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OF A HYPERPLANE ARRANGEMENT

FEDERICO ARDILA

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# COMPUTING THE TUTTE POLYNOMIAL OF A HYPERPLANE ARRANGEMENT

FEDERICO ARDILA

**We define and study the Tutte polynomial of a hyperplane arrangement. We introduce a method for computing the Tutte polynomial by solving a related enumerative problem. As a consequence, we obtain new formulas for the generating functions enumerating alternating trees, labelled trees, semiorders and Dyck paths.**

## 1. Introduction

Much work has been devoted in recent years to studying hyperplane arrangements and, in particular, their characteristic polynomials. The polynomial  $\chi_{\mathcal{A}}(q)$  is a very powerful invariant of the arrangement  $\mathcal{A}$ ; it arises very naturally in many different contexts. Two of the many important and beautiful results about the characteristic polynomial of an arrangement are the following.

**Theorem 1.1** [Zaslavsky 1975]. *Let  $\mathcal{A}$  be a hyperplane arrangement in  $\mathbb{R}^n$ . The number of regions into which  $\mathcal{A}$  dissects  $\mathbb{R}^n$  is equal to  $(-1)^n \chi_{\mathcal{A}}(-1)$ . The number of regions which are relatively bounded is equal to  $(-1)^n \chi_{\mathcal{A}}(1)$ .*

**Theorem 1.2** [Orlik and Solomon 1980]. *Let  $\mathcal{A}$  be a hyperplane arrangement in  $\mathbb{C}^n$ , and let  $M_{\mathcal{A}} = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} H$  be its complement. Then the Poincaré polynomial of the cohomology ring of  $M_{\mathcal{A}}$  is given by:*

$$\sum_{k \geq 0} \text{rank } H^k(M_{\mathcal{A}}, \mathbb{Z}) q^k = (-q)^n \chi_{\mathcal{A}}(-1/q).$$

Several authors have worked on computing the characteristic polynomials of specific hyperplane arrangements. This work has led to some very nice enumerative results; see for example [Athanasiadis 1996; Postnikov and Stanley 2000].

Somewhat surprisingly, nothing has been said about the Tutte polynomial of a hyperplane arrangement. Graphs and matroids have a Tutte polynomial associated with them, which generalizes the characteristic polynomial and arises naturally

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in numerous enumerative problems in both areas. Many interesting invariants of graphs and matroids can be computed immediately from this polynomial.

The present paper aims to define and investigate the Tutte polynomial of a hyperplane arrangement. This paper is devoted to purely enumerative questions, particularly the computation of Tutte polynomials of specific arrangements. We addressed the matroid-theoretic aspects of this investigation in [Ardila 2004].

In Section 2 we introduce the basic notions of hyperplane arrangements that we will need. In Section 3 we define the Tutte polynomial of a hyperplane arrangement, and we present a finite field method for computing it. This is done in terms of the coboundary polynomial, a simple transformation of the Tutte polynomial. We recover several known results about the characteristic and Tutte polynomials of graphs and representable matroids, and derive other consequences of this method. In Section 4 we compute the Tutte polynomials of Coxeter arrangements, threshold arrangements, and generic deformations of the braid arrangement. In Section 5 we focus on a large family of deformations of the braid arrangement, where the computation of Tutte polynomials is related to the enumeration of classical combinatorial objects. As a consequence, we obtain several purely enumerative results about alternating trees, parking functions, semiorders, and Dyck paths.

## 2. Hyperplane arrangements

We recall some of the basic concepts of hyperplane arrangements. For a more thorough introduction, see [Orlik and Terao 1992] or [Stanley 2004].

Given a field  $\mathbb{k}$  and a positive integer  $n$ , an *affine hyperplane* in  $\mathbb{k}^n$  is an  $(n - 1)$ -dimensional affine subspace of  $\mathbb{k}^n$ . If we fix a system of coordinates  $x_1, \dots, x_n$  on  $\mathbb{k}^n$ , a hyperplane can be seen as the set of points that satisfy a certain equation  $c_1x_1 + \dots + c_nx_n = c$ , where  $c_1, \dots, c_n, c \in \mathbb{k}$  and not all  $c_i$ s are equal to 0. A *hyperplane arrangement*  $\mathcal{A}$  in  $\mathbb{k}^n$  is a finite set of affine hyperplanes of  $\mathbb{k}^n$ . We will refer to hyperplane arrangements simply as *arrangements*. We will mostly be interested in arrangements in  $\mathbb{R}^n$ , but we will find it useful to work over other fields as well.

We will say that an arrangement  $\mathcal{A}$  is *central* if the hyperplanes in  $\mathcal{A}$  have a nonempty intersection. (Sometimes we will call an arrangement *affine* to emphasize that it does not need to be central.) Similarly, we will say that a subset (or *subarrangement*)  $\mathcal{B} \subseteq \mathcal{A}$  of hyperplanes is *central* if the hyperplanes in  $\mathcal{B}$  have a nonempty intersection.

The *rank function*  $r_{\mathcal{A}}$  is defined for each central subset  $\mathcal{B}$  by  $r_{\mathcal{A}}(\mathcal{B}) = n - \dim \bigcap \mathcal{B}$ . This function can be extended to a function  $r_{\mathcal{A}} : 2^{\mathcal{A}} \rightarrow \mathbb{N}$ , by defining the rank of a noncentral subset  $\mathcal{B}$  to be the largest rank of a central subset of  $\mathcal{B}$ . The *rank* of  $\mathcal{A}$  is  $r_{\mathcal{A}}(\mathcal{A})$ , and it is denoted  $r_{\mathcal{A}}$ .

Alternatively, if the hyperplane  $H$  has defining equation  $c_1x_1 + \cdots + c_nx_n = c$ , associate its normal vector  $v = (c_1, \dots, c_n)$  to it. Then define  $r_{\mathcal{A}}(\{H_1, \dots, H_k\})$  to be the dimension of the span of the corresponding vectors  $v_1, \dots, v_k$  in  $\mathbb{R}^n$ . It is easy to see that these two definitions of the rank function agree. In particular, this means that the resulting function  $r_{\mathcal{A}} : 2^{\mathcal{A}} \rightarrow \mathbb{N}$  is the rank function of a matroid. We will usually omit the subscripts when the underlying arrangement is clear, and simply write  $r(\mathcal{B})$  and  $r$  for  $r_{\mathcal{A}}(\mathcal{B})$  and  $r_{\mathcal{A}}$ , respectively.

(Note that there is another natural way to extend  $r_{\mathcal{A}}$  to the rank function of a matroid; see [Ardila 2004].)

The rank function gives us natural definitions of the usual concepts of matroid theory, such as independent sets, bases, flats, and circuits, in the context of hyperplane arrangements. All of this is done more naturally in the broader context of *semimatroids* in [Ardila 2004].

To each hyperplane arrangement  $\mathcal{A}$  we assign a partially ordered set, called the *intersection poset* of  $\mathcal{A}$  and denoted  $L_{\mathcal{A}}$ . It consists of the nonempty intersections  $H_{i_1} \cap \cdots \cap H_{i_k}$ , ordered by reverse inclusion. This poset is graded, with rank function  $r(H_{i_1} \cap \cdots \cap H_{i_k}) = r_{\mathcal{A}}(\{H_{i_1}, \dots, H_{i_k}\})$ , and a unique minimal element  $\hat{0} = \mathbb{R}^n$ . We will call two arrangements  $\mathcal{A}_1$  and  $\mathcal{A}_2$  *combinatorially isomorphic* or simply *isomorphic*, and write  $\mathcal{A}_1 \cong \mathcal{A}_2$ , if  $L_{\mathcal{A}_1} \cong L_{\mathcal{A}_2}$ . Isomorphic arrangements may be defined over different fields.

The *characteristic polynomial* of  $\mathcal{A}$  is

$$\chi_{\mathcal{A}}(q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{n-r(x)}.$$

where  $\mu$  denotes the Möbius function of  $L_{\mathcal{A}}$  [Stanley 1997, Section 3.7].

Let  $\mathcal{A}$  be an arrangement and let  $H$  be a hyperplane in  $\mathcal{A}$ . The arrangement  $\mathcal{A} - \{H\}$  (or simply  $\mathcal{A} - H$ ), obtained by removing  $H$  from the arrangement, is called the *deletion of  $H$  in  $\mathcal{A}$* . It is an arrangement in  $\mathbb{R}^n$ . The arrangement  $\mathcal{A}/H = \{H' \cap H \mid H' \in \mathcal{A} - H, H' \cap H \neq \emptyset\}$ , consisting of the intersections of the other hyperplanes with  $H$ , is called the *contraction of  $H$  in  $\mathcal{A}$* . It is an arrangement in  $H$ .

However, some technical difficulties can arise. In a hyperplane arrangement  $\mathcal{A}$ , contracting a hyperplane  $H$  may give us repeated hyperplanes  $H_1$  and  $H_2$  in the arrangement  $\mathcal{A}/H$ . Say we want to contract  $H_1$  in  $\mathcal{A}/H$ . In passing to the contraction  $(\mathcal{A}/H)/H_1$ , the hyperplane  $H_2$  of  $\mathcal{A}/H$  becomes the “hyperplane”  $H_2 \cap H_1 = H_1$  in the “arrangement”  $(\mathcal{A}/H)/H_1$ . But this is not a hyperplane in  $H_1$ .

