IDEAL STRUCTURE OF C^*-ALGEBRAS ASSOCIATED WITH C^*-CORRESPONDENCES

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We study the ideal structure of C^*-algebras arising from C^*-correspondences. We prove that gauge-invariant ideals of our C^*-algebras are parameterized by certain pairs of ideals of original C^*-algebras. We show that our C^*-algebras have a nice property that should be possessed by a generalization of crossed products. Applications to crossed products by Hilbert C^*-bimodules and relative Cuntz–Pimsner algebras are also discussed.

Introduction

For a C^*-algebra A, a C^*-correspondence over A is a (right) Hilbert A-module with a left action of A. Since endomorphisms (or families of endomorphisms) of A define C^*-correspondences over A, we can regard C^*-correspondences as (multivalued) generalizations of automorphisms or endomorphisms. This point of view has the same philosophy as the idea that certain topological correspondences are generalizations of continuous maps [Katsura 2004a, Section 1].

A crossed product by an automorphism is a C^*-algebra which has an original C^*-algebra as a C^*-subalgebra, and reflects many aspects of the automorphism. For example, the set of ideals of the crossed product that are invariant under the dual action of the one-dimensional torus $\mathbb{T}$ corresponds bijectively to the set of ideals of the original C^*-algebra that are invariant under the automorphism. As C^*-correspondences are generalizations of endomorphisms, a natural problem is to define "crossed products" by C^*-correspondences. There is plenty of evidence that the construction given in [Katsura 2003a] for the C^*-algebra $\mathbb{O}_X$ from a C^*-correspondence X is the right one. One piece of evidence given there is that this construction generalizes many constructions that were or were not considered as generalizations of crossed products. We are going to explain another piece of

evidence. For a C^*-correspondence X, we can naturally define a notion of repre-sentations of X (Definition 2.7). Thus one C^*-algebra which is naturally associated with a C^*-correspondence X is a C^*-algebra $\mathcal{T}_X$ having a universal property with respect to representations of X (Definition 3.1). This C^*-algebra $\mathcal{T}_X$ is none other than the (augmented) Cuntz–Toeplitz algebra defined in [Pimsner 1997]. When a C^*-correspondence X is defined by an automorphism, the C^*-algebra $\mathcal{T}_X$ is isomorphic to the Toeplitz extension of the crossed product by the automorphism defined in [Pimsner and Voiculescu 1980]. This C^*-algebra is too large to reflect the information in X. In order to get crossed products, we have to go to a quotient of $\mathcal{T}_X$. There are two ways to proceed. One is to define the *covariance* of repre-sentations of a C^*-correspondence X, and define a crossed product by X so that it has the universal property with respect to covariant representations of X. This kind of method has been used in many papers, and we define our C^*-algebra $\mathcal{O}_X$ along this line (Definitions 3.4 and 3.5). The other way is to list up the properties of $\mathcal{T}_X$ that the crossed product should have, and define a crossed product by X to be the smallest quotient of $\mathcal{T}_X$ among the quotients satisfying these properties. For this method, the following two properties seem to be reasonable:

 (i) The original C^*-algebra is embedded into the crossed product,
(ii) There exists a "dual action" of $\mathbb{T}$ on the crossed product.

In this paper, we show that these two methods give the same C^*-algebra $\mathcal{O}_X$ (Propo-sition 7.14). This indicates that the C^*-algebra $\mathcal{O}_X$ is the right one for a "crossed product" by a C^*-correspondence X. We note that Cuntz–Pimsner algebras do not satisfy the property (i) above when the left action of the C^*-correspondence is not injective, and that the C^*-algebra $\mathcal{O}_X$ is isomorphic to the Cuntz–Pimsner algebra when the left action of the C^*-correspondence is injective.

The "dual action" of $\mathbb{T}$ on the C^*-algebra $\mathcal{O}_X$ is called the gauge action. The main purpose of this paper is to describe the all ideals of the C^*-algebra $\mathcal{O}_X$ associated with a C^*-correspondence X that are invariant under the gauge action. We define invariance of ideals of A with respect to a C^*-correspondence X over A (Definition 4.8). Unlike the case of crossed products by automorphisms, we need extra ideals of A other than invariant ideals to describe all gauge-invariant ideals of $\mathcal{O}_X$. Similar facts were observed in many papers ([Bates et al. 2002; Drinen and Tomforde 2005; Katsura 2003b; Katsura 2006a] to name a few) for C^*-algebras arising from graphs or topological graphs. We introduce a notion of O-pairs, which are pairs consisting of invariant ideals and extra ideals of A, and show that gauge-invariant ideals are parameterized by O-pairs (Theorem 8.6).

This paper is organized as follows. In Sections 1 and 2, we fix notation and gather results on Hilbert C^*-modules and C^*-correspondences. In Section 3, we give the definition of our C^*-algebras $\mathcal{O}_X$ constructed from C^*-correspondences X.

In Sections 4 and 5, we introduce and study invariance of ideals, T-pairs and O-pairs. These are related to representations of C^*-correspondences. In Section 6, we construct a C^*-correspondence X_ω from a T-pair ω, and in Section 7 we prove that this C^*-correspondence X_ω has a certain universal property. As a corollary, we give an alternative definition of our C^*-algebras $\mathcal{O}_X$ described above (Proposition 7.14). In Section 8, we prove the main theorem (Theorem 8.6) which says that the set of all gauge-invariant ideals of $\mathcal{O}_X$ corresponds bijectively to the set of all O-pairs of X. We also see that a quotient of $\mathcal{O}_X$ by a gauge-invariant ideal falls into the class of our C^*-algebras. In Section 9, we see that every gauge-invariant ideals have hereditary and full C^*-subalgebras which are isomorphic to C^*-algebras associated with C^*-correspondences. As a consequence of the study of crossed products by Hilbert C^*-bimodules in Section 10, all gauge-invariant ideals themselves are shown to be isomorphic to C^*-algebras associated with C^*-correspondences. In Section 11, we apply our investigation to the relative Cuntz–Pimsner algebras defined in [Muhly and Solel 1998].

We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers, and by $\mathbb{C}$ the set of complex numbers. We denote by $\mathbb{T}$ the group consisting of complex numbers whose absolute values are 1. We use a convention that $\gamma(A, B) = \{\gamma(a, b) \in D \mid a \in A, b \in B\}$ for a map $\gamma : A \times B \to D$ such as inner products, multiplications or representations. We denote by $\overline{\operatorname{span}}\{\cdots\}$ the closure of linear span of $\{\cdots\}$. The Hewitt–Cohen factorization theorem can be stated as follows:

Lemma. *Let A be a C^*-algebra, X be a Banach space, and $\pi : A \to B(X)$ a bounded homomorphism from A to the Banach algebra $B(X)$ of the bounded operators on X. Then we have $\pi(A)X = \overline{\operatorname{span}}(\pi(A)X)$.*

We use this result just to abbreviate the notation and arguments. Readers not familiar with the theorem may use $\overline{\operatorname{span}}(\pi(A)X)$ instead of $\pi(A)X$; the two spaces are actually the same (for a proof, see [Raeburn and Williams 1998, Proposition 2.33] for example).

1. Hilbert C^*-modules

Definition 1.1. Let A be a C^*-algebra. A (right) *Hilbert A-module X* is a linear space with a right action of the C^*-algebra A and an A-valued inner product $\langle \cdot, \cdot \rangle_X$ satisfying certain conditions such that X is complete with respect to the norm defined by $\|\xi\|_X = \|\langle \xi, \xi \rangle_X\|^{1/2}$ for $\xi \in X$.

For a precise definition of Hilbert C^*-modules, consult [Lance 1995]. We do not assume that a Hilbert A-module X is full. Thus $\overline{\operatorname{span}}\langle X, X \rangle_X$ can be a proper ideal of A, where an ideal of a C^*-algebra always means a closed two-sided ideal, except in the proof of Lemma 4.6.

Definition 1.2. For a Hilbert A-module X, we denote by $\mathscr{L}(X)$ the C^*-algebra of all adjointable operators on X. For $\xi, \eta \in X$, the operator $\theta_{\xi,\eta} \in \mathscr{L}(X)$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle_X$ for $\zeta \in X$. We define the ideal $\mathscr{K}(X)$ of $\mathscr{L}(X)$ by

$$\mathscr{K}(X) = \overline{\mathrm{span}} \{ \theta_{\xi,\eta} \in \mathscr{L}(X) \mid \xi, \eta \in X \}.$$

We fix a C^*-algebra A and a Hilbert A-module X throughout this section.

Proposition 1.3. *Let I be an ideal of A. For $\xi \in X$, the following are equivalent*:

 (i) $\xi \in XI$.
 (ii) $\langle \eta, \xi \rangle_X \in I$ *for all $\eta \in X$.*
(iii) $\langle \xi, \xi \rangle_X \in I$.
(iv) *There exist $\eta \in X$ and a positive element $a \in I$ such that $\xi = \eta a$.*

Proof. Clearly (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). For $\xi \in X$ with $\langle \xi, \xi \rangle_X \in I$, we can find $\eta \in X$ such that $\xi = \eta a$ for $a = (\langle \xi, \xi \rangle_X)^{1/3} \in I$ ([Lance 1995, Lemma 4.4]). This proves (iii) $\Rightarrow$ (iv). $\qquad\square$

Corollary 1.4. *For an ideal I of A, XI is a closed linear subspace of X which is invariant by the right action of A and by the left action of $\mathscr{L}(X)$.*

Proof. Since the set of $\xi \in X$ satisfying condition (ii) in Proposition 1.3 is a closed linear space, we see that XI is a closed linear space (this also follows from the Cohen factorization theorem). The rest of the statement is easy to verify. $\qquad\square$

By this corollary, XI is a Hilbert A-submodule of X. We can and will consider $\mathscr{K}(XI)$ as a subalgebra of $\mathscr{K}(X)$ by

$$\mathscr{K}(XI) = \overline{\mathrm{span}} \{ \theta_{\xi,\eta} \in \mathscr{K}(X) \mid \xi, \eta \in XI \} \subset \mathscr{K}(X)$$

(see [Fowler et al. 2003, Lemma 2.6 (1)] for the proof). Note that XI is also considered as a Hilbert I-module. For an ideal I of A, we denote by X_I the quotient space X/XI. Both of the natural quotient maps $A \to A/I$ and $X \to X_I$ are denoted by $[\,\cdot\,]_I$. The space X_I has an A/I-valued inner product $\langle \cdot, \cdot \rangle_{X_I}$ and a right action of A/I so that

$$\big\langle [\xi]_I, [\zeta]_I \big\rangle_{X_I} = \big[\langle \xi, \zeta \rangle_X \big]_I, \quad [\xi]_I [a]_I = [\xi a]_I$$

for $\xi, \zeta \in X$ and $a \in A$. By Proposition 1.3, $\eta \in X_I$ satisfies $\langle \eta, \eta \rangle_{X_I} = 0$ only when $\eta = 0$. Hence $\|\eta\|_{X_I} = \|\langle \eta, \eta \rangle_{X_I}\|^{1/2}$ defines a norm on X_I.

Lemma 1.5. *For $\eta \in X_I$, there exists $\xi \in X$ such that $\eta = [\xi]_I$ and $\|\eta\|_{X_I} = \|\xi\|_X$.*

Proof. Clearly $[\,\cdot\,]_I$ is a norm-decreasing map. Thus it suffices to find $\xi \in X$ such that $[\xi]_I = \eta$ and $\|\xi\|_X \leq \|\eta\|_{X_I}$ for $\eta \in X_I$. Set $C = \|\eta\|_{X_I}^2 = \|\langle \eta, \eta \rangle_{X_I}\|$. Let f, g be functions on $\mathbb{R}_+ = [0, \infty)$ defined by

$$f(r) = \begin{cases} 1 & (0 \leq r \leq C) \\ \sqrt{C/r} & (r > C) \end{cases} , \quad g(r) = \min\{r, C\}.$$

Then we have $g(r) = r f(r)^2$ and $g(r) \leq C$ for $r \in \mathbb{R}_+$. Take $\xi_0 \in X$ with $\eta = [\xi_0]_I$. Set $a = f(\langle \xi_0, \xi_0 \rangle_X) \in \widetilde{A}$ and $\xi = \xi_0 a \in X$ where $\widetilde{A}$ is the unitization of A. We have $\langle \xi, \xi \rangle_X = a^* \langle \xi_0, \xi_0 \rangle_X a = g(\langle \xi_0, \xi_0 \rangle_X)$. Hence we get $\|\xi\|_X \leq C^{1/2} = \|\eta\|_{X_I}$. Since f is 1 on $[0, C]$, we have

$$[a]_I = f([\langle \xi_0, \xi_0 \rangle_X]_I) = f(\langle \eta, \eta \rangle_{X_I}) = 1.$$

Therefore we see that $[\xi]_I = [\xi_0]_I [a]_I = \eta$. We are done. $\square$

By this lemma, the norm $\|\cdot\|_{X_I}$ of X_I coincides with the quotient norm of $[\,\cdot\,]_I \colon X \to X_I$ (see [Fowler et al. 2003, Lemma 2.1] for another proof). Hence X_I is complete, and so it is a Hilbert A/I-module.

Since XI is closed under the action of $\mathcal{L}(X)$, we can define a map $\mathcal{L}(X) \to \mathcal{L}(X_I)$, which is also denoted by $[\,\cdot\,]_I$, so that $[S]_I [\xi]_I = [S\xi]_I$ for $S \in \mathcal{L}(X)$ and $\xi \in X$. By definition, $S \in \mathcal{L}(X)$ satisfies $[S]_I = 0$ if and only if $S\xi \in XI$ for all $\xi \in X$, which is equivalent to the condition that $\langle \eta, S\xi \rangle \in I$ for all $\xi, \eta \in X$ by Proposition 1.3.

Lemma 1.6. *For $\xi, \eta \in X$, we have $[\theta_{\xi,\eta}]_I = \theta_{[\xi]_I, [\eta]_I}$. The restriction of the map $[\,\cdot\,]_I \colon \mathcal{L}(X) \to \mathcal{L}(X_I)$ to $\mathcal{K}(X)$ is a surjection onto $\mathcal{K}(X_I)$ whose kernel is $\mathcal{K}(XI)$.*

Proof. The first assertion is easily verified by the definition. This implies that the restriction of the map $[\,\cdot\,]_I$ to $\mathcal{K}(X)$ is a surjection onto $\mathcal{K}(X_I)$, and that $\mathcal{K}(XI)$ is in the kernel of $[\,\cdot\,]_I$. We will show that if $k \in \mathcal{K}(X)$ satisfies that $[k]_I = 0$, then $k \in \mathcal{K}(XI)$.

There exists an approximate unit $\{u_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{K}(X)$ such that for each $\lambda \in \Lambda$, u_λ is a finite linear sum of elements in the form $\theta_{\xi,\eta}$. Take $k \in \mathcal{K}(X)$ with $[k]_I = 0$. Since we have $k = \lim k u_\lambda$, to prove $k \in \mathcal{K}(XI)$ it suffices to show that $k\theta_{\xi,\eta} \in \mathcal{K}(XI)$ for arbitrary $\xi, \eta \in X$. Since $k\xi \in XI$, we can find $\xi_0 \in X$ and a positive element $a_0 \in I$ such that $k\xi = \xi_0 a_0$ by Proposition 1.3. Then we have

$$k\theta_{\xi,\eta} = \theta_{k\xi,\eta} = \theta_{\xi_0 a_0, \eta} = \theta_{\xi_0 \sqrt{a_0}, \eta \sqrt{a_0}} \in \mathcal{K}(XI),$$

as needed. $\square$

See also [Fowler et al. 2003, Lemma 2.6 (2), (3)]. Note that it often happens that $[S]_I \in \mathcal{K}(X_I)$ even if $S \notin \mathcal{K}(X)$. This observation plays an important role in our analysis after Section 5. Note also that though three maps $[\,\cdot\,]_I \colon A \to A/I$, $[\,\cdot\,]_I \colon X \to X_I$ and $[\,\cdot\,]_I \colon \mathcal{K}(X) \to \mathcal{K}(X_I)$ are always surjective, the map

$[\,\cdot\,]_I \colon \mathcal{L}(X) \to \mathcal{L}(X_I)$ need not be surjective, because Tietze's extension theorem fails in general.

Take two ideals I and I' of A such that $I \subset I'$. Then I'/I is an ideal of A/I and $(A/I)/(I'/I) \ni \big[[a]_I\big]_{I'/I} \mapsto [a]_{I'} \in A/I'$ gives a well-defined isomorphism. By this isomorphism, we will identify $(A/I)/(I'/I)$ with A/I'. Thus the quotient map $[\,\cdot\,]_{I'} \colon A \to A/I'$ coincides with the composition of $[\,\cdot\,]_I \colon A \to A/I$ and $[\,\cdot\,]_{I'/I} \colon A/I \to A/I'$. Similarly we will identify $(X_I)_{I'/I}$ with $X_{I'}$ so that $[\,\cdot\,]_{I'} = [\,\cdot\,]_{I'/I} \circ [\,\cdot\,]_I$ holds for both $X \to X_{I'}$ and $\mathcal{L}(X) \to \mathcal{L}(X_{I'})$. It is easy to see the following.

Lemma 1.7. *We have $(XI')_I = X_I(I'/I)$ in X_I.*

Now take two ideals I_1 and I_2 of A. It is well-known that the ideal $I_1 \cap I_2$ coincides with $I_1 I_2$, and that $I_1 + I_2$ is an ideal of A. It is easy to see that the natural map $I_1/(I_1 \cap I_2) \to (I_1 + I_2)/I_2$ is an isomorphism. The pull-back C^*-algebra B of the two quotient maps $[\,\cdot\,]_{(I_1+I_2)/I_1} \colon A/I_1 \to A/(I_1+I_2)$ and $[\,\cdot\,]_{(I_1+I_2)/I_2} \colon A/I_2 \to A/(I_1+I_2)$ is defined by

$$B = \big\{ (b_1, b_2) \in A/I_1 \oplus A/I_2 \mid [b_1]_{(I_1+I_2)/I_1} = [b_2]_{(I_1+I_2)/I_2} \in A/(I_1+I_2) \big\}.$$

It is not difficult to see the following (see the proof of Proposition 1.10).

