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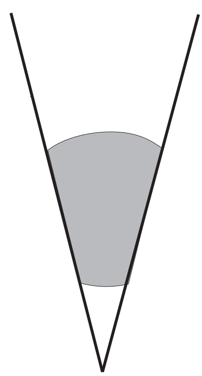
We consider capillary problems that arise physically when the equilibrium surface of a fluid with fixed volume is situated in a cone. By using a variational approach in the space of functions of bounded variation on the sphere S^n, we obtain regularity results for a certain class of relative minima of the energy functional provided the volume is large enough. This special class of relative minima can be described by radial graphs.

1. Introduction



We investigate equilibrium configurations of nonwetting liquids in a conical vessel. The remarkable feature of such a vessel is that, in contrast to the classical cylinder, drops can be in equilibrium therein provided the angle of aperture of the cone is not too large; this was already mentioned in [Minkowski 1907].

The diagram on the right, also taken from Minkowski's article, suggests that the surface of the drop in equilibrium consists of an upper and lower membrane that admit a central projection onto an open part $\Omega \times \{1\} \mathring{\subset} \mathbb{R}^n \times \{1\}$, with zero as the center of projection. Moreover the membranes should be separable by an horizontal plane. Therefore, the class of admissible functions can be described by $K_\delta := \left\{(u, v) \in (BV_+(\Omega))^2 \mid u \le \delta \le v\right\}$, with $\delta > 0$. This easy observation was the starting point for Schindelmayer, a student of Bemelmans, who proved the existence of such functions [Schindelmayr 1999]. Earlier Ambrazevicius [1982] had dealt with conical surfaces (although, as far as I know, not with drops); his approach rests on barrier methods for partial differential equations.

Following Schindelmayer, we use direct methods from the calculus of variations which in particular lead to stable configurations. More precisely, we are interested

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in minima of the auxiliary functional

$$\begin{aligned}(1)\quad \mathcal{F}_{\Omega,\mu}(u,v) := \frac{1}{n}\int_\Omega \sqrt{|\nabla u|^2+(\operatorname{div}(ux))^2} + \frac{1}{n}\int_\Omega \sqrt{|\nabla v|^2+(\operatorname{div}(vx))^2} \\ + \frac{1}{n}\int_\Omega \Big(\kappa(v^{1+2/n}-u^{1+2/n}) - \mu(v^{1+1/n}-u^{1+1/n})\Big)\,dx \\ + \frac{\sigma}{n}\int_{\partial\Omega}(v-u)\sqrt{1+(x\nu)^2}\,d\mathcal{H}^{n-1}(x)\end{aligned}$$

in the class K_δ, with $\kappa, \mu, \delta > 0$ and $0 < \sigma < 1$.

Our actual intention is to prove the existence and regularity of relative minimizers of the energy functional (see [Finn 1986; Giusti 1981]), rather than of $\mathcal{F}_{\Omega,\mu}(\cdot\,,\cdot)$; see Theorem 1.5. But this can only be accomplished by using properties of the functional defined in (1).

Physically, for $w \in BV_+(\Omega)$,

$$\mathcal{A}_\Omega(w) := \frac{1}{n}\int_\Omega \sqrt{|\nabla w|^2+(\operatorname{div}(wx))^2}$$

describes the normalized free surface energy of the fluid $\big\{\varrho(x,1)\mid 0\le\varrho\le\sqrt[n]{w(x)},\ x\in\Omega\big\}$ in the closure of the cone $C_\Omega := \{\varrho(x,1)\mid \varrho>0,\ x\in\Omega\}$, whereas

$$(2)\qquad \mathcal{H}_{\Omega,\mu}(w) := -n\int_\Omega d\mathcal{H}^n(x)\int_0^{\sqrt[n]{w(x)}} H(\lambda,x)\lambda^n\,d\lambda,$$

with

$$H(\lambda,x) := -\kappa\frac{n+2}{n^2}\lambda + \mu\frac{n+1}{n^2},$$

stands for the volume of the liquid, weighted by $-nH$. This abbreviation is made because H represents in differential geometry the (generalized) mean curvature with respect to the inner normal N of the surface; that is, $\langle N(p), p\rangle < 0$ on the surface. From a physical viewpoint, the mean curvature term

$$\mathcal{H}_{\Omega,\mu}(w) = \frac{\kappa}{n}\int_\Omega w(x)^{1+2/n}d\mathcal{H}^n(x) - \frac{\mu}{n}\int_\Omega w(x)^{1+1/n}d\mathcal{H}^n(x)$$

consists of two parts: the gravitational energy and a volume term involving a Lagrange multiplier, whose introduction is natural because the volume of the fluid is kept fixed. Finally, the boundary part

$$\sigma\mathcal{A}_{\partial\Omega}(w) := \frac{\sigma}{n}\int_{\partial\Omega} w(x)\sqrt{1+(x\nu)^2}\,d\mathcal{H}^{n-1}(x)$$

gives, up to a constant, the wetting energy between the liquid and the boundary wall of the cone C_Ω. (See [Finn 1986] for more information.) Therefore, the functional

$$\mathscr{E}_\Omega(u,v)=\mathscr{A}_\Omega(u)+\mathscr{A}_\Omega(v)+\mathscr{G}_\Omega(u,v)+\sigma\mathscr{A}_{\partial\Omega}(v-u) \tag{3}$$

represents the total energy of a liquid drop of the form

$$\{\lambda(x,1)\mid \sqrt[n]{u(x)}\le\lambda\le\sqrt[n]{v(x)}\};$$

the summand

$$\mathscr{G}_\Omega(u,v):=\frac{\kappa}{n}\int_\Omega(v^{1+2/n}-u^{1+2/n})\,dx$$

stands for the gravitational energy of the liquid drop. Equation (1) can thus be abbreviated to

$$\mathscr{F}_{\Omega,\mu}(u,v)=\mathscr{A}_\Omega(u)+\mathscr{A}_\Omega(v)+\mathscr{H}_{\Omega,\mu}(v)-\mathscr{H}_{\Omega,\mu}(u)+\sigma\mathscr{A}_{\partial\Omega}(v-u).$$