Therefore, the class of hyperplane arrangements, as defined, is not closed under deletion and contraction. This is problematic when we want to mirror matroid-theoretic results in this context. There is an artificial solution to this problem: we can consider multisets  $\{H_1, \dots, H_k\}$  of subspaces of a vector space  $V$ , where each

$H_i$  has dimension  $\dim V - 1$  or  $\dim V$ . In other words, we allow repeated hyperplanes, and we allow the full space  $V$  to be regarded as a “hyperplane”, mirroring a loop of a matroid. This class of objects is closed under deletion and contraction, but it is somewhat awkward to work with. A better solution is to think of arrangements as members of the class of semimatroids: a class that is also closed under deletion and contraction, and is more natural matroid-theoretically. We develop this point of view in [Ardila 2004]. However, such issues will be irrelevant in this paper, which focuses on purely enumerative aspects of arrangements.

### 3. Computing the Tutte polynomial

Athanasiadis [1996] introduced a powerful method for computing the characteristic polynomial of a subspace arrangement, based on ideas of Crapo and Rota [1970]. He reduced the computation of characteristic polynomials to an enumeration problem in a vector space over a finite field. He used this method to compute explicitly the characteristic polynomial of several families of hyperplane arrangements, obtaining very nice enumerative results. As should be expected, this method only works when the equations defining the hyperplanes of the arrangement have integer (or rational) coefficients. Such an arrangement will be called a  $\mathbb{Z}$ -arrangement.

Reiner discovered an elegant interpretation for the Tutte polynomial  $T_M$  of a representable matroid  $M$  (Equation (3) in [1999]), and asked whether it could be used to compute  $T_M$  for some nontrivial families of matroids. Compared to all the work that has been done on computing characteristic polynomials explicitly, virtually nothing is known about computing Tutte polynomials.

Our Theorem 3.3 below gives a new method for computing Tutte polynomials of hyperplane arrangements. Our approach does not use Reiner’s result; it is closer to Athanasiadis’ method. After proving the theorem, we present some of its consequences, using it in Section 4 to compute explicitly the Tutte polynomials of several families of arrangements.

#### *The Tutte and coboundary polynomials.*

**Definition 3.1.** The *Tutte polynomial* of a hyperplane arrangement  $\mathcal{A}$  is

$$(3-1) \quad T_{\mathcal{A}}(x, y) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (x-1)^{r-r(\mathcal{B})} (y-1)^{|\mathcal{B}|-r(\mathcal{B})},$$

where the sum is over all central subsets  $\mathcal{B} \subseteq \mathcal{A}$ .

If  $\mathcal{A}$  is central and  $M(\mathcal{A})$  is its associated matroid, this definition coincides with the usual definition of the Tutte polynomial of the matroid  $M(\mathcal{A})$ .

It will be useful for us to consider a simple transformation of the Tutte polynomial, first considered by Crapo [1969] in the context of matroids.

**Definition 3.2.** The *coboundary polynomial*  $\overline{\chi}_{\mathcal{A}}(q, t)$  of an arrangement  $\mathcal{A}$  is

$$(3-2) \quad \overline{\chi}_{\mathcal{A}}(q, t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} q^{r-r(\mathcal{B})} (t-1)^{|\mathcal{B}|}.$$

It is easy to check that

$$\overline{\chi}_{\mathcal{A}}(q, t) = (t-1)^r T_{\mathcal{A}} \left( \frac{q+t-1}{t-1}, t \right)$$

and

$$T_{\mathcal{A}}(x, y) = \frac{1}{(y-1)^r} \overline{\chi}_{\mathcal{A}}((x-1)(y-1), y).$$

Therefore, computing the coboundary polynomial of an arrangement is essentially equivalent to computing its Tutte polynomial. The results in this paper can be presented more elegantly in terms of the coboundary polynomial.

Also, recall Whitney's theorem [Stanley 2004, Theorem 2.4], which states that

$$\chi_{\mathcal{A}}(q) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (-1)^{|\mathcal{B}|} q^{n-r(\mathcal{B})}.$$

This allows us to express the characteristic polynomial in terms of the coboundary polynomial:

$$\chi_{\mathcal{A}}(q) = q^{n-r} \overline{\chi}_{\mathcal{A}}(q, 0).$$

**The finite field method.** Let  $\mathcal{A}$  be a  $\mathbb{Z}$ -arrangement in  $\mathbb{R}^n$ , and let  $q$  be a prime power. The arrangement  $\mathcal{A}$  induces an arrangement  $\mathcal{A}_q$  in the vector space  $\mathbb{F}_q^n$ . If we consider the equations defining the hyperplanes of  $\mathcal{A}$ , and regard them as equations over  $\mathbb{F}_q$ , they define the hyperplanes of  $\mathcal{A}_q$ .

Say that  $\mathcal{A}$  *reduces correctly* over  $\mathbb{F}_q$  if the arrangements  $\mathcal{A}$  and  $\mathcal{A}_q$  are isomorphic. This does not always happen; sometimes the hyperplanes of  $\mathcal{A}$  do not even become hyperplanes in  $\mathcal{A}_q$ . For example, the hyperplane  $2x + 2y = 1$  in  $\mathbb{R}^2$  becomes the empty "hyperplane"  $0 = 1$  in  $\mathbb{F}_2^2$ . Sometimes independence is not preserved. For example, the independent hyperplanes  $2x + y = 0$  and  $y = 0$  in  $\mathbb{R}^2$  become the same hyperplane in  $\mathbb{F}_2^2$ .

However, if  $q$  is a power of a large enough prime,  $\mathcal{A}$  will reduce correctly over  $\mathbb{F}_q$ . To have  $\mathcal{A} \cong \mathcal{A}_q$ , we need central and independent subarrangements to be preserved. Cramer's rule lets us rephrase these conditions, in terms of certain determinants (formed by the coefficients of the hyperplanes in  $\mathcal{A}$ ) being zero or nonzero. If  $q$  is a power of a prime which is larger than all these determinants, we will guarantee that  $\mathcal{A}$  reduces correctly over  $\mathbb{F}_q$ .

**Theorem 3.3.** *Let  $\mathcal{A}$  be a  $\mathbb{Z}$ -arrangement in  $\mathbb{R}^n$ . Let  $q$  be a power of a large enough prime, and let  $\mathcal{A}_q$  be the induced arrangement in  $\mathbb{F}_q^n$ . Then*

$$(3-3) \quad q^{n-r} \bar{\chi}_{\mathcal{A}}(q, t) = \sum_{p \in \mathbb{F}_q^n} t^{h(p)}$$

where  $h(p)$  denotes the number of hyperplanes of  $\mathcal{A}_q$  that  $p$  lies on.

*Proof.* Let  $q$  be a power of a large enough prime, so that  $\mathcal{A}$  reduces correctly over  $\mathbb{F}_q$ . For each  $\mathcal{B} \subseteq \mathcal{A}$ , let  $\mathcal{B}_q$  be the subarrangement of  $\mathcal{A}_q$  induced by it. For each  $p \in \mathbb{F}_q^n$ , let  $H(p)$  be the set of hyperplanes of  $\mathcal{A}_q$  that  $p$  lies on. From (3-2) we have

$$\begin{aligned} q^{n-r} \bar{\chi}_{\mathcal{A}}(q, t) &= \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} q^{n-r(\mathcal{B})} (t-1)^{|\mathcal{B}|} = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} q^{\dim \cap \mathcal{B}} (t-1)^{|\mathcal{B}|} \\ &= \sum_{\substack{\mathcal{B}_q \subseteq \mathcal{A}_q \\ \text{central}}} |\cap \mathcal{B}_q| (t-1)^{|\mathcal{B}_q|} = \sum_{\substack{\mathcal{B}_q \subseteq \mathcal{A}_q \\ \text{central}}} \sum_{p \in \cap \mathcal{B}_q} (t-1)^{|\mathcal{B}_q|} \\ &= \sum_{p \in \mathbb{F}_q^n} \sum_{\mathcal{B}_q \subseteq H(p)} (t-1)^{|\mathcal{B}_q|} = \sum_{p \in \mathbb{F}_q^n} (1 + (t-1))^{h(p)}, \end{aligned}$$

as desired. □

In principle, [Theorem 3.3](#) only computes  $\bar{\chi}_{\mathcal{A}}(q, t)$  when  $q$  is a power of a large enough prime. In practice, however, when we compute the right-hand side of (3-3) for large prime powers  $q$ , we will get a polynomial function in  $q$  and  $t$ . Since the left-hand side is also a polynomial, these two polynomials must be equal.

[Theorem 3.3](#) reduces the computation of coboundary polynomials (and hence Tutte polynomials) to enumerating points in the finite vector space  $\mathbb{F}_q^n$ , according to a certain statistic. This method can be extremely useful when the hyperplanes of the arrangement are defined by simple equations. We illustrate this in [Section 4](#).

We remark that [Theorem 3.3](#) was also obtained by Welsh and Whittle [1999, Theorem 7.1]. Also, since the characteristic polynomial of  $\mathcal{A}$  is given by  $\chi_{\mathcal{A}}(q) = q^{n-r} \bar{\chi}_{\mathcal{A}}(q, 0)$ , the special case  $t = 0$  is the following result:

**Theorem 3.4** [[Athanasiadis 1996](#), Theorem 2.2]. *If  $\mathcal{A}$  and  $q$  are as in [Theorem 3.3](#), then  $\chi_{\mathcal{A}}(q)$  is the number of points in  $\mathbb{F}_q^n$  which are not on any of the hyperplanes of  $\mathcal{A}_q$ ; that is,*

$$\chi_{\mathcal{A}}(q) = |\mathbb{F}_q^n - \mathcal{A}_q|.$$

**Special cases and applications.** We now show how the finite field method of [Theorem 3.3](#) can be used to give straightforward proofs of four known facts ([Theorems 3.5, 3.6, 3.7, 3.8](#)) and an apparently new result ([Theorem 3.9](#)) on Tutte polynomials.