Lemma 1.8. *The map*

$$\Pi \colon A/(I_1 \cap I_2) \ni b \mapsto \big([b]_{I_1/(I_1 \cap I_2)}, [b]_{I_2/(I_1 \cap I_2)}\big) \in B$$

is an isomorphism.

We will show analogous statements for Hilbert modules and sets of operators on them. Define a linear space Y by

$$Y = \big\{ (\eta_1, \eta_2) \in X_{I_1} \oplus X_{I_2} \mid [\eta_1]_{(I_1+I_2)/I_1} = [\eta_2]_{(I_1+I_2)/I_2} \in X_{I_1+I_2} \big\}.$$

We define a B-valued inner product on Y by

$$\big\langle (\eta_1, \eta_2), (\eta_1', \eta_2') \big\rangle_Y = \big(\langle \eta_1, \eta_1' \rangle_{X_{I_1}}, \langle \eta_2, \eta_2' \rangle_{X_{I_2}} \big) \in B,$$

for $(\eta_1, \eta_2), (\eta_1', \eta_2') \in Y$. Clearly Y is complete with respect to the norm defined by the inner product. If we define a right action of B on Y by

$$(\eta_1, \eta_2)(b_1, b_2) = (\eta_1 b_1, \eta_2 b_2) \in Y$$

for $(\eta_1, \eta_2) \in Y$, $(b_1, b_2) \in B$, then we can easily see that Y is a Hilbert B-module.

Lemma 1.9. *The restriction of the quotient map $[\,\cdot\,]_{I_2/(I_1 \cap I_2)} \colon X_{I_1 \cap I_2} \to X_{I_2}$ to $X_{I_1 \cap I_2}\big(I_1/(I_1 \cap I_2)\big)$ is a bijection onto $X_{I_2}\big((I_1 + I_2)/I_2\big)$.*

Proof. By Lemma 1.7, we have $X_{I_1 \cap I_2}\big(I_1/(I_1 \cap I_2)\big) = (XI_1)_{I_1 \cap I_2}$. It is easy to see that the surjection $[\,\cdot\,]_{I_2/(I_1 \cap I_2)} \colon (XI_1)_{I_1 \cap I_2} \to (XI_1)_{I_2}$ is injective. It is also easy to see that $(XI_1)_{I_2} = \big(X(I_1 + I_2)\big)_{I_2}$. We have $\big(X(I_1 + I_2)\big)_{I_2} = X_{I_2}\big((I_1 + I_2)/I_2\big)$ by Lemma 1.7. This completes the proof. $\qquad\square$

Proposition 1.10. *By Π in Lemma 1.8, we can consider $X_{I_1 \cap I_2}$ as a Hilbert B-module. Then the map*

$$T \colon X_{I_1 \cap I_2} \ni \eta \mapsto \big([\eta]_{I_1/(I_1 \cap I_2)}, [\eta]_{I_2/(I_1 \cap I_2)}\big) \in Y$$

is an isomorphism of Hilbert B-modules.

Proof. Clearly T preserves inner products and right actions. This implies that T is isometric. It remains to show that T is surjective. Take $(\eta_1, \eta_2) \in Y$. Since $[\,\cdot\,]_{I_1/(I_1 \cap I_2)} \colon X_{I_1 \cap I_2} \to X_{I_1}$ is surjective, we can find $\eta' \in X_{I_1 \cap I_2}$ with $[\eta']_{I_1/(I_1 \cap I_2)} = \eta_1$. Since $[\eta_2]_{(I_1 + I_2)/I_2} = [\eta_1]_{(I_1 + I_2)/I_1} = [\eta']_{(I_1 + I_2)/(I_1 \cap I_2)}$, we have

$$\eta_2 - [\eta']_{I_2/(I_1 \cap I_2)} \in \ker([\,\cdot\,]_{(I_1 + I_2)/I_2}) = X_{I_2}\big((I_1 + I_2)/I_2\big).$$

By Lemma 1.9, we can find $\eta'' \in X_{I_1 \cap I_2}\big(I_1/(I_1 \cap I_2)\big)$ with

$$[\eta'']_{I_2/(I_1 \cap I_2)} = \eta_2 - [\eta']_{I_2/(I_1 \cap I_2)}.$$

Set $\eta = \eta' + \eta'' \in X_{I_1 \cap I_2}$. We see that

$$[\eta]_{I_1/(I_1 \cap I_2)} = [\eta']_{I_1/(I_1 \cap I_2)} + 0 = \eta_1,$$
$$[\eta]_{I_2/(I_1 \cap I_2)} = [\eta']_{I_2/(I_1 \cap I_2)} + [\eta'']_{I_2/(I_1 \cap I_2)} = \eta_2.$$

Therefore $T(\eta) = (\eta_1, \eta_2)$. Thus T is surjective. $\qquad\square$

Proposition 1.11. *Define a C^*-algebra $\mathcal{M}$ by*

$$\mathcal{M} = \big\{(S_1, S_2) \in \mathcal{L}(X_{I_1}) \oplus \mathcal{L}(X_{I_2}) \,\big|\, [S_1]_{(I_1 + I_2)/I_1} = [S_2]_{(I_1 + I_2)/I_2} \in \mathcal{L}(X_{I_1 + I_2})\big\}.$$

Then the map

$$\Psi \colon \mathcal{L}(X_{I_1 \cap I_2}) \ni S \mapsto \big([S]_{I_1/(I_1 \cap I_2)}, [S]_{I_2/(I_1 \cap I_2)}\big) \in \mathcal{M}$$

is an isomorphism, and its restriction to $\mathcal{K}(X_{I_1 \cap I_2})$ is an isomorphism onto the C^*-subalgebra $\mathcal{K}$ of $\mathcal{M}$ defined by

$$\mathcal{K} = \big\{(k_1, k_2) \in \mathcal{K}(X_{I_1}) \oplus \mathcal{K}(X_{I_2}) \,\big|\, [k_1]_{(I_1 + I_2)/I_1} = [k_2]_{(I_1 + I_2)/I_2} \in \mathcal{K}(X_{I_1 + I_2})\big\}.$$

Proof. Take $(S_1, S_2) \in \mathcal{M}$ and define $\Psi'(S_1, S_2) \in \mathcal{L}(X_{I_1 \cap I_2})$. For $\xi \in X_{I_1 \cap I_2}$, we have

$$[S_1[\xi]_{I_1/(I_1 \cap I_2)}]_{(I_1 + I_2)/I_1} = [S_2[\xi]_{I_2/(I_1 \cap I_2)}]_{(I_1 + I_2)/I_2}.$$

Hence by Proposition 1.10, there exists a unique element $\eta \in X_{I_1 \cap I_2}$ with

$$[\eta]_{I_1/(I_1 \cap I_2)} = S_1[\xi]_{I_1/(I_1 \cap I_2)}, \quad \text{and} \quad [\eta]_{I_2/(I_1 \cap I_2)} = S_2[\xi]_{I_2/(I_1 \cap I_2)}.$$

We define $\Psi'(S_1, S_2)\colon X_{I_1 \cap I_2} \to X_{I_1 \cap I_2}$ by $\Psi'(S_1, S_2)\xi = \eta$ where η is the unique element satisfying the two equations above. Then, using Lemma 1.8, we see that

$$\langle \Psi'(S_1, S_2)\xi, \xi' \rangle_{X_{I_1 \cap I_2}} = \langle \xi, \Psi'(S_1^*, S_2^*)\xi' \rangle_{X_{I_1 \cap I_2}}$$

for every $\xi, \xi' \in X_{I_1 \cap I_2}$. Thus $\Psi'(S_1, S_2) \in \mathcal{L}(X_{I_1 \cap I_2})$ for all $(S_1, S_2) \in \mathcal{M}$. It is easy to see that $\Psi'\colon \mathcal{M} \to \mathcal{L}(X_{I_1 \cap I_2})$ is a $*$-homomorphism, and gives the inverse of Ψ. Hence $\Psi\colon \mathcal{L}(X_{I_1 \cap I_2}) \to \mathcal{M}$ is an isomorphism.

Clearly the restriction of Ψ on $\mathcal{K}(X_{I_1 \cap I_2})$ is an injection into $\mathcal{K}$. We will show that this is surjective. By Lemma 1.9, we can see that the restriction of the map $[\,\cdot\,]_{I_2/(I_1 \cap I_2)}\colon \mathcal{K}(X_{I_1 \cap I_2}) \to \mathcal{K}(X_{I_2})$ to

$$\ker([\,\cdot\,]_{I_1/(I_1 \cap I_2)}) = \mathcal{K}\big(X_{I_1 \cap I_2}\big(I_1/(I_1 \cap I_2)\big)\big)$$

is a bijection onto

$$\ker([\,\cdot\,]_{(I_1 + I_2)/I_2}) = \mathcal{K}\big(X_{I_2}\big((I_1 + I_2)/I_2\big)\big).$$

Take $(k_1, k_2) \in \mathcal{K}$. Since the map $[\,\cdot\,]_{I_1/(I_1 \cap I_2)}\colon \mathcal{K}(X_{I_1 \cap I_2}) \to \mathcal{K}(X_{I_1})$ is surjective, we can find $k' \in \mathcal{K}(X_{I_1 \cap I_2})$ with $[k']_{I_1/(I_1 \cap I_2)} = k_1$. Then we see that

$$k_2 - [k']_{I_2/(I_1 \cap I_2)} \in \ker([\,\cdot\,]_{(I_1 + I_2)/I_2}).$$

Thus there exists a unique element

$$k'' \in \ker([\,\cdot\,]_{I_1/(I_1 \cap I_2)}) \subset \mathcal{K}(X_{I_1 \cap I_2})$$

with $[k'']_{I_2/(I_1 \cap I_2)} = k_2 - [k']_{I_2/(I_1 \cap I_2)}$. Now it is easy to see that $k = k' + k'' \in \mathcal{K}(X_{I_1 \cap I_2})$ satisfies $\Psi(k) = (k_1, k_2)$. We are done. $\square$

Corollary 1.12. *If $S \in \mathcal{L}(X_{I_1 \cap I_2})$ satisfies*

$$[S]_{I_1/(I_1 \cap I_2)} \in \mathcal{K}(X_{I_1}), \quad [S]_{I_2/(I_1 \cap I_2)} \in \mathcal{K}(X_{I_2}),$$

then $S \in \mathcal{K}(X_{I_1 \cap I_2})$.

Proof. Clear by Proposition 1.11. $\square$

2. *C*-correspondences and representations*

Definition 2.1. For a C^*-algebra A, we say that X is a C^*-*correspondence* over A when X is a Hilbert A-module and a $*$-homomorphism $\varphi_X\colon A \to \mathcal{L}(X)$ is given.

We refer to φ_X as the left action of a C^*-correspondence X. C^*-correspondences can be considered as generalizations of automorphisms or endomorphisms. In fact, we can associate a C^*-correspondence X_φ with each endomorphism φ as follows.

Definition 2.2. Let A be a C^*-algebra and $\varphi \colon A \to A$ be an endomorphism. We define a C^*-correspondence X_φ such that it is isomorphic to A as Banach spaces, its inner product is defined by $\langle \xi, \eta \rangle_X = \xi^* \eta$, right action is multiplication and left action is given by $\varphi_{X_\varphi}(a)\xi = \varphi(a)\xi$. We denote X_{id_A} by A, and call it the *identity correspondence* over A.

Note that the left action φ_A of the identity correspondence A gives an isomorphism from A to $\mathcal{K}(A)$.

Definition 2.3. A *morphism* from a C^*-correspondence X over a C^*-algebra A to a C^*-correspondence Y over a C^*-algebra B is a pair (Π, T) consisting of a $*$-homomorphism $\Pi \colon A \to B$ and a linear map $T \colon X \to Y$ satisfying

 (i) $\langle T(\xi), T(\eta) \rangle_Y = \Pi\big(\langle \xi, \eta \rangle_X\big)$ for $\xi, \eta \in X$,
 (ii) $\varphi_Y\big(\Pi(a)\big)T(\xi) = T\big(\varphi_X(a)\xi\big)$ for $a \in A$, $\xi \in X$.

A morphism (Π, T) is said to be *injective* if a $*$-homomorphism Π is injective.

A morphism is called a semicovariant homomorphism in [Schweizer 2001]. For a morphism (Π, T) from X to Y, we can see that $T(\xi)\Pi(a) = T(\xi a)$ and $\|T(\xi)\|_Y \le \|\xi\|_X$ for $a \in A$ and $\xi \in X$ by the same argument as in [Katsura 2004b, Section 2]. We also see that T is isometric for an injective morphism (Π, T).

Definition 2.4. For a morphism (Π, T) from a C^*-correspondence X over A to a C^*-correspondence Y over B, we define a $*$-homomorphism $\Psi_T \colon \mathcal{K}(X) \to \mathcal{K}(Y)$ by $\Psi_T(\theta_{\xi,\eta}) = \theta_{T(\xi),T(\eta)}$ for $\xi, \eta \in X$.

For the well-definedness of a $*$-homomorphism Ψ_T, see, for example, [Kajiwara et al. 1998, Lemma 2.2]. Note that Ψ_T is injective for an injective morphism (Π, T). The following two lemmas are easily verified.

Lemma 2.5. *For a morphism (Π, T) from a C^*-correspondence X over A to a C^*-correspondence Y over B, we have $\varphi_Y(\Pi(a))\Psi_T(k) = \Psi_T(\varphi_X(a)k)$ and $\Psi_T(k)T(\xi) = T(k\xi)$ for $a \in A$, $\xi \in X$ and $k \in \mathcal{K}(X)$.*

Lemma 2.6. *Let X, Y, Z be C^*-correspondences, and (Π_1, T_1), (Π_2, T_2) be morphisms from X to Y and from Y to Z, respectively. Then its composition $(\Pi_2 \circ \Pi_1, T_2 \circ T_1)$ is a morphism from X to Z, and we have $\Psi_{T_2 \circ T_1} = \Psi_{T_2} \circ \Psi_{T_1}$.*

Definition 2.7. A *representation* of a C^*-correspondence X over A on a C^*-algebra B is a pair (π, t) consisting of a $*$-homomorphism $\pi \colon A \to B$ and a linear map $t \colon X \to B$ satisfying

 (i) $t(\xi)^* t(\eta) = \pi\big(\langle \xi, \eta \rangle_X\big)$ for $\xi, \eta \in X$,
 (ii) $\pi(a)t(\xi) = t\big(\varphi_X(a)\xi\big)$ for $a \in A$, $\xi \in X$.

We denote by $C^*(\pi, t)$ the C^*-algebra generated by the images of π and t in B. We define a $*$-homomorphism $\psi_t \colon \mathcal{K}(X) \to C^*(\pi, t)$ by $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^* \in C^*(\pi, t)$ for $\xi, \eta \in X$.

Representations of a C^*-correspondence X on a C^*-algebra B are precisely the morphisms from X to the identity correspondence over B, and we have $\varphi_B \circ \psi_t = \Psi_t$. Note that we get $\pi(a)\psi_t(k) = \psi_t(\varphi_X(a)k)$ and $\psi_t(k)t(\xi) = t(k\xi)$ for $k \in \mathcal{K}(X)$, $a \in A$ and $\xi \in X$.

Definition 2.8. A representation (π, t) of X is said to *admit a gauge action* if for each $z \in \mathbb{T}$, there exists a $*$-homomorphism $\beta_z \colon C^*(\pi, t) \to C^*(\pi, t)$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(\xi)) = z t(\xi)$ for all $a \in A$ and $\xi \in X$.

If it exists, such a $*$-homomorphism β_z is unique and $\beta \colon \mathbb{T} \to \operatorname{Aut}(C^*(\pi, t))$ is a strongly continuous homomorphism.

3. C^*-algebras associated with C^*-correspondences

In this section, we review the constructions of the C^*-algebras $\mathcal{T}_X$ and $\mathbb{O}_X$ from a C^*-correspondence X. These C^*-algebras were introduced by Pimsner in [Pimsner 1997], and modified in [Katsura 2003a].

Definition 3.1. For a C^*-correspondence X over a C^*-algebra A, we denote by $\mathcal{T}_X$ the C^*-algebra generated by the universal representation.

The universal representation can be obtained by taking a direct sum of sufficiently many representations. By universality, we have a surjection $\mathcal{T}_X \to C^*(\pi, t)$ for every representation (π, t) of X. The C^*-algebra $\mathcal{T}_X$ is too large to reflect the informations of X, and so we will take a certain quotient of $\mathcal{T}_X$ to get the nice C^*-algebra $\mathbb{O}_X$.

Definition 3.2. For an ideal I of a C^*-algebra A, we define $I^\perp \subset A$ by

$$I^\perp = \{a \in A \mid ab = 0 \text{ for all } b \in I\}.$$

Note that $I^\perp$ is the largest ideal of A satisfying $I \cap I^\perp = 0$.

Definition 3.3. For a C^*-correspondence X over A, we define an ideal J_X of A by

$$J_X = \varphi_X^{-1}\big(\mathcal{K}(X)\big) \cap \big(\ker \varphi_X\big)^\perp.$$

The ideal J_X is the largest ideal to which the restriction of φ_X is an injection into $\mathcal{K}(X)$.

Definition 3.4. A representation (π, t) of X is said to be *covariant* if we have $\pi(a) = \psi_t(\varphi_X(a))$ for all $a \in J_X$.

Definition 3.5. For a C^*-correspondence X over a C^*-algebra A, the C^*-algebra $\mathbb{O}_X$ is defined by $\mathbb{O}_X = C^*(\pi_X, t_X)$ where (π_X, t_X) is the universal covariant representation of X.