Remark 1.1. In case of the condition

$$|\Omega|<\sigma\mathscr{A}_{\partial\Omega}(1)=\frac{\sigma}{n}\int_{\partial\Omega}\sqrt{1+(x\nu)^2}d\mathscr{H}^{n-1}(x) \tag{4}$$

we have $u_\mu\neq 0$ for every minimizer (u_μ,v_μ) of $\mathscr{F}_{\Omega,\mu}(u,v)$ in the class K_δ. This can be seen by comparing u_μ with an sufficiently small $0<\lambda<\delta$; see [Schindelmayr 1999]. In Theorem 1.5 we can even prove that $u_\mu>0$, together with the necessity of condition (4) in the case where $\Omega=B_r(0)$ with radius $r<1$.

We prove here the existence, uniqueness, and regularity of solutions (u,v) belonging to the energy functional (3). For this purpose, we make use of the fact that the upper and lower membrane are naturally given as radial graphs over some open part of S^n. Indeed, the transformation $Z:\Omega\to S^n$,

$$x\mapsto\frac{(x,1)}{\sqrt{1+|x|^2}},$$

of the independent variable x leads to

$$\mathscr{A}_{Z(\Omega)}(\tilde v):=\frac{1}{n}\int_{Z(\Omega)}\sqrt{|\nabla_{S^n}\tilde v|^2+(n\tilde v)^2}=\frac{1}{n}\int_\Omega\sqrt{|\nabla v|^2+(\mathrm{div} vx)^2}=\mathscr{A}_\Omega(v)$$

if $\tilde v(Z(x)):=(1+|x|^2)^{n/2}v(x)$. For a detailed discussion of functions of bounded variation on submanifolds, see [Schwab 2004].

Moreover,

$$\mathscr{H}_{\Omega,\mu}(v)=\mathscr{H}_{Z(\Omega),\mu}(\tilde v):=-n\int_{Z(\Omega)}d\mathscr{H}^n(z)\int_0^{\sqrt[n]{\tilde v}}H(\lambda z)\,\lambda^n d\lambda$$

with $H(\lambda z):=H(\lambda,Z^{-1}(z))$ for every $(\lambda,z)\in\mathbb{R}_{>0}\times Z(\Omega)$, and finally

$$\mathcal{A}_{\partial\Omega}(v) = \mathcal{A}_{\partial Z(\Omega)}(\tilde{v}) := \frac{1}{n}\int_{\partial Z(\Omega)} \tilde{v}\, d\mathcal{H}^{n-1}(z).$$

Now, we can consider the following more general problem on an open part of the unit sphere: Minimize

$$\mathcal{F}_{\Omega,\mu}(u, v) = \mathcal{A}_{\Omega}(u) + \mathcal{A}_{\Omega}(v) + \mathcal{H}_{\Omega,\mu}(v) - \mathcal{H}_{\Omega,\mu}(u) + \sigma\mathcal{A}_{\partial\Omega}(v-u)$$

in the class $K_\delta := \{(u, v) \in (BV_+(\Omega))^2 \mid u \le \delta \le v\}$ with $\kappa, \mu, \delta > 0$. In this setting, $\Omega \subset S^n$ is assumed to be a Lipschitz domain in S^n fulfilling an interior sphere condition (ISC):

Definition 1.2. An open domain Ω of S^n is said to be a Lipschitz domain if either $\Omega = S^n$ or if there is a point $p_S \in S^n \setminus \overline{\Omega}$ such that the image of Ω under the central projection

$$\pi_S : S^n \setminus \{p_S\} \to T_{p_N} S^n$$

onto the northern tangential hyperplane at $p_N := -p_S$ is (strongly) Lipschitz. The domain Ω fulfills an ISC if its central projection satisfies an interior sphere condition in $\mathbb{R}^n$. (See [Gerhardt 1975].)

Remark 1.3. Although we tackle a problem that includes a wider class of domains than treated in (1), its physical relevance seems to be restricted to the case where Ω is a part of an open hemisphere; i.e., Ω can be viewed as a domain of $\mathbb{R}^n$. Furthermore, the ISC will lead to an natural structure condition for our domain (see Lemma 1.4), which will prove to be more suitable than the one used in [Schindelmayr 1999]. But this fact can only be seen if one considers more appropriate coordinates — in this case, polar coordinates.

Lemma 1.4. *Let Ω be a Lipschitz domain of S^n, $n \ge 2$, that fulfills an ISC. Then*

$$\int_{\partial\Omega} |v|\, d\mathcal{H}^{n-1} \le \int_{\Omega_\varepsilon} |\nabla_{S^n} v| + C\int_{\Omega} |v|\, d\mathcal{H}^n$$

for every $v \in BV(\Omega)$ and $\varepsilon > 0$, where $C = C(\Omega, n, \varepsilon, \pi_S)$ and

$$\Omega_\varepsilon := \{x \in \Omega \mid d(x, \partial\Omega) < \varepsilon\}$$

for the euclidean distance d.

