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REAL FORMS OF COMPLEX LIE SUPERALGEBRAS AND COMPLEX ALGEBRAIC SUPERGROUPS

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The paper concerns two versions of the notion of real forms of Lie superalgebras. One is the standard approach, where a real form of a complex Lie superalgebra is a real Lie superalgebra whose complexification is the original complex Lie superalgebra. The second arises from considering A-points of a Lie superalgebra over a commutative complex superalgebra A equipped with superconjugation. The first kind of real form can be obtained as the set of fixed points of an antilinear involutive automorphism; the second is related to an automorphism ϕ such that ϕ^2 is the identity on the even part and the negative identity on the odd part. The generalized notion of real forms is then introduced for complex algebraic supergroups.

1. Introduction

There are two equivalent points of view for defining a complex Lie superalgebra. The standard one treats a complex Lie superalgebra as a supervector space with a superbracket. The other defines Lie superalgebra as a representable functor. The equivalence of the two definitions is a corollary of the so-called "even rules" [Deligne and Morgan 1999]. In studying any particular aspect of the theory of Lie superalgebras, we can choose to work either with the standard (supervector) Lie superalgebra or with the functorial definition. We argue in this paper that the functorial language is particularly well adapted to the problem of defining and classifying real forms of simple Lie superalgebras.

Real forms were defined and classified by Serganova [1983] in the standard framework of complex vector superspaces with a somewhat puzzling conclusion that simple complex Lie superalgebras have no compact real forms. The functorial point of view avoids this puzzle by defining two kinds of real forms, which we call standard and graded. In the standard approach a real form is defined as a real Lie superalgebra whose complexification is the original complex Lie superalgebra. It can be seen easily that every standard real form is naturally associated to an antilinear involutive automorphism of the complex Lie superalgebra [Serganova 1983]. The

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notion of a graded real form is characterized by antilinear automorphisms that are not involutive, but rather graded involutive. It turns out, remarkably, that Serganova classified in her paper also the graded involutive automorphisms, although she did not interpret them as real forms. Using her results, we show that our more general definition of real forms solves the puzzle, in the sense that every simple complex Lie superalgebra possesses precisely one compact (graded) real form.

The functorial point of view turns out to be useful also for defining real forms of complex algebraic supergroups. We do this for subsupergroups of $\mathrm{GL}_{m|n}$. We show that a real form of a supergroup induces a real form on its corresponding Lie superalgebra. This raises a question: What are the lifts of Serganova automorphisms to the corresponding supergroups? We answer this question completely for the series $\mathrm{SL}_{m|n}$ and $\mathrm{OSp}_{m|2n}$.

2. Graded and standard real forms of complex Lie superalgebras

By a "superalgebra" we mean a commutative and associative superalgebra over $\mathbb{C}$. The even and odd parts of a superalgebra A are denoted respectively by A_0 and A_1. Elements of A_0 and A_1 are called homogeneous and the parity of an homogeneous element x is $|x| = 0$ for $x \in A_0$ or $|x| = 1$ for $x \in A_1$. A superalgebra is called commutative if $xy = (-1)^{|x||y|}yx$ for homogeneous elements x, y. By $\mathcal{J}_A^{\text{odd}}$ we denote the ideal in A generated by the odd part. We say that a superalgebra A is reduced if $A/\mathcal{J}_A^{\text{odd}}$ has no nilpotent elements.

We denote by $\mathcal{A}$ the category of reduced, finitely generated complex superalgebras and by $\mathcal{S}$ the category of sets. We restrict our category A to reduced and finitely generated superalgebra because we are only concerned with the Lie superalgebra that come from affine algebraic supergroups.

Let $\mathbb{O} : \mathcal{A} \to \mathcal{S}$ be the functor sending each superalgebra A on its even part: $\mathbb{O}(A) = A_0$. Here is the definition of a complex Lie superalgebra in the functorial setting, as given in [Fioresi and Lledó 2004]:

Definition 2.1. A Lie superalgebra is a representable functor

$$\begin{aligned} \mathcal{G}_V : \mathcal{A} &\to \mathcal{S} \\ A &\mapsto \mathcal{G}_V(A) \end{aligned}$$

satisfying these conditions:

(i) For each superalgebra A we have an A_0-module structure on $\mathcal{G}_V(A)$ that is functorial in A; in other words, there is a natural transformation $\mathbb{O} \times \mathcal{G}_V \to \mathcal{G}_V$.

(ii) For each superalgebra A, we have a Lie-bracket $[\ ,\ ]_A$ on $\mathcal{G}_V(A)$ which is A_0-linear and functorial in A; in other words, there is a natural transformation $[\ ,\ ] : \mathcal{G}_V \times \mathcal{G}_V \to \mathcal{G}_V$ which is $\mathbb{O}$-linear and satisfies a commutative diagram corresponding to the antisymmetric property and Jacobi identity.

The statement that a Lie superalgebra is representable means that there exists a supervector space V (which is unique) such that $\mathcal{G}_V(A) = (A \otimes_{\mathbb{C}} V)_0$. Following the even rules principle, V inherits a superbracket from the Lie algebra structure of the $\mathcal{G}_V(A)$ for all A. For clarity, we explain the even rules principle to the extent needed in this article. (For a proof see [Deligne and Morgan 1999, Theorem 1.7.1 and Corollary 1.7.3, pp. 56–57] or [Varadarajan 2004].) Let f be a bilinear map from $V \times V$ to V. We define a B_0-bilinear map $f_B : \mathcal{G}_V(B) \times \mathcal{G}_V(B) \to \mathcal{G}_V(B)$ by

$$f_B(b_1 \otimes v_1, b_2 \otimes v_2) = (-1)^{N(N-1)/2} b_1 b_2 \otimes f(v_1, v_2), \tag{1}$$

where $b_1, b_2 \in B$ and $v_1, v_2 \in V$ are such that $|b_i| = |w_i|$ and there are N odd elements among the b_i. The collection of maps f_B is functorial in B. This means that the diagram