*Colorings of graphs.* Given a graph  $G$  on  $[n]$ , we associate to it an arrangement  $\mathcal{A}_G$  in  $\mathbb{R}^n$ . It consists of the hyperplanes  $x_i = x_j$ , for all  $1 \leq i < j \leq n$  such that  $ij$  is an edge in the graph  $G$ . It is easy to see that the matroid of  $\mathcal{A}_G$  is the cycle matroid of  $G$ , so  $T_{\mathcal{A}_G}(x, y)$  coincides with the (graph-theoretic) Tutte polynomial  $T_G(x, y)$ . We can define the coboundary polynomial for a graph as we did for arrangements, and then  $\bar{\chi}_G(q, t) = \bar{\chi}_{\mathcal{A}_G}(q, t)$  also.

We shall now interpret [Theorem 3.3](#) in this framework. It is easy to see that the rank of  $\mathcal{A}_G$  is equal to  $n - c$ , where  $c$  is the number of connected components of  $G$ . Therefore the left-hand side of (3-3) is  $q^c \bar{\chi}_G(q, t)$  in this case.

To interpret the right-hand side, notice that each point  $p \in \mathbb{F}_q^n$  corresponds to a  $q$ -coloring of the vertices of  $G$ . The point  $p = (p_1, \dots, p_n)$  will correspond to the coloring  $\kappa_p$  of  $G$  which assigns color  $p_i$  to vertex  $i$ . A hyperplane  $x_i = x_j$  contains  $p$  when  $p_i = p_j$ . This happens precisely when edge  $ij$  is *monochromatic* in  $\kappa_p$ ; that is, when its two ends have the same color. Therefore, applying [Theorem 3.3](#) to the arrangement  $\mathcal{A}_G$ , we recover the following known result:

**Theorem 3.5** [[Brylawski and Oxley 1992](#), Proposition 6.3.26]. *Let  $G$  be a graph with  $c$  connected components. Then*

$$q^c \bar{\chi}_G(q, t) = \sum_{\substack{q\text{-colorings} \\ \kappa \text{ of } G}} t^{\text{mono}(\kappa)},$$

where  $\text{mono}(\kappa)$  is the number of monochromatic edges in  $\kappa$ .

*Linear codes.* Given positive integers  $n \geq r$ , an  $[n, r]$  linear code  $C$  over  $\mathbb{F}_q$  is an  $r$ -dimensional subspace of  $\mathbb{F}_q^n$ . A generator matrix for  $C$  is an  $r \times n$  matrix  $U$  over  $\mathbb{F}_q$ , the rows of which form a basis for  $C$ . It is not difficult to see that the isomorphism class of the matroid on the columns of  $U$  depends only on  $C$ . We shall denote the corresponding matroid  $M_C$ .

The elements of  $C$  are called *codewords*. The weight  $w(v)$  of a codeword is the cardinality of its support; that is, the number of nonzero coordinates of  $v$ . The *codeweight polynomial* of  $C$  is

$$(3-4) \quad A(C, q, t) = \sum_{v \in C} t^{w(v)}.$$

The translation of [Theorem 3.3](#) to this setting is the following.

**Theorem 3.6** [[Greene 1976](#)]. *For any linear code  $C$  over  $\mathbb{F}_q$ ,*

$$A(C, q, t) = t^n \bar{\chi}_{M_C}\left(q, \frac{1}{t}\right).$$

*Proof.* Let  $\mathcal{A}_C$  be the central arrangement corresponding to the columns of  $U$ . (We can call it  $\mathcal{A}_C$  because, as stated above, its isomorphism class depends only on  $C$ .)



This is a rank  $r$  arrangement in  $\mathbb{F}_q^r$  such that  $\bar{\chi}_{M_C}(q, \frac{1}{t}) = \bar{\chi}_{\mathcal{A}_C}(q, \frac{1}{t})$ . Comparing (3-4) with [Theorem 3.3](#), it remains to prove that

$$\sum_{v \in C} t^{w(v)} = \sum_{p \in \mathbb{F}_q^r} t^{n-h(p)}.$$

To do this, consider the bijection  $\phi : \mathbb{F}_q^r \rightarrow C$  determined by right multiplication by  $U$ . If  $u_1, \dots, u_r$  are the row vectors of  $U$ , then  $\phi$  sends  $p = (p_1, \dots, p_r) \in \mathbb{F}_q^r$  to the codeword  $v_p = p_1 u_1 + \dots + p_r u_r \in C$ . For  $1 \leq i \leq n$ ,  $p$  lies on the hyperplane determined by the  $i$ th column of  $U$  if and only if the  $i$ th coordinate of  $v_p$  is equal to zero. Therefore  $h(p) = n - w(v_p)$ . This completes the proof.  $\square$

*Deletion-contraction.* The point of view of [Theorem 3.3](#) can be used to give a simple enumerative proof of the deletion-contraction formula for the Tutte polynomial of an arrangement. Once again, this formula is more natural in the context of semi-matroids, as shown in [[Ardila 2004](#)]. For the moment, leaving matroid-theoretical issues aside, we only wish to present a special case of it as an application.

**Theorem 3.7.** *Let  $\mathcal{A}$  be a hyperplane arrangement, and let  $H$  be a hyperplane in  $\mathcal{A}$  such that  $r_{\mathcal{A}}(\mathcal{A} - H) = r_{\mathcal{A}}$ . Then  $T_{\mathcal{A}}(x, y) = T_{\mathcal{A}-H}(x, y) + T_{\mathcal{A}/H}(x, y)$ .*

*Proof.* Because there will be several arrangements involved, let  $h(\mathcal{B}, p)$  denote the number of hyperplanes in  $\mathcal{B}_q$  that  $p$  lies on. Then

$$\begin{aligned} q^{n-r} \bar{\chi}_{\mathcal{A}}(q, t) &= \sum_{p \in \mathbb{F}_q^n} t^{h(\mathcal{A}, p)} = \sum_{p \in \mathbb{F}_q^n - H} t^{h(\mathcal{A}, p)} + \sum_{p \in H} t^{h(\mathcal{A}, p)} \\ &= \sum_{p \in \mathbb{F}_q^n - H} t^{h(\mathcal{A}-H, p)} + \sum_{p \in H} t^{h(\mathcal{A}-H, p)+1} \\ &= \sum_{p \in \mathbb{F}_q^n} t^{h(\mathcal{A}-H, p)} + (t-1) \sum_{p \in H} t^{h(\mathcal{A}-H, p)} \\ &= q^{n-r} \bar{\chi}_{\mathcal{A}-H}(q, t) + (t-1) q^{(n-1)-(r-1)} \bar{\chi}_{\mathcal{A}/H}(q, t). \end{aligned}$$

We conclude that  $\bar{\chi}_{\mathcal{A}}(q, t) = \bar{\chi}_{\mathcal{A}-H}(q, t) + (t-1) \bar{\chi}_{\mathcal{A}/H}(q, t)$ , which is equivalent to the deletion-contraction formula for Tutte polynomials.  $\square$

*A Möbius formula.* The finite field method, when combined with the Möbius inversion formula for posets, naturally gives an alternative formula for the coboundary polynomial. This formula, in the context of matroids, is due to Crapo [[1968](#)].

**Theorem 3.8.** *For an arrangement  $\mathcal{A}$  and an affine subspace  $x$  in the intersection poset  $L_{\mathcal{A}}$ , let  $h(x)$  be the number of hyperplanes of  $\mathcal{A}$  containing  $x$ . Then*

$$\bar{\chi}_{\mathcal{A}}(q, t) = \sum_{x \leq y \text{ in } L_{\mathcal{A}}} \mu(x, y) q^{r-r(y)} t^{h(x)}.$$

*Proof.* Consider the arrangement  $\mathcal{A}$  restricted to  $\mathbb{F}_q^n$ , where  $q$  is a power of a large enough prime, so that  $\mathcal{A}$  reduces correctly over  $\mathbb{F}_q$ . Given  $x \in L_{\mathcal{A}_q}$ , let  $P(x)$  be the set of points in  $\mathbb{F}_q^n$  which are contained in  $x$ , and are not contained in any  $y$  such that  $y > x$  in  $L_{\mathcal{A}_q}$ . Then the set  $x$  is partitioned by the sets  $P(y)$  for  $y \geq x$ , so we have

$$q^{\dim x} = |x| = \sum_{y \geq x} |P(y)|.$$

By the Möbius inversion formula [Stanley 1997, Proposition 3.7.1] we have

$$|P(x)| = \sum_{y \geq x} \mu(x, y) q^{\dim y}.$$

Now, from Theorem 3.3 we know that

$$q^{n-r} \bar{\chi}_{\mathcal{A}}(q, t) = \sum_{x \in L_{\mathcal{A}}} \sum_{p \in P(x)} t^{h(p)} = \sum_{x \in L_{\mathcal{A}}} |P(x)| t^{h(x)} = \sum_{x \leq y \text{ in } L_{\mathcal{A}}} \mu(x, y) q^{n-r(y)} t^{h(x)},$$

as desired.  $\square$

*A probabilistic interpretation.* In matroid reliability and percolation problems, one starts with a fixed matroid  $M$ . Each element of the ground set of  $M$  has a certain probability of being deleted, independently of the other elements. One then asks for the probability that the retained elements satisfy a certain property. See [Brylawski and Oxley 1992, Section 6.3.E] for more on this subject. The following theorem is similar in spirit to these results, and it may be applied to the analogous questions concerning hyperplane arrangements.