Proof. Let X be the inverse of the central projection π_S mentioned in Definition 1.2. Set $\tilde{\Omega} := \pi_S(\Omega)$ and let λ be the conformal factor associated to X, so that $|DX(v)| = \lambda|v|$ for every $v \in \mathbb{R}^n$, or in local coordinates $\lambda^2 = \langle X_{,i}, X_{,i}\rangle$. There exists for a given $\varepsilon > 0$ a $\delta > 0$ such that $X(\tilde{\Omega}_\delta) \subset \Omega_\varepsilon$. Applying a lemma from [Gerhardt 1975] to $(v \circ X)\lambda^{n-1}$, we conclude that

$$\begin{aligned}\int_{\partial\Omega}|v|\,d\mathcal{H}^{n-1} &= \int_{\partial\tilde{\Omega}}|v\circ X|\lambda^{n-1}\,d\mathcal{H}^{n-1}\\ &\le \int_{\tilde{\Omega}_\delta}|D((v\circ X)\lambda^{n-1})| + C_1(\delta,\partial\tilde{\Omega})\int_{\tilde{\Omega}}|(v\circ X)\lambda^{n-1}|\,dx\\ &\le \int_{\tilde{\Omega}_\delta}|D(v\circ X)\lambda^{n-1}| + C_2(\varepsilon,n,\tilde{\Omega},\lambda)\int_{\tilde{\Omega}}|(v\circ X)\lambda^{n}|\,dx\\ &\le \int_{\Omega_\varepsilon}|\nabla_{S^n}v| + C_2(\varepsilon,n,\Omega,\pi_S)\int_{\Omega}|v|\,d\mathcal{H}^n,\end{aligned}$$

by virtue of the relation $|D(v\circ X)| = \lambda|(\nabla_{S^n}v\circ X)|$. □

After this preparation step, we formulate our main theorem:

Theorem 1.5. *Let Ω be a Lipschitz domain of S^n satisfying an ISC, and let δ, κ, σ be given positive constants, with $0 < \sigma < 1$.*

(i) *There exists a volume $V_0 > 0$ such that, for every $V \ge V_0$, the energy functional*

$$\mathcal{E}_\Omega(u,v) := \mathcal{A}_\Omega(u) + \mathcal{A}_\Omega(v) + \mathcal{G}_\Omega(u,v) + \sigma\mathcal{A}_{\partial\Omega}(v-u)$$

attains its minimum at exactly one (u,v) in the class

$$\mathcal{C} := K_\delta \cap \left\{(u,v)\in(BV_+(\Omega))^2 \;\middle|\; \int_\Omega (v^{1+1/n} - u^{1+1/n})\,dx = V\right\}.$$

This minimum is regular in the sense that $u \in C^\omega(\{x\in\Omega \mid u(x)>0\})$ and $v\in C^\omega(\Omega)$.

Moreover, there exists $\mu > 0$ such that (u,v) is also the unique minimum of $\mathcal{F}_{\Omega,\mu}(\cdot,\cdot)$ in the class K_δ.

(ii) *If, in addition, $\overline{\Omega}$ is included in an open hemisphere — say $\overline{\Omega}\subset H(p_N) := \{z\in S^n \mid \langle z, p_n\rangle > 0\}$ — with $|\pi_S(\Omega)| < \omega_n$ and if $\frac{1}{n}\mathcal{H}^{n-1}(\partial\Omega) \le \sigma\mathcal{A}_{\partial\Omega}(1)$, then we have $u > 0$ in Ω, which implies $u\in C^\omega(\Omega)$.*

2. L_∞-estimates

In the following, we are mainly concerned with the functional $\mathcal{F}_{\Omega,\mu}$. It is clear that this functional only makes sense if μ is a Lagrange multiplier belonging to the variational problem of minimizing $\mathcal{E}_\Omega(u,v)$ for (u,v) defined in the nonconvex set $\mathcal{C}$; we justify this choice later in Section 4. A great advantage of considering $\mathcal{F}_{\Omega,\mu}$ is that we can prove this result:

Theorem 2.1. *Let (u,v) be a minimizer of $\mathcal{F}_{\Omega,\mu}$ in K_δ and define*

$$k_\mu := \left(\frac{(n+1)\,\mu}{(n+2)\,\kappa}\right)^n.$$

Then $0 \le u \le v \le k_\mu$. *If we assume that* $\overline{\Omega} \subset H(p_N)$, $\frac{1}{n}\mathscr{H}^{n-1}(\partial\Omega) \le \sigma \mathscr{A}_{\partial\Omega}(1)$, *and* $|\pi_S(\Omega)| < \omega_n$, *then* $u > 0$ *almost everywhere in* Ω.

Remark 2.2. Set

$$K_{\delta,1} := \{v \in BV_+(\Omega) \mid v \ge \delta\}, \quad K_{\delta,2} := \{u \in BV_+(\Omega) \mid u \le \delta\}.$$

A pair (u, v) minimizes $\mathscr{F}_{\Omega,\mu}(\cdot\,,\cdot)$ over K_δ if and only if v minimizes the functional $\mathscr{A}_\Omega(\cdot) + \mathscr{H}_\Omega(\cdot) + \sigma\mathscr{A}_{\partial\Omega}$ over $K_{\delta,1}$ and, simultaneously, u minimizes the functional $\mathscr{A}_\Omega(\cdot) - \mathscr{H}_\Omega(\cdot) - \sigma\mathscr{A}_{\partial\Omega}$ over $K_{\delta,2}$.

