A of $\mathbb{R}$ is called *syndetic* if $\mathbb{R} = \{a + k \mid a \in A,\ k \in K\}$ for some compact set K of $\mathbb{R}$. A flow φ_t on a compact metric space $\mathfrak{M}$ is called *almost periodic* if, for any $\varepsilon > 0$, there is a syndetic set A such that $d(z, \varphi_a(z)) < \varepsilon$ for any $z \in \mathfrak{M}$ and $a \in A$, where d is a metric of $\mathfrak{M}$. If the whole $\mathfrak{M}$ is a minimal set, then the flow is called *minimal*.

Almost-periodic minimal flows on a compact metric space $\mathfrak{M}$ are already classified in the topological sense. In fact, they are equivalent to equicontinuous minimal flows, see [Auslander 1988, Theorem, p. 36]. Furthermore, there are invariant metrics on $\mathfrak{M}$, and $\mathfrak{M}$ is a compact abelian group, see [Auslander 1988, Exercises, p. 45]. However, the infinitesimal behavior of almost-periodic minimal flows is still complicated, as Example 1.4 indicates (see also [Nakayama 2001]).

With Theorem 1.2, if φ_t has no dominated splitting, we find two types of orbits. Thus, we deduce Theorem 1.3 by using the next lemma:

Lemma 5.1. *Assume that $\varphi_t|_{\mathfrak{M}}$ is an almost-periodic minimal flow and $\psi_t|_{\widetilde{\mathfrak{M}}}$ has exactly two minimal sets. If, for some $i \in \{1, 2\}$, we have both*:

(1) *ψ_t has a positive semiorbit in E_i that does not approach $\mathfrak{N}_1$ (that is, $\overline{O_+(z,s)}$ and $\mathfrak{N}_1$ are disjoint for some (z, s)), and*

(2) *ψ_t has a positive semiorbit in E_i that does not approach $\mathfrak{N}_2$ (that is, $\overline{O_+(z,s)}$ and $\mathfrak{N}_2$ are disjoint for some (z,s));*

then $\psi_t|_{E_i}$ has a dense orbit.

Proof. We only prove this lemma for $\psi_t|_{E_1}$.

Let W_1 and W_2 be arbitrary open sets in E_1. We only have to show that

$$\Big(\bigcup_{t\in\mathbb{R}} \psi_t(W_2)\Big) \cap W_1 \neq \varnothing.$$

We choose open sets U_1 and U_2 in $\mathfrak{M}$ and open intervals $I_1 = (\alpha_1, \beta_1)$ and $I_2 = (\alpha_2, \beta_2)$, so that $\overline{U_1 \times I_1} \subset W_1$ and $\overline{U_2 \times I_2} \subset W_2$. Then it is enough to show that $\big(\bigcup_{t\in\mathbb{R}} \psi_t(U_2 \times I_2)\big) \cap (U_1 \times I_1) \neq \varnothing$.

First, we claim that there are a connected open set V_2 contained in U_2 and a syndetic set A, such that $\varphi_a(V_2)$ is contained in U_1 for any $a \in A$. Let z_3 be a point of U_1. There is an $\varepsilon > 0$ such that the ε-ball $B_\varepsilon(z_3)$ with center z_3 is contained in U_1. By the minimality of φ_t, there is a $t_1 \in \mathbb{R}$ such that $\varphi_{t_1}(z_3)$ is contained in U_2. Let V_2 be a connected component of $U_2 \cap \varphi_{t_1}(B_{\varepsilon/2}(z_3))$. For any $y \in V_2$, we have $d(\varphi_{-t_1}(y), z_3) < \varepsilon/2$. Since φ_t is almost periodic, there is a syndetic set A' such that $d(\varphi_a(x), x) < \varepsilon/2$ for any $x \in \mathfrak{M}$ and $a \in A'$. In particular, $d(\varphi_{a-t_1}(y), \varphi_{-t_1}(y)) < \varepsilon/2$ for any $a \in A'$. Hence, $d(\varphi_{a-t_1}(y), z_3) < \varepsilon$, which implies that $\varphi_{a-t_1}(y)$ is contained in U_1. Since $\{a - t_1 \mid a \in A'\}$ is also syndetic, the claim follows. Hence, for any $t \in \mathbb{R}$, there is a $u \in [0, C_1]$ satisfying $\varphi_{t+u}(V_2) \subset U_1$.