**Theorem 3.9.** *Let  $\mathcal{A}$  be an arrangement and let  $0 \leq t \leq 1$  be a real number. Let  $\mathcal{B}$  be a random subarrangement of  $\mathcal{A}$ , obtained by independently removing each hyperplane from  $\mathcal{A}$  with probability  $t$ . Then the expected characteristic polynomial  $\chi_{\mathcal{B}}(q)$  of  $\mathcal{B}$  is  $q^{n-r} \bar{\chi}_{\mathcal{A}}(q, t)$ .*

*Proof.* We have

$$\begin{aligned} E[\chi_{\mathcal{B}}(q)] &= \sum_{\mathcal{C} \subseteq \mathcal{A}} Pr[\mathcal{B} = \mathcal{C}] \chi_{\mathcal{C}}(q) = \sum_{\mathcal{C} \subseteq \mathcal{A}} Pr[\mathcal{B} = \mathcal{C}] |\mathbb{F}_q^n - \mathcal{C}_q| \\ &= \sum_{p \in \mathbb{F}_q^n} \sum_{\substack{\mathcal{C} \subseteq \mathcal{A} \\ p \notin \mathcal{C}_q}} Pr[\mathcal{B} = \mathcal{C}], \end{aligned}$$

where in the second step we have used Theorem 3.4.

Recall that  $H(p)$  denotes the set of hyperplanes in  $\mathcal{A}_q$  containing  $p$ . Then

$$E[\chi_{\mathcal{B}}(q)] = \sum_{p \in \mathbb{F}_q^n} Pr[\mathcal{B}_q \cap H(p) = \emptyset] = \sum_{p \in \mathbb{F}_q^n} t^{h(p)},$$

which is precisely what we wanted to show.  $\square$

#### 4. Computing coboundary polynomials

In this section we use [Theorem 3.3](#) to compute the coboundary polynomials of several families of arrangements. As remarked on page 4, this is essentially the same as computing their Tutte polynomials.

**Coxeter arrangements.** To illustrate how our finite field method works, we start by presenting some simple examples.

Let  $\Phi$  be an irreducible crystallographic root system in  $\mathbb{R}^n$ , and let  $W$  be its associated Weyl group. The Coxeter arrangement of type  $W$  consists of the hyperplanes  $(\alpha, x) = 0$  for each  $\alpha \in \Phi^+$ , with the standard inner product on  $\mathbb{R}^n$ . See [[Björner and Brenti 2005](#)] or [[Humphreys 1990](#)] for an introduction to root systems and Weyl groups, and [[Björner et al. 1993](#), Section 2.3] or [[Orlik and Terao 1992](#), Chapter 6] for more information on Coxeter arrangements.

In this section we compute the coboundary polynomials of the Coxeter arrangements of type  $A_n$ ,  $B_n$  and  $D_n$ . (The arrangement of type  $C_n$  is the same as the arrangement of type  $B_n$ .) The best way to state our results is to compute the exponential generating function for the coboundary polynomials of each family.

The following three theorems have never been stated explicitly in the literature in this form. [Theorem 4.1](#) is equivalent to a result of Tutte [[1954](#)], who computed the Tutte polynomial of the complete graph. It is also an immediate consequence of a more general theorem of Stanley [[1998a](#), (15)]. [Theorems 4.2](#) and [4.3](#) are implicit in the work of Zaslavsky [[1995](#)].

**Theorem 4.1.** *Let  $\mathcal{A}_n$  be the Coxeter arrangement (known as the braid arrangement) of type  $A_{n-1}$  in  $\mathbb{R}^n$ , consisting of the hyperplanes  $x_i = x_j$  for  $1 \leq i < j \leq n$ . We have*

$$1 + q \sum_{n \geq 1} \bar{\chi}_{\mathcal{A}_n}(q, t) \frac{x^n}{n!} = \left( \sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!} \right)^q.$$

*Proof.* For  $n \geq 1$  we have  $q \bar{\chi}_{\mathcal{A}_n}(q, t) = \sum_{p \in \mathbb{F}_q^n} t^{h(p)}$  for all powers of a large enough prime  $q$ , according to [Theorem 3.3](#). For each  $p \in \mathbb{F}_q^n$ , if we let  $A_k = \{i \in [n] \mid p_i = k\}$  for  $0 \leq k \leq q-1$ , then  $h(p) = \binom{|A_0|}{2} + \dots + \binom{|A_{q-1}|}{2}$ . Thus

$$q \bar{\chi}_{\mathcal{A}_n}(q, t) = \sum_{A_0 \cup \dots \cup A_{q-1} = [n]} t^{\binom{|A_0|}{2} + \dots + \binom{|A_{q-1}|}{2}}$$

summing over all weak ordered  $q$ -partitions of  $[n]$ ; that is, ordered lists of  $q$  pairwise disjoint, possibly empty sets whose union is  $[n]$ . The compositional formula for exponential generating functions [[Bergeron et al. 1998](#); [Stanley 1999](#), Proposition 5.1.3] implies the desired result.  $\square$

**Theorem 4.2.** *Let  $\mathcal{B}_n$  be the Coxeter arrangement of type  $B_n$  in  $\mathbb{R}^n$ , consisting of the hyperplanes  $x_i = x_j$  and  $x_i + x_j = 0$  for  $1 \leq i < j \leq n$ , and the hyperplanes  $x_i = 0$  for  $1 \leq i \leq n$ . We have*

$$\sum_{n \geq 0} \bar{\chi}_{\mathcal{B}_n}(q, t) \frac{x^n}{n!} = \left( \sum_{n \geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!} \right)^{\frac{q-1}{2}} \left( \sum_{n \geq 0} t^{n^2} \frac{x^n}{n!} \right).$$

*Proof.* Let  $q$  be a power of a large enough prime, and let  $s = \frac{q-1}{2}$ . Now for each  $p \in \mathbb{F}_q^n$ , if we let  $B_k = \{i \in [n] \mid p_i = k \text{ or } p_i = q - k\}$  for  $0 \leq k \leq s$ , we have that  $h(p) = |B_0|^2 + \binom{|B_1|}{2} + \cdots + \binom{|B_s|}{2}$ . Also, given a weak ordered partition  $(B_0, \dots, B_s)$  of  $[n]$ , there are  $2^{|B_1| + \cdots + |B_s|}$  points of  $p$  which correspond to it: for each  $i \in B_k$  with  $k \neq 0$ , we get to choose whether  $p_i$  is equal to  $k$  or to  $q - k$ . Therefore

$$q \bar{\chi}_{\mathcal{B}_n}(q, t) = \sum_{B_0 \cup \cdots \cup B_s = [n]} t^{|B_0|^2} \left( 2^{|B_1|} t^{\binom{|B_1|}{2}} \right) \cdots \left( 2^{|B_s|} t^{\binom{|B_s|}{2}} \right),$$

and the compositional formula for exponential generating functions implies the theorem.  $\square$

The proof of the next theorem is very similar to that of [Theorem 4.2](#).

**Theorem 4.3.** *Let  $\mathcal{D}_n$  be the Coxeter arrangement of type  $D_n$  in  $\mathbb{R}^n$ , consisting of the hyperplanes  $x_i = x_j$  and  $x_i + x_j = 0$  for  $1 \leq i < j \leq n$ . We have*

$$\sum_{n \geq 0} \bar{\chi}_{\mathcal{D}_n}(q, t) \frac{x^n}{n!} = \left( \sum_{n \geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!} \right)^{\frac{q-1}{2}} \left( \sum_{n \geq 0} t^{n(n-1)} \frac{x^n}{n!} \right).$$

Setting  $t = 0$  in [Theorems 4.1](#), [4.2](#) and [4.3](#), it is easy to recover the formulas for the characteristic polynomials of the above arrangements:

$$\begin{aligned} \chi_{\mathcal{A}_n}(q) &= q(q-1)(q-2) \cdots (q-n+1), \\ \chi_{\mathcal{B}_n}(q) &= (q-1)(q-3) \cdots (q-2n+1), \\ \chi_{\mathcal{D}_n}(q) &= (q-1)(q-3) \cdots (q-2n+3)(q-n+1), \end{aligned}$$

which are well known; see for example [\[Stanley 2004\]](#).

### Two more examples.

**Theorem 4.4.** *Let  $\mathcal{A}_n^\#$  be a generic deformation of the arrangement  $\mathcal{A}_n$ , consisting of the hyperplanes  $x_i - x_j = a_{ij}$  ( $1 \leq i < j \leq n$ ), where the  $a_{ij}$  are generic real*

numbers<sup>1</sup>. For  $n \geq 1$ ,

$$q \bar{\chi}_{\mathcal{A}_n^\#}(q, t) = \sum_{\substack{F \text{ forest} \\ \text{on } [n]}} q^{n-e(F)} (t-1)^{e(F)},$$

where  $e(F)$  denotes the number of edges of  $F$ . Also,

$$1 + q \sum_{n \geq 1} \bar{\chi}_{\mathcal{A}_n^\#}(q, t) \frac{x^n}{n!} = \exp\left(\frac{q}{t-1} \sum_{n \geq 0} n^{n-2} \frac{x^n (t-1)^n}{n!}\right).$$

*Proof.* It is possible to prove [Theorem 4.4](#) using our finite field method, as we did in the previous section. However, it will be easier to proceed directly from (3-2), the definition of the coboundary polynomial.