By universality, for any covariant representation (π, t) of a C^*-correspondence X, there exists a $*$-homomorphism $\rho_{(\pi,t)} \colon \mathbb{O}_X \to C^*(\pi, t)$ such that $\pi = \rho_{(\pi,t)} \circ \pi_X$ and $t = \rho_{(\pi,t)} \circ t_X$. Again by universality, the universal covariant representation (π_X, t_X) admits a gauge action. We denote it by $\gamma \colon \mathbb{T} \curvearrowright \mathbb{O}_X$. When we consider $\mathbb{O}_X$ as a generalization of crossed products by automorphisms, the gauge action γ is regarded as the dual action of $\mathbb{T}$. If a covariant representation (π, t) admits a gauge action β, then we have $\beta_z \circ \rho_{(\pi,t)} = \rho_{(\pi,t)} \circ \gamma_z$ for each $z \in \mathbb{T}$. In [Katsura 2004b, Proposition 4.11], we saw that the universal covariant representation (π_X, t_X) is injective. The following gauge-invariant uniqueness theorem says that two conditions, admitting a gauge action and being injective, characterize the universal one (π_X, t_X) among all covariant representations.

Theorem 3.6 [Katsura 2004b, Theorem 6.4]. *For a covariant representation (π, t) of a C^*-correspondence X, the map $\rho_{(\pi,t)} \colon \mathbb{O}_X \to C^*(\pi, t)$ is an isomorphism if and only if (π, t) is injective and admits a gauge action.*

In Proposition 7.14, we see that the universal covariant representation (π_X, t_X) is the smallest one among injective representations admitting gauge actions.

Remark 3.7. A morphism (Π, T) from a C^*-correspondence X to a C^*-correspondence Y gives us a $*$-homomorphism $\mathcal{T}_X \to \mathcal{T}_Y$. This also gives a $*$-homomorphism $\mathbb{O}_X \to \mathbb{O}_Y$ when the morphism (Π, T) is covariant, that is, we have $\Pi(a) \in J_Y$ and $\varphi_Y(\Pi(a)) = \Psi_T(\varphi_X(a))$ for all $a \in J_X$. We do not use these facts explicitly.

4. Invariant ideals

In this section, we introduce the notion of invariant ideals with respect to C^*-correspondences. Let us take a C^*-correspondence X over a C^*-algebra A, and fix them until the end of Section 9.

Definition 4.1. For an ideal I of A, we define $X(I)$, $X^{-1}(I) \subset A$ by

$$X(I) = \overline{\mathrm{span}} \left\{ \langle \eta, \varphi_X(a)\xi \rangle \in A \mid a \in I, \xi, \eta \in X \right\},$$
$$X^{-1}(I) = \{ a \in A \mid \langle \eta, \varphi_X(a)\xi \rangle_X \in I \text{ for all } \xi, \eta \in X \}.$$

Clearly $X(I)$ is an ideal of A. We also see that $X^{-1}(I)$ is an ideal because it is the kernel of the composition of φ_X and the map $[\,\cdot\,]_I \colon \mathcal{L}(X) \to \mathcal{L}(X_I)$. For a C^*-correspondence X_φ defined from an endomorphism $\varphi \colon A \to A$, we see that $X_\varphi(I)$ is the ideal generated by $\varphi(I)$, and $X_\varphi^{-1}(I) = \varphi^{-1}(I)$ for an ideal I of A. It is easy to see that we have $X(I_1) \subset X(I_2)$ and $X^{-1}(I_1) \subset X^{-1}(I_2)$ for two ideals I_1, I_2 of A with $I_1 \subset I_2$. For an ideal I, we have $X(X^{-1}(I)) \subset I$ and $X^{-1}(X(I)) \supset I$. These inclusions are proper in general, because we always have $X(I) \subset \overline{\mathrm{span}} \langle X, X \rangle_X$ and

$X^{-1}(I) \supset \ker \varphi_X$. The inclusions

$$X(X^{-1}(I)) \subset I \cap \overline{\text{span}}\langle X, X \rangle_X, \qquad X^{-1}(X(I)) \supset I + \ker \varphi_X$$

still can be proper as we will see in Examples 4.3 and 4.12.

Lemma 4.2. *For two ideals I_1, I_2 of A, we have*

$$X(I_1 \cap I_2) \subset X(I_1) \cap X(I_2), \qquad X^{-1}(I_1 \cap I_2) = X^{-1}(I_1) \cap X^{-1}(I_2),$$

$$X(I_1 + I_2) = X(I_1) + X(I_2), \quad and \ \ X^{-1}(I_1 + I_2) \supset X^{-1}(I_1) + X^{-1}(I_2).$$

Proof. Clear by the definitions. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Both inclusions in Lemma 4.2 can be proper in general (see Examples 4.3 and 4.12).

Example 4.3. Let A be $\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$, and $\varphi\colon A \to A$ be an endomorphism defined by $\varphi\big((\lambda, \mu, T)\big) = (0, 0, \text{diag}\{\lambda, \mu\})$. This endomorphism gives us a C^*-correspondence $X = X_\varphi$ over A. Let us define three ideals I_1, I_2 and I_3 of A by $I_1 = \mathbb{C} \oplus 0 \oplus 0$, $I_2 = 0 \oplus \mathbb{C} \oplus 0$ and $I_3 = 0 \oplus 0 \oplus M_2(\mathbb{C})$. We see that $\ker \varphi_X = \ker \varphi = I_3$ and $\varphi_X^{-1}(\mathcal{K}(X)) = A$. Hence we get $J_X = I_1 + I_2$. We have $X(I_1) = X(I_2) = I_3$. However clearly we have $X(I_1 \cap I_2) = X(0) = 0$. This gives an example of a proper inclusion $X(I_1 \cap I_2) \subset X(I_1) \cap X(I_2)$. Since $X^{-1}(I_3) = A$, we have two proper inclusions $X^{-1}(X(I_i)) \supset I_i + \ker \varphi_X$ for $i = 1, 2$. We see that there exist no nontrivial invariant ideals of A (see Definition 4.8), and the C^*-algebra $\mathcal{O}_X$ is isomorphic to a simple C^*-algebra $M_6(\mathbb{C})$.

For an increasing family $\{I_n\}_{n \in \mathbb{N}}$ of ideals of a C^*-algebra D, we denote by $\lim_{n \to \infty} I_n$ the ideal of D defined by

$$\lim_{n \to \infty} I_n = \overline{\bigcup_{n \in \mathbb{N}} I_n}.$$

Proposition 4.4. *Let $\{I_n\}_{n \in \mathbb{N}}$ be an increasing family of ideals of A. Then we have $X(\lim_{n \to \infty} I_n) = \lim_{n \to \infty} X(I_n)$.*

Proof. Clear by the definition of $X(\cdot)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The analogous statement of Proposition 4.4 for X^{-1} is not valid as the next example shows.

Example 4.5. Let $A = C((0, 1])$. We define a C^*-correspondence X over A which is isomorphic to A as Hilbert A-modules and its left action $\varphi_X\colon A \to \mathcal{L}(X)$ is defined by $\varphi_X(f) = f(1) \, \text{id}_X$ for $f \in A$. For each $n \in \mathbb{N}$, we define an ideal I_n of A by $I_n = C((2^{-n}, 1])$. We have $\lim_{n \to \infty} I_n = A$. It is not difficult to see that $X^{-1}(I_n) = C((0, 1))$ for every $n \in \mathbb{N}$. Hence we get $\lim_{n \to \infty} X^{-1}(I_n) = C((0, 1))$. However, we have $X^{-1}(\lim_{n \to \infty} I_n) = X^{-1}(A) = A$. The C^*-algebra

$\mathbb{O}_X$ is isomorphic to the universal C^*-algebra generated by a contractive scaling element (see [Katsura 2006b]).

Though we do not have $X^{-1}(\lim_{n\to\infty} I_n) = \lim_{n\to\infty} X^{-1}(I_n)$ in general, we can prove Proposition 4.7, which suffices for the further investigation. For the proof of Proposition 4.7, we need the following general fact.

Lemma 4.6. *Let D be a C^*-algebra, and $\{I_n\}_{n\in\mathbb{N}}$ be an increasing family of ideals of D. For each C^*-subalgebra B of D, we have $B\cap(\lim_{n\to\infty} I_n) = \lim_{n\to\infty}(B\cap I_n)$.*

Proof. Set $I_\infty = \lim_{n\to\infty} I_n$. Clearly we have $B \cap I_\infty \supset \lim_{n\to\infty}(B \cap I_n)$. Take a positive element $x \in B\cap I_\infty$. For $\varepsilon > 0$, let $f_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function defined by $f_\varepsilon(t) = \max\{0, t-\varepsilon\}$. Then we have $\|x - f_\varepsilon(x)\| \leq \varepsilon$. Since $\bigcup_{n\in\mathbb{N}} I_n$ is a dense ideal in I_∞, we have $f_\varepsilon(x) \in \bigcup_{n\in\mathbb{N}} I_n$ (see [Pedersen 1979, Theorem 5.6.1]). Thus x is approximated by elements $f_\varepsilon(x) \in B\cap\bigcup_{n\in\mathbb{N}} I_n = \bigcup_{n\in\mathbb{N}}(B \cap I_n)$. This shows that $x \in \lim_{n\to\infty}(B\cap I_n)$. Therefore $B\cap(\lim_{n\to\infty} I_n) = \lim_{n\to\infty}(B\cap I_n)$. $\square$

Note that Lemma 4.6 is not valid when I_n's are just C^*-subalgebras.

Proposition 4.7. *Let $\{I_n\}_{n\in\mathbb{N}}$ be an increasing family of ideals of A. For each ideal J of A with $\varphi_X(J) \subset \mathcal{K}(X)$, we have $J\cap X^{-1}(\lim_{n\to\infty} I_n) = \lim_{n\to\infty}(J\cap X^{-1}(I_n))$.*

Proof. Set $I_\infty = \lim_{n\to\infty} I_n$. First note that we have

$$J \cap X^{-1}(I) = \{a \in J \mid \varphi_X(a) \in \mathcal{K}(XI)\}$$

for an ideal I of A by Lemma 1.6. Take $a \in J \cap X^{-1}(I_\infty)$ and $\varepsilon > 0$. It is easy to see that $\mathcal{K}(XI_\infty) = \lim_{n\to\infty} \mathcal{K}(XI_n)$. By Lemma 4.6, we have $\varphi_X(J) \cap \mathcal{K}(XI_\infty) = \lim_{n\to\infty}(\varphi_X(J) \cap \mathcal{K}(XI_n))$. Since $\varphi_X(a) \in \varphi_X(J)\cap\mathcal{K}(XI_\infty)$, we can find $n \in \mathbb{N}$ and $k \in \varphi_X(J)\cap\mathcal{K}(XI_n)$ such that $\|\varphi_X(a) - k\| < \varepsilon$. Then we can find $x \in J$ with $\|x\| < \varepsilon$ and $\varphi_X(x) = \varphi_X(a) - k$. Set $j = a - x \in J$. We have $\varphi_X(j) = k \in \mathcal{K}(XI_n)$. Thus we get $j \in J\cap X^{-1}(I_n)$ and $\|a - j\| < \varepsilon$. Therefore we get $J\cap X^{-1}(I_\infty) \subset \lim_{n\to\infty}(J \cap X^{-1}(I_n))$. The converse inclusion is obvious. $\square$

Definition 4.8. An ideal I of A is said to be *positively invariant* if $X(I) \subset I$, *negatively invariant* if $J_X \cap X^{-1}(I) \subset I$, and *invariant* if I is both positively and negatively invariant.

In [Kajiwara et al. 1998; Fowler et al. 2003; Schweizer 2001], a positively invariant ideal is called X-invariant. It is clear that I is positively invariant if and only if $I \subset X^{-1}(I)$. It is also equivalent to $\varphi_X(I)X \subset XI$ by Proposition 1.3. Clearly A is an invariant ideal. We also see that 0 is invariant because $X(0) = 0$ and $J_X \cap X^{-1}(0) = J_X \cap \ker\varphi_X = 0$.

Proposition 4.9. *Let $\{I_n\}_{n\in\mathbb{N}}$ be an increasing family of ideals of A. If I_n is positively invariant (negatively invariant, invariant), then $\lim_{n\to\infty} I_n$ is also.*

Proof. Clear by Proposition 4.4 and Proposition 4.7. $\qquad \square$

Proposition 4.10. *If two ideals I_1, I_2 are positively invariant, then their intersection $I_1 \cap I_2$ is also positively invariant. The same is true for negative invariance.*

Proof. Clear by Lemma 4.2. $\qquad \square$

Corollary 4.11. *The intersection of two invariant ideals is invariant.*

By Lemma 4.2, we see that if two ideals I_1, I_2 are positively invariant, then so is their sum $I_1 + I_2$. However, the sum of two negatively invariant ideals need not be negatively invariant. Moreover, the sum of two invariant ideals can fail to be negatively invariant as we will see in the next example.

Example 4.12. Let A be $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, and X be $\mathbb{C} \oplus \mathbb{C}$ which is a Hilbert A-module by the operations $\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle_X = (\overline{\xi_1}\xi_2, \overline{\eta_1}\eta_2, 0)$ and $(\xi, \eta)(\lambda, \mu, \nu) = (\xi\lambda, \eta\mu)$. We define a left action $\varphi_X \colon A \to \mathcal{L}(X)$ by $\varphi_X((\lambda, \mu, \nu)) = \nu \, \mathrm{id}_X$. We define three ideals I_1, I_2 and I_3 of A by $I_1 = \mathbb{C} \oplus 0 \oplus 0$, $I_2 = 0 \oplus \mathbb{C} \oplus 0$ and $I_3 = 0 \oplus 0 \oplus \mathbb{C}$. We have $J_X = I_3$. An easy computation shows that $X(I_1) = X(I_2) = 0$ and $X^{-1}(I_1) = X^{-1}(I_2) = I_1 + I_2$. Thus both I_1 and I_2 are invariant ideals. However we have $X(I_1 + I_2) = 0$ and $X^{-1}(I_1 + I_2) = A$. Thus $I_1 + I_2$ is positively invariant, but not negatively invariant. We also have proper inclusions

$$
\begin{aligned}
A &= X^{-1}(I_1 + I_2) \;\supset\; X^{-1}(I_1) + X^{-1}(I_2) \;=\; I_1 + I_2 \\
0 &= X(X^{-1}(I_i)) \;\subset\; I_i \cap \overline{\mathrm{span}}\langle X, X \rangle_X \;=\; I_i \qquad (i = 1, 2).
\end{aligned}
$$

We have $\mathbb{O}_X \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, and the two nontrivial invariant ideals I_1, I_2 correspond to the two nontrivial ideals of $\mathbb{O}_X$.

Definition 4.13. Let us take an ideal I of A. We define ideals $X^n(I)$ for $n \in \mathbb{N}$ by $X^0(I) = I$ and $X^{n+1}(I) = X(X^n(I))$. We also define ideals $X_{-n}(I)$ for $n \in \mathbb{N}$ by $X_0(I) = I$, $X_{-1}(I) = I + J_X \cap X^{-1}(I)$ and $X_{-(n+1)}(I) = X_{-1}(X_{-n}(I))$ for $n \geq 1$.

Note that we have $I \subset X_{-1}(I)$, hence $X_{-n}(I) \subset X_{-(n+1)}(I)$ for every $n \in \mathbb{N}$

Definition 4.14. For an ideal I of A, we define ideals $X^\infty(I)$, $X_{-\infty}(I)$ and $X_{-\infty}^\infty(I)$ of A by

$$
X^\infty(I) = \sum_{n=0}^{\infty} X^n(I) = \lim_{k \to \infty} \sum_{n=0}^{k} X^n(I), \quad X_{-\infty}(I) = \lim_{n \to \infty} X_{-n}(I),
$$

$$
\text{and } \; X_{-\infty}^\infty(I) = X_{-\infty}(X^\infty(I)).
$$

Lemma 4.15. *If an ideal I is positively invariant, so are $X_{-n}(I)$ for $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. Let us take a positively invariant ideal I. From

$$
X_{-1}(I) = I + J_X \cap X^{-1}(I) \subset X^{-1}(I) \subset X^{-1}(X_{-1}(I))
$$

we see that $X_{-1}(I)$ is positively invariant. By using this fact, we can prove inductively that $X_{-n}(I)$ is positively invariant for all $n \in \mathbb{N}$. Finally $X_{-\infty}(I)$ is positively invariant by Proposition 4.9. $\qquad\square$

Proposition 4.16. *For an ideal I of A, the ideal $X^\infty(I)$ ($X_{-\infty}(I)$, $X_{-\infty}^\infty(I)$) is the smallest positively invariant (negatively invariant, invariant) ideal containing I.*

Proof. For each $k \in \mathbb{N}$, we have

$$X\left(\sum_{n=0}^{k} X^n(I)\right) = \sum_{n=0}^{k} X^{n+1}(I) \subset X^\infty(I).$$

Hence by Proposition 4.4, we have $X(X^\infty(I)) \subset X^\infty(I)$. Thus $X^\infty(I)$ is positively invariant. If I' is a positively invariant ideal containing I, then we can prove inductively $X^n(I) \subset I'$ for all $n \in \mathbb{N}$. Hence we have $X^\infty(I) \subset I'$. Thus $X^\infty(I)$ is the smallest positively invariant ideal containing I.

For each $n \in \mathbb{N}$, we have $J_X \cap X^{-1}(X_{-n}(I)) \subset X_{-(n+1)}(I) \subset X_{-\infty}(I)$. Hence by Proposition 4.7, we have $J_X \cap X^{-1}(X_{-\infty}(I)) \subset X_{-\infty}(I)$. Thus $X_{-\infty}(I)$ is negatively invariant. If I' is a negatively invariant ideal containing I, then we can prove inductively $X_{-n}(I) \subset I'$ for all $n \in \mathbb{N}$. Hence we have $X_{-\infty}(I) \subset I'$. Thus $X_{-\infty}(I)$ is the smallest negatively invariant ideal containing I.

Combining the above argument with Lemma 4.15, we see that $X_{-\infty}^\infty(I)$ is the smallest invariant ideal containing I. $\qquad\square$

5. T-pairs and O-pairs

In this section, we introduce the notion of T-pairs and O-pairs of the C^*-correspondence X over A. These are related to representations of X.