Proof of Theorem 2.1. To show the first claim we compare v with $w := \min(v, k_\mu)$ and use the fact that $f(t) := \kappa\, t^{1+2/n} - \mu\, t^{1+1/n}$ attains its minimum in k_μ. Moreover, f is a nondecreasing function for $t \ge k_\mu$. Thus, we have $\mathscr{A}_\Omega(v) + \mathscr{H}_\Omega(v) + \sigma\mathscr{A}_{\partial\Omega}(v) = \mathscr{A}_\Omega(w) + \mathscr{H}_\Omega(w) + \sigma\mathscr{A}_{\partial\Omega}(w)$, from which we conclude that $\mathscr{H}_\Omega(v) = \mathscr{H}_\Omega(w)$ and consequently $v \le k_\mu$. To derive the second assertion in the theorem, we can assume that the problem is situated in $\mathbb{R}^n$ (see Section 1). Then we use $w := \max(u, \lambda)$ as a comparison function, where $\lambda \in [\lambda_1, \lambda_2]$ with $0 < \lambda_1 < \lambda_2$ such that $\mu \sqrt[n]{\lambda_2} < \varepsilon < \delta$ and where $\varepsilon > 0$ is sufficiently small. We thus have

$$\mathscr{A}_{A(\lambda)}(u) \le \lambda\,|A(\lambda)| + \mu \sqrt[n]{\lambda}\,\lambda\,|A(\lambda)| - \sigma\mathscr{A}_{\partial\Omega}((\lambda - u)^+),$$

where $A(\lambda) := \{x \in \Omega \mid u(x) < \lambda\}$. which implies that

$$\begin{aligned} |(\lambda - u)^+|_{L^{n/(n-1)}(\Omega)} &\le c(n)|\nabla(\lambda - u)^+|(\mathbb{R}^n) \\ &= c(n)|\nabla(\lambda - u)^+|(\Omega) + c(n)\int_{\partial\Omega}(\lambda - u)^+ d\mathscr{H}^{n-1} \\ &\le n\,c(n)(1+\varepsilon)\lambda|A(\lambda)|, \end{aligned}$$

by the Sobolev inequality in $\mathbb{R}^n$: $|f|_{L^{n/(n-1)}(\Omega)} \le c(n)|\nabla f|(\mathbb{R}^n)$ with isoperimetric constant $c(n) := n^{-1}\omega_n{}^{-1/n}$ and by the assumption $\frac{1}{n}\mathscr{H}^{n-1}(\partial\Omega) \le \sigma\mathscr{A}_{\partial\Omega}(1)$. Consequently

$$y(\lambda) := \int_\Omega (\lambda - u)^+ dx \le (1+\varepsilon)n\,c(n)\lambda y'(\lambda)^{1+1/n}$$

for almost every $\lambda \in [\lambda_1, \lambda_2]$, where we have used the Hölder inequality. Now suppose that $y(\lambda_1) > 0$. By integration, we have

$$\begin{aligned} &(1+\varepsilon)^{n/(n+1)}\Big(\frac{|\Omega|}{\omega_n}\Big)^{1/(n+1)} y(\lambda_1)^{1/(n+1)} \\ &\qquad \le \bigg((1+\varepsilon)^{n/(n+1)}\Big(\frac{|\Omega|}{\omega_n}\Big)^{1/(n+1)} - 1\bigg)\lambda_2{}^{1/(n+1)} + \lambda_1{}^{1/(n+1)}. \end{aligned}$$

Now choose ε so small that $(1+\varepsilon)^{n/(n+1)}(|\Omega|/\omega_n)^{1/(n+1)} < 1$. We see at once that $y(\lambda_1) = 0$ for sufficiently small λ_1. □

Remark 2.3. By virtue of the definition of H in (2) and our inequality $u \le v \le k_\mu$, the mean curvature of the upper membrane is nonnegative, that is, $H\big(\sqrt[n]{v(x)}x\big) \ge 0$, and that of lower membrane is nonpositive, $-H\big(\sqrt[n]{u(x)}x\big) \le 0$. This is as expected, lending support to our variational approach.

Remark 2.4. The inequality $v \le k_\mu$ depends essentially on the sign of σ. In case of $\sigma < 0$ and δ sufficiently small ($\delta \le (\mu/\kappa)^n$), we have no longer drops as stable configurations. Indeed, $\mathcal{F}_{\Omega,\mu}(u,v) \ge \mathcal{A}_\Omega(v) + \mathcal{H}_{\Omega,\mu}(v) + \sigma\mathcal{A}_{\partial\Omega}(v)$ for any solution (u,v) of our problem. From this we see that only the upper membrane remains as a solution; i.e., $u = 0$.

Example 2.5. Let Ω be a ball with radius $r < 1$, so the angle of aperture of the corresponding cone is less than $90°$. Then we can conclude that $u > 0$ if $\sigma \ge 1/\sqrt{2}$. Moreover, this estimate is sharp. In fact, if $u > 0$ and $\sigma < 1/\sqrt{2}$, we can find $r > 0$ such that $r/\sqrt{1+r^2} > \sigma$. Hence, for any minimizer (u,v) of $\mathcal{F}_{\Omega,\mu}$ with $\Omega = B_r(0)$, we have

$$\begin{aligned}\mathcal{F}_{\Omega,\mu}(u,v) &\ge \int_\Omega \operatorname{div}(ux)\,dx - \sigma\int_{\partial\Omega} u\sqrt{1+(x\nu)^2} + \mathcal{A}_\Omega(v) + \mathcal{H}_{\Omega,\mu}(v) + \sigma\mathcal{A}_{\partial\Omega}(v)\\ &= \mathcal{A}_\Omega(v) + \mathcal{H}_{\Omega,\mu}(v) + \sigma\mathcal{A}_{\partial\Omega}(v) + \int_{\partial\Omega} u(r - \sigma\sqrt{1+r^2})\,d\mathcal{H}^{n-1}\\ &> \mathcal{A}_\Omega(v) + \mathcal{H}_{\Omega,\mu}(v) + \sigma\mathcal{A}_{\partial\Omega}(v) = \mathcal{F}_{\Omega,\mu}(0,v).\end{aligned}$$

3. Existence

Theorem 3.1. *For any $\mu > 0$ there exists a solution (u_μ, v_μ) to the problem of minimizing $\mathcal{F}_{\Omega,\mu}(\cdot,\cdot)$ over K_δ. If v is assumed large enough, that is, if we restrict to the class*

$$K_\delta \cap \{(u,v) \mid u \le \delta < k_\mu/2^n < v\},$$

the solution is unique.