$$\begin{array}{ccc} \mathcal{G}_V(B) \times \mathcal{G}_V(B) & \xrightarrow{f_B} & \mathcal{G}_V(B) \\ \Big\downarrow {\scriptstyle (\mathcal{G}_V(f), \mathcal{G}_V(f))} & & \Big\downarrow {\scriptstyle \mathcal{G}_V(f)} \\ G_{\mathbb{C}} \times G_{\mathbb{C}} & \xrightarrow[f_C]{} & G_{\mathbb{C}} \end{array}$$

is commutative for any complex superalgebras B, C, where the morphism $\mathcal{G}_V(f)$ associated to $f : B \to C$ is given by $\mathcal{G}_V(f)(b \otimes v) = f(b) \otimes v$ for $b \in B$, $v \in V$. The even rules principle claims that any functorial collection f_B of B_0-linear maps

$$f_B : \mathcal{G}_V(B) \times \mathcal{G}_V(B) \to \mathcal{G}_V(B)$$

comes from a unique $\mathbb{C}$-linear map $f : V \times V \to V$; that is, the maps f_B and f are linked by the formula (1). Thus a Lie superbracket [,] comes from the functorial collection of Lie bracket $[\ ,\ ]_A$ and , vice versa, a Lie superbracket gives rise to a system of functorial brackets. The relation between [,] and $[\ ,\ ]_A$ is

$$[a \otimes v, b \otimes w]_A = (-1)^{|b||v|} ab \otimes [v, w], \tag{2}$$

where $a, b \in A$, $v, w \in V$ which fulfill $|a| = |v|$ and $|b| = |w|$.

We next introduce the notion of a real structure of a complex Lie superalgebra. For this, the category $\mathcal{A}$ from Definition 2.1 must be constructed as the category of superalgebras with conjugation. In particular, the morphism in $\mathcal{A}$ must respect (commute with) conjugation. In contrast to the nonsuper case, there are two kinds of conjugation on the commutative associative superalgebra:

Definition 2.2 [Deligne and Morgan 1999]. Let A be a complex superalgebra. A map $a \to \bar{a}$ with $a, \bar{a} \in A$ is called a *standard* conjugation if

$$\overline{\lambda a} = \bar{\lambda}\bar{a}, \quad \overline{ab} = \bar{a}\bar{b}, \quad \bar{\bar{a}} = a \qquad \text{for } \lambda \in \mathbb{C},\ a, b \in A.$$

Definition 2.3. Let A be a complex superalgebra. A map $a \to \tilde{a}$ with $a, \tilde{a} \in A$ is called a *graded* conjugation if

$$\widetilde{\lambda a} = \bar{\lambda}\tilde{a}, \quad \widetilde{ab} = \tilde{a}\tilde{b}, \quad \tilde{\tilde{a}} = (-1)^{|a|}a \qquad \text{for } \lambda \in \mathbb{C},\ a, b \in A.$$

Remark 2.4. We denote the conjugate of $a \in A$ by $\hat{a}$ when the distinction between graded and standard conjugation does not matter. We say that an element $a \in A$ is real if it satisfies $a = \hat{a}$. The superalgebra A^{real} is the superalgebra consisting of the real elements of A.

To give the definition of the real structure of a Lie superalgebra in the functorial setting we need some conventions. In the case of a supervector space $V = V_0 + V_1$, a map $f : V \to V$ is said to be even if it respects parity; that is, if it maps V_0 into V_0 and V_1 into V_1. The A_0-module $(A \otimes V)_0$ also has a grading, given by the decomposition

$$(A \otimes V)_0 = A_0 \otimes V_0 + A_1 \otimes V_1. \tag{3}$$

The elements of $A_0 \otimes V_0$ are called even and those of $A_1 \otimes V_1$ are called odd. A map f on $(A \otimes V)_0$ is even when it preserves the grading above.

Definition 2.5. A real structure of a Lie superalgebra is a natural transformation $\Phi : \mathcal{G} \to \mathcal{G}$ such that for $a, b \in A_0$ and $x, y \in \mathcal{G}(A)$ we have

$$\begin{aligned} \Phi_A(ax + by) &= \hat{a}\Phi_A(x) + \hat{b}\Phi_A(y), \\ \Phi_A(\Phi_A(x)) &= x, \\ \Phi_A([x, y]_A) &= [\Phi_A(x), \Phi_A(y)]_A; \end{aligned}$$

we also demand that Φ_A be an even map. The real structure is called *standard* if $\hat{}$ represents the standard conjugation, and *graded* if $\hat{}$ is the graded conjugation.

Theorem 2.6. (a) *A standard real structure* $(\mathcal{G}_V, \Phi)$ *comes from a unique map* $\phi : V \to V$ *such that*

$$\phi(\lambda x + \mu y) = \bar{\lambda}\phi(x) + \bar{\mu}\phi(y),\ \phi^2(x) = x,\ \phi([x, y]) = [\phi(x), \phi(y)], \tag{4}$$

with $\lambda, \mu \in \mathbb{C}$ *and* $x, y \in V$.

(b) *A graded real structure* $(\mathcal{G}_V, \Phi)$ *comes from a unique map* $\phi : V \to V$ *such that*

$$\phi(\lambda x + \mu y) = \bar{\lambda}\phi(x) + \bar{\mu}\phi(y),\ \phi^2(x) = (-1)^{|x|}x,\ \phi([x, y]) = [\phi(x), \phi(y)], \tag{5}$$

with $\lambda, \mu \in \mathbb{C}$ *and* $x, y \in V$.

(c) *Conversely, a map* ϕ *on* V *fulfilling condition* (4) (*resp.* (5)) *gives rise to a standard* (*resp. graded*) *real form* $(\mathcal{G}_V, \Phi)$.

Proof. We prove (b) and (c). The proof of (a) is very similar to that of (b).