Definition 5.1. For an ideal I of A, we define an ideal $J(I)$ of A by

$$J(I) = \left\{a \in A \,\middle|\, [\varphi_X(a)]_I \in \mathcal{K}(X_I), \ aX^{-1}(I) \subset I\right\}.$$

For a positively invariant ideal I, we can define a map $\varphi_{X_I} : A/I \to \mathcal{L}(X_I)$ so that $\varphi_{X_I}([a]_I) = [\varphi_X(a)]_I$ because $a \in I$ implies $[\varphi_X(a)]_I = 0$. Thus in this case, X_I is a C^*-correspondence over A/I. It is clear that the pair $([\,\cdot\,]_I, [\,\cdot\,]_I)$ of the quotient maps $A \to A/I$ and $X \to X_I$ is a morphism from X to X_I.

Lemma 5.2. *For a positively invariant ideal I, we have $X^{-1}(I) = [\,\cdot\,]_I^{-1}(\ker \varphi_{X_I})$, $J(I) = [\,\cdot\,]_I^{-1}(J_{X_I})$ and $X^{-1}(I) \cap J(I) = I$.*

Proof. We have

$$X^{-1}(I) = \ker([\,\cdot\,]_I \circ \varphi_X) = \ker(\varphi_{X_I} \circ [\,\cdot\,]_I) = [\,\cdot\,]_I^{-1}(\ker \varphi_{X_I}).$$

We also see that $[\varphi_X(a)]_I \in \mathcal{K}(X_I)$ if and only if $\varphi_{X_I}([a]_I) \in \mathcal{K}(X_I)$. Since $X^{-1}(I) = [\cdot]_I^{-1}(\ker \varphi_{X_I})$, the condition $aX^{-1}(I) \subset I$ for $a \in A$ is equivalent to $[a]_I \ker \varphi_{X_I} = 0$. Hence $a \in J(I)$ if and only if

$$[a]_I \in \varphi_{X_I}^{-1}\big(\mathcal{K}(X_I)\big) \cap \big(\ker \varphi_{X_I}\big)^\perp = J_{X_I}.$$

Thus we get $J(I) = [\cdot]_I^{-1}(J_{X_I})$. Finally,

$$X^{-1}(I) \cap J(I) = [\cdot]_I^{-1}(\ker \varphi_{X_I} \cap J_{X_I}) = [\cdot]_I^{-1}(0) = I. \qquad \square$$

Note that Lemma 5.2 implies that $X^{-1}(I)/I = \ker \varphi_{X_I}$ and $J(I)/I = J_{X_I}$ for a positively invariant ideal I. Note also that $X^{-1}(0) = \ker \varphi_X$ and $J(0) = J_X$.

Proposition 5.3. *An ideal I is negatively invariant if and only if $J_X \subset J(I)$.*

Proof. For $a \in J_X$, we have $\varphi_X(a) \in \mathcal{K}(X)$. Hence $[\varphi_X(a)]_I \in \mathcal{K}(X_I)$. Thus $J_X \subset J(I)$ if and only if $J_X X^{-1}(I) \subset I$. This is equivalent to the negative invariance of I because $J_X X^{-1}(I) = J_X \cap X^{-1}(I)$. $\qquad \square$

Note that $I_1 \subset I_2$ need not imply $J(I_1) \subset J(I_2)$ in general as the following example shows.

Example 5.4 (compare Example 4.12). Let $A \cong \mathbb{C}^3$ be the C^*-algebra generated by three mutually orthogonal projections p_0, p_1 and p_2. Let X be the ℓ^∞-direct sum of two Hilbert spaces $\mathbb{C}$, whose base is denoted by s_0, and $\ell^2(\mathbb{N})$, whose base is denoted by $\{s_k\}_{k=1}^\infty$. We define an inner product $\langle \cdot, \cdot \rangle_X : X \times X \to A$ by $\langle s_0, s_0 \rangle_X = p_0$, $\langle s_k, s_k \rangle_X = p_1$ for $k = 1, 2, \ldots$, and $\langle s_k, s_l \rangle_X = 0$ for $k \neq l$. The right action of A on X is defined by

$$s_k p_i = \begin{cases} s_0 & \text{for } k = i = 0, \\ s_k & \text{for } k \geq 1, i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then X becomes a Hilbert A-module. We define a left action $\varphi_X : A \to \mathcal{L}(X)$ by $\varphi_X(p_0) = \varphi_X(p_1) = 0$, and $\varphi_X(p_2) = \mathrm{id}_X$. Now we get a C^*-correspondence X over A. This C^*-correspondence is defined from the following graph;

(see [Katsura 2004a]). Let us define ideals of A by

$$I_0 = \mathbb{C}p_0, \quad I_1 = \mathbb{C}p_1, \quad I_{01} = \mathbb{C}p_0 + \mathbb{C}p_1 \quad \text{and} \quad I_{12} = \mathbb{C}p_1 + \mathbb{C}p_2.$$

Since $\ker \varphi_X = \varphi_X^{-1}(\mathcal{K}(X)) = I_{01}$, we have $J_X = 0$. Hence all ideals are negatively invariant. Since $X(I_1) = X(I_{01}) = 0$, both I_1 and I_{01} are invariant. By straightforward computation, we get $J(I_1) = I_{12}$ and $J(I_{01}) = I_{01}$. Thus two ideals I_1, I_{01} satisfy that $I_1 \subset I_{01}$ and $J(I_1) \not\subset J(I_{01})$. We can see that $\mathbb{O}_X$ is isomorphic to the direct sum of $M_2(\mathbb{C})$ and the unitization $\widetilde{K}$ of the C^*-algebra K of compact operators on $\ell^2(\mathbb{N})$. There exist six O-pairs (see Definition 5.12) which correspond to six ideals of $\mathbb{O}_X \cong M_2(\mathbb{C}) \oplus \widetilde{K}$;

$$
\begin{array}{ccccccccc}
(0,0) & \subset & (I_1, I_1) & \subset & (I_1, I_{12}) & & 0 & \subset & K & \subset & \widetilde{K} \\
\cap & & \cap & & \cap & \leftrightsquigarrow & \cap & & \cap & & \cap \\
(I_0, I_0) & \subset & (I_{01}, I_{01}) & \subset & (A, A) & & M_2(\mathbb{C}) & \subset & M_2(\mathbb{C}) \oplus K & \subset & \mathbb{O}_X.
\end{array}
$$

This example also shows that $J(I_1 \cap I_2) \subset J(I_1) \cap J(I_2)$ does not hold in general for two ideals I_1, I_2 of A. However, the converse inclusion $J(I_1) \cap J(I_2) \subset J(I_1 \cap I_2)$ always holds.

Proposition 5.5. *For two ideals I_1, I_2 of A, we have $J(I_1) \cap J(I_2) \subset J(I_1 \cap I_2)$.*

Proof. Take $a \in J(I_1) \cap J(I_2)$. Since $[\varphi_X(a)]_{I_1} \in \mathcal{K}(X_{I_1})$ and $[\varphi_X(a)]_{I_2} \in \mathcal{K}(X_{I_2})$, we have $[\varphi_X(a)]_{I_1 \cap I_2} \in \mathcal{K}(X_{I_1 \cap I_2})$ by Corollary 1.12. We get $aX^{-1}(I_1 \cap I_2) \subset I_1 \cap I_2$ from

$$
aX^{-1}(I_1 \cap I_2) \subset aX^{-1}(I_1) \subset I_1, \quad aX^{-1}(I_1 \cap I_2) \subset aX^{-1}(I_2) \subset I_2.
$$

Hence $a \in J(I_1 \cap I_2)$. Thus we have $J(I_1) \cap J(I_2) \subset J(I_1 \cap I_2)$. $\qquad\square$

Definition 5.6. Let X be a C^*-correspondence over a C^*-algebra A. A T-*pair* of X is a pair $\omega = (I, I')$ of ideals I, I' of A such that I is positively invariant and $I \subset I' \subset J(I)$.

Definition 5.7. Let $\omega_1 = (I_1, I_1')$ and $\omega_2 = (I_2, I_2')$ be T-pairs. We write $\omega_1 \subset \omega_2$ if $I_1 \subset I_2$ and $I_1' \subset I_2'$. We denote by $\omega_1 \cap \omega_2$ the pair $(I_1 \cap I_2, I_1' \cap I_2')$.

Proposition 5.8. *For two T-pairs $\omega_1 = (I_1, I_1')$, $\omega_2 = (I_2, I_2')$, their intersection $\omega_1 \cap \omega_2 = (I_1 \cap I_2, I_1' \cap I_2')$ is a T-pair.*

Proof. By Proposition 4.10, $I_1 \cap I_2$ is a positively invariant ideal. By Proposition 5.5, we have

$$
I_1 \cap I_2 \subset I_1' \cap I_2' \subset J(I_1) \cap J(I_2) \subset J(I_1 \cap I_2).
$$

Hence $\omega_1 \cap \omega_2$ is a T-pair. $\qquad\square$

T-pairs arise from representations.

Definition 5.9. For a representation (π, t) of X, we define $I_{(\pi,t)}$, $I'_{(\pi,t)} \subset A$ by

$$
I_{(\pi,t)} = \ker \pi, \quad I'_{(\pi,t)} = \pi^{-1}\big(\psi_t(\mathcal{K}(X))\big).
$$

The pair $\big(I_{(\pi,t)}, I'_{(\pi,t)}\big)$ is denoted by $\omega_{(\pi,t)}$.

Clearly $I_{(\pi,t)}$ is an ideal of A. By the remark before Definition 2.8, we see that $I'_{(\pi,t)}$ is also an ideal of A.

Lemma 5.10. *For a representation (π, t) of a C^*-correspondence X over a C^*-algebra A, we have the following.*

(i) *$I_{(\pi,t)}$ is positively invariant.*

(ii) *$\ker t = XI_{(\pi,t)}$.*

(iii) *There exists an injective representation $(\dot{\pi}, \dot{t})$ of the C^*-correspondence $X_{I_{(\pi,t)}}$ on $C^*(\pi, t)$ such that $(\pi, t) = (\dot{\pi} \circ [\,\cdot\,]_{I_{(\pi,t)}}, \dot{t} \circ [\,\cdot\,]_{I_{(\pi,t)}})$.*

(iv) *$a \in I'_{(\pi,t)}$ implies $[\varphi_X(a)]_{I_{(\pi,t)}} \in \mathcal{K}(X_{I_{(\pi,t)}})$ and $\pi(a) = \psi_{\dot{t}}([\varphi_X(a)]_{I_{(\pi,t)}})$.*

(v) *For an element $a \in A$ with $\varphi_X(a) \in \mathcal{K}(X)$, we have $\pi(a) = \psi_t(\varphi_X(a))$ if and only if $a \in I'_{(\pi,t)}$.*

Proof.

(i) For $a \in I_{(\pi,t)}$ and $\xi, \eta \in X$, we have $\langle \eta, \varphi_X(a)\xi \rangle_X \in I_{(\pi,t)}$ because

$$\pi\big(\langle \eta, \varphi_X(a)\xi \rangle_X\big) = t(\eta)^* t(\varphi_X(a)\xi) = t(\eta)^* \pi(a) t(\xi) = 0.$$

Hence $X(I_{(\pi,t)}) \subset I_{(\pi,t)}$. Thus $I_{(\pi,t)}$ is positively invariant.

(ii) For $\xi \in X$, we have

$$\xi \in \ker t \Longleftrightarrow t(\xi) = 0 \Longleftrightarrow t(\xi)^* t(\xi) = 0$$
$$\Longleftrightarrow \pi(\langle \xi, \xi \rangle_X) = 0 \Longleftrightarrow \langle \xi, \xi \rangle_X \in I_{(\pi,t)} \Longleftrightarrow \xi \in XI_{(\pi,t)}.$$

(iii) Obvious by the definition of $I_{(\pi,t)}$ and (ii).

(iv) Since $a \in I'_{(\pi,t)}$, we can find $k \in \mathcal{K}(X)$ with $\pi(a) = \psi_t(k)$. For $\xi \in X$, we have

$$t(\varphi_X(a)\xi) = \pi(a) t(\xi) = \psi_t(k) t(\xi) = t(k\xi).$$

Hence $\big(\varphi_X(a) - k\big)\xi \in \ker t = XI_{(\pi,t)}$ for all $\xi \in X$. This implies that $[\varphi_X(a)]_{I_{(\pi,t)}} = [k]_{I_{(\pi,t)}} \in \mathcal{K}(X_{I_{(\pi,t)}})$ and

$$\pi(a) = \psi_t(k) = \psi_{\dot{t}}([k]_{I_{(\pi,t)}}) = \psi_{\dot{t}}([\varphi_X(a)]_{I_{(\pi,t)}}).$$

(v) If $\pi(a) = \psi_t(\varphi_X(a))$, then $a \in I'_{(\pi,t)}$. For $a \in I'_{(\pi,t)}$ with $\varphi_X(a) \in \mathcal{K}(X)$, we have $\pi(a) = \psi_{\dot{t}}([\varphi_X(a)]_{I_{(\pi,t)}}) = \psi_t(\varphi_X(a))$ by (iv). $\qquad \square$

Proposition 5.11. *For a representation (π, t) of X, the pair $\omega_{(\pi,t)}$ is a T-pair.*

Proof. By Lemma 5.10 (i), $I_{(\pi,t)}$ is positively invariant. Clearly we have $I_{(\pi,t)} \subset I'_{(\pi,t)}$. Take $a \in I'_{(\pi,t)}$. We have $[\varphi_X(a)]_{I_{(\pi,t)}} \in \mathcal{K}(X_{I_{(\pi,t)}})$ by Lemma 5.10 (iv). Take $b \in X^{-1}(I_{(\pi,t)})$. Since $ab \in I'_{(\pi,t)}$, we have $\pi(ab) = \psi_{\dot{t}}\big([\varphi_X(ab)]_{I_{(\pi,t)}}\big)$ by Lemma 5.10 (iv). We see $[\varphi_X(ab)]_{I_{(\pi,t)}} = 0$ because $ab \in X^{-1}(I_{(\pi,t)})$. Hence $\pi(ab) = 0$. Thus we get $ab \in \ker \pi = I_{(\pi,t)}$. This shows $a \in J(I_{(\pi,t)})$. Hence we get $I'_{(\pi,t)} \subset J(I_{(\pi,t)})$. Thus $\omega_{(\pi,t)} = \big(I_{(\pi,t)}, I'_{(\pi,t)}\big)$ is a T-pair. $\qquad \square$

We will see that all T-pairs come from representations (Proposition 6.12). In the same way as in the proof of Proposition 5.11, we can see that for a morphism (Π, T) from a C^*-correspondence X to a C^*-correspondence Y, the pair $\omega_{(\Pi,T)} = \left(I_{(\Pi,T)}, I'_{(\Pi,T)}\right)$ defined by

$$I_{(\Pi,T)} = \ker \Pi, \quad I'_{(\Pi,T)} = (\varphi_Y \circ \Pi)^{-1}\left(\Psi_T(\mathcal{K}(X))\right)$$

is a T-pair.

Definition 5.12. A T-pair $\omega = (I, I')$ satisfying $J_X \subset I'$ is called an *O-pair*.

It is clear that the intersection $\omega_1 \cap \omega_2$ of two O-pairs ω_1, ω_2 is an O-pair.

Lemma 5.13. *A pair $\omega = (I, I')$ of ideals of A is an O-pair if and only if I is invariant and $I + J_X \subset I' \subset J(I)$.*

Proof. For an O-pair $\omega = (I, I')$, we have $I + J_X \subset I' \subset J(I)$. Thus we get $J_X \subset J(I)$. Now Proposition 5.3 implies that I is negatively invariant. Therefore I is an invariant ideal. The converse is obvious. □

For a C^*-correspondence $X = C_d(E^1)$ arising from a topological graph E, an O-pair (I, I') is in the form $(C_0(E^0 \setminus X^0), C_0(E^0 \setminus Z))$ where (X^0, Z) is an admissible pair of closed sets of E^0 defined in [Katsura 2006a].
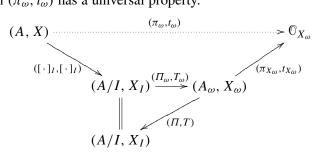
Proposition 5.14. *A representation (π, t) is covariant if and only if the pair $\omega_{(\pi,t)}$ is an O-pair.*

Proof. If (π, t) is covariant, then clearly $J_X \subset I'_{(\pi,t)}$. Thus $\omega_{(\pi,t)}$ is an O-pair. Conversely, if $\omega_{(\pi,t)}$ is an O-pair, then for $a \in J_X \subset I'_{(\pi,t)}$, we have $\pi(a) = \psi_t(\varphi_X(a))$ by Lemma 5.10 (v). Hence (π, t) is covariant. □

By Proposition 5.14, we have $\omega_{(\pi,t)} = (0, J_X)$ for all injective covariant representations (π, t).

6. C^*-correspondences associated with T-pairs

Take a T-pair $\omega = (I, I')$ of X and fix it throughout this section. In this section, we construct a C^*-algebra A_ω, a C^*-correspondence X_ω over A_ω and a representation (π_ω, t_ω) of X on the C^*-algebra $\mathcal{O}_{X_\omega}$. In the next section, we will see that this representation (π_ω, t_ω) has a universal property.

Definition 6.1. For a T-pair $\omega = (I, I')$ of a C^*-correspondence X over A, we define a C^*-algebra A_ω and a Hilbert A_ω-module X_ω by

$$A_\omega = \big\{ (b, b') \in A/I \oplus A/I' \ \big|\ [b]_{J(I)/I} = [b']_{J(I)/I'} \in A/J(I) \big\},$$
$$X_\omega = \big\{ (\eta, \eta') \in X_I \oplus X_{I'} \ \big|\ [\eta]_{J(I)/I} = [\eta']_{J(I)/I'} \in X_{J(I)} \big\},$$

where the operations are defined as in Section 1.