Proof. Take (u,v) in K_δ. Since $u \le \delta$, one sees easily that

$$\mathcal{A}_\Omega(u) + \mathcal{A}_\Omega(v) + \mathcal{H}_{\Omega,\mu}(v) - C(\delta) \le \mathcal{F}_{\Omega,\mu}(u,v),$$

where $C(\delta)$ is a positive constant. This inequality shows that $\inf \mathcal{F}_{\Omega,\mu} > -\infty$ and that any minimizing sequence (u_n, v_n) in K_δ has bounded BV-norm; i.e., $\mathcal{A}_\Omega(u_n) + \mathcal{A}_\Omega(v_n) < \infty$. From the Sobolev imbedding (see [Schwab 2004]), we conclude that the sequences above are precompact in any Lebesgue space $L_q(\Omega)$, for $1 \le q < n/(n-1)$. Hence, for convenience, we can assume that there exists $(u,v) \in L_{n+1/n}(\Omega)^2$ such that $u_n \to u$ and $v_n \to v$ in this space. The existence conclusion follows by using the lower semicontinuity of $\mathcal{F}_{\Omega,\mu}(\cdot,\cdot)$, proved in the next theorem.

To prove uniqueness in the restricted space, note that the function h given by

$$v \mapsto -\int_0^{\sqrt[n]{v}} H(\lambda x)\lambda^n d\lambda$$

is strictly convex for $v > k_\mu/2^n$ and strictly concave for $v < k_\mu/2^n$; this follows at once from the relation

$$\begin{aligned} h''(v) &= -\frac{\partial}{\partial v}\big(H(\sqrt[n]{v}x)\sqrt[n]{v}\big) = -\frac{\partial}{\partial \lambda}(H(\lambda x)\lambda)\big|_{\lambda=\sqrt[n]{v}}\, v^{(1/n)-1} \\ &= \frac{v^{(1/n)-1}}{n}\big(2\kappa(n+2)\sqrt[n]{v} - \mu(n+1)\big) > 0, \end{aligned}$$

the latter inequality being equivalent to $\sqrt[n]{v} > \mu(n+1)/(2\kappa(n+2)) = \frac{1}{2}\sqrt[n]{k_\mu}$. This implies the strict convexity of $v \mapsto \mathcal{H}_{\Omega,\mu}(v)$ in $\{v \in BV_+(\Omega) \mid v > k_\mu/2^n\}$ and $u \mapsto -\mathcal{H}_{\Omega,\mu}(u)$ in $\{u \in BV_+(\Omega) \mid u < k_\mu/2^n\}$. The conclusion follows thanks to Remark 2.2. □

Theorem 3.2. *The functional $\mathcal{F}_{\Omega,\mu}$ is lower semicontinuous with respect to convergence in $L_{n+1/n}(\Omega)^2 \cap K_\delta$.*

Proof. We use an idea from [Gerhardt 1975]. Let (u_n, v_n) be a sequence converging in $L_{n+1/n}(\Omega)^2 \cap K_\delta$ to a function (u, v), and suppose to the contrary that $\liminf_{n\to\infty} \mathcal{F}_{\Omega,\mu}(u_n, v_n) < \mathcal{F}_{\Omega,\mu}(u, v)$. For simplicity we can assume that there is an $\alpha > 0$ such that $\mathcal{F}_{\Omega,\mu}(u_n, v_n) < \mathcal{F}_{\Omega,\mu}(u, v) - \alpha$ for all $n \in \mathbb{N}$. Taking into account Lemma 1.4 applied to $v - v_n$ and $u_n - u$, we obtain for $\varepsilon > 0$ the inequality

$$\begin{aligned} &\mathcal{A}_\Omega(v_n) + \mathcal{A}_\Omega(u_n) + \mathcal{H}_{\Omega,\mu}(v_n) - \mathcal{H}_{\Omega,\mu}(u_n) + \alpha \\ &\qquad \le \mathcal{A}_\Omega(v) + \mathcal{A}_\Omega(u) + \mathcal{H}_{\Omega,\mu}(v) - \mathcal{H}_{\Omega,\mu}(u) \\ &\qquad\qquad + \mathcal{A}_{\Omega_\varepsilon}(v - v_n) + \mathcal{A}_{\Omega_\varepsilon}(u - u_n) + C_\varepsilon \textstyle\int_\Omega (|v - v_n| + |u - u_n|) \end{aligned}$$

We conclude that

$$\begin{aligned} &\mathcal{A}_{\Omega\setminus\bar{\Omega}_\varepsilon}(v_n) + \mathcal{A}_{\Omega\setminus\bar{\Omega}_\varepsilon}(u_n) + \mathcal{H}_{\Omega,\mu}(v_n) - \mathcal{H}_{\Omega,\mu}(u_n) + \alpha \\ &\qquad \le \mathcal{A}_\Omega(v) + \mathcal{A}_\Omega(u) + \mathcal{H}_{\Omega,\mu}(v) - \mathcal{H}_{\Omega,\mu}(u) \\ &\qquad\qquad + \mathcal{A}_{\Omega_\varepsilon}(v) + \mathcal{A}_{\Omega_\varepsilon}(u) + C_\varepsilon \textstyle\int_\Omega (|v - v_n| + |u - u_n|). \end{aligned}$$