(b) Let $(\mathcal{G}_V, \Phi)$ be a graded real form. We first show that Φ gives rise to a map $\phi : V \to V$, which means that $\Phi(a \otimes v) = \tilde{a} \otimes \phi(v)$ with $a \in A$, $v \in V$ and $|a| = |v|$. The crucial property is that Φ is a natural transformation. Thus for two superalgebras A, B and a morphism of superalgebras $p : A \to B$ we have the commutative diagram

$$\begin{array}{ccc} \mathcal{G}_V(A) & \xrightarrow{\Phi_A} & \mathcal{G}_V(A) \\ {\scriptstyle \mathcal{G}_V(p)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}_V(p)} \\ \mathcal{G}_V(B) & \xrightarrow[\Phi_B]{} & \mathcal{G}_V(B). \end{array}$$

We say that this diagram is associated to the triple (A, B, p). The morphism p clearly fixes the orientation of the diagram. First we want to extract a map $\phi : V \to V$ from Φ_A for $A = \mathbb{C}[\theta, \tilde{\theta}]$.[1] For this we have to evaluate Φ_A only on $1 \otimes v$ and $\theta \otimes w$, where $|v| = 0$, $|w| = 1$, $\lambda \in \mathbb{C}$. It is not necessary to compute $\Phi_A(\tilde{\theta} \otimes w)$ because the graded conjugation is a superalgebra morphism and Φ is a natural transformation. To evaluate $\Phi_A(1 \otimes v)$ we use the commutative diagram associated to the triplet $(\mathbb{C}, A, p)$, where p is the morphism that injects the algebra of complex numbers into A. We find that $\Phi_A(1 \otimes v) = 1 \otimes \Phi_{\mathbb{C}}(v)$. Let $\pi : A \to \mathbb{C}$ be the canonical projection $\pi(\theta) = \pi(\tilde{\theta}) = 0$. From the commutative diagram associated to $(A, \mathbb{C}, \pi)$ and the fact that Φ_A is an even map we deduce the existence of $x, y \in V_1$ such that

$$\Phi_A(\theta \otimes w) = \theta \otimes x + \tilde{\theta} \otimes y.$$

In defining the morphism of superalgebra $q : A \to D$ with $A = \mathbb{C}[\theta, \tilde{\theta}]$, $D = \mathbb{C}[\theta, \tilde{\theta}, \eta, \tilde{\eta}]$ (such that $q(\theta) = \theta$, $q(\tilde{\theta}) = \tilde{\theta}$), we obtain from the diagram (A, D, q) the equality

$$\Phi_D(\theta \otimes w) = \theta \otimes x + \tilde{\theta} \otimes y.$$

By multiplying this last equation by $\eta\tilde{\theta}$ we find $\eta\tilde{\theta}\theta \otimes x = 0$, hence $x = 0$. Thus

$$\Phi_D(\theta \otimes w) = \tilde{\theta} \otimes y.$$

Then from the commutative diagram associated to the triplet (D, A, Π), where Π is defined by $\Pi(\eta) = \Pi(\tilde{\eta}) = 0$, we deduce

$$\Phi_A(\theta \otimes w) = \tilde{\theta} \otimes y$$

[1] A is an example of a Grassmann algebra with a graded conjugation. A Grassmann algebra is a superalgebra $\mathbb{C}[\theta_1, \ldots, \theta_n]$ over $\mathbb{C}$ generated by n odd elements θ_i that anticommute, that is, satisfy $\theta_i\theta_j + \theta_j\theta_i = 0$ for all i; in particular, $\theta_i^2 = 0$. The notation $\mathbb{C}[\theta, \tilde{\theta}]$ indicates how the graded conjugation acts on the generators.

Now we define ϕ by $\phi(w)=y$ and $\phi(v)=\Phi_{\mathbb{C}}(v)$ for $v\in V_0$, $w\in V_1$. We have to prove that this map ϕ gives rise to Φ_C for any superalgebra C. In other words, we have to show that $\Phi_C(c\otimes v)=\tilde{c}\otimes\phi(v)$ for $|c|=|v|$. For $c\in C_0$ the commutative diagram associated to $(\mathbb{C}[\theta,\tilde{\theta}], C, m(1)=1)$ implies that $\Phi_C(1\otimes v)=1\otimes\phi(v)$ and thus, by C_0-antilinearity, $\Phi_C(c\otimes v)=\tilde{c}\otimes\phi(v)$. For $c\in C_1$ we define a morphism of superalgebras from $\mathbb{C}[\theta,\tilde{\theta}]$ to C by $n(\theta)=c$, $n(\tilde{\theta})=\tilde{c}$ and we obtain the desired equality from the commutativity of the diagram $(\mathbb{C}[\theta,\tilde{\theta}], C, n)$. The final step is to show that ϕ satisfies (4). Antilinearity is clear from the C_0-antilinearity of Φ_C (which implies $\mathbb{C}$-antilinearity) and the linearity of the tensor product. From the involutivity of Φ_C and the definition of ϕ $(\Phi(c\otimes v)=\tilde{c}\otimes\phi(v))$ we find that $\tilde{\tilde{c}}\otimes\phi^2(v)=c\otimes v$. Therefore $\phi^2(v)=(-1)^{|v|}v$, since $|c|=|v|$. That ϕ is a morphism of Lie superalgebras comes from equation (1) together with the fact that Φ_C is a morphism of Lie algebras. This proves the claim.