To each subarrangement  $\mathcal{B}$  of  $\mathcal{A}_n^\#$  we can assign a graph  $G_{\mathcal{B}}$  on the vertex set  $[n]$ , by letting edge  $ij$  be in  $G_{\mathcal{B}}$  if and only if the hyperplane  $x_i - x_j = a_{ij}$  is in  $\mathcal{B}$ . Since the  $a_{ij}$ s are generic, the subarrangement  $\mathcal{B}$  is central if and only if the corresponding graph  $G_{\mathcal{B}}$  is a forest. For such a  $\mathcal{B}$ , it is clear that  $|\mathcal{B}| = r(\mathcal{B}) = e(G_{\mathcal{B}})$ . Hence,

$$\bar{\chi}_{\mathcal{A}_n^\#}(q, t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A}_n^\# \\ \text{central}}} q^{r-r(\mathcal{B})} (t-1)^{|\mathcal{B}|} = \sum_{\substack{F \text{ forest} \\ \text{on } [n]}} q^{(n-1)-e(F)} (t-1)^{e(F)},$$

proving the first claim. Now let  $c(F) = n - e(F)$  be the number of connected components of  $F$ . We have

$$\begin{aligned} 1 + q \sum_{n \geq 1} \bar{\chi}_{\mathcal{A}_n^\#}(q, t) \frac{x^n}{n!} &= \sum_{n \geq 0} \sum_{\substack{F \text{ forest} \\ \text{on } [n]}} \left(\frac{q}{t-1}\right)^{c(F)} \frac{x^n (t-1)^n}{n!} \\ &= \exp\left(\frac{q}{t-1} \sum_{n \geq 0} n^{n-2} \frac{x^n (t-1)^n}{n!}\right), \end{aligned}$$

using the exponential formula for exponential generating functions, and the fact that there are  $n^{n-2}$  trees on  $n$  labeled vertices [[Stanley 1999](#)].  $\square$

**Theorem 4.5.** *The threshold arrangement  $\mathcal{T}_n$  in  $\mathbb{R}^n$  consists of the hyperplanes  $x_i + x_j = 0$ , for  $1 \leq i < j \leq n$ . For all  $n \geq 0$  we have*

$$\bar{\chi}_{\mathcal{T}_n}(q, t) = \sum_{\substack{G \text{ graph} \\ \text{on } [n]}} q^{bc(G)} (t-1)^{e(G)},$$

<sup>1</sup>The  $a_{ij}$  are “generic” if no  $n$  of the hyperplanes have a nonempty intersection, and any nonempty intersection of  $k$  hyperplanes has rank  $k$ . This can be achieved, for example, by requiring that the  $a_{ij}$ s are linearly independent over the rational numbers. Almost all choices of the  $a_{ij}$ s are generic.

where  $bc(G)$  is the number of connected components of  $G$  which are bipartite, and  $e(G)$  is the number of edges of  $G$ . Also,

$$\sum_{n \geq 0} \bar{\chi}_{\mathcal{T}_n}(q, t) \frac{x^n}{n!} = \left( \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} t^{k(n-k)} \frac{x^n}{n!} \right)^{\frac{q-1}{2}} \left( \sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!} \right).$$

*Proof.* Once again, the proof of the first claim is easier using the definition of the coboundary polynomial. Every subarrangement  $\mathcal{B}$  of  $\mathcal{T}_n$  is central, and we can assign to it a graph  $G_{\mathcal{B}}$  as in the proof of [Theorem 4.4](#). In view of (3-2), we only need to check that  $r(\mathcal{B}) = n - bc(G_{\mathcal{B}})$  and  $|\mathcal{B}| = e(G_{\mathcal{B}})$ . The second claim is trivial. To prove the first one, we show that  $\dim(\cap \mathcal{B}) = bc(G_{\mathcal{B}})$ .

Consider a point  $p$  in  $\cap \mathcal{B}$ . We know that, if  $ab$  is an edge in  $G_{\mathcal{B}}$ , then  $p_a = -p_b$ . If vertex  $i$  is in a connected component  $C$  of  $G_{\mathcal{B}}$ , then the value of  $p_i$  determines the value of  $p_j$  for all  $j$  in  $C$ :  $p_j = p_i$  if there is a path of even length between  $i$  and  $j$ , and  $p_j = -p_i$  if there is a path of odd length between  $i$  and  $j$ . If  $C$  is bipartite, this determines the values of the  $p_j$ s consistently. If  $C$  is not bipartite, take a cycle of odd length and a vertex  $k$  in it. We get that  $p_k = -p_k$ , so  $p_k = 0$ ; therefore we must have  $p_j = 0$  for all  $j \in C$ .

Therefore, to specify a point  $p$  in  $\cap \mathcal{B}$ , we split  $G_{\mathcal{B}}$  into its connected components. We know that  $p_i = 0$  for all  $i$  in connected components which are not bipartite. To determine the remaining coordinates of  $p$  we have to specify the value of  $p_j$  for exactly one  $j$  in each bipartite connected component. Therefore  $\dim(\cap \mathcal{B}) = bc(G_{\mathcal{B}})$ , as desired.

From this point, it is possible to prove the second claim of [Theorem 4.5](#) using the compositional formula for exponential generating functions, in the same way that we proved [Theorem 4.4](#). However, the work involved is considerable, and it is much simpler to use our finite field method, [Theorem 3.3](#), in this case. The proof that we obtain is very similar to the proofs of [Theorems 4.1, 4.2 and 4.3](#), so we omit the details. □

### 5. Deformations of the braid arrangement

This section concerns the deformations of the braid arrangement of the form

$$(5-1) \quad x_i - x_j = a_1, \dots, a_k \quad 1 \leq i < j \leq n,$$

where  $A = \{a_1, \dots, a_k\}$  is a fixed set of integers. Such arrangements have been studied extensively by Athanasiadis [[2000](#)] and Postnikov and Stanley [[2000](#)]. In this section we study the problem of finding their coboundary polynomials.

We proceed as follows. In the next subsection we introduce the family of graded  $A$ -graphs, and show in [Proposition 5.6](#) that enumerating them is equivalent to the

problem at hand. By understanding the structure of those graphs, we obtain [Theorem 5.7](#), a formula for the generating function of our coboundary polynomials.

The formula provided by [Theorem 5.7](#) is not very explicit, as one might expect from the fact that it applies to such a large family of arrangements. However, we show that for  $A \subseteq \{-1, 0, 1\}$ ,  $A$ -graphs possess additional structure, which makes it possible to obtain very explicit answers. This is done starting on page [19](#) for the Linal, Shi, semiorder, and Catalan arrangements. As a consequence, we also obtain new formulas for the generating functions of alternating trees, labelled trees, semiorders, and Dyck paths.

### *Enumerating graphs to compute coboundary polynomials.*

**Definition 5.1.** An *exponential sequence of arrangements*  $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \dots)$  is a sequence of arrangements satisfying the following properties:

- (1)  $\mathcal{E}_n$  is an arrangement in  $\mathbb{k}^n$ , for some fixed field  $\mathbb{k}$ .
- (2) Every hyperplane in  $\mathcal{E}_n$  is parallel to some hyperplane in the braid arrangement  $\mathcal{A}_n$ .
- (3) For any subset  $S$  of  $[n]$ , the subarrangement  $\mathcal{E}_n^S \subseteq \mathcal{E}_n$ , which consists of the hyperplanes in  $\mathcal{E}_n$  of the form  $x_i - x_j = c$  with  $i, j \in S$ , is isomorphic to the arrangement  $\mathcal{E}_{|S|}$ .

The special case  $t = 0$  of the next result is due to Stanley [[1996](#), Theorem 1.2]; we omit the proof, which is an easy extension of his.

**Theorem 5.2.** Let  $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \dots)$  be an exponential sequence of arrangements. Then

$$1 + q \sum_{n \geq 1} \bar{\chi}_{\mathcal{E}_n}(q, t) \frac{x^n}{n!} = \left( \sum_{n \geq 0} \bar{\chi}_{\mathcal{E}_n}(1, t) \frac{x^n}{n!} \right)^q.$$

The most natural examples of exponential sequences of arrangements are the following. Fix a set  $A$  of  $k$  distinct integers  $a_1 < \dots < a_k$ . Let  $\mathcal{E}_n$  be the arrangement in  $\mathbb{R}^n$  consisting of the hyperplanes

$$(5-2) \quad x_i - x_j = a_1, \dots, a_k \quad 1 \leq i < j \leq n.$$

Then  $(\mathcal{E}_0, \mathcal{E}_1, \dots)$  is an exponential sequence of arrangements and [Theorem 5.2](#) applies to this case. In fact, we can say much more about this type of arrangement.

After proving the results in this section, we found out that Postnikov and Stanley [[2000](#)] had used similar techniques in computing the characteristic polynomials of these types of arrangements. Therefore, for consistency, we will use the terminology that they introduced.

**Definition 5.3.** A *graded graph* is a triple  $G = (V_G, E_G, h_G)$ , where  $V_G$  is a linearly ordered set of vertices (usually  $V_G = [n]$ ),  $E_G$  is a set of undirected edges, and  $h_G$  is a function  $h_G : V \rightarrow \mathbb{N}$ , called a *grading*.

We will drop the subscripts when the underlying graded graph is clear. We will refer to  $h(v)$  as the *height* of vertex  $v$ . The *height* of  $G$ , denoted  $h(G)$ , is the largest height of a vertex of  $G$ .

**Definition 5.4.** Let  $G$  be a graded graph and  $r$  be a nonnegative integer. Let the  $r$ th *level* of  $G$  be the set of vertices  $v$  such that  $h(v) = r$ . Say  $G$  is *planted* if each one of its connected components has a vertex on the 0th level.