Now, observe that $(u, v) \mapsto \mathcal{H}_{\Omega,\mu}(v) - \mathcal{H}_{\Omega,\mu}(u)$ is lower semicontinuous with respect to convergence in $L_{n+1/n}(\Omega)^2 \cap K_\delta$. Indeed, for any sequence (u_k, v_k) converging in that space, we have

$$\begin{aligned} &\liminf_{k\to\infty}(\mathcal{H}_{\Omega,\mu}(v_k) - \mathcal{H}_{\Omega,\mu}(u_k)) \\ &\qquad = \liminf_{k\to\infty} \frac{\kappa}{n}\int_\Omega {v_k}^{1+2/n} - {u_k}^{1+2/n} - \lim_{k\to\infty}\frac{\mu}{n}\int_\Omega ({v_k}^{1+1/n} - {u_k}^{1+1/n}) \\ &\qquad \ge \mathcal{H}_{\Omega,\mu}(v) - \mathcal{H}_{\Omega,\mu}(u), \end{aligned}$$

where we have used Fatou's Lemma applied to the first integral. Next, $\mathcal{A}_\Omega(\,\cdot\,)$ is lower semicontinuous with respect to convergence in $L_1(\Omega)$; see [Schwab 2004]. Then, by letting $n \to \infty$, we get

$$\mathcal{A}_{\Omega\setminus\bar{\Omega}_\varepsilon}(v) + \mathcal{A}_{\Omega\setminus\bar{\Omega}_\varepsilon}(u) + \alpha \le \mathcal{A}_\Omega(v) + \mathcal{A}_\Omega(u) + \mathcal{A}_{\Omega_\varepsilon}(v) + \mathcal{A}_{\Omega_\varepsilon}(u).$$

One sees at once that $\varepsilon \to 0$ leads to a contradiction. □

4. Regularity

We now deal with the interior smoothness of local minimizers. To prove the desired regularity results we rely on the following theorem, reformulated slightly from [Schwab 2005]. Using also the results obtained in the proof of Theorem 3.1, we will infer regularity from the L_∞-estimates stated in Proposition 4.2 below.

Theorem 4.1. *Let $H : C_\Omega \to \mathbb{R}$ be a function in $C^\omega(C_\Omega)$ defined on the cone $C_\Omega := \{\varrho x \mid \varrho > 0, x \in \Omega\}$, and suppose that*

$$\frac{\partial}{\partial\rho}(\rho H(\rho x)) \le 0$$

for every $x \in \Omega$ and $\rho \in\,]r, R[$, where $0 \le r < R \le \infty$. Then any local minimizer v of the functional $\mathcal{A}_\Omega(\,\cdot\,) + \mathcal{H}_\Omega(\,\cdot\,)$ in the set of $BV_{+,loc}(\Omega)$ functions bounded from below by r and from above by R is of class

$$C^\omega\big(\{x \in \Omega \mid r < v(x) < R\}\big).$$

Proposition 4.2. *Let $k > 0$ and $\delta > 0$ be given. There exists an $\mu_0 > 0$ such that, for any $\mu \ge \mu_0$ and any solution (v_μ, u_μ) of the problem of minimizing $\mathcal{F}_{\Omega,\mu}(\,\cdot\,,\cdot\,)$ over K_δ, we have*

$$v_\mu > k_\mu/2^n \ge \delta > u_\mu.$$

To prove the proposition we state an interpolation lemma, which can be verified using a contradiction argument based on the compactness result of [Schwab 2004].

Lemma 4.3. *Let $\Omega \subset S^n$ be a Lipschitz domain and take $\varepsilon > 0$. There exists a constant $C(\varepsilon)$ such that*

$$|v|_{L_q(\Omega)} \le \varepsilon\,|\nabla_{S^n} v|(\Omega) + C(\varepsilon)|v|_{L_1(\Omega)}$$

whenever $1 \le q < n/(n-1)$ and $v \in BV_+(\Omega)$.

proof of Proposition 4.2. Let $\mu > 0$ be so large that $\delta < k_\mu/2^n$ and let $\alpha \in\,]\frac{1}{2}, 1[$ be a constant. We already know that v_μ is a solution of $\mathcal{A}_\Omega(\,\cdot\,)+\mathcal{H}_\Omega(\,\cdot\,)+\sigma\mathcal{A}_{\partial\Omega}(\,\cdot\,)$ in the class $K_{\delta,1}$. By comparing v_μ with $w_\mu := \max(v_\mu, \lambda)$, where $\lambda \in\,]k_\mu/2^n, \alpha^n k_\mu[$, we see that

$$\mathcal{A}_{A(\lambda)\cap\Omega}(v_\mu) + \frac{1}{n}\int_{A(\lambda)} \mu(\lambda^{1+1/n} - {v_\mu}^{1+1/n}) - \kappa(\lambda^{1+2/n} - {v_\mu}^{1+2/n}) \\ - \sigma\mathcal{A}_{\partial\Omega\cap A(\lambda)}(\lambda - v_\mu),$$

which is bounded above by $\lambda|A(\lambda)|$ on the set $A(\lambda) := \{x \in \bar{\Omega} \mid v_\mu(x) < \lambda\}$.