(c) Let (V,ϕ) be a graded real form. For each superalgebra A we define a collection of maps $\Phi_A:\mathcal{G}_V(A)\to\mathcal{G}_V(A)$ by

$$\Phi_A(a\otimes v)=\tilde{a}\otimes\phi(v) \tag{6}$$

for all $a\in A$ and $v\in V$. From this formula, we find that

$$\mathcal{G}_V(f)(\Phi_A(a\otimes v))=\Phi_B(\mathcal{G}_V(f)(a\otimes v)), \tag{7}$$

thanks to the definition of $\mathcal{G}_V(f)$ and the equality $f(\tilde{a})=\widetilde{f(a)}$. The equality (7) means that the collection of maps Φ_A is functorial in A. The property of antilinearity in A_0 comes from the definition of Φ_A in (6), the linearity of the tensor product and the structure of A_0-module on $\mathcal{G}_V(A)$. Then (6) implies that, for $a\in A$ and $v\in V$ with $|a|=|v|$,

$$\Phi_A(\Phi_A(a\otimes v))=\Phi_A(\tilde{a}\otimes\phi(v))=\tilde{\tilde{a}}\otimes\phi(\phi(v))=(-1)^{|a|+|v|}a\otimes v=a\otimes v;$$

that is, Φ_A is involutive. From equation (1) and the fact that ϕ is a morphism of Lie superalgebra we deduce easily that Φ_A is a morphism of the Lie algebra $\mathcal{G}_V(A)$. Hence Φ is a graded real structure. □

In the *standard* setting we can associate to each *real structure* Φ the corresponding *real form*, which is by definition the real Lie superalgebra $V^\phi=\{v\in V,\ \phi(v)=v\}$ obtained as the fixed point set of the automorphism ϕ. However the corresponding notion of a graded real form as the fixed point set of ϕ is more subtle, because it turns out that $V^\phi=\{v\in V,\ \phi(v)=v\}$ is a trivial Lie superalgebra (its odd elements were killed by the requirement $\phi(v)=v$). The correct point of view to solve this difficulty is again functorial, since it treats the notion of the real form on the same footing for both the standard and the graded real structure:

Definition 2.7. Let Φ be a real structure of $\mathcal{G}_V$. The real form associated to Φ is the functor $\mathcal{G}_V^\Phi$

$$\mathcal{G}_V^\Phi : \mathcal{A} \to \mathcal{S},$$
$$A \mapsto \mathcal{G}_V^\Phi(A),$$

where $\mathcal{G}_V^\Phi(A) = \{w \in \mathcal{G}_V(A), \Phi_A(w) = w\}$ is an A_0^{real}-Lie algebra that associates to a morphism of superalgebras $f : A \to B$ the restriction $\mathcal{G}_V^\Phi(f) = \mathcal{G}_V(f)_{\mathcal{G}_V^\Phi(A)}$ of $\mathcal{G}_V(f)$ to the set of fixed points of Φ_A. The real form is called standard (graded) when the real structure is standard (graded).

Remark 2.8. This definition is consistent because $\mathcal{G}_V^\Phi$ is a functor, as can be easily demonstrated. Moreover, $\mathcal{G}_V^\Phi(f)$ sends elements of $\mathcal{G}_V^\Phi(A)$ into $\mathcal{G}_V^\Phi(B)$ because Φ is a natural transformation.

That we can extract a real Lie superalgebra V^ϕ from a standard real structure, whereas this is impossible for a graded real structure, has some consequences for the representability of the real form. In fact:

Theorem 2.9. (1) *If Φ is a standard real structure, the functor $\mathcal{G}_V^\Phi$ is represented by V^ϕ; that is, $\mathcal{G}_V^\Phi(A) = (A^{\text{real}} \otimes V^\phi)_0$.*

(2) *If Φ is a graded real structure, the functor $\mathcal{G}_V^\Phi$ is not representable.*

Proof. (1) Suppose A is equipped with a standard conjugation and Φ is a standard real structure. Let $a \otimes v \in \mathcal{G}_V(A)$ be such that $\Phi_A(a \otimes v) = a \otimes v$. We know from Theorem 2.6 that a standard real structure Φ comes from a unique map ϕ satisfying (3). Every element v of V can be uniquely decomposed as $v = v_1 + i v_2$, with $\phi(v_1) = v_1$ and $\phi(v_2) = v_2$. Similarly, every element a of A has a unique decomposition $a = a_1 + i a_2$, with $\bar{a}_1 = a_1$ and $\bar{a}_2 = a_2$. Thus we have

$$a \otimes v = \frac{a \otimes v + \Phi_A(a \otimes v)}{2} = a_1 \otimes v_1 - a_2 \otimes v_2.$$

From this it is clear that the fixed points of Φ_A are elements of $(A^{\text{real}} \otimes V^\phi)_0$.

(2) Now let A and V be equipped, respectively, with a graded conjugation $a \to \tilde{a}$ and a map ϕ coming from a graded real structure Φ on $\mathcal{G}_V$. It is clear that no odd element of A can be real for a graded conjugation and no odd vector of V can be a fixed point of ϕ. So there are some fixed points of Φ_A in $\mathcal{G}_V(A)$ that are not elements of $(A^{\text{real}} \otimes V^\phi)_0$: for instance, $a \otimes v + \tilde{a} \otimes \phi(v)$ with $a \in A_1$ and $v \in V_1$. □

Denote by $(\mathcal{G}_V^\Phi)_0$ the restriction of the functor $\mathcal{G}_V^\Phi$ to its even part with respect to the grading defined by the decomposition (3), i.e., $(\mathcal{G}_V^\Phi)_0(A) = (\mathcal{G}_V^\Phi(A))_0$. This restricted functor is representable for both standard and graded real forms; the proof is similar to that of Theorem 2.9. The representative of $(\mathcal{G}_V^\Phi)_0$ is an ordinary Lie algebra $(V^\phi)_0$. It makes sense, therefore, to ask whether $(V^\Phi)_0$ is compact.

Remarkably, it follows from Serganova's classification [1983, Tables 3 and 6] that there is exactly one (graded) automorphism ϕ for each simple complex Lie superalgebra such that $(V^{\phi})_0$ is always compact. This shows the utility of our interpretation of graded involutive automorphisms as real forms.

3. Graded and standard real forms of complex algebraic supergroup

The goal of this section is to introduce the notions of real structure and real form for subsupergroups of $\mathrm{GL}_{m|n}$.