**Definition 5.5.** If  $u < v$  are connected by edge  $e$  in a graded graph  $G$ , let the *type* of  $e$  be  $s(e) = h(u) - h(v)$ . Say  $G$  is an *A-graph* if the types of all edges of  $G$  are in  $A = \{a_1, \dots, a_k\}$ .

Recall that, for a graph  $G$ , we let  $e(G)$  be the number of edges and  $c(G)$  be the number of connected components of  $G$ .

**Proposition 5.6.** Let  $A = \{a_1, \dots, a_k\}$ , and let  $\mathcal{E}_n$  be the arrangement

$$x_i - x_j = a_1, \dots, a_k \quad 1 \leq i < j \leq n.$$

Then, for  $n \geq 1$ ,

$$q \bar{\chi}_{\mathcal{E}_n}(q, t) = \sum_G q^{e(G)} (t-1)^{c(G)},$$

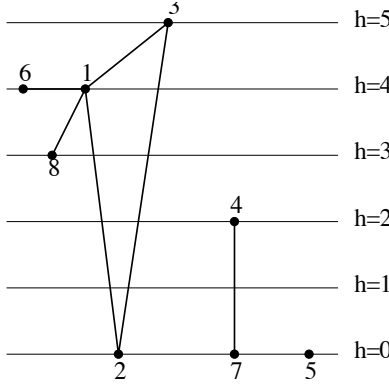
where the sum is over all planted graded  $A$ -graphs on  $[n]$ .

*Proof.* We associate to each planted graded  $A$ -graph  $G = (V, E, h)$  on  $[n]$  a central subarrangement  $\mathcal{A}_G$  of  $\mathcal{E}_n$ . It consists of the hyperplanes  $x_i - x_j = h(i) - h(j)$ , for each  $i < j$  such that  $ij$  is an edge in  $G$ . This is a subarrangement of  $\mathcal{E}_n$  because  $h(i) - h(j)$ , the type of edge  $ij$ , is in  $A$ . It is central because the point  $(h(1), \dots, h(n)) \in \mathbb{R}^n$  belongs to all these hyperplanes.

*Example.* Consider an arrangement  $\mathcal{E}_8$  in  $\mathbb{R}^8$ , with a subarrangement consisting of the hyperplanes  $x_1 - x_2 = 4$ ,  $x_1 - x_3 = -1$ ,  $x_1 - x_6 = 0$ ,  $x_1 - x_8 = 1$ ,  $x_2 - x_3 = -5$  and  $x_4 - x_7 = 2$ . [Figure 1](#) shows the planted graded  $A$ -graph corresponding to this subarrangement.

This is in fact a bijection between planted graded  $A$ -graphs on  $[n]$  and central subarrangements of  $\mathcal{E}_n$ . To see this, take a central subarrangement  $\mathcal{A}$ . We will recover the planted graded  $A$ -graph  $G$  that it came from. For each pair  $(i, j)$  with  $1 \leq i < j \leq n$ ,  $\mathcal{A}$  can have at most one hyperplane of the form  $x_i - x_j = a_t$ . If this hyperplane is in  $\mathcal{A}$ , we must put edge  $ij$  in  $G$ , and demand that the heights  $h(i)$  and  $h(j)$  satisfy  $h(i) - h(j) = a_t$ . When we do this for all the hyperplanes in  $\mathcal{A}$ , the height requirements that we introduce are consistent, because  $\mathcal{A}$  is central.





**Figure 1.** The planted graded  $A$ -graph for the subarrangement of  $\mathcal{E}_8$  in the example from the proof of Proposition 5.6.

However, these requirements do not fully determine the heights of the vertices; they only determine the relative heights within each connected component of  $G$ . Since we want  $G$  to be planted, we demand that the vertices with the lowest height in each connected component of  $G$  should have height 0. This does determine  $G$  completely, and clearly  $\mathcal{A} = \mathcal{A}_G$ .

With this bijection in hand, and keeping (3-2) in mind, it remains to show that  $r(\mathcal{A}_G) = n - c(G)$  and  $|\mathcal{A}_G| = e(G)$ . The second of these claims is trivial. We omit the proof of the first one which is very similar to, and simpler than, that of  $r(\mathcal{B}) = n - bc(G_{\mathcal{B}})$  in our proof of Theorem 4.5.  $\square$

**Theorem 5.7.** Let  $A = \{a_1, \dots, a_n\}$  and let  $\mathcal{E}_n$  be the arrangement

$$x_i - x_j = a_1, \dots, a_k \quad 1 \leq i < j \leq n.$$

Let

$$(5-3) \quad A_r(t, x) = \sum_{n \geq 0} \left( \sum_{f: [n] \rightarrow [r]} t^{a(f)} \right) \frac{x^n}{n!},$$

where the inside sum is over all functions  $f : [n] \rightarrow [r]$ , and

$$a(f) = \#\{(i, j) \mid 1 \leq i < j \leq n, f(i) - f(j) \in A\}.$$

Then

$$(5-4) \quad 1 + q \sum_{n \geq 1} \bar{\chi}_{\mathcal{E}_n}(q, t) \frac{x^n}{n!} = \left( \lim_{r \rightarrow \infty} \frac{A_r(t, x)}{A_{r-1}(t, x)} \right)^q.$$

**Remark.** The limit in (5-4) is a limit in the sense of convergence of formal power series. Let  $F_1(t, x), F_2(t, x), \dots$  be a sequence of formal power series. We say that  $\lim_{n \rightarrow \infty} F_n(t, x) = F(t, x)$  if for all  $a$  and  $b$  there exists a constant  $N(a, b)$  such that, for all  $n$  larger than  $N(a, b)$ , the coefficient of  $t^a x^b$  in  $F_n(t, x)$  is equal

to the coefficient of  $t^a x^b$  in  $F(t, x)$ . For more on this notion of convergence, see [Niven 1969] or [Stanley 1997, Section 1.1].

*Proof of Theorem 5.7.* Let  $v(G)$  be the number of vertices of graph  $G$ . First we prove that

$$(5-5) \quad A_r(t, x) = \sum_G (t-1)^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all graded  $A$ -graphs  $G$  on  $[n]$  of height less than  $r$ . The coefficient of  $\frac{x^n}{n!}$  in the right-hand side of (5-5) is  $\sum_G (t-1)^{e(G)}$ , summing over all graded  $A$ -graphs  $G$  on  $[n]$  with height less than  $r$ . We have

$$\begin{aligned} \sum_G (t-1)^{e(G)} &= \sum_{h:[n] \rightarrow [0, r-1]} \sum_{G: h_G=h} (t-1)^{e(G)} \\ &= \sum_{h:[n] \rightarrow [0, r-1]} (1 + (t-1))^{a(h)} = \sum_{f:[n] \rightarrow [r]} t^{a(f)}. \end{aligned}$$

The only tricky step here is the second: if we want all graded  $A$ -graphs  $G$  on  $[n]$  with a specified grading  $h$ , we need to consider the possible choices of edges of the graph. Any edge  $ij$  can belong to the graph, as long as  $h(i) - h(j) \in A$ , so there are  $a(h)$  possible edges.

Equation (5-5) suggests the following definitions. Let

$$B_r(t, x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded  $A$ -graphs  $G$  of height less than  $r$ , and let

$$B(t, x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded  $A$ -graphs  $G$ .

The equation

$$(5-6) \quad 1 + q \sum_{n \geq 1} \bar{\chi}_{\mathcal{E}_n}(q, t) \frac{x^n}{n!} = B(t-1, x)^q,$$

follows from Proposition 5.6, using either Theorem 5.2 or the compositional formula for exponential generating functions.

Now we claim that  $B(t, x) = \lim_{r \rightarrow \infty} B_r(t, x)$ . In a planted graded  $A$ -graph  $G$  with  $e$  edges and  $v$  vertices, each vertex has a path of length at most  $v$  that connects it to a vertex on the 0th level. Therefore  $h(G) \leq v \cdot \max(|a_1|, \dots, |a_k|)$ , so the coefficients of  $t^e \frac{x^v}{v!}$  in  $B_r(t, x)$  and  $B(t, x)$  are equal for  $r > v \cdot \max(|a_1|, \dots, |a_k|)$ .

Then it is not difficult to show that

$$(5-7) \quad B(t-1, x) = \lim_{r \rightarrow \infty} B_r(t-1, x).$$

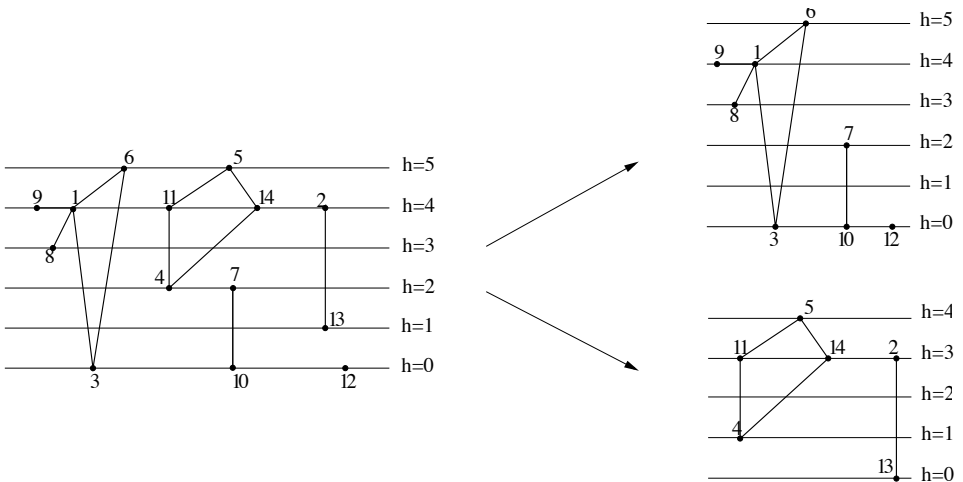
Here it is necessary to check that  $B(t-1, x)$  is, indeed, a formal power series. This follows from the observation that the coefficient of  $\frac{x^n}{n!}$  in  $B(t, x)$  is a polynomial in  $t$  of degree at most  $\binom{n}{2}$ . Once again, see [Stanley 1997, Section 1.1] for more information on these technical details.