Now consider the function $h(t) := (\mu/n)t^{1+1/n} - (\kappa/n)t^{1+2/n}$. Then

$$h'(t) = \mu\frac{n+1}{n^2}t^{1/n} - \kappa\frac{n+2}{n^2}t^{2/n},$$

and one easily sees that $h''(t) \le 0$ if and only if $t \ge k_\mu/2^n$. Therefore

$$h'(t) \ge \begin{cases} h'(\delta) & \text{if } \delta \le t \le k_\mu/2^n, \\ h'(\alpha^n k_\mu) & \text{if } k_\mu/2^n \le t \le \alpha^n k_\mu. \end{cases}$$

This implies that

$$h'(t) \ge \min\left(\delta^{1/n}\mu\frac{n+1}{n^2} - \delta^{2/n}\kappa\frac{n+2}{n^2},\ \alpha(1-\alpha)\frac{\mu^2}{\kappa}\frac{(n+1)^2}{n^2(n+2)}\right) =: C_1(\mu)$$

for $t \in [\delta, \alpha^n k_\mu]$. Clearly $C_1(\mu)$ is nondecreasing in μ and $C_1(\mu) > 0$ by the assumption on μ made above. In view of Lemma 1.4, then, we obtain

$$\mathcal{A}_{\partial\Omega\cap A(\lambda)}(\lambda - v_\mu) = \mathcal{A}_{\partial\Omega}((\lambda - v_\mu)^+) \le \int_\Omega \nabla_{S^n}((\lambda - v_\mu)^+) + C_2\int_\Omega (\lambda - v_\mu)^+.$$

Thus, by the Mean Value Theorem,

$$(1-\sigma)\int_\Omega \nabla_{S^n}((\lambda - v_\mu)^+) + (C_1(\mu) - C_3)\int_\Omega (\lambda - v_\mu)^+ \le \lambda|A(\lambda)|.$$

From now on we proceed much as in [Schindelmayr 1999]. The main idea is to involve the interpolation Lemma 4.3. Choose μ_0 large enough that $\varepsilon(C_1(\mu_0) - C_3)$ exceeds $(1-\sigma)C(\varepsilon)$, where ε is a constant to be determined later; also assume $1 < p < n/(n-1)$. For every $\mu \ge \mu_0$, the interpolation inequality implies

$$\left(\int_\Omega \left((\lambda - v_\mu)^+\right)^p dx\right)^{1/p} \le \frac{\varepsilon}{1-\sigma}\lambda|A(\lambda)|,$$

and by using the Hölder inequality we get

$$y(\lambda) := \int_\Omega (\lambda - v_\mu)^+ dx \le \frac{\varepsilon}{1-\sigma}\lambda|A(\lambda)|^{1+(p-1)/p}.$$

Now take into account that y is absolutely continuous with $y'(\lambda) = |A(\lambda)|$ a.e. It follows that, for almost every $\lambda \in]k_\mu/2^n, \alpha^n k_\mu[$,

$$y(\lambda)^{p/(2p-1)} \le \left(\frac{\varepsilon}{1-\sigma}\right)^{p/(2p-1)} \lambda^{p/(2p-1)} y'(\lambda)$$

and we want to show that $y(\lambda_1)=0$ for a certain $\lambda_1 > k_\mu/2^n$. For this purpose, let $\lambda_2 := \alpha^n k_\mu$ and $\lambda_1 = \gamma^n k_\mu$, where $\frac{1}{2} < \gamma < \alpha$. Furthermore, suppose on the contrary that $y(\lambda_1) > 0$. Integrating from λ_1 to λ_2, we obtain, with $g := (p-1)/(2p-1)$:

$$\left(\frac{\varepsilon}{1-\sigma}\right)^{-p/(2p-1)}\left({\lambda_2}^g - {\lambda_1}^g\right) \le y(\lambda_2)^g - y(\lambda_1)^g,$$

whence

$$y(\lambda_1)^g \le {k_\mu}^g\left((|\Omega|\alpha^n)^g - \left(\frac{\varepsilon}{1-\sigma}\right)^{-p/(2p-1)}\left((\alpha^n)^g - (1/2^n)^g\right)\right).$$

Because $q > 0$ and $\varepsilon^{-p/(2p-1)} \to \infty$ for $\varepsilon \to 0$, we arrive at the contradiction $y(\lambda_1) < 0$.

Finally, the case $u_\mu < \delta$ can be treated as in [Schindelmayr 1999]. □

Lemma 4.4. *The map* $L : [\mu_0, \infty[\to L_{1+1/n}(\Omega)^2$ *given by* $\mu \mapsto (u_\mu, v_\mu)$ *and the map*

$$V : [\mu_0, \infty[\to [V(\mu_0), \infty[\quad \textit{given by} \quad \mu \mapsto \int_\Omega ({v_\mu}^{1+1/n} - {u_\mu}^{1+1/n})\, dx$$

are continuous.

Proof. Note that the volume is a nondecreasing function of the Lagrange multiplier μ; see [Schindelmayr 1999, 4.6.4]. Let (μ_k) be a sequence converging to $\mu > \mu_0$ and let (u_{μ_k}, v_{μ_k}) the unique solution of the problem of minimizing $\mathscr{F}_{\Omega,\mu_k}(\,\cdot\,,\cdot\,)$ over K_δ (see Theorem 3.1). Using the definition of $\mathscr{F}_{\Omega,\mu}(\,\cdot\,,\cdot\,)$, we have