In order to define the supergroup $\mathrm{GL}_{m|n}$, we introduce some notations. We denote by $A^{m|n}$ the free A-supermodule generated by m even $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ and n odd generators $\boldsymbol{e}_{m+1}, \dots, \boldsymbol{e}_{m+n}$ such that $\boldsymbol{a} \in A^{m|n}$ is of the form $\boldsymbol{a} = a_1\boldsymbol{e}_1 + \cdots + a_{m+n}\boldsymbol{e}_{m+n}$. An even morphism $T : A^{m|n} \to A^{m|n}$ can be represented by a supermatrix $\mathcal{T}$ of size $(m+n) \times (m+n)$:

$$\mathcal{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

where the matrices P, S have even entries from A and are respectively of size $m \times m$, $n \times n$; the matrices Q, R have odd entries and are respectively of size $n \times m$, $m \times n$. The endomorphisms and automorphisms of $A^{m|n}$ are denoted respectively by $\mathrm{gl}_{m|n}(A)$ and $\mathrm{GL}_{m|n}(A)$. The supermatrices $\mathcal{T}$ of $\mathrm{GL}_{m|n}(A)$ are such that the Berezinian or superdeterminant

$$\operatorname{sdet} \mathcal{T} = \det(P - QS^{-1}R)\det(S^{-1})$$

is invertible in A. A necessary and sufficient condition for $\operatorname{sdet} \mathcal{T}$ to be invertible is the invertibility of P and S. The functor of the linear affine algebraic supergroup is

$$\begin{aligned} \mathrm{GL}_{m|n} : \mathcal{A} &\to \mathcal{S}, \\ A &\mapsto \mathrm{GL}_{m|n}(A). \end{aligned}$$

Definition 3.1. A complex (linear affine) algebraic supergroup is a functor

$$\begin{aligned} G_{\mathbb{C}} : \mathcal{A} &\to \mathcal{S}, \\ A &\mapsto G_{\mathbb{C}}(A), \end{aligned}$$

where $G_{\mathbb{C}}(A) = \{x \in \mathrm{GL}_{m|n}(A) : \mathcal{P}_l(x) = 0 \text{ for } l = 1, \dots, k\}$ and the polynomials $\mathcal{P}_l$ are such that $G_{\mathbb{C}}(A)$ is a group.

Remark 3.2. Two examples are the $\mathrm{SL}_{m|n}$ and $\mathrm{OSp}_{m|2n}$ series of supergroups, where the polynomials $\mathcal{P}_l$ are defined in equations (9), (10), respectively.

Let A be a superalgebra and ε a formal indeterminate. Let $A(\varepsilon)$ be the superalgebra of dual numbers defined by $A(\varepsilon) = (A \oplus \varepsilon A)/(\varepsilon^2)$. There are three useful morphisms: $i : A \to A(\varepsilon)$, defined by $i(x) = x + \varepsilon 0$; $p : A(\varepsilon) \to A$, defined by

$p(x+\varepsilon y)=x$; and $v_a : A(\varepsilon) \to A(\varepsilon)$, defined by $v_a(x+\varepsilon y)=x+\varepsilon ay$ for $a \in A_0$, $x, y \in A$.

Definition 3.3. A real structure on a complex algebraic supergroup $G_{\mathbb{C}}$ is a natural transformation $\Sigma : G_{\mathbb{C}} \to G_{\mathbb{C}}$ satisfying

$$\Sigma_A(xy) = \Sigma_A(x)\Sigma_A(y),$$
$$\Sigma_A{}^2(x) = x,$$
$$\Sigma_{A(\varepsilon)}(G_{\mathbb{C}}(v_a)(z)) = G_{\mathbb{C}}(v_{\hat{a}})(\Sigma_{A(\varepsilon)}(z)) \tag{8}$$

for all $x, y \in G_{\mathbb{C}}(A)$ and $z \in \mathrm{Ker}(G_{\mathbb{C}}(p))$. The real structure is called standard (graded) when the superalgebra A is equipped with a standard (graded) conjugation.

Remark 3.4. The requirement (8) means that the map induced by Σ on the Lie superalgebra of $G_{\mathbb{C}}$ is antilinear.

Definition 3.5. Let Σ be a real structure of $G_{\mathbb{C}}$. The *real form* associated to Σ is the functor

$$G_{\mathbb{C}}^{\Sigma} : \mathcal{A} \to \mathcal{S}, \qquad A \mapsto G_{\mathbb{C}}^{\Sigma}(A),$$

where $G_{\mathbb{C}}^{\Sigma}(A) = \{x \in G_{\mathbb{C}}(A) : \Sigma_A(x) = x\}$ is a group associating to a morphism $f : A \to B$ of superalgebras the restriction $G_{\mathbb{C}}^{\Sigma}(f) = G_{\mathbb{C}}(f)_{G_{\mathbb{C}}^{\Sigma}}$ of $G_{\mathbb{C}}(f)$ to the set of fixed points of Σ_A. The real form is said standard (graded) when the real structure is standard (graded).

Definition 3.6 [Fioresi and Lledó 2004]. The Lie superalgebra associated to the algebraic supergroup $G_{\mathbb{C}}$ is the functor

$$\mathrm{Lie}(G_{\mathbb{C}}) : \mathcal{A} \to \mathcal{S}, \qquad A \mapsto \mathrm{Lie}(G_{\mathbb{C}})(A) = \mathrm{Ker}(G_{\mathbb{C}}(p)),$$

where $G_{\mathbb{C}}(p) : G_{\mathbb{C}}(A(\varepsilon)) \to G_{\mathbb{C}}(A)$ is the morphism associated to $p : A(\varepsilon) \to A$.

Remark 3.7. As explained in [Fioresi and Lledó 2004], $\mathrm{Ker}(G_{\mathbb{C}}(p))$ has an A_0-module structure with a Lie bracket.