The set

$$\big\{z \in \mathfrak{M} \;\big|\; \overline{O_+(z,s)} \cap \mathfrak{N}_1 = \varnothing \text{ for any } (z,s) \in E_1\big\}$$

is a nonempty invariant set in $\mathfrak{M}$, by Lemmas 2.2 and 2.3. Hence, it is dense in $\mathfrak{M}$, because $\varphi_t|_{\mathfrak{M}}$ is minimal. Furthermore, the set

$$\big\{z \;\big|\; \overline{O_+(z,s)} \cap \mathfrak{N}_2 = \varnothing \text{ for any } (z,s) \in E_1\big\}$$

is also dense in $\mathfrak{M}$. Thus, there are points (z_1, s_1) and (z_2, s_2) of $V_2 \times I_2$ such that $\overline{O_+(z_1,s_1)} \cap \mathfrak{N}_1 = \varnothing$ and $\overline{O_+(z_2,s_2)} \cap \mathfrak{N}_2 = \varnothing$. Since ψ_t contains no minimal set in E_1, we have $\overline{O_+(z_1,s_1)} \cap \mathfrak{N}_2 \neq \varnothing$ and $\overline{O_+(z_2,s_2)} \cap \mathfrak{N}_1 \neq \varnothing$.

Set

$$\begin{aligned} F_2 &= \big\{(z,s) \in E_1 \;\big|\; z \in \overline{U}_1,\ s \le \beta_1\big\}, \\ W_2 &= E_1 \setminus \big\{\psi_t(z,s) \;\big|\; (z,s) \in F_2,\ -C_1 \le t \le 0\big\}. \end{aligned}$$

Then, $W_2 \cup K_2$ is a neighborhood of $\mathfrak{N}_2$ in $E_1 \cup K_2$, where

$$K_i = \big\{(z,s) \in PX \;\big|\; \xi_i^-(z) \le s \le \xi_i^+(z)\big\},$$

as in Section 1. If $(z,s) \in W_2$, then, for $t \in [0, C_1]$, $\psi_t(z,s)$ is not contained in F_2. If we further assume that $\varphi_t(z) \in U_1$, then $p_2\psi_t(z,s) > \beta_1$.

We claim that there exists a $C_2 > 0$ such that, for any $t \ge 0$, there is a $u \in [0, C_2]$ such that $\psi_{t+u}(z_1, s_1) \in W_2$. If not, there is a sequence $\{t'_n\}_{n=1,2,\dots}$ with $t'_n \ge 0$, such that $\psi_{t'_n+u}(z_1, s_1) \notin W_2$ for $0 \leq u \leq n$. Let (z_0, s_0) be an accumulating point of $\{\psi_{t'_n}(z_1, s_1)\}_{n=1,2,\ldots}$. The positive semiorbit starting from (z_0, s_0) is disjoint from W_2. Thus, $\overline{O_+(z_0, s_0)} \cap \mathfrak{N}_2 = \varnothing$. On the other hand, $O_+(z_1, s_1)$ is disjoint from a neighborhood of $\mathfrak{N}_1$. Hence, $\psi_{t'_n+u}(z_1, s_1)$ is disjoint from this neighborhood for $0 \leq u \leq n$. Thus, $\overline{O_+(z_0, s_0)} \cap \mathfrak{N}_1 = \varnothing$. Therefore, the ω-limit set of (z_0, s_0) contains a minimal set different from $\mathfrak{N}_1$ and $\mathfrak{N}_2$, which contradicts the assumption.