Next, we show that

$$(5-8) \quad B_r(t-1, x) = A_r(t, x)/A_{r-1}(t, x)$$

or, equivalently, that  $A_r(t, x) = B_r(t-1, x)A_{r-1}(t, x)$ . The multiplication formula for exponential generating functions [Stanley 1999, Proposition 5.1.1] and (5-5) give us a combinatorial interpretation of this identity. We need to show that the ways of putting the structure of a graded  $A$ -graph  $G$  with  $h(G) < r$  on  $[n]$  can be put in correspondence with the ways of doing the following: first splitting  $[n]$  into two disjoint sets  $S_1$  and  $S_2$ , then putting the structure of a *planted* graded  $A$ -graph  $G_1$  with  $h(G_1) < r$  on  $S_1$ , and then putting the structure of a graded  $A$ -graph  $G_2$  with  $h(G_2) < r-1$  on  $S_2$ . We also need that, in that correspondence,  $(t-1)^{e(G)} = (t-1)^{e(G_1)}(t-1)^{e(G_2)}$ .

We do this as follows. Let  $G$  be a graded  $A$ -graph  $G$  with  $h(G) < r$ . Let  $G_1$  be the union of the connected components of  $G$  which contain a vertex on the 0th level. Put a grading on  $G_1$  by defining  $h_{G_1}(v) = h_G(v)$  for  $v \in G_1$ . Let  $G_2 = G - G_1$ . It is clear that  $h_G(v) \geq 1$  for all  $v \in G_2$ ; therefore we can put a grading on  $G_2$  by defining  $h_{G_2}(v) = h_G(v) - 1$  for  $v \in G_2$ . Then  $G_1$  is a planted



**Figure 2.** The decomposition of a graded  $A$ -graph.

graded  $A$ -graph with  $h(G_1) < r$ ,  $G_2$  is a graded  $A$ -graph with  $h(G_2) < r - 1$ , and  $(t - 1)^{e(G)} = (t - 1)^{e(G_1)}(t - 1)^{e(G_2)}$ . It is clear how to recover  $G$  from  $G_1$  and  $G_2$ . [Figure 2](#) illustrates this bijection with an example.

Now we just have to put together [\(5-6\)](#), [\(5-7\)](#) and [\(5-8\)](#) to complete the proof of [Theorem 5.7](#).  $\square$

**Subarrangements of the Catalan arrangement.** The Catalan arrangement  $C_n$  in  $\mathbb{R}^n$  consists of the hyperplanes

$$x_i - x_j = -1, 0, 1 \quad 1 \leq i < j \leq n.$$

When the arrangement in [Theorem 5.7](#) is a subarrangement of the Catalan arrangement, we can say more about the power series  $A_r$  of [\(5-3\)](#). Let

$$A(t, x, y) = \sum_r A_r(t, x) y^r = \sum_{n \geq 0} \sum_{r \geq 0} \left( \sum_{f: [n] \rightarrow [r]} t^{a(f)} \right) \frac{x^n}{n!} y^r$$

and let

$$(5-9) \quad S(t, x, y) = \sum_{n \geq 0} \sum_{r \geq 0} \left( \sum_{f: [n] \rightarrow [r]} t^{a(f)} \right) \frac{x^n}{n!} y^r$$

where the inner sum is over all *surjective* functions  $f : [n] \twoheadrightarrow [r]$ . The following proposition reduces the computation of  $A(t, x, y)$  to the computation of  $S(t, x, y)$ , which is often easier in practice.

**Proposition 5.8.** *If  $A \subseteq \{-1, 0, 1\}$  in the notation of [Theorem 5.7](#), we have*

$$A(t, x, y) = \frac{S(t, x, y)}{1 - yS(t, x, y)}$$

*Proof.* Once again, we think of this as an identity about exponential generating functions in the variable  $x$ . Fix  $n, r$ , and a function  $f : [n] \rightarrow [r]$ . Write  $[r] - \text{Im } f = \{m_1, m_1 + m_2, \dots, m_1 + \dots + m_{k-1}\}$ , so the image of  $f$  is  $M_1 \cup \dots \cup M_k = \{1, \dots, m_1 - 1\} \cup \{m_1 + 1, \dots, m_1 + m_2 - 1\} \cup \dots \cup \{m_1 + \dots + m_{k-1} + 1, \dots, m_1 + \dots + m_k - 1\}$ . Here  $m_1, \dots, m_k$  are arbitrary positive integers such that  $m_1 + \dots + m_k - 1 = r$ . For  $1 \leq i \leq k$ , let  $f_i$  be the restriction of  $f$  to  $f^{-1}(M_i)$ ; it maps  $f^{-1}(M_i)$  surjectively to  $M_i$ . Then we can “decompose”  $f$  in a unique way into the  $k$  *surjective* functions  $f_1, \dots, f_k$ . The weight  $w(f)$  corresponding to  $f$  in  $A(t, x, y)$  is  $t^{a(f)} y^r$ , while the weight  $w(f_i)$  corresponding to  $f_i$  in  $S(t, x, y)$  is  $t^{a(f_i)} y^{m_i - 1}$ .

Now observe that  $a(f) = a(f_1) + \dots + a(f_k)$ : whenever we have a pair of numbers  $1 \leq i < j \leq n$  counted by  $a(f)$ , since  $f(i) - f(j) \in \{-1, 0, 1\}$ , we know that  $f(i)$  and  $f(j)$  must be in the same  $M_h$ . Therefore  $i$  and  $j$  are in the same  $f^{-1}(M_h)$ , and this pair is also counted in  $a(f_h)$ . We also have that  $r =$

$(m_1 - 1) + \cdots + (m_k - 1) + (k - 1)$ . Therefore  $w(f) = w(f_1) \cdots w(f_k) y^{k-1}$ . It follows from the compositional formula for exponential generating functions that

$$A(t, x, y) = \sum_{k \geq 1} S(t, x, y)^k y^{k-1} = \frac{S(t, x, y)}{1 - yS(t, x, y)},$$

as desired.  $\square$

Considering the different subsets of  $\{-1, 0, 1\}$ , we get six nonisomorphic subarrangements of the Catalan arrangement. They come from the subsets  $\emptyset$ ,  $\{0\}$ ,  $\{1\}$ ,  $\{0, 1\}$ ,  $\{-1, 1\}$  and  $\{-1, 0, 1\}$ . The corresponding subarrangements are the empty arrangement, the braid arrangement, the Linial arrangement, the Shi arrangement, the semiorder arrangement, and the Catalan arrangement, respectively. The empty arrangement is trivial, and the braid arrangement was already treated in detail starting on page 10. We now have a technique that lets us talk about the remaining four arrangements under the same framework. We will do this in the remainder of this section.

*The Linial arrangement.* The Linial arrangement  $\mathcal{L}_n$  consists of the hyperplanes  $x_i - x_j = 1$  for  $1 \leq i < j \leq n$ . This arrangement was first considered by Linial and Ravid. It was later studied by Athanasiadis [1996] and Postnikov and Stanley [2000], who independently computed the characteristic polynomial of  $\mathcal{L}_n$ :

$$\chi_{\mathcal{L}_n}(q) = \frac{q}{2^n} \sum_{k=0}^n \binom{n}{k} (q-k)^{n-1}.$$

They also put the regions of  $\mathcal{L}_n$  in bijection with several different sets of combinatorial objects. Perhaps the simplest such set is the set of *alternating trees* on  $[n+1]$ : the trees such that every vertex is either larger or smaller than all its neighbors.

Now we present the consequences of Proposition 5.8 and Propositions 5.6 and 5.7 for the Linial arrangement. Say that a poset  $P$  on  $[n]$  is *naturally labeled* if  $i < j$  in  $P$  implies  $i < j$  in  $\mathbb{Z}^+$ .

**Proposition 5.9.** *For all  $n \geq 1$  we have*

$$q \bar{\chi}_{\mathcal{L}_n}(q, t) = \sum_P q^{c(P)} (t-1)^{e(P)}$$

where the sum is over all naturally labeled, graded posets  $P$  on  $[n]$ . Here  $c(P)$  and  $e(P)$  denote the number of components and edges of the Hasse diagram of  $P$ , respectively.