From Definition 3.1 we deduce that $\mathrm{Ker}(G_{\mathbb{C}}(p))$ is the set of even supermatrices N such that $\mathcal{P}_l(1+\varepsilon N) = 0$ for all $l = 1, \ldots, k$. From [Fioresi and Lledó 2004] we deduce that the Lie bracket on $\mathrm{Ker}(G_{\mathbb{C}}(p))$ is simply the commutator $[M, N] = MN - NM$. Moreover $\mathrm{Ker}(G_{\mathbb{C}}(p))$ is an abelian group with respect to the addition of supermatrices and the action A_0 on $\mathrm{Ker}(G_{\mathbb{C}}(p))$ corresponds to the multiplication of the supermatrix entries by an element of A_0. Thus $\mathrm{Ker}(G_{\mathbb{C}}(p))$ is also a A_0-module.

Now we prove that the real structure of a supergroup gives rise to a real structure of its corresponding Lie superalgebra.

Theorem 3.8. *Let $G_{\mathbb{C}}$ be an algebraic supergroup and Σ a real structure of $G_{\mathbb{C}}$. For every superalgebra A, we have*

$$\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon M)=\mathrm{Id}+\varepsilon\Phi_A(M) \quad \textit{for all } M\in G_{\mathbb{C}}(A).$$

The collection of maps Φ_A define a real structure on $\mathrm{Lie}(G_{\mathbb{C}})$.

Proof. Let $G_{\mathbb{C}}(p):G_{\mathbb{C}}(A(\varepsilon))\to G_{\mathbb{C}}(A)$ be the morphism induced by the morphism $p:A(\varepsilon)\to A$ such that $p(a+\varepsilon b)=a$ for all $a,b\in A$. Then $\Sigma_{A(\varepsilon)}$ is a morphism from $G_{\mathbb{C}}(A(\varepsilon))$ to $G_{\mathbb{C}}(A)$; by functoriality, we have $G_{\mathbb{C}}(p)\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon M)=\Sigma_A(G_{\mathbb{C}}(p)(\mathrm{Id}+\varepsilon M))=\Sigma_A(\mathrm{Id})=\mathrm{Id}$. We deduce that $\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon M)$ lies in $\mathrm{Ker}(G_{\mathbb{C}}(p))$. Therefore we have defined a map Φ_A such that

$$\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon M)=\mathrm{Id}+\Phi_A(M).$$

To show that Φ is a real structure on $\mathrm{Lie}(G_{\mathbb{C}})$, we have to prove that each map Φ_A is antilinear, involutive and is a morphism of Lie algebras. We conclude that $\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon(M+N))=\mathrm{Id}+\varepsilon\Phi_A(M+N)$ and $\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon aM)=\mathrm{Id}+\Phi_A(aM)$ with $a\in A_0$ and $M,N\in G_{\mathbb{C}}(A)$. Furthermore we have

$$\begin{aligned}\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon(M+N))&=\Sigma_{A(\varepsilon)}((\mathrm{Id}+\varepsilon M)(\mathrm{Id}+\varepsilon N))\\&=\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon M)\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon N)\\&=(\mathrm{Id}+\varepsilon\Phi_A(M))(\mathrm{Id}+\varepsilon\Phi_A(N))\\&=\mathrm{Id}+\varepsilon(\Phi_A(M)+\Phi_A(N)),\end{aligned}$$

$$\begin{aligned}\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon aM)&=\Sigma_{A(\varepsilon)}(G_{\mathbb{C}}(v_a)(\mathrm{Id}+\varepsilon M))\\&=G_{\mathbb{C}}(v_{\hat{a}})(\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon M))\\&=G_{\mathbb{C}}(v_{\hat{a}})(\mathrm{Id}+\varepsilon\Phi_A(M))\\&=\mathrm{Id}+\varepsilon\hat{a}\Phi_A(M).\end{aligned}$$

These equations establish the antilinearity of Φ_A. The involutivity of Φ_A is deduced easily from the involutivity of $\Sigma_{A(\varepsilon)}$.

It remains to show that Φ_A is a morphism of Lie algebras. Let $A(\varepsilon,\eta)$ be the superalgebra of polynomials in the indeterminates ε and η with coefficients in A and such that $\varepsilon^2=0$, $\eta^2=0$, $\varepsilon\eta-\eta\varepsilon=0$. We have two morphisms $p_\varepsilon:A(\varepsilon,\eta)\to A(\eta)$ and $p_\eta:A(\varepsilon,\eta)\to A(\varepsilon)$, defined by $p_\varepsilon(a+\varepsilon b+\eta c+\varepsilon\eta d)=a+\eta c$ and $p_\eta(a+\varepsilon b+\eta c+\varepsilon\eta d)=a+\varepsilon b$. Each induces a morphism of groups via the functor $G_{\mathbb{C}}$. By functoriality we deduce that $\Sigma_{A(\varepsilon,\eta)}(\mathrm{Id}+\varepsilon M)=\mathrm{Id}+\varepsilon\Phi_A(M)$ and $\Sigma_{A(\varepsilon,\eta)}(\mathrm{Id}+\eta M)=\mathrm{Id}+\eta\Phi_A(M)$ for $M\in G_{\mathbb{C}}(A)$.

Finally, from the equalities

$$\begin{aligned}\Sigma_{A(\varepsilon,\eta)}&(\mathrm{Id}+\varepsilon\eta[M,N]_A)\\ &=\Sigma_{A(\varepsilon,\eta)}((\mathrm{Id}+\varepsilon M)(\mathrm{Id}+\eta N)(\mathrm{Id}-\varepsilon M)(\mathrm{Id}-\eta N))\\ &=(\mathrm{Id}+\varepsilon\Phi_A(M))(\mathrm{Id}+\eta\Phi_A(N))(\mathrm{Id}-\varepsilon\Phi_A(M))(\mathrm{Id}-\eta\Phi_A(N))\\ &=\mathrm{Id}+\varepsilon\eta[\Phi_A(M),\Phi_A(N)]_A,\end{aligned}$$

we deduce that Φ_A is a morphism of Lie algebra. □

We end this section with a theorem that shows that the Lie superalgebra of the real form $G_{\mathbb{C}}^{\Sigma}$ is the same thing as the real form of $\mathrm{Lie}(G_{\mathbb{C}})$.