Note that A_ω is a pull-back C^*-algebra of two surjections $[\cdot]_{J(I)/I} \colon A/I \to A/J(I)$ and $[\cdot]_{J(I)/I'} \colon A/I' \to A/J(I)$.

Definition 6.2. We define a $*$-homomorphism $\Psi_\omega \colon \mathscr{L}(X_I) \to \mathscr{L}(X_\omega)$ by

$$\Psi_\omega(S)(\eta, \eta') = (S\eta, [S]_{I'/I}\eta') \in X_\omega$$

for $S \in \mathscr{L}(X_I)$ and $(\eta, \eta') \in X_\omega$.

Definition 6.3. We define a left action $\varphi_{X_\omega} \colon A_\omega \to \mathscr{L}(X_\omega)$ by

$$\varphi_{X_\omega}\big((b, b')\big) = \Psi_\omega\big(\varphi_{X_I}(b)\big),$$

for $(b, b') \in A_\omega$. Thus X_ω is a C^*-correspondence over A_ω.

Definition 6.4. We set

$$\Pi_\omega \colon A/I \ni b \mapsto (b, [b]_{I'/I}) \in A_\omega, \quad T_\omega \colon X_I \ni \eta \mapsto (\eta, [\eta]_{I'/I}) \in X_\omega.$$

Lemma 6.5. *We have* $\varphi_{X_\omega} \circ \Pi_\omega = \Psi_\omega \circ \varphi_{X_I}$, *and* $T_\omega(S\eta) = \Psi_\omega(S)T_\omega(\eta)$ *for* $S \in \mathscr{L}(X_I)$ *and* $\eta \in X_I$.

Proof. Clear by the definitions.   □

From this lemma, we easily get the following.

Proposition 6.6. *The pair* (Π_ω, T_ω) *is an injective morphism from* X_I *to* X_ω, *and the map* $\Psi_{T_\omega} \colon \mathscr{K}(X_I) \to \mathscr{K}(X_\omega)$ *coincides with the restriction of* Ψ_ω *to* $\mathscr{K}(X_I)$.

The next proposition is also easy to see from the definitions.

Proposition 6.7. *For a* T-pair $\omega = (I, I')$ *with* $I' = J(I)$, *the morphism* (Π_ω, T_ω) *from* X_I *to* X_ω *is an isomorphism.*

To compute $J_{X_\omega} \subset A_\omega$, we need the following lemma.

Lemma 6.8. *A pair* (Π, T) *of maps defined by*

$$\Pi \colon A_\omega \ni (b, b') \mapsto b \in A/I, \quad T \colon X_\omega \ni (\eta, \eta') \mapsto \eta \in X_I,$$

is a morphism from X_ω *to* X_I *satisfying* $\Pi \circ \Pi_\omega = \mathrm{id}_{A/I}$ *and* $T \circ T_\omega = \mathrm{id}_{X_I}$. *A $*$-homomorphism* $\Psi \colon \mathscr{L}(X_\omega) \ni S \mapsto T \circ S \circ T_\omega \in \mathscr{L}(X_I)$ *satisfies that* $\Psi \circ \Psi_\omega = \mathrm{id}_{\mathscr{L}(X_I)}$ *and the restriction of* Ψ *to* $\mathscr{K}(X_\omega)$ *coincides with* $\Psi_T \colon \mathscr{K}(X_\omega) \to \mathscr{K}(X_I)$.

Proof. It is clear that (Π, T) is a morphism satisfying $\Pi \circ \Pi_\omega = \mathrm{id}_{A/I}$ and $T \circ T_\omega = \mathrm{id}_{X_I}$. By Lemma 6.5, we have

$$\Psi(\Psi_\omega(S))\eta = T\big(\Psi_\omega(S)T_\omega(\eta)\big) = T\big(T_\omega(S\eta)\big) = S\eta,$$

for $S \in \mathscr{L}(X_I)$ and $\eta \in X_I$. This proves $\Psi \circ \Psi_\omega = \mathrm{id}_{\mathscr{L}(X_I)}$. For $(\eta_1, \eta_1'), (\eta_2, \eta_2') \in X_\omega$ and $\eta \in X_I$, we have

$$\begin{aligned}
\Psi\big(\theta_{(\eta_1,\eta_1'),(\eta_2,\eta_2')}\big)\eta &= T\big(\theta_{(\eta_1,\eta_1'),(\eta_2,\eta_2')}T_\omega(\eta)\big) \\
&= T\big((\eta_1\langle\eta_2,\eta\rangle_{X_I}, \eta_1'\langle\eta_2', [\eta]_{I'/I}\rangle_{X_{I'}})\big) \\
&= \eta_1\langle\eta_2,\eta\rangle_{X_I} \\
&= \theta_{\eta_1,\eta_2}(\eta).
\end{aligned}$$

Hence we have $\Psi\big(\theta_{(\eta_1,\eta_1'),(\eta_2,\eta_2')}\big) = \theta_{\eta_1,\eta_2}$. This shows that the restriction of Ψ to $\mathscr{K}(X_\omega)$ coincides with $\Psi_T \colon \mathscr{K}(X_\omega) \to \mathscr{K}(X_I)$. $\qquad\square$

Proposition 6.9. *We have*

$$\begin{aligned}
\ker\varphi_{X_\omega} &= \big\{(b,b') \in A_\omega \mid b \in \ker\varphi_{X_I}\big\}, \\
\varphi_{X_\omega}^{-1}\big(\mathscr{K}(X_\omega)\big) &= \big\{(b,b') \in A_\omega \mid b \in \varphi_{X_I}^{-1}\big(\mathscr{K}(X_I)\big)\big\}, \\
J_{X_\omega} &= \big\{(b,b') \in A_\omega \mid b \in J_{X_I}, \ b' = 0\big\}.
\end{aligned}$$

Proof. Since $\Psi \circ \Psi_\omega = \mathrm{id}_{\mathscr{L}(X_I)}$ by Lemma 6.8, we have $\Psi\big(\varphi_{X_\omega}((b,b'))\big) = \varphi_{X_I}(b)$ for $(b,b') \in A_\omega$. Hence for $(b,b') \in A_\omega$, we have that $\varphi_{X_\omega}((b,b')) = 0$ if and only if $\varphi_{X_I}(b) = 0$. This proves the first equality. The second one follows similarly because we have $\Psi_\omega\big(\mathscr{K}(X_I)\big) \subset \mathscr{K}(X_\omega)$ and $\Psi\big(\mathscr{K}(X_\omega)\big) \subset \mathscr{K}(X_I)$. We will prove the third equality. It is easy to see that for $b \in J_{X_I}$, we have

$$(b,0) \in \varphi_{X_\omega}^{-1}\big(\mathscr{K}(X_\omega)\big) \cap (\ker\varphi_{X_\omega})^\perp = J_{X_\omega}.$$

Take $(b,b') \in J_{X_\omega}$, and we will prove that $b \in J_{X_I}$ and $b' = 0$. Since $\varphi_{X_\omega}((b,b')) \in \mathscr{K}(X_\omega)$, we have $\varphi_{X_I}(b) \in \mathscr{K}(X_I)$. For any $b_0 \in \ker\varphi_{X_I} \subset A/I$, we have $\Pi_\omega(b_0) = (b_0, [b_0]_{I'/I}) \in \ker\varphi_{X_\omega}$. Hence $(b,b')(b_0, [b_0]_{I'/I}) = 0$. This implies that $b \in (\ker\varphi_{X_I})^\perp$. Hence $b \in J_{X_I}$. Since $J_{X_I} = J(I)/I$ by Lemma 5.2, we have

$$[b']_{J(I)/I'} = [b]_{J(I)/I} = 0.$$

Hence $(0, b'^*) \in A_\omega$. Since $(0, b'^*) \in \ker\varphi_{X_\omega}$, we have $(b,b')(0, b'^*) = 0$. This implies $b' = 0$. Thus we get $J_{X_\omega} = \{(b,b') \in A_\omega \mid b \in J_{X_I}, \ b' = 0\}$. $\qquad\square$

We have $J_{X_\omega} = \big\{(b,b') \in A_\omega \mid b' = 0\big\}$ because for $b \in A/I$ we have $(b,0) \in A_\omega$ if and only if $b \in J_{X_I}$.

Definition 6.10. We define a $*$-homomorphism $\pi_\omega \colon A \to \mathbb{O}_{X_\omega}$ and a linear map $t_\omega \colon X \to \mathbb{O}_{X_\omega}$ by

$$\pi_\omega(a) = \pi_{X_\omega}\big(\Pi_\omega([a]_I)\big), \quad t_\omega(\xi) = t_{X_\omega}\big(T_\omega([\xi]_I)\big)$$

for $a \in A$ and $\xi \in X$, where $(\pi_{X_\omega}, t_{X_\omega})$ is the universal covariant representation of the C^*-correspondence X_ω on $\mathbb{O}_{X_\omega}$.

Proposition 6.11. *The pair (π_ω, t_ω) is a representation of X on $\mathbb{O}_{X_\omega}$, which admits a gauge action and satisfies $C^*(\pi_\omega, t_\omega) = \mathbb{O}_{X_\omega}$.*

Proof. Since (π_ω, t_ω) is a composition of morphisms, it is a representation. Clearly the gauge action of $\mathbb{O}_{X_\omega}$ gives a gauge action for the representation (π_ω, t_ω). We will prove $C^*(\pi_\omega, t_\omega) = \mathbb{O}_{X_\omega}$. Since $\mathbb{O}_{X_\omega}$ is generated by the images of π_{X_ω} and t_{X_ω}, it suffices to show that

$$\pi_{X_\omega}(A_\omega), t_{X_\omega}(X_\omega) \subset C^*(\pi_\omega, t_\omega).$$

Take $(b, b') \in A_\omega$. Choose $a \in A$ with $[a]_{I'} = b'$. We have $b - [a]_I \in J(I)/I = J_{X_I}$ because $[b]_{J(I)/I} = [b']_{J(I)/I'} = [a]_{J(I)}$. Thus we have $\varphi_{X_I}(b - [a]_I) \in \mathcal{K}(X_I)$. Hence there exists $k \in \mathcal{K}(X)$ such that $[k]_I = \varphi_{X_I}(b - [a]_I)$. Since $(b - [a]_I, 0) \in J_{X_\omega}$ by Proposition 6.9, we have

$$\pi_{X_\omega}\big((b - [a]_I, 0)\big) = \psi_{t_{X_\omega}}\big(\varphi_{X_\omega}((b - [a]_I, 0))\big) = \psi_{t_{X_\omega}}\big(\Psi_\omega(\varphi_{X_I}(b - [a]_I))\big)$$
$$= \psi_{t_{X_\omega}}\big(\Psi_{t_\omega}([k]_I)\big) = \psi_{t_\omega}(k).$$

Therefore we get

$$\pi_{X_\omega}\big((b, b')\big) = \pi_{X_\omega}\big(([a]_I, [a]_{I'})\big) + \pi_{X_\omega}\big((b - [a]_I, 0)\big)$$
$$= \pi_\omega(a) + \psi_{t_\omega}(k) \in C^*(\pi_\omega, t_\omega).$$

Thus we have shown that $\pi_{X_\omega}(A_\omega) \subset C^*(\pi_\omega, t_\omega)$.

Take $(\eta, \eta') \in X_\omega$. Choose $\xi \in X$ with $[\xi]_{I'} = \eta'$. As above, we get $\eta - [\xi]_I \in X_I J_{X_I}$. Choose $\xi' \in X$ and $b \in J_{X_I}$ with $\eta - [\xi]_I = [\xi']_I b$. Then we have $(\eta - [\xi]_I, 0) = T_\omega([\xi']_I)(b, 0)$. Hence we get

$$t_{X_\omega}\big((\eta, \eta')\big) = t_{X_\omega}\big(([\xi]_I, [\xi]_{I'})\big) + t_{X_\omega}\big((\eta - [\xi]_I, 0)\big)$$
$$= t_\omega(\xi) + t_{X_\omega}\big(T_\omega([\xi']_I)\big)\pi_{X_\omega}\big((b, 0)\big)$$
$$= t_\omega(\xi) + t_\omega(\xi')\pi_{X_\omega}\big((b, 0)\big) \in C^*(\pi_\omega, t_\omega),$$

because $\pi_{X_\omega}\big((b, 0)\big) \in C^*(\pi_\omega, t_\omega)$ as shown above. This completes the proof. $\qquad\square$

Proposition 6.12. *For a T-pair $\omega = (I, I')$, we have $\omega_{(\pi_\omega, t_\omega)} = \omega$.*

Proof. Since the maps $\Pi_\omega\colon A_I \to A_\omega$ and $\pi_{X_\omega}\colon A_\omega \to \mathbb{O}_{X_\omega}$ are injective, we have

$$I_{(\pi_\omega, t_\omega)} = \ker \pi_\omega = \ker([\,\cdot\,]_I) = I.$$

For $a \in I'$, we have $[a]_I \in I'/I \subset J(I)/I = J_{X_I}$. Since $\Pi_\omega([a]_I) = ([a]_I, 0) \in J_{X_I}$, we have

$$\pi_\omega(a) = \pi_{X_\omega}\big(\Pi_\omega([a]_I)\big) = \psi_{t_{X_\omega}}\big(\varphi_{X_\omega}(([a]_I, 0))\big).$$

We see $\varphi_{X_I}([a]_I) \in \mathcal{K}(X_I)$ from $[a]_I \in J_{X_I}$. Hence by the definition of φ_{X_ω} we get

$$\varphi_{X_\omega}\big(([a]_I, 0)\big) = \Psi_{T_\omega}\big(\varphi_{X_I}([a]_I)\big) \in \Psi_{T_\omega}\big(\mathcal{K}(X_I)\big).$$

Since $\mathcal{K}(X_I) = [\mathcal{K}(X)]_I$, we have

$$\pi_\omega(a) \in \psi_{t_{X_\omega}}\big(\Psi_{T_\omega}\big([\mathcal{K}(X)]_I\big)\big) = \psi_{t_\omega}\big(\mathcal{K}(X)\big).$$

Hence $a \in I'_{(\pi_\omega, t_\omega)}$. We have shown that $I' \subset I'_{(\pi_\omega, t_\omega)}$. Conversely take $a \in I'_{(\pi_\omega, t_\omega)}$. Since

$$\pi_{X_\omega}\big(\Pi_\omega([a]_I)\big) = \pi_\omega(a) \in \psi_{t_\omega}\big(\mathcal{K}(X)\big) \subset \psi_{t_{X_\omega}}\big(\mathcal{K}(X_\omega)\big),$$

we have $\Pi_\omega([a]_I) \in J_{X_\omega}$. Hence by Proposition 6.9, we have $[a]_{I'} = 0$. This means $a \in I'$. Thus we get $I'_{(\pi_\omega, t_\omega)} \subset I'$. Therefore $I'_{(\pi_\omega, t_\omega)} = I'$. We have shown that $\omega_{(\pi_\omega, t_\omega)} = \omega$. $\qquad\square$
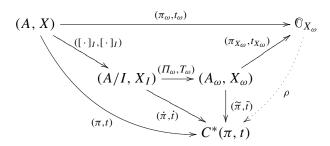
By Proposition 6.12, we see that all T-pairs come from representations.

7. C^*-algebras generated by representations of C^*-correspondences

In this section, we prove the following theorem.

Theorem 7.1. *Let X be a C^*-correspondence over a C^*-algebra A, and (π, t) be a representation of X. If a T-pair ω of X satisfies $\omega \subset \omega_{(\pi,t)}$, then there exists a unique surjective $*$-homomorphism $\rho\colon \mathbb{O}_{X_\omega} \to C^*(\pi, t)$ such that $\pi = \rho \circ \pi_\omega$ and $t = \rho \circ t_\omega$. The surjection ρ is an isomorphism if and only if $\omega = \omega_{(\pi,t)}$ and (π, t) admits a gauge action.*

Take a representation (π, t) of a C^*-correspondence X and a T-pair $\omega = (I, I')$ of X satisfying $\omega \subset \omega_{(\pi,t)}$. In order to get a $*$-homomorphism $\rho\colon \mathbb{O}_{X_\omega} \to C^*(\pi, t)$, we will construct a covariant representation $(\widetilde{\pi}, \widetilde{t})$ of the C^*-correspondence X_ω on $C^*(\pi, t)$. Since $I \subset I_{(\pi,t)} = \ker \pi$, we can define a representation $(\dot{\pi}, \dot{t})$ of a C^*-correspondence X_I over A/I on $C^*(\pi, t)$ such that $\dot{\pi}([a]_I) = \pi(a)$ for $a \in A$ and $\dot{t}([\xi]_I) = t(\xi)$ for $\xi \in X$ as in Lemma 5.10 (iii). It is easy to see that $I_{(\dot{\pi}, \dot{t})} = I_{(\pi,t)}/I$ and $I'_{(\dot{\pi}, \dot{t})} = I'_{(\pi,t)}/I$.

Definition 7.2. Let $(b, b') \in A_\omega$. Take $d \in A/I$ with $[d]_{I'/I} = b'$. Define $\widetilde{\pi}\big((b, b')\big) \in C^*(\pi, t)$ by

$$\widetilde{\pi}\big((b, b')\big) = \dot{\pi}(d) + \psi_i\big(\varphi_{X_I}(b - d)\big) \in C^*(\pi, t).$$

Note that this definition makes sense because $b - d \in J(I)/I = J_{X_I}$ implies $\varphi_{X_I}(b - d) \in \mathcal{K}(X_I)$. Note also that $\widetilde{\pi}\big((b, b')\big) \in C^*(\pi, t)$ does not depend on the choice of $d \in A/I$ with $[d]_{I'/I} = b'$ because we have $\dot{\pi}(d_1 - d_2) = \psi_i\big(\varphi_{X_I}(d_1 - d_2)\big)$ if $d_1 - d_2 \in I'/I \subset I'_{(\pi,t)}/I = I'_{(\dot{\pi},i)}$ by Lemma 5.10 (v).