$$\begin{aligned}
&\mathscr{A}_\Omega(u_{\mu_k}) + \mathscr{A}_\Omega(v_{\mu_k}) + \mathscr{H}_{\Omega,\mu_k}(v_{\mu_k}) + C_1(\delta)\\
&\qquad \le \mathscr{F}_{\Omega,\mu}(u_{\mu_k}, v_{\mu_k}) - \mathscr{H}_{\Omega,\mu}(v_{\mu_k}) + \mathscr{H}_{\Omega,\mu_k}(v_{\mu_k})\\
&\qquad = \mathscr{F}_{\Omega,\mu_k}(u_{\mu_k}, v_{\mu_k}) + \mathscr{H}_{\Omega,\mu_k}(u_{\mu_k}) - \mathscr{H}_{\Omega,\mu}(u_{\mu_k})\\
&\qquad \le \mathscr{F}_{\Omega,\mu_k}(u_\mu, v_\mu) + \mathscr{H}_{\Omega,\mu_k}(u_{\mu_k}) - \mathscr{H}_{\Omega,\mu}(u_{\mu_k})\\
&\qquad \le \mathscr{F}_{\Omega,\mu_k}(u_\mu, v_\mu) + C_2(\delta).
\end{aligned}$$

The constants $C_1(\delta)$ and $C_2(\delta)$ do not depend on k. From this we see that (u_{μ_k}, v_{μ_k}) has bounded BV-norm and that it provides a minimizing sequence for $\mathscr{F}_{\Omega,\mu}$. The latter observation follows by taking into account the relation

$$\mathscr{H}_{\Omega,\mu_k}(w) - \mathscr{H}_{\Omega,\mu}(w) = \frac{\mu-\mu_k}{n}\int_\Omega w^{1+1/n}$$

for $w \in BV(\Omega)$, with an argument applied to every subsequence $(u_{\mu_{k(l)}}, v_{\mu_{k(l)}})$ with convergent $\mathscr{F}_{\Omega,\mu}(u_{\mu_{k(l)}}, v_{\mu_{k(l)}})$. Thus, we get the desired result by the uniqueness part in Theorem 3.1 and the lower semicontinuity statement in Theorem 3.2. □

To complete the proof of Theorem 1.5, it remains to show that the solution (u_μ, v_μ) for $\mu \geq \mu_0$ provides also a minimum for $\mathscr{E}_\Omega(\cdot\,,\cdot)$ in the class

$$\mathscr{C} := K_\delta \cap \left\{(u, v) \in (BV_+(\Omega))^2 \,\middle|\, \int_\Omega (v^{1+1/n} - u^{1+1/n})\, dx = V\right\}$$

for $V \geq V_0 := V(\mu_0)$. But now this is fairly easy. Indeed, for a given volume $V \geq V(\mu_0)$, we obtain a $\mu \geq \mu_0$ such that $V = V(\mu)$ by the Intermediate Value Theorem. Using the relation $\mathscr{F}_{\Omega,\mu}(u_\mu, v_\mu) = \mathscr{E}_\Omega(u_\mu, v_\mu) - \mu V(\mu)$, we see that (u_μ, v_μ) is also a solution of $\mathscr{E}_\Omega(\cdot\,,\cdot)$ in the class of comparison functions $\mathscr{C}$. Any further solution (u, v) of $\mathscr{E}_\Omega(\cdot\,,\cdot)$ in $\mathscr{C}$ is a minimum of $\mathscr{F}_{\Omega,\mu}(\cdot\,,\cdot)$ in K_δ too, because $\mathscr{E}_\Omega(u, v) = \mathscr{E}_\Omega(u_\mu, v_\mu)$. Thus Proposition 4.2 yields the estimate $v > k_\mu/2$ and consequently, by Theorem 3.1, one obtains $(u, v) = (u_\mu, v_\mu)$, which proves the existence and uniqueness part of Theorem 1.5. Regularity follows, as already mentioned, from Theorem 4.1 in connection with Proposition 4.2 by applying Theorem 4.1 to the cases $r = 0$, $R = \delta$ and $r = k_\mu/2$, $R = \infty$.

References

[Ambrazevicius 1982] A. P. Ambrazevicius, "Finding the form of the surface of a liquid in a conical container for a given volume of the liquid, I", *Liet. Mat. Rink.* (*Litov. Mat. Sb.*) **22**:1 (1982), 17–24. In Russian; translated in *Lithuanian Math. J.* **22**:1 (1982), 1-6. Zbl 0565.35111

[Finn 1986] R. Finn, *Equilibrium capillary surfaces*, vol. 284, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, New York, 1986. MR 88f:49001 Zbl 0583.35002

[Gerhardt 1975] C. Gerhardt, "On the capillarity problem with constant volume", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **2**:2 (1975), 303–320. MR 52 #11720 Zbl 0321.76010

[Giusti 1981] E. Giusti, "The equilibrium configuration of liquid drops", *J. Reine Angew. Math.* **321** (1981), 53–63. MR 82c:49037 Zbl 0438.76078

[Minkowski 1907] H. Minkowski, "Kapillarität", pp. 558–613 in *Enzyklopädie der mathematischen Wissenschaften*, vol. 1, Heft 4, 1907. JFM 38.0794.12

[Schindelmayr 1999] G. Schindelmayr, *Kapillarität im Kegel und in allgemeineren skalierten Gebieten*, Dissertation, Rheinisch-Westfälische Technische Hochschule Aachen, 1999. published by Shaker Verlag, ISBN 3-8265-4867-1.

[Schwab 2004] D. Schwab, "Hypersurfaces of prescribed mean curvature in central projection, I", *Arch. Math.* (*Basel*) **82**:3 (2004), 245–262. MR 2005k:49109 Zbl 1067.49028

[Schwab 2005] D. Schwab, "Hypersurfaces of prescribed mean curvature in central projection, II", *Arch. Math.* (*Basel*) **84**:2 (2005), 171–182. MR 2006e:49084 Zbl 1071.49024

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