Theorem 3.9. *Let Σ be the real structure of the algebraic supergroup $G_{\mathbb{C}}$ and let Φ be the corresponding real structure on the Lie superalgebra* $\mathrm{Lie}(G_{\mathbb{C}})$. *The real form* $(\mathrm{Lie}(G_{\mathbb{C}}))^{\Phi}$, *defined by* $(\mathrm{Lie}(G_{\mathbb{C}}))^{\Phi}(A)=\{x\in\mathrm{Lie}(G_{\mathbb{C}}):\Phi_A(x)=x\}$, *is then the Lie superalgebra of the real form $G_{\mathbb{C}}^{\Sigma}$.*

Proof. By definition, $\mathrm{Lie}(G_{\mathbb{C}}^{\Sigma})(A)=\{X\in G_{\mathbb{C}}^{\Sigma}(A(\varepsilon)):G_{\mathbb{C}}(p)(X)=\mathrm{Id},\ \Sigma_A(X)=X\}$. The elements X of $G_{\mathbb{C}}^{\Sigma}(A)$ that fulfill the condition $G_{\mathbb{C}}(p)(X)=\mathrm{Id}$ are the even supermatrices of the form $\mathrm{Id}+\varepsilon M$ with $M\in\mathrm{gl}_{m|n}(A)$. Therefore, $\mathrm{Lie}(G_{\mathbb{C}}^{\Sigma})(A)=\{M\in\mathrm{gl}_{m|n}(A):\Sigma_{A(\varepsilon)}(\mathrm{Id}+\varepsilon M)=\mathrm{Id}+\varepsilon M\}=\{M\in M_{m|n}(A):\Phi_A(M)=M\}$, which proves the theorem. □

4. Lifting the Serganova automorphisms to the algebraic supergroups $\mathrm{SL}_{m|n}$ and $\mathrm{OSp}_{m|2n}$

Before defining the algebraic supergroups $\mathrm{SL}_{m|n}$ and $\mathrm{OSp}_{m|2n}$ we introduce some notations. Let $J_{m,n}$ be the supermatrix $\mathrm{diag}(1_m,J_n)$, where

$$\mathrm{diag}(M,N)=\begin{pmatrix}M&0\\0&N\end{pmatrix},\qquad J_n=\begin{pmatrix}0&1_n\\-1_n&0\end{pmatrix},$$

with 1_n the unit square matrix of order n. The supertranspose and the Π-transpose of an even supermatrix are defined by

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}^{st}=\begin{pmatrix}A^t&-C^t\\B^t&D^t\end{pmatrix},\qquad \Pi\begin{pmatrix}A&B\\C&D\end{pmatrix}=\begin{pmatrix}D&C\\B&A\end{pmatrix},$$

where M^t means the usual transpose of the matrix M. The supertrace of a supermatrix is given by

$$\mathrm{str}\begin{pmatrix}A&B\\C&D\end{pmatrix}=\mathrm{tr}\,A-\mathrm{tr}\,D,$$

where tr A is the usual trace of the matrix A. The functor $\mathrm{SL}_{m|n}$ is defined by

$$\begin{aligned}\mathrm{SL}_{m|n} &: \mathcal{A} \to \mathcal{S},\\ & A \mapsto \mathrm{SL}_{m|n}(A) = \{M \in \mathrm{GL}_{m|n}(A) : \mathrm{sdet}(M) = 1\},\end{aligned} \tag{9}$$

and the functor $\mathrm{OSp}_{m|2n}$ is

$$\begin{aligned}\mathrm{OSp}_{m|2n} &: \mathcal{A} \to \mathcal{S},\\ & A \mapsto \mathrm{OSp}_{m|2n}(A) = \{M \in \mathrm{GL}_{m|2n}(A) : M^{st} J_{m,n} M = J_{m,n}\}.\end{aligned} \tag{10}$$

Their corresponding Lie superalgebras are the functors $\mathrm{sl}_{m|n}$ and $\mathrm{osp}_{m|2n}$ that associate to a superalgebra A the sets $\mathrm{sl}_{m|n}(A) = \{X \in \mathrm{gl}_{m|n}(A) : \mathrm{str}(X) = 0\}$ and $\mathrm{osp}_{m|2n}(A) = \{X \in \mathrm{gl}_{m|2n}(A) : X^{st} J_{m,n} + J_{m,n} X = 0\}$.

To describe the Serganova automorphisms for these Lie superalgebras we need to introduce the following conventions:

$$\delta_\lambda \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & \lambda B \\ \lambda^{-1} C & D \end{pmatrix}, \qquad {I_n}^l = \mathrm{diag}(1_l, -1_{n-l}),$$
$$\mathrm{Ad}(M)X = MXM^{-1}, \qquad c(N) = \overline{N},$$

where M, X are supermatrices of the same order, while N is a supermatrix filled with complex numbers and the bar on N indicates that all the entries of N are conjugated.

Now we write the automorphisms from [Serganova 1983, Table 3], which give rise to the standard real structure of $\mathrm{sl}_{m|n}$. They are of four types:

$$\begin{aligned}
\sigma_1(M) &= -\,\mathrm{str} \circ \mathrm{Ad}(\mathrm{diag}({I_m}^p, {I_m}^q)) \circ c \circ \delta_i(M), \\
\sigma_2(M) &= \mathrm{Ad}(\mathrm{diag}(J_m, J_n)) \circ c(M) && (m, n \text{ even}), \\
\sigma_3(M) &= \Pi \circ c(M) && (m = n), \\
\sigma_4(M) &= -\,\mathrm{str} \circ \Pi \circ c(M) && (m = n \text{ even}),
\end{aligned}$$

with $M \in \mathrm{sl}_{m|n}(\mathbb{C})$.