Lemma 7.3. *The map $\widetilde{\pi} \colon A_\omega \to C^*(\pi, t)$ is a $*$-homomorphism.*

Proof. It is obvious that $\widetilde{\pi}$ is a $*$-preserving linear map. We will show $\widetilde{\pi}$ is multiplicative. Take $(b_1, b'_1), (b_2, b'_2) \in A_\omega$. Take $d_1, d_2 \in A/I$ with $[d_1]_{I'/I} = b'_1$, $[d_2]_{I'/I} = b'_2$. Since

$$\dot{\pi}(d)\psi_i(\varphi_{X_I}(b)) = \psi_i(\varphi_{X_I}(db))$$

for $d \in A/I$ and $b \in J(I)/I = J_{X_I}$, we have

$$\begin{aligned}
\widetilde{\pi}&\big((b_1, b'_1)\big)\widetilde{\pi}\big((b_2, b'_2)\big) \\
&= \big(\dot{\pi}(d_1) + \psi_i(\varphi_{X_I}(b_1 - d_1))\big)\big(\dot{\pi}(d_2) + \psi_i(\varphi_{X_I}(b_2 - d_2))\big) \\
&= \dot{\pi}(d_1 d_2) + \psi_i\big(\varphi_{X_I}\big(d_1(b_2 - d_2) + (b_1 - d_1)d_2 + (b_1 - d_1)(b_2 - d_2)\big)\big) \\
&= \dot{\pi}(d_1 d_2) + \psi_i\big(\varphi_{X_I}(b_1 b_2 - d_1 d_2)\big) \\
&= \widetilde{\pi}\big((b_1 b_2, b'_1 b'_2)\big) \\
&= \widetilde{\pi}\big((b_1, b'_1)(b_2, b'_2)\big).
\end{aligned}$$

Hence $\widetilde{\pi}$ is a $*$-homomorphism. $\square$

Proposition 7.4. *The map $\widetilde{\pi} \colon A_\omega \to C^*(\pi, t)$ is injective if and only if $\omega = \omega_{(\pi,t)}$.*

Proof. Suppose that $\widetilde{\pi}$ is injective. For $a \in I_{(\pi,t)}$, we have $([a]_I, [a]_{I'}) \in A_\omega$ and

$$\widetilde{\pi}\big(([a]_I, [a]_{I'})\big) = \dot{\pi}([a]_I) = \pi(a) = 0.$$

Hence $([a]_I, [a']_I) = 0$. This implies $a \in I$. Thus we get $I_{(\pi,t)} = I$. For $a \in I'_{(\pi,t)}$, we have $[a]_I \in I'_{(\pi,t)}/I \subset J(I_{(\pi,t)})/I = J(I)/I$. Hence we get $(0, [a]_{I'}) \in A_\omega$. We

also get $\varphi_{X_I}([a]_I) \in \mathcal{K}(X_I)$. Since $[a]_I \in I'_{(\pi,t)}/I = I'_{(\dot\pi,\dot\imath)}$, we have

$$\widetilde{\pi}\big((0, [a]_{I'})\big) = \dot\pi([a]_I) - \psi_i(\varphi_{X_I}([a]_I)) = 0,$$

by Lemma 5.10 (v). Since $\widetilde{\pi}$ is injective, we have $(0, [a]_{I'}) = 0$. This implies $a \in I'$. Thus we get $I'_{(\pi,t)} = I'$. Therefore if $\widetilde{\pi}$ is injective, then $\omega = \omega_{(\pi,t)}$.

Conversely assume $\omega = \omega_{(\pi,t)}$. Take $(b, b') \in A_\omega$ with $\widetilde{\pi}\big((b,b')\big) = 0$. Take $d \in A/I$ with $[d]_{I'/I} = b'$. Then we have $\dot\pi(d) = \psi_i(\varphi_{X_I}(d - b))$. Hence $d \in I'_{(\pi,t)}/I = I'/I$. Therefore we have $b' = 0$. We also have $\psi_i(\varphi_{X_I}(b)) = 0$. Since $I = I_{(\pi,t)}$, the map $\dot\imath$ is injective. Hence ψ_i is also injective. Therefore we have $b \in \ker \varphi_{X_I}$. We also have $b \in J(I)/I = J_{X_I}$ because $[b]_{J(I)/I} = [b']_{J(I)/I'} = 0$. Hence $b = 0$. We have proved that $\widetilde{\pi}$ is injective. $\qquad\square$

Definition 7.5. Let $\zeta \in X_I J_{X_I}$. Take $\eta \in X_I$ and $b \in J_{X_I}$ such that $\zeta = \eta b$. We define $\bar{t}(\zeta) = \dot\imath(\eta)\psi_i(\varphi_{X_I}(b)) \in C^*(\pi, t)$.

Lemma 7.6. *The map $\bar{t} \colon X_I J_{X_I} \to C^*(\pi, t)$ is a well-defined linear map satisfying that $\dot\imath(\eta)^*\bar{t}(\zeta) = \psi_i\big(\varphi_X(\langle\eta, \zeta\rangle_{X_I})\big)$ for all $\zeta \in X_I J_{X_I}$ and $\eta \in X_I$, and $\bar{t}(\zeta_1)^*\bar{t}(\zeta_2) = \psi_i\big(\varphi_{X_I}(\langle\zeta_1, \zeta_2\rangle_{X_I})\big)$ for all $\zeta_1, \zeta_2 \in X_I J_{X_I}$.*

Proof. Take $\eta_1, \eta_2 \in X_I$, $b_1, b_2 \in J_{X_I}$, and define $\zeta_1, \zeta_2 \in X_I J_{X_I}$ by $\zeta_1 = \eta_1 b_1$, $\zeta_2 = \eta_2 b_2$. We have

$$\dot\imath(\eta_1)^*\dot\imath(\eta_2)\psi_i(\varphi_{X_I}(b_2)) = \dot\pi(\langle\eta_1, \eta_2\rangle_{X_I})\psi_i(\varphi_{X_I}(b_2))$$
$$= \psi_i\big(\varphi_{X_I}(\langle\eta_1, \eta_2\rangle_{X_I} b_2)\big) = \psi_i\big(\varphi_{X_I}(\langle\eta_1, \zeta_2\rangle_{X_I})\big).$$

A similar computation shows that

$$\big(\dot\imath(\eta_1)\psi_i(\varphi_{X_I}(b_1))\big)^*\big(\dot\imath(\eta_2)\psi_i(\varphi_{X_I}(b_2))\big) = \psi_i\big(\varphi_{X_I}(\langle\zeta_1, \zeta_2\rangle_{X_I})\big).$$

For $\zeta \in X_I J_{X_I}$, take $\eta_1, \eta_2 \in X_I$ and $b_1, b_2 \in J_{X_I}$ such that $\zeta = \eta_1 b_1 = \eta_2 b_2$. Set $x = \dot\imath(\eta_1)\psi_i(\varphi_{X_I}(b_1)) - \dot\imath(\eta_2)\psi_i(\varphi_{X_I}(b_2)) \in C^*(\pi, t)$. We have $x^*x = 0$ because for $i, j = 1, 2$,

$$\big(\dot\imath(\zeta_i)\psi_i(\varphi_{X_I}(b_i))\big)^*\big(\dot\imath(\zeta_j)\psi_i(\varphi_{X_I}(b_j))\big) = \psi_i\big(\varphi_{X_I}(\langle\zeta, \zeta\rangle_{X_I})\big).$$

This shows $\bar{t}$ is well-defined. We can check the linearity of $\bar{t}$ in a similar fashion. The two equalities in the statement had been already checked. $\qquad\square$

Lemma 7.7. *We have*

$$\dot\pi(b)\bar{t}(\zeta) = \bar{t}(\varphi_{X_I}(b)\zeta), \quad \psi_i(k)\bar{t}(\zeta) = \bar{t}(k\zeta),$$

for $b \in A/I$, $k \in \mathcal{K}(X_I)$, and $\zeta \in X_I J_{X_I}$.

Proof. Take $\eta \in X_I$ and $d \in J_{X_I}$ with $\zeta = \eta d$. Then we have

$$
\begin{aligned}
\dot\pi(b)\bar{t}(\zeta) &= \dot\pi(b)\dot{t}(\eta)\psi_i(\varphi_{X_I}(d)) & \psi_i(k)\bar{t}(\zeta) &= \psi_i(k)\dot{t}(\eta)\psi_i(\varphi_{X_I}(d)) \\
&= \dot{t}(\varphi_{X_I}(b)\eta)\psi_i(\varphi_{X_I}(d)) & &= \dot{t}(k\eta)\psi_i(\varphi_{X_I}(d)) \\
&= \bar{t}\big((\varphi_{X_I}(b)\eta)d\big) & &= \bar{t}\big((k\eta)d\big) \\
&= \bar{t}(\varphi_{X_I}(b)\zeta), & &= \bar{t}(k\zeta). \qquad \square
\end{aligned}
$$

Lemma 7.8. *For* $\zeta \in X_I(I'/I)$, *we have* $\bar{t}(\zeta) = \dot{t}(\zeta)$.

Proof. Choose $\eta \in X_I$ and $b \in I'/I \subset J(I)/I = J_{X_I}$ such that $\zeta = \eta b$. Since $b \in I'/I \subset I'_{(\pi,t)}/I = I'_{(\dot\pi,\dot{t})}$, we have $\dot\pi(b) = \psi_i(\varphi_{X_I}(b))$ by Lemma 5.10 (v). Hence, we get

$$
\bar{t}(\zeta) = \dot{t}(\eta)\psi_i(\varphi_{X_I}(b)) = \dot{t}(\eta)\dot\pi(b) = \dot{t}(\eta b) = \dot{t}(\zeta).
$$

$\square$

Definition 7.9. Let $(\eta, \eta') \in X_\omega$. Take $\zeta \in X_I$ such that $[\zeta]_{I'/I} = \eta'$. Define $\tilde{t}\big((\eta, \eta')\big) \in C^*(\pi, t)$ by

$$
\tilde{t}\big((\eta, \eta')\big) = \dot{t}(\zeta) + \bar{t}(\eta - \zeta) \in C^*(\pi, t).
$$

Note that $\eta - \zeta \in X_I J_{X_I}$ and that $\tilde{t}\colon X_\omega \to C^*(\pi, t)$ is a well-defined linear map by Lemma 7.8.

Proposition 7.10. *The pair* $(\tilde\pi, \tilde{t})$ *is a representation of the C*-correspondence* X_ω *on* $C^*(\pi, t)$ *such that* $\dot\pi = \tilde\pi \circ \Pi_\omega$ *and* $\dot{t} = \tilde{t} \circ T_\omega$.

Proof. It is easy to see that $\dot\pi = \tilde\pi \circ \Pi_\omega$ and $\dot{t} = \tilde{t} \circ T_\omega$. We will check that the pair $(\tilde\pi, \tilde{t})$ satisfies the two conditions in Definition 2.7. Take $(\eta_1, \eta_1'), (\eta_2, \eta_2') \in X_\omega$. Choose $\zeta_1, \zeta_2 \in X_I$ with $[\zeta_1]_{I'/I} = \eta_1'$, $[\zeta_2]_{I'/I} = \eta_2'$. By Lemma 7.6, we have

$$
\begin{aligned}
\tilde{t}&\big((\eta_1, \eta_1')\big)^* \tilde{t}\big((\eta_2, \eta_2')\big) \\
&= \big(\dot{t}(\zeta_1) + \bar{t}(\eta_1 - \zeta_1)\big)^* \big(\dot{t}(\zeta_2) + \bar{t}(\eta_2 - \zeta_2)\big) \\
&= \dot\pi\big(\langle \zeta_1, \zeta_2 \rangle_{X_I}\big) \\
&\quad + \psi_i\big(\varphi_X(\langle \zeta_1, \eta_2 - \zeta_2 \rangle_{X_I} + \langle \eta_1 - \zeta_1, \zeta_2 \rangle_{X_I} + \langle \eta_1 - \zeta_1, \eta_2 - \zeta_2 \rangle_{X_I})\big) \\
&= \dot\pi\big(\langle \zeta_1, \zeta_2 \rangle_{X_I}\big) + \psi_i\big(\varphi_X(\langle \eta_1, \eta_2 \rangle_{X_I} - \langle \zeta_1, \zeta_2 \rangle_{X_I})\big) \\
&= \tilde\pi\big((\langle \eta_1, \eta_2 \rangle_{X_I}, \langle \eta_1', \eta_2' \rangle_{X_I})\big) \\
&= \tilde\pi\big(\langle (\eta_1, \eta_1'), (\eta_2, \eta_2') \rangle_{X_\omega}\big).
\end{aligned}
$$

This proves condition (i) in Definition 2.7. We check condition (ii). Take $(b, b') \in A_\omega$ and $(\eta, \eta') \in X_\omega$. Choose $d \in A/I$ and $\zeta \in X_I$ with $[d]_{I'/I} = b'$ and $[\zeta]_{I'/I} = \eta'$.

By Lemma 7.7, we have

$$\widetilde{\pi}\big((b,b')\big)\tilde{t}\big((\eta,\eta')\big)$$
$$=\big(\dot{\pi}(d)+\psi_i(\varphi_{X_I}(b-d))\big)\big(i(\zeta)+\bar{t}(\eta-\zeta)\big)$$
$$=\dot{\pi}(d)i(\zeta)+\psi_i(\varphi_{X_I}(b-d))i(\zeta)$$
$$\quad+\dot{\pi}(d)\bar{t}(\eta-\zeta)+\psi_i(\varphi_{X_I}(b-d))\bar{t}(\eta-\zeta)$$
$$=i(\varphi_{X_I}(d)\zeta)+i(\varphi_{X_I}(b-d)\zeta)+\bar{t}(\varphi_{X_I}(d)(\eta-\zeta))+\bar{t}(\varphi_{X_I}(b-d)(\eta-\zeta))$$
$$=i(\varphi_{X_I}(b)\zeta)+\bar{t}(\varphi_{X_I}(b)(\eta-\zeta)).$$

On the other hand, we have

$$\varphi_{X_\omega}((b,b'))(\eta,\eta')=\big(\varphi_{X_I}(b)\eta,[\varphi_{X_I}(b)]_{I'/I}\eta'\big)=\big(\varphi_{X_I}(b)\eta,[\varphi_{X_I}(b)\zeta]_{I'/I}\big).$$

Hence we get

$$\tilde{t}\big(\varphi_{X_\omega}((b,b'))(\eta,\eta')\big)=i\big(\varphi_{X_I}(b)\zeta\big)+\bar{t}\big(\varphi_{X_I}(b)(\eta-\zeta)\big).$$

Thus we have $\widetilde{\pi}\big((b,b')\big)\tilde{t}\big((\eta,\eta')\big)=\tilde{t}\big(\varphi_{X_\omega}((b,b'))(\eta,\eta')\big)$. We are done. $\square$

Proposition 7.11. *The representation $(\widetilde{\pi},\tilde{t})$ is covariant.*

Proof. Take $(b,0)\in J_{X_\omega}$. By definition, we have $\widetilde{\pi}((b,0))=\psi_i(\varphi_{X_I}(b))$. Since $\varphi_{X_\omega}((b,0))=\Psi_{T_\omega}(\varphi_{X_I}(b))$, we have

$$\psi_{\tilde{t}}\big(\varphi_{X_\omega}((b,0))\big)=\psi_{\tilde{t}}\big(\Psi_{T_\omega}(\varphi_{X_I}(b))\big)=\psi_{\tilde{t}\circ T_\omega}(\varphi_{X_I}(b))=\psi_i(\varphi_{X_I}(b)).$$

Hence we get $\widetilde{\pi}((b,0))=\psi_{\tilde{t}}\big(\varphi_{X_\omega}((b,0))\big)$ for every element $(b,0)\in J_{X_\omega}$. This completes the proof. $\square$

Lemma 7.12. *The representation $(\widetilde{\pi},\tilde{t})$ of X_ω is injective if and only if $\omega=\omega_{(\pi,t)}$. It admits a gauge action if and only if so does (π,t).*

Proof. The first assertion follows from Proposition 7.4. If a representation (π,t) admits a gauge action β, then β is also a gauge action for the representation $(\widetilde{\pi},\tilde{t})$ because $\beta_z(\psi_t(k))=\psi_t(k)$ for all $k\in\mathcal{K}(X)$ and $z\in\mathbb{T}$. The converse is obvious. $\square$

   Now we are ready to prove the main theorem of this section.

Proof of Theorem 7.1. Define $\rho=\rho_{(\widetilde{\pi},\tilde{t})}\colon\mathcal{O}_{X_\omega}\to C^*(\pi,t)$. Since $\dot{\pi}=\widetilde{\pi}\circ\Pi_\omega$ and $i=\tilde{t}\circ T_\omega$, we have $\pi=\rho\circ\pi_\omega$ and $t=\rho\circ t_\omega$. This implies that ρ is surjective. The uniqueness follows from $C^*(\pi_\omega,t_\omega)=\mathcal{O}_{X_\omega}$ which was proved in Proposition 6.11. Finally by Lemma 7.12 and Theorem 3.6, ρ is an isomorphism if and only if $\omega=\omega_{(\pi,t)}$ and (π,t) admits a gauge action. $\square$

Corollary 7.13. *Let X be a C^*-correspondence over a C^*-algebra A and (π,t) be a representation of X which admits a gauge action. Then the C^*-algebra $C^*(\pi,t)$ is naturally isomorphic to the C^*-algebra $\mathcal{O}_{X_{\omega_{(\pi,t)}}}$.*

We finish this section with a characterization of the C^*-algebra $\mathcal{O}_X$ without using J_X and the notion of covariance.