We construct a neighborhood W_1 of $\mathfrak{N}_1$, similarly to the construction of W_2 as a neighborhood of $\mathfrak{N}_2$, by using the constant $C_1 + C_2$. Set

$$\begin{aligned} F_1 &= \left\{(z, s) \in E_1 \;\middle|\; z \in \overline{U}_1, s \geq \alpha_1\right\}, \\ W_1 &= E_1 \setminus \left\{\psi_t(z, s) \;\middle|\; (z, s) \in F_1, -(C_1 + C_2) \leq t \leq 0\right\}. \end{aligned}$$

Then, $W_1 \cup K_1$ is a neighborhood of $\mathfrak{N}_1$ in $E_1 \cup K_1$ such that, if $(z, s) \in W_1$ and $\varphi_t(z) \in U_1$ for some $0 \leq t \leq C_1 + C_2$, then $p_2 \psi_t(z, s) < \alpha_1$.

We will choose t (as $t = t_2 + t_3 + t_4$) so that $\psi_t(U_2 \times I_2) \cap (U_1 \times I_1) \neq \varnothing$. First, we choose $t_2 \geq 0$ so that $\psi_{t_2}(z_2, s_2) \in W_1$. By the choice of C_2, there is a $t_3 \in [0, C_2]$ such that $\psi_{t_2+t_3}(z_1, s_1) \in W_2$. Finally, we take $t_4 \in [0, C_1]$ so that $\varphi_{t_2+t_3+t_4}(V_2)$ is contained in U_1. Since $\psi_{t_2}(z_2, s_2) \in W_1$ and $\varphi_{t_2+t_3+t_4}(z_2) \in U_1$, we have $p_2 \psi_{t_2+t_3+t_4}(z_2, s_2) < \alpha_1$. On the other hand, $p_2 \psi_{t_2+t_3+t_4}(z_1, s_1) > \beta_1$, because $\psi_{t_2+t_3}(z_1, s_1) \in W_2$ and $\varphi_{t_2+t_3+t_4}(z_1) \in U_1$. Therefore, $\psi_{t_2+t_3+t_4}$ maps an arc contained in $V_2 \times I_2$ that joins (z_1, s_1) and (z_2, s_2) onto an arc contained in $\pi_P^{-1}(U_1)$ which intersects $U_1 \times I_1$. Thus, $\psi_{t_2+t_3+t_4}(V_2 \times I_2) \cap (U_1 \times I_1) \neq \varnothing$. $\square$

Now, at the end of the paper, we comment on the "cocycle condition" for the projective flows with exactly two minimal sets $\mathfrak{N}_1$ and $\mathfrak{N}_2$: Assume that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are the image of two (continuous) sections, and change the trivialization of $PX = \mathfrak{M} \times \mathbb{P}^1 = \mathfrak{M} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]/\sim$ so that $\mathfrak{N}_1 = \mathfrak{M} \times \{0\}$ and $\mathfrak{N}_2 = \mathfrak{M} \times \left\{\frac{\pi}{2}\right\}$. Then, for $z \in \mathfrak{M}$, the restriction of ψ_t to the fiber can be written

$$\begin{pmatrix} 1 & 0 \\ 0 & a_t(z) \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & 0 \\ 0 & a_t(z) \end{pmatrix} \begin{pmatrix} 1 \\ \rho(s) \end{pmatrix} = \begin{pmatrix} 1 \\ \rho(s') \end{pmatrix}$$

if $\psi_t(z, s) = (\varphi_t(z), s')$. Thus, $a_{t_1+t_2}(z) = a_{t_2}(\varphi_{t_1}(z)) a_{t_1}(z)$ for $t_1, t_2 \in \mathbb{R}$. Now, ψ_t has exactly two minimal sets, $\mathfrak{N}_1$ and $\mathfrak{N}_2$. Thus, $\left|\sum_{i=0}^{n} \log a_1(\varphi_i(z))\right| = |\log a_n(z)|$ is not bounded. By Gottschalk and Hedlund [1955, Theorem 14.11], there is no continuous function $h : \mathfrak{M} \to \mathbb{R}$ such that $h(\varphi_1(z)) - h(z) = \log a_1(z)$ if φ_1 is minimal.

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Received June 2, 2005. Revised August 27, 2005.

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