The automorphisms from [Serganova 1983, Table 6], coming from the graded real structure of $\mathrm{sl}_{m|n}$, are

$$\begin{aligned}
\omega_1(M) &= c \circ \mathrm{Ad}(\mathrm{diag}(1_m, J_n)) \circ (M) && (n \text{ even}), \\
\omega_2(M) &= -st \circ c \circ \mathrm{Ad}(\mathrm{diag}({I_m}^p, {I_n}^q))(M), \\
\omega_3(M) &= c \circ \Pi \circ \delta_i(M) && (m = n),
\end{aligned}$$

with $M \in \mathrm{sl}_{m|n}(\mathbb{C})$.

In functorial language, the standard real structures $\bar{\sigma}_l$ ($l = 1, 2, 3, 4$) associated to the σ_l are given by the same formula, except that c (complex conjugation) is replaced by the standard conjugation $a \to \bar{a}$ of the superalgebra A. For the graded real structure $\tilde{\omega}_k$ ($k = 1, 2, 3$), complex conjugation is replaced by the graded conjugation $a \to \tilde{a}$.

The lifts of the $\tilde{\phi}$ are the following *standard* real structures of the supergroup $\mathrm{SL}_{m|n}$:

$$\begin{aligned}
\Sigma_1(X) &= (-\bar{\sigma}_1(X))^{-1}, \\
\Sigma_2(X) &= \bar{\sigma}_2(X) && (m, n \text{ even}), \\
\Sigma_3(X) &= \bar{\sigma}_3(X) && (m = n), \\
\Sigma_4(X) &= (-\bar{\sigma}_4(X))^{-1} && (m = n \text{ even}),
\end{aligned}$$

for $X \in \mathrm{SL}_{m|n}(A)$. The lifts of the *graded* real structure of the supergroup $\mathrm{SL}_{m|n}$ are, in turn,

$$\begin{aligned}
\Omega_1(X) &= \tilde{\omega}_1(X) && (n \text{ even}), \\
\Omega_2(X) &= (-\tilde{\omega}_2(X))^{-1}, \\
\Omega_3(X) &= \tilde{\omega}_3(X) && (m = n),
\end{aligned}$$

for $X \in \mathrm{SL}_{m|n}(A)$.

Remark 4.1. It is easy to prove that the Σ_k (Ω_l) are indeed the standard (graded) real forms of $\mathrm{SL}_{m|n}$ in the sense of Definition 3.3. It is no more difficult to find that these real structures satisfy the equalities

$$\begin{aligned}
\Sigma_l(\mathrm{Id} + \varepsilon M) &= \mathrm{Id} + \varepsilon \bar{\sigma}_l(M), \\
\Omega_l(\mathrm{Id} + \varepsilon M) &= \mathrm{Id} + \varepsilon \tilde{\omega}_l(M),
\end{aligned}$$

which means that they are the lifts of $\bar{\sigma}_k$, $\tilde{\omega}_l$.

Now we turn to the Serganova automorphisms of $\mathrm{OSp}_{m|2n}$. The Lie superalgebra automorphisms which come from *standard* real structures are (see [Serganova 1983, Table 3])

$$\begin{aligned}
\xi_1(M) &= \mathrm{Ad}(\mathrm{diag}(I_m{}^p, 1_{2n})) \circ c(M), \\
\xi_2(M) &= \mathrm{Ad}(\mathrm{diag}(J_m, I_n{}^p, I_n{}^p))(M) \quad (m \text{ even}),
\end{aligned}$$

for $M \in \mathrm{osp}_{m|2n}(\mathbb{C})$. The Lie superalgebra automorphisms giving rise to the *graded* real structure are (see [Serganova 1983, Table 6])

$$\begin{aligned}
\psi_1(M) &= c \circ \mathrm{Ad}(\mathrm{diag}(I_m{}^p, d(I_n{}^q, I_n{}^q)) \circ J_{2n}) \circ (M), \\
\psi_2(M) &= c \circ \mathrm{Ad}(\mathrm{diag}(J_m, 1_{2n})) \circ (M) && (m \text{ even}),
\end{aligned}$$

for $M \in \mathrm{osp}_{m|2n}(\mathbb{C})$.

To switch to the functorial language, we again replace the complex conjugation in the automorphisms $\xi_{1,2}$ ($\psi_{1,2}$) by the standard (graded) conjugation and we denote the corresponding standard (graded) real structure by $\bar{\xi}_{1,2}$ ($\tilde{\psi}_{1,2}$). The lifts to the supergroup $\mathrm{OSp}_{m|n}$ are respectively

$$\begin{aligned}\Xi_1(X) &= \bar{\xi}_1(X),\\ \Xi_2(X) &= \bar{\xi}_2(X) \quad (m \text{ even}),\\ \Psi_1(X) &= \tilde{\psi}_1(X),\\ \Psi_2(X) &= \tilde{\psi}_2(X) \quad (m \text{ even}),\end{aligned}$$

for $X \in \mathrm{OSp}_{m,2n}(A)$.

Remark 4.2. It is easy to prove that the Ξ_k (Ψ_l) are indeed the standard (graded) real forms of $\mathrm{OSp}_{m|2n}$ in the sense of Definition 3.3. It is no more difficult to find that these real structures satisfy the equalities

$$\begin{aligned}\Xi_l(\mathrm{Id} + \varepsilon M) &= \mathrm{Id} + \varepsilon \bar{\xi}_l(M),\\ \Psi_l(\mathrm{Id} + \varepsilon M) &= \mathrm{Id} + \varepsilon \tilde{\psi}_l(M),\end{aligned}$$

which means that they are the lifts of $\bar{\xi}_k$, $\tilde{\psi}_l$.

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