Proposition 7.14. *If a representation (π, t) is injective and admits a gauge action, then there exists a surjection $\overline{\rho} \colon C^*(\pi, t) \to \mathcal{O}_X$ with $\pi_X = \overline{\rho} \circ \pi$ and $t_X = \overline{\rho} \circ t$.*

Proof. Set $\omega = \omega_{(\pi,t)} = (I_{(\pi,t)}, I'_{(\pi,t)})$. Since (π, t) is injective, we have $I_{(\pi,t)} = 0$ and $I'_{(\pi,t)} \subset J(0) = J_X$. Hence we get $\omega \subset (0, J_X) = \omega_{(\pi_X, t_X)}$. Thus by Theorem 7.1, there exists a surjective $*$-homomorphism $\rho \colon \mathcal{O}_{X_\omega} \to \mathcal{O}_X$ with $\pi_X = \rho \circ \pi_\omega$ and $t_X = \rho \circ t_\omega$. Since (π, t) admits a gauge action, the C^*-algebra $C^*(\pi, t)$ is isomorphic to $\mathcal{O}_{X_\omega}$ by Corollary 7.13. This completes the proof. $\qquad\square$

By Proposition 7.14, we can define $\mathcal{O}_X$ to be the smallest C^*-algebra among C^*-algebras generated by injective representations admitting gauge actions. Theorem 3.6 tells us that the covariance of representations characterizes the representation (π_X, t_X) among injective representations admitting gauge actions.

8. Structure of gauge-invariant ideals of $\mathcal{O}_X$

We say that an ideal of $\mathcal{O}_X$ is gauge-invariant if it is globally invariant under the gauge action γ. In this section, we analyze structure of gauge-invariant ideals of $\mathcal{O}_X$.

Definition 8.1. For an ideal P of $\mathcal{O}_X$, we define $I_P, I'_P \subset A$ by

$$\pi_X(I_P) = \pi_X(A) \cap P, \quad \pi_X(I'_P) = \pi_X(A) \cap \big(P + \psi_{t_X}(\mathcal{K}(X))\big).$$

We set $\omega_P = (I_P, I'_P)$.

Proposition 8.2. *For an ideal P of $\mathcal{O}_X$, denote by σ_P a natural surjection from $\mathcal{O}_X$ to $\mathcal{O}_X/P$. Then we have $\omega_P = \omega_{(\sigma_P \circ \pi_X, \sigma_P \circ t_X)}$. Hence ω_P is an O-pair.*

Proof. Clear by the definitions. $\qquad\square$

Definition 8.3. Let ω be an O-pair of X. The representation (π_ω, t_ω) of X on $\mathcal{O}_{X_\omega}$ is covariant by Proposition 5.14 and Proposition 6.12. Hence there exists a surjection $\rho_{(\pi_\omega, t_\omega)} \colon \mathcal{O}_X \to \mathcal{O}_{X_\omega}$. We define $P_\omega = \ker \rho_{(\pi_\omega, t_\omega)}$.

Lemma 8.4. *For an O-pair ω, the ideal P_ω of $\mathcal{O}_X$ is gauge-invariant and satisfies $\omega_{P_\omega} = \omega$.*

Proof. Clear by the definitions. $\qquad\square$

Proposition 8.5. *For a gauge-invariant ideal P of $\mathcal{O}_X$, we have $P = P_{\omega_P}$ and $\mathcal{O}_X/P \cong \mathcal{O}_{X_{\omega_P}}$.*

Proof. If P is gauge-invariant, the representation $(\sigma_P \circ \pi_X, \sigma_P \circ t_X)$ admits a gauge action, where $\sigma_P \colon \mathcal{O}_X \to \mathcal{O}_X/P$ is a natural surjection. Hence by the definition of ω_P and Theorem 7.1, we have an isomorphism $\rho \colon \mathcal{O}_{X_{\omega_P}} \to \mathcal{O}_X/P$ such that $(\rho \circ \pi_{\omega_P}, \rho \circ t_{\omega_P}) = (\sigma_P \circ \pi_X, \sigma_P \circ t_X)$. Hence $\mathcal{O}_X/P \cong \mathcal{O}_{X_{\omega_P}}$ and $P = P_{\omega_P}$. $\qquad\square$

Now we get the following.

Theorem 8.6. *The set of all gauge-invariant ideals of $\mathcal{O}_X$ corresponds bijectively to the set of all O-pairs of X by $P \mapsto \omega_P$ and $\omega \mapsto P_\omega$. These maps preserve inclusions and intersections.*

In the case that C^*-correspondences are defined from graphs, or more generally from topological graphs, Theorem 8.6 was proved in [Bates et al. 2002] and [Katsura 2006a].

Corollary 8.7 [Muhly and Tomforde 2004, Theorem 6.4]. *If $A = J_X + \ker \varphi_X$, then $P \mapsto I_P$ is a bijection from the set of all gauge-invariant ideals of $\mathcal{O}_X$ to the set of all invariant ideals of A with respect to X.*

Proof. By Theorem 8.6 and Lemma 5.2, it suffices to show that $J_{X_I} \subset [J_X]_I$ for all invariant ideals I of A. Let I be an invariant ideal. Since $A = J_X + \ker \varphi_X$, we have $A/I = [J_X]_I + [\ker \varphi_X]_I$. Hence we get $([\ker \varphi_X]_I)^\perp = [J_X]_I$. Since $\ker \varphi_{X_I} \supset [\ker \varphi_X]_I$, we obtain

$$J_{X_I} \subset (\ker \varphi_{X_I})^\perp \subset ([\ker \varphi_X]_I)^\perp = [J_X]_I. \qquad \square$$

Note that the assumption $A = J_X + \ker \varphi_X$ is equivalent to the assumption in [Muhly and Tomforde 2004, Theorem 6.4]. This is also equivalent to saying that $A \cong A_1 \oplus A_2$ and $\varphi_X \colon A \to \mathcal{L}(X)$ is the composition of the natural surjection $A \to A_1$ and an embedding $A_1 \hookrightarrow \mathcal{K}(X)$. This assumption is not necessary for the map $P \mapsto I_P$ to be bijective, as we will see in Sections 9 and 10.

We finish this section with the following result on the gauge-invariant ideals of $\mathcal{T}_X$.

Proposition 8.8. *The set of all gauge-invariant ideals of $\mathcal{T}_X$ corresponds bijectively to the set of all T-pairs of X such that inclusions and intersections are preserved.*

Proof. The set of all gauge-invariant ideals of $\mathcal{T}_X$ corresponds bijectively to the "set" of all representations of X admitting gauge actions if we consider two representations (π, t) and (π', t') to be the same when there exists a (necessarily unique) isomorphism $\rho \colon C^*(\pi, t) \to C^*(\pi', t')$ such that $\rho \circ \pi = \pi'$ and $\rho \circ t = t'$. Under this identification, the "set" of all representations of X admitting gauge actions corresponds bijectively to the set of all T-pairs of X by $(\pi, t) \mapsto \omega_{(\pi,t)}$ defined in Definition 5.9, and $\omega \mapsto (\pi_\omega, t_\omega)$ defined in Definition 6.10 by Proposition 6.12 and Theorem 7.1. This completes the proof. $\qquad \square$

9. Gauge invariant ideals and strong Morita equivalence.

In this section, we prove that each gauge-invariant ideal P of the C^*-algebra $\mathcal{O}_X$ is strongly Morita equivalent to the C^*-algebra $\mathcal{O}_{Y_P}$ for a certain C^*-correspondence

Y_P. In the next section, we will see that in fact we can find a C^*-correspondence Y'_P such that P is isomorphic to $\mathbb{O}_{Y'_P}$.

For a positively invariant ideal I of A, we have $\varphi_X(I)X \subset XI$. Hence the closed linear subspace $Y_I = \varphi_X(I)X$ of X is naturally considered as a C^*-correspondence over I.

Lemma 9.1. *For a positively invariant ideal I of A, we have $\ker \varphi_{Y_I} = I \cap \ker \varphi_X$ and $\varphi_{Y_I}^{-1}(\mathcal{K}(Y_I)) = I \cap \varphi_X^{-1}(\mathcal{K}(X))$.*

Proof. Take $a \in \ker \varphi_{Y_I}$. For $\xi \in X$, we have $\varphi_X(a)\varphi_X(a^*)\xi = 0$ because $\varphi_X(a^*)\xi \in Y_I$. Hence we have $aa^* \in \ker \varphi_X$. Thus we get $a \in I \cap \ker \varphi_X$. This shows $\ker \varphi_{Y_I} \subset I \cap \ker \varphi_X$. Since the converse inclusion is obvious, we get $\ker \varphi_{Y_I} = I \cap \ker \varphi_X$.

Take $a \in \varphi_{Y_I}^{-1}(\mathcal{K}(Y_I))$. Set $k = \varphi_{Y_I}(a) \in \mathcal{K}(Y_I) \subset \mathcal{K}(X)$. Since we have $\varphi_X(a)\varphi_X(b)\xi = k\varphi_X(b)\xi$ for $b \in I$ and $\xi \in X$, we get $\varphi_X(a)\varphi_X(a)^* = k\varphi_X(a)^*$. We also get $\varphi_X(a)k^* = kk^*$ because $k \in \mathcal{K}(Y_I)$. Thus $(\varphi_X(a) - k)(\varphi_X(a) - k)^* = 0$. Hence $\varphi_X(a) = k \in \mathcal{K}(X)$. Since the converse inclusion is obvious, we have $\varphi_{Y_I}^{-1}(\mathcal{K}(Y_I)) = I \cap \varphi_X^{-1}(\mathcal{K}(X))$. $\square$

Proposition 9.2. *For a positively invariant ideal I of A, we have $J_{Y_I} = I \cap J_X$.*

Proof. Since $\ker \varphi_{Y_I} \subset \ker \varphi_X$, we have $(\ker \varphi_{Y_I})^\perp \supset (\ker \varphi_X)^\perp$. By Lemma 9.1, we have $(\ker \varphi_{Y_I})^\perp \cap I \cap \ker \varphi_X = 0$. Hence $(\ker \varphi_{Y_I})^\perp \cap I \subset (\ker \varphi_X)^\perp$. Thus we have $(\ker \varphi_{Y_I})^\perp \cap I = (\ker \varphi_X)^\perp \cap I$. From this equality and Lemma 9.1, we get

$$\begin{aligned} J_{Y_I} &= \varphi_{Y_I}^{-1}(\mathcal{K}(Y_I)) \cap (\ker \varphi_{Y_I})^\perp \\ &= I \cap \varphi_X^{-1}(\mathcal{K}(X)) \cap (\ker \varphi_{Y_I})^\perp \\ &= I \cap \varphi_X^{-1}(\mathcal{K}(X)) \cap (\ker \varphi_X)^\perp \\ &= I \cap J_X. \end{aligned}$$
$\square$

Proposition 9.3. *For a positively invariant ideal I of A, the C^*-subalgebra generated by $\pi_X(I)$ and $t_X(Y_I)$ is isomorphic to $\mathbb{O}_{Y_I}$, and it is the smallest hereditary C^*-subalgebra in $\mathbb{O}_X$ containing $\pi_X(I)$.*

Proof. Let B be the C^*-subalgebra of $\mathbb{O}_X$ generated by $\pi_X(I)$ and $t_X(Y_I)$. Clearly the restrictions of π_X and t_X to I and Y_I give an injective representation (π, t) of Y_I on $\mathbb{O}_X$ which admits a gauge action. It is also clear that $C^*(\pi, t) = B$. By Proposition 9.2, this representation (π, t) is covariant. Thus B is isomorphic to $\mathbb{O}_{Y_I}$ by Theorem 3.6.

Since we have $\pi_X(I)t_X(X)\pi_X(I) = t_X(Y_I)\pi_X(I) = t_X(Y_I)$, B is contained in the C^*-subalgebra $\pi_X(I)\mathbb{O}_X\pi_X(I)$. By [Katsura 2004b, Proposition 2.7], $\mathbb{O}_X$ is the closure of the linear span of elements in the form

$$t_X(\xi_1) \cdots t_X(\xi_n)\pi_X(a)t_X(\eta_m)^* \cdots t_X(\eta_1)^*$$

for $a \in A$ and $\xi_k, \eta_l \in X$. Using the fact that $\pi_X(I)t_X(X) = t_X(Y_I)\pi_X(I)$, we can prove by induction on n that $\pi_X(b)t_X(\xi_1) \cdots t_X(\xi_n)\pi_X(a) \in B$ for $b \in I$, $a \in A$ and $\xi_k \in X$. Hence $\pi_X(I)\mathbb{O}_X\pi_X(I)$ is contained in B. Thus we have shown that $B = \pi_X(I)\mathbb{O}_X\pi_X(I)$ which is the smallest hereditary C^*-subalgebra containing $\pi_X(I)$. $\qquad\square$

Proposition 9.4. *For an ideal I of A, the ideal of $\mathbb{O}_X$ generated by $\pi_X(I)$ is P_ω where $\omega = (X^\infty_{-\infty}(I), X^\infty_{-\infty}(I) + J_X)$.*

Proof. Let P be the ideal of $\mathbb{O}_X$ generated by $\pi_X(I)$. Since $I \subset I_P$ and I_P is invariant, we have $X^\infty_{-\infty}(I) \subset I_P$ by Proposition 4.16. Hence we have $X^\infty_{-\infty}(I) + J_X \subset I_P + J_X \subset I'_P$. Therefore we get $\omega \subset (I_P, I'_P) = \omega_P$. Since $\pi_X(I) \subset \pi_X(X^\infty_{-\infty}(I)) \subset P_\omega$ implies $P \subset P_\omega$, we have $\omega_P \subset \omega_{P_\omega} = \omega$. Thus we get $\omega_P = \omega$. Since $\pi_X(I)$ is closed under the gauge action, the ideal P is gauge-invariant. Hence we have $P = P_{\omega_P} = P_\omega$ by Proposition 8.5. $\qquad\square$

Proposition 9.5. *Let I be a positively invariant ideal of A. For an O-pair $\omega = (X_{-\infty}(I), X_{-\infty}(I) + J_X)$, the gauge-invariant ideal P_ω is strongly Morita equivalent to the C^*-algebra $\mathbb{O}_{Y_I}$.*

Proof. By Proposition 9.3 and Proposition 9.4, the C^*-subalgebra generated by $\pi_X(I)$ and $t_X(Y_I)$ is isomorphic to $\mathbb{O}_{Y_I}$ and is a hereditary and full C^*-subalgebra of P_ω which is the ideal generated by $\pi_X(I)$. Thus P_ω is strongly Morita equivalent to the C^*-algebra $\mathbb{O}_{Y_I}$. $\qquad\square$

Corollary 9.6. *Let X be a C^*-correspondence over a C^*-algebra A. Define a C^*-correspondence Y over A by $Y = \varphi_X(A)X$. Then $\mathbb{O}_Y$ is strongly Morita equivalent to $\mathbb{O}_X$.*

Proof. Apply Proposition 9.5 to the invariant ideal A. $\qquad\square$

The C^*-correspondence Y defined in the above corollary is nondegenerate; that is, $\varphi_Y(A)Y = Y$. Thus, by Corollary 9.6, we can exchange a given C^*-correspondence to a nondegenerate one so that the C^*-algebras constructed by them are strongly Morita equivalent (we used this fact in [Katsura 2004b, Appendix C]).

By Proposition 9.5, gauge-invariant ideals P satisfying that $I'_P = I_P + J_X$ are shown to be strongly Morita equivalent to the C^*-algebra $\mathbb{O}_{Y_{I_P}}$ of the C^*-correspondence Y_{I_P}. To deal with all gauge-invariant ideals of $\mathbb{O}(X)$, we need the following argument.

Let us define a C^*-algebra $\widetilde{A}$ and a Banach space $\widetilde{X}$ by

$$\widetilde{A} = \pi_X(A) + \psi_{t_X}(\mathcal{K}(X)) \subset \mathbb{O}_X,$$
$$\widetilde{X} = \overline{\mathrm{span}}\left(t_X(X) + t_X(X)\psi_{t_X}(\mathcal{K}(X))\right) \subset \mathbb{O}_X.$$

If we define the left and right actions of $\widetilde{A}$ on $\widetilde{X}$ as multiplication, and the inner product by $\langle \xi, \eta \rangle_{\widetilde{X}} = \xi^*\eta$, $\widetilde{X}$ becomes a C^*-correspondence over $\widetilde{A}$. Since the

embeddings $\widetilde{A} \hookrightarrow \mathcal{O}_X$ and $\widetilde{X} \hookrightarrow \mathcal{O}_X$ give an injective representation of $\widetilde{X}$, we have an injective $*$-homomorphism from $\mathcal{K}(\widetilde{X})$ onto $\overline{\operatorname{span}}(\widetilde{X}\widetilde{X}^*) \subset \mathcal{O}_X$. Thus we can identify $\mathcal{K}(\widetilde{X})$ with $\overline{\operatorname{span}}(\widetilde{X}\widetilde{X}^*)$.

Lemma 9.7. *We have* $J_{\widetilde{X}} = \psi_{t_X}(\mathcal{K}(X)) \subset \widetilde{A}$.