*Proof.* There is an obvious bijection between Hasse diagrams of naturally labeled graded posets on  $[n]$  and planted graded  $\{1\}$ -graphs on  $[n]$ . The result then follows immediately from Proposition 5.6.  $\square$

**Theorem 5.10.** *Let*

$$A_r(t, x) = \sum_{n \geq 0} \left( \sum_{f: [n] \rightarrow [r]} t^{id(f)} \right) \frac{x^n}{n!},$$

where  $id(f)$  denotes the number of inverse descents of the word  $f(1) \dots f(n)$ , that is, the number of pairs  $(i, j)$  with  $1 \leq i < j \leq n$  such that  $f(i) - f(j) = 1$ . Then

$$1 + q \sum_{n \geq 1} \bar{\chi}_{\mathcal{L}_n}(q, t) \frac{x^n}{n!} = \left( \lim_{r \rightarrow \infty} \frac{A_r(t, x)}{A_{r-1}(t, x)} \right)^q.$$

*Proof.* This is immediate from [Theorem 5.7](#). □

Recall that the *descents* of a permutation  $\sigma = \sigma_1 \dots \sigma_r \in S_r$  are the indices  $i$  such that  $\sigma_i > \sigma_{i+1}$ . For more information about descents, see [[Stanley 1997](#), Section 1.3], for example. We call  $id(f)$  the number of inverse descents, because they generalize descents in the following way. If  $\pi : [r] \rightarrow [r]$  is a permutation, then  $id(\pi)$  is the number of descents of the permutation  $\pi^{-1}$ .

We can use [Theorem 5.10](#) to say more about the characteristic polynomial of  $\mathcal{L}_n$  which, as discussed on [page 4](#), is given by  $\chi_{\mathcal{L}_n}(q) = q \bar{\chi}_{\mathcal{L}_n}(q, 0)$ .

**Theorem 5.11.** *Let*

$$(5-10) \quad \frac{1 + ye^{x(1+y)}}{1 - y^2e^{x(1+y)}} = \sum_{r \geq 0} A_r(x)y^r.$$

Then we have

$$\sum_{n \geq 0} \chi_{\mathcal{L}_n}(q) \frac{x^n}{n!} = \left( \lim_{r \rightarrow \infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

In particular, if  $f_n$  is the number of alternating trees on  $[n + 1]$ , we have

$$\sum_{n \geq 0} (-1)^n f_n \frac{x^n}{n!} = \lim_{r \rightarrow \infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

*Proof.* In view of [Theorem 5.10](#) and [Proposition 5.8](#), we compute  $S(0, x, y)$ . From [\(5-9\)](#), the coefficient of  $\frac{x^n}{n!}y^r$  in  $S(0, x, y)$  is equal to the number of surjective functions  $f : [n] \rightarrow [r]$  with no inverse descents. These are just the nondecreasing surjective functions  $f : [n] \rightarrow [r]$ . For  $n \geq 1$  and  $r \geq 1$  there are  $\binom{n-1}{r-1}$  such functions, and for  $n = r = 0$  there is one such function. In the other cases there are none. Therefore

$$S(0, x, y) = 1 + \sum_{n \geq 1} \sum_{r \geq 1} \binom{n-1}{r-1} \frac{x^n}{n!} y^r = 1 + \sum_{n \geq 1} \frac{x^n}{n!} y(1+y)^{n-1} = \frac{1 + ye^{x(1+y)}}{1 + y}.$$

[Proposition 5.8](#) then implies that

$$A(0, x, y) = \frac{1 + ye^{x(1+y)}}{1 - y^2e^{x(1+y)}},$$

in agreement with [\(5-10\)](#), and the theorem follows.  $\square$

*The Shi arrangement.* The Shi arrangement  $\mathcal{S}_n$  consists of the hyperplanes  $x_i - x_j = 0, 1$  for  $1 \leq i < j \leq n$ . Shi [[1986](#), Chapter 7; [1987](#)] first considered this arrangement, and showed that it has  $(n+1)^{n-1}$  regions. Headley [[1994](#), Chapter VI; [1997](#)] later computed the characteristic polynomial of  $\mathcal{S}_n$ :

$$\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}.$$

Stanley [[1996](#); [1998b](#)] gave a nice bijection between regions of the Shi arrangement and parking functions of length  $n$ . Parking functions were first introduced by Konheim and Weiss [[1966](#)]; for more information, see [[Stanley 1999](#), Exercise 5.49].

For the Shi arrangement, we can say the following.

**Theorem 5.12.** *Let*

$$A_r(x) = \sum_{n=0}^r (r-n)^n \frac{x^n}{n!}.$$

*Then we have*

$$\sum_{n \geq 0} \chi_{\mathcal{S}_n}(q) \frac{x^n}{n!} = \left( \lim_{r \rightarrow \infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

*In particular, we have*

$$\sum_{n \geq 0} (-1)^n (n+1)^{n-1} \frac{x^n}{n!} = \lim_{r \rightarrow \infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

*Proof.* We proceed in the same way that we did in [Theorem 5.11](#). In this case, we need to compute the number of surjective functions  $f : [n] \rightarrow [r]$  such that  $f(i) - f(j)$  is never equal to 0 or 1 for  $i < j$ . These are just the surjective, strictly increasing functions. There is only one of them when  $n = r$ , and there are none when  $n \neq r$ . Hence

$$S(0, x, y) = \sum_{n \geq 0} \frac{x^n}{n!} y^n = e^{xy}.$$

The rest follows easily by computing  $A(0, x, y)$  and  $A_r(x)$  explicitly.  $\square$

*The semiorder arrangement.* The semiorder arrangement  $\mathcal{J}_n$  consists of the hyperplanes  $x_i - x_j = -1, 1$  for  $1 \leq i < j \leq n$ . A *semiorder on  $[n]$*  is a poset  $P$  on  $[n]$  for which there exist  $n$  unit intervals  $I_1, \dots, I_n$  of  $\mathbb{R}$ , such that  $i < j$  in  $P$  if and only if  $I_i$  is disjoint from  $I_j$  and to the left of it. It is known [Scott and Suppes 1958] that a poset is a semiorder if and only if it does not contain a subposet isomorphic to  $\mathbf{3} + \mathbf{1}$  or  $\mathbf{2} + \mathbf{2}$ . We are interested in semiorders because the number of regions of  $\mathcal{J}_n$  is equal to the number of semiorders on  $[n]$ , as shown in [Postnikov and Stanley 2000] and [Stanley 1996].

**Theorem 5.13.** *Let*

$$\frac{1 - y + ye^x}{1 - y + y^2 - y^2e^x} = \sum_{r \geq 0} A_r(x)y^r.$$

*Then we have*

$$\sum_{n \geq 0} \chi_{\mathcal{J}_n}(q) \frac{x^n}{n!} = \left( \lim_{r \rightarrow \infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

*In particular, if  $i_n$  is the number of semiorders on  $[n]$ , we have*

$$\sum_{n \geq 0} (-1)^n i_n \frac{x^n}{n!} = \lim_{r \rightarrow \infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

*Proof.* In this case,  $S(0, x, y)$  counts surjective functions  $f : [n] \rightarrow [r]$  such that  $f(i) - f(j)$  is never equal to 1 for  $i \neq j$ . Such a function has to be constant; so it can only exist (and is unique) if  $n \geq 1$  and  $r = 1$  or if  $n = r = 0$ . Thus

$$S(0, x, y) = 1 + (e^x - 1)y$$

and the rest follows easily. □

*The Catalan arrangement.* The Catalan arrangement  $C_n$  consists of the hyperplanes  $x_i - x_j = -1, 0, 1$  for  $1 \leq i < j \leq n$ . Stanley [1996] observed that the number of regions of this arrangement is  $n!C_n$ , where  $C_n = \frac{1}{n+1} \binom{2n}{n}$  is the  $n$ -th Catalan number. For (much) more information on the Catalan numbers; see [Stanley 1999, Chapter 6], especially Exercise 6.19.

**Theorem 5.14.** *Let*

$$A_r(x) = \sum_{n=0}^{\lfloor \frac{r+1}{2} \rfloor} \binom{r-n+1}{n} x^n.$$

*Then we have*

$$\sum_{n \geq 0} \chi_{C_n}(q) \frac{x^n}{n!} = \left( \lim_{r \rightarrow \infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$



In particular,

$$(5-11) \quad \frac{\sqrt{1+4x}-1}{2x} = \sum_{n \geq 0} (-1)^n C_n x^n = \lim_{r \rightarrow \infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

*Proof.* If  $f : [n] \rightarrow [r]$  is a surjective function such that  $f(i) - f(j)$  is never equal to  $-1, 0$  or  $1$  for  $i \neq j$ , then  $n = r = 0$  or  $n = r = 1$ . Thus  $S(x, y, 0) = 1 + xy$ , and the rest of the proof is straightforward.  $\square$

The polynomial  $A_r(x)$  is a simple transformation of the *Fibonacci polynomial*. The number of words of length  $r$ , consisting of 0s and 1s, which do not contain two consecutive 1s, is equal to  $F_{r+2}$ , the  $(r+2)$ th Fibonacci number. It is easy to see that the polynomial  $A_r(x)$  counts those words according to the number of 1s they contain. In particular,  $A_r(1) = F_{r+2}$ .

We close with an amusing observation. Irresponsibly<sup>2</sup> plugging  $x = 1$  into (5-11), we obtain an unconventional “proof” of the asymptotic rate of growth of Fibonacci numbers:

$$\frac{\sqrt{5}-1}{2} = \lim_{r \rightarrow \infty} \frac{F_{r-1}}{F_r}.$$

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<sup>2</sup>We are not necessarily justified in doing this, since we have only proved equality in (5-11) as formal power series!

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FEDERICO ARDILA  
MATHEMATICS DEPARTMENT  
SAN FRANCISCO STATE UNIVERSITY  
1600 HOLLOWAY AVENUE  
SAN FRANCISCO, CA 94110  
UNITED STATES

[federico@math.sfsu.edu](mailto:federico@math.sfsu.edu)  
<http://math.sfsu.edu/federico>