Proof. By the identification above, the restriction of $\varphi_{\widetilde{X}}$ to the ideal $\psi_{t_X}(\mathcal{K}(X))$ of $\widetilde{A}$ is just the embedding $\psi_{t_X}(\mathcal{K}(X)) \hookrightarrow \mathcal{K}(\widetilde{X})$. Hence we have $\psi_{t_X}(\mathcal{K}(X)) \subset J_{\widetilde{X}}$. We will prove the converse inclusion. Take $\pi_X(a) + \psi_{t_X}(k) \in J_{\widetilde{X}}$. Then we have $\pi_X(a) \in J_{\widetilde{X}}$. Let $\{u_\lambda\}$ be an approximate unit of $\psi_{t_X}(\mathcal{K}(X))$. It is not difficult to see that $\{\varphi_{\widetilde{X}}(u_\lambda)\}$ is an approximate unit of $\mathcal{K}(\widetilde{X})$ (see [Katsura 2004b, Lemma 5.10]). Since $\varphi_{\widetilde{X}}(\pi_X(a)) \in \mathcal{K}(\widetilde{X})$, we have

$$\varphi_{\widetilde{X}}(\pi_X(a)) = \lim_\lambda \varphi_{\widetilde{X}}(\pi_X(a))\varphi_{\widetilde{X}}(u_\lambda) = \lim_\lambda \varphi_{\widetilde{X}}(\pi_X(a)u_\lambda) \in \varphi_{\widetilde{X}}\big(\psi_{t_X}(\mathcal{K}(X))\big).$$

Hence there exists $k \in \mathcal{K}(X)$ with $\varphi_{\widetilde{X}}(\pi_X(a)) = \varphi_{\widetilde{X}}(\psi_{t_X}(k))$. Therefore we have

$$\begin{aligned} t_X(\varphi_X(a)\xi) = \pi_X(a)t_X(\xi) &= \varphi_{\widetilde{X}}(\pi_X(a))t_X(\xi) \\ &= \varphi_{\widetilde{X}}(\psi_{t_X}(k))t_X(\xi) = \psi_{t_X}(k)t_X(\xi) = t_X(k\xi) \end{aligned}$$

for each $\xi \in X$. Hence we obtain $\varphi_X(a) = k \in \mathcal{K}(X)$. For $b \in \ker\varphi_X$ we have $\pi_X(b) \in \ker\varphi_{\widetilde{X}}$. Therefore we get $\pi_X(ab) = 0$ for all $b \in \ker\varphi_X$. Thus $a \in \varphi_X^{-1}(\mathcal{K}(X)) \cap (\ker\varphi_X)^\perp = J_X$. Therefore $\pi_X(a) + \psi_{t_X}(k) = \psi_{t_X}(\varphi_X(a) + k) \in \psi_{t_X}(\mathcal{K}(X))$. This shows $J_{\widetilde{X}} \subset \psi_{t_X}(\mathcal{K}(X))$. Thus we get $J_{\widetilde{X}} = \psi_{t_X}(\mathcal{K}(X))$. $\square$

Proposition 9.8. *The natural inclusions* $\widetilde{A} \hookrightarrow \mathcal{O}_X$ *and* $\widetilde{X} \hookrightarrow \mathcal{O}_X$ *induce an isomorphism* $\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$.

Proof. It is clear that the pair (π, t) of the inclusions $\pi: \widetilde{A} \hookrightarrow \mathcal{O}_X$ and $t: \widetilde{X} \hookrightarrow \mathcal{O}_X$ is an injective representation of $\widetilde{X}$ admitting a gauge action and satisfying $C^*(\pi, t) = \mathcal{O}_X$. By Lemma 9.7, the representation (π, t) is covariant. Hence we have an isomorphism $\rho_{(\pi,t)}: \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$ by Theorem 3.6. $\square$

Proposition 9.9. *For a gauge-invariant ideal* P *of* $\mathcal{O}_X$, *we set* $\widetilde{I} = \widetilde{A} \cap P$. *Then* P *is strongly Morita equivalent to the* C^*-*algebra* $\mathcal{O}_{Y_{\widetilde{I}}}$ *where* $Y_{\widetilde{I}} = \varphi_{\widetilde{X}}(\widetilde{I})\widetilde{X}$ *is a* C^*-*correspondence over* $\widetilde{I}$.

Proof. Since $\widetilde{I}$ is the intersection of $\widetilde{A}$ and the ideal P of $\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, the ideal $\widetilde{I}$ is an invariant ideal of $\widetilde{A}$. Let $\widetilde{P}$ be the ideal in $\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ generated by $\widetilde{I}$. By Proposition 9.5, $\widetilde{P}$ is strongly Morita equivalent to the C^*-algebra $\mathcal{O}_{Y_{\widetilde{I}}}$. We will show that $\widetilde{P} = P$. To do so, it suffices to see $\omega_{\widetilde{P}} = \omega_P$ by Theorem 8.6 because both $\widetilde{P}$ and P are gauge-invariant. Since $\widetilde{I} \subset P$, we have $\widetilde{P} \subset P$. Hence $\omega_{\widetilde{P}} \subset \omega_P$. We have

$$\pi_X(A) \cap P = \pi_X(A) \cap \widetilde{A} \cap P = \pi_X(A) \cap \widetilde{I} \subset \pi_X(A) \cap \widetilde{P}.$$

Similarly,

$$\pi_X(A) \cap \left(P + \psi_{t_X}(\mathcal{K}(X))\right) = \pi_X(A) \cap \left(\widetilde{A} \cap P + \psi_{t_X}(\mathcal{K}(X))\right)$$
$$= \pi_X(A) \cap \left(\widetilde{I} + \psi_{t_X}(\mathcal{K}(X))\right)$$
$$\subset \pi_X(A) \cap \left(\widetilde{P} + \psi_{t_X}(\mathcal{K}(X))\right).$$

Hence we get $\omega_P \subset \omega_{\widetilde{P}}$. Thus $\omega_{\widetilde{P}} = \omega_P$. □

Remark 9.10. As we saw in the proof of Proposition 9.9, we can see that gauge-invariant ideals of $\mathcal{O}_X$ are distinguished by their intersection with $\widetilde{A}$. By Proposition 9.9, the set of all gauge-invariant ideals of $\mathcal{O}_{\widetilde{X}}$ corresponds bijectively to the set of all invariant ideals of $\widetilde{A}$ even though the C^*-correspondence $\widetilde{X}$ does not satisfy the assumption in Corollary 8.7 in general.

Proposition 9.9 shows that every gauge-invariant ideal of $\mathcal{O}_X$ is strongly Morita equivalent to the C^*-algebra $\mathcal{O}_Y$ for some C^*-correspondences Y. In the next section, we will see that for every gauge-invariant ideal P of $\mathcal{O}_X$ we can find a C^*-correspondence Y so that P is isomorphic to $\mathcal{O}_Y$.

10. Crossed products by Hilbert C^*-bimodules

For a C^*-algebra A, a *Hilbert A-bimodule* is a C^*-correspondence X over A together with a left inner product $_X\langle \cdot, \cdot \rangle \colon X \times X \to A$ such that $\varphi_X(_X\langle \xi, \eta \rangle) = \theta_{\xi, \eta}$ for $\xi, \eta \in X$ (for details, see [Abadie et al. 1998], for example). We have

$$J_X = \overline{\operatorname{span}}\{_X\langle \xi, \eta \rangle \in A \mid \xi, \eta \in X\}.$$

A C^*-correspondence X has a left inner product so that it becomes a Hilbert A-bimodules if and only if we have $\varphi_X(J_X) = \mathcal{K}(X)$, and in this case a left inner product is uniquely determined by the structure of C^*-correspondence as $_X\langle \xi, \eta \rangle = (\varphi_X|_{J_X})^{-1}(\theta_{\xi, \eta}) \in J_X$ (see [Katsura 2003a, Lemma 3.4]).

For a general C^*-correspondence X over A, an ideal I of A is positively invariant if and only if $\varphi_X(I)X \subset XI$. For Hilbert C^*-bimodules, we get an analogous statement for negative invariance. Let us fix a Hilbert A-bimodule X whose left inner product is denoted by $_X\langle \cdot, \cdot \rangle$.

Lemma 10.1. *An ideal I of A is negatively invariant if and only if $\varphi_X(I)X \supset XI$.*

Proof. Let I be a negatively invariant ideal of A. Take $\xi \in X$ and $a \in I$. For arbitrary $\eta \in X$, we have $\varphi_X(_X\langle \xi a, \eta \rangle) = \theta_{\xi a, \eta} \in \mathcal{K}(XI)$. Since $_X\langle \xi a, \eta \rangle \in J_X$, the negative invariance of I implies $_X\langle \xi a, \eta \rangle \in I$ for arbitrary $\eta \in X$. Similarly to the proof of Proposition 1.3, we can prove $\xi a \in \varphi_X(I)X$. Thus we have $\varphi_X(I)X \supset XI$. Conversely, assume that an ideal I satisfies $\varphi_X(I)X \supset XI$. For $\xi, \eta \in XI$, we can find $\xi' \in X$ and $a \in I$ with $\xi = \varphi_X(a)\xi'$. Therefore we have $_X\langle \xi, \eta \rangle =$

$_X\langle \varphi_X(a)\xi', \eta \rangle = a(_X\langle \xi', \eta \rangle) \in I$.   Hence we can see that $(\varphi_X|_{J_X})^{-1}(k) \in I$ for $k \in \mathcal{K}(XI)$. Therefore for $a \in J_X$ with $\varphi_X(a) \in \mathcal{K}(XI)$ we have $a \in I$. This shows that I is negatively invariant. $\qquad\square$

Proposition 10.2. *An ideal I of A is invariant if and only if $\varphi_X(I)X = XI$.*

Proof. Clear from Lemma 10.1. $\qquad\square$

Proposition 10.3. *For an invariant ideal I of A, the C^*-correspondence X_I defined in Section 5 has a left inner product $_{X_I}\langle \cdot, \cdot \rangle$ such that $_{X_I}\langle [\xi]_I, [\eta]_I \rangle = [_X\langle \xi, \eta \rangle]_I$ for $\xi, \eta \in X$.*

Proof. Since $\varphi_X(I)X = XI$, it is not difficult to see that the left inner product of X_I described above is well-defined, and satisfies the required conditions. $\qquad\square$

Corollary 10.4. *For an invariant ideal I of A, we have $J_{X_I} = [J_X]_I$.*

Proof. By Proposition 10.3, we have

$$J_{X_I} = \overline{\mathrm{span}}\{_{X_I}\langle \xi', \eta' \rangle \in A/I \mid \xi', \eta' \in X_I\}$$
$$= \overline{\mathrm{span}}\{[_X\langle \xi, \eta \rangle]_I \in A/I \mid \xi, \eta \in X\} = [J_X]_I.$$

$\qquad\square$

Proposition 10.5. *For an invariant ideal I, the C^*-subalgebra of $\mathcal{O}_X$ generated by $\pi_X(I)$ and $t_X(XI)$ is an ideal.*

Proof. This follows from the fact that $XI = \varphi_X(I)X = \varphi_X(I)XI$. $\qquad\square$

Theorem 10.6. *Let X be a Hilbert A-bimodule. For an ideal P of $\mathcal{O}_X$, we define an ideal I_P of A by $\pi_X(I_P) = \pi_X(A) \cap P$. Then the map $P \mapsto I_P$ gives a one-to-one correspondence from the set of all gauge-invariant ideals P of $\mathcal{O}_X$ to the set of ideals I of A satisfying $\varphi_X(I)X = XI$. We also have isomorphisms $P \cong \mathcal{O}_{XI_P}$ and $\mathcal{O}_X/P \cong \mathcal{O}_{X_{I_P}}$ for a gauge-invariant ideal P.*

Proof. By Corollary 10.4, we have $I' = I + J_X$ for all O-pair $\omega = (I, I')$. Thus the first assertion follows from Theorem 8.6 and Proposition 10.2. The second assertion follows from Proposition 8.5, Proposition 9.3 and Proposition 10.5. $\qquad\square$

Both XI and X_I are Hilbert C^*-bimodules. Thus the class of C^*-algebras associated with Hilbert C^*-bimodules behave well. We will see that this class is same as the one of C^*-algebras associated with C^*-correspondences, which we are studying in this paper.

Let us take a C^*-algebra A and a C^*-correspondence X over A. We define a C^*-algebra $\bar{A}$ and a Banach space $\bar{X}$ by

$$\bar{A} = \mathcal{O}_X^\gamma, \quad \bar{X} = \{x \in \mathcal{O}_X \mid \gamma_z(x) = zx \text{ for all } z \in \mathbb{T}\}.$$

Remark 10.7. In a similar way as in the proof of [Katsura 2004b, Proposition 5.7], we can prove that $\bar{X} = \overline{\text{span}}(t_X(X)\mathbb{O}_X^\gamma)$. We do not use this fact.

It is easy to see that $\bar{X}$ is a Hilbert $\bar{A}$-bimodule where the inner products are defined by

$$\langle \xi, \eta \rangle_{\bar{X}} = \xi^* \eta, \quad {}_{\bar{X}}\langle \xi, \eta \rangle = \xi \eta^*,$$

for $\xi, \eta \in \bar{X}$, and the left and right actions are multiplication.

Proposition 10.8 (compare [Abadie et al. 1998, Theorem 3.1]). *The natural embedding of $\bar{A}$ and $\bar{X}$ into $\mathbb{O}_X$ gives an isomorphism $\mathbb{O}_{\bar{X}} \cong \mathbb{O}_X$.*

Proof. By Theorem 3.6, it suffices to check that the embedding of $\bar{A}$ and $\bar{X}$ into $\mathbb{O}_X$ is an injective covariant representation admitting a gauge action. These conditions are easily checked. □

Corollary 10.9. *Let X be a C^*-correspondence over a C^*-algebra A, and P be a gauge-invariant ideal of $\mathbb{O}_X$. If we set $I = P \cap \bar{A}$, then P is isomorphic to $\mathbb{O}_{\bar{X}I}$.*

Proof. Combine Theorem 10.6 and Proposition 10.8. □

We remark that in order to compute the K-groups of gauge-invariant ideals, Proposition 9.5 and Proposition 9.9 seem to be more useful than Corollary 10.9.

11. Relative Cuntz–Pimsner algebras

In this last section, we apply the results obtained above to the relative Cuntz–Pimsner algebras introduced in [Muhly and Solel 1998]. Recall that for a C^*-correspondence X over a C^*-algebra A, and an ideal J of A with $\varphi_X(J) \subset \mathcal{K}(X)$, the *relative Cuntz–Pimsner algebra* $\mathbb{O}(J, X)$ is generated by the image of a representation (π, t) which is universal among representations satisfying $\pi(a) = \psi_t(\varphi_X(a))$ for $a \in J$ (see [Muhly and Solel 1998, Theorem 2.19]). We will show that every relative Cuntz–Pimsner algebra is isomorphic to $\mathbb{O}_{X'}$ for some C^*-correspondences X'. In particular, every Cuntz–Pimsner algebra and Toeplitz algebra introduced in [Pimsner 1997], including augmented ones, is in the class of our C^*-algebras.

By universality, the representation (π, t) of X on $\mathbb{O}(J, X)$ admits a gauge action. Hence by Corollary 7.13 we see that $\mathbb{O}(J, X)$ is isomorphic to $\mathbb{O}_{X_{\omega_{(\pi,t)}}}$. We will express $\omega_{(\pi,t)}$ in terms of a C^*-correspondence X over A and an ideal J of A.

Now let us take a C^*-correspondence X over a C^*-algebra A, and an ideal J of A with $\varphi_X(J) \subset \mathcal{K}(X)$. We inductively define an increasing family of ideals $\{J_{-n}\}_{n \in \mathbb{N}}$ by $J_0 = 0$ and $J_{-(n+1)} = J_{-n} + J \cap X^{-1}(J_{-n})$. We set $J_{-\infty} = \lim_{n \to \infty} J_{-n}$. We denote by ω_J the pair $(J_{-\infty}, J)$ of ideals of A. Since $X^{-1}(0) = \ker \varphi_X$, we have $J_{-1} = J \cap \ker \varphi_X$. It is easy to see that $J_{-\infty} = 0$ if and only if $J \cap \ker \varphi_X = 0$.

Lemma 11.1. *The pair $\omega_J = (J_{-\infty}, J)$ is a T-pair of X.*

Proof. Clearly $J_0 = 0$ is positively invariant. We can prove that J_{-n} is positively invariant for all $n \in \mathbb{N}$ by induction with respect to $\mathbb{N}$, as in Lemma 4.15. Hence $J_{-\infty}$ is positively invariant. Again by induction, we see that $J_{-\infty} \subset J$. Since $J \cap X^{-1}(J_{-n}) \subset J_{-(n+1)} \subset J_{-\infty}$ for all n, we have $J \cap X^{-1}(J_{-\infty}) \subset J_{-\infty}$ by Proposition 4.7. Since $\varphi_X(J) \subset \mathcal{K}(X)$ by assumption, we have $\left[\varphi_X(J)\right]_{J_{-\infty}} \subset \mathcal{K}(X_{J_{-\infty}})$. Hence we get $J \subset J(J_{-\infty})$. Thus we have $J_{-\infty} \subset J \subset J(J_{-\infty})$. We are done. $\qquad\square$

Lemma 11.2. *If a T-pair $\omega = (I, I')$ satisfies $J \subset I'$, then $\omega_J \subset \omega$.*

Proof. We will prove $J_{-n} \subset I$ by induction on n. For $n = 0$, it is trivial. Assume $J_{-n} \subset I$. We have

$$J \cap X^{-1}(J_{-n}) \subset I' \cap X^{-1}(I) \subset J(I) \cap X^{-1}(I) = I$$

by Lemma 5.2. Hence

$$J_{-(n+1)} = J_{-n} + J \cap X^{-1}(J_{-n}) \subset I.$$

We have shown that $J_{-n} \subset I$ for all n. This implies that $J_{-\infty} \subset I$. Hence we have $\omega_J \subset \omega$. $\qquad\square$

Proposition 11.3. *The relative Cuntz–Pimsner algebra $\mathcal{O}(J, X)$ is isomorphic to the C^*-algebra $\mathcal{O}_{X_{\omega_J}}$ of the C^*-correspondence X_{ω_J}.*

Proof. Let us denote by (π, t) the universal representation of X on $\mathcal{O}(J, X)$ satisfying $\pi(a) = \psi_t(\varphi_X(a))$ for all $a \in J$, and by $(\pi_{\omega_J}, t_{\omega_J})$ the representation of X on the C^*-algebra $\mathcal{O}_{X_{\omega_J}}$ defined in Section 6. By Proposition 6.12, we have $I'_{(\pi_{\omega_J}, t_{\omega_J})} = J$. Hence by Lemma 5.10 (v), we have $\pi_{\omega_J}(a) = \psi_{t_{\omega_J}}(\varphi_X(a))$ for all $a \in J$. By the universal property of $\mathcal{O}(J, X)$, there exists a $*$-homomorphism $\rho \colon \mathcal{O}(J, X) \to \mathcal{O}_{X_{\omega_J}}$ such that $\pi_{\omega_J} = \rho \circ \pi$ and $t_{\omega_J} = \rho \circ t$. On the other hand, $J \subset I'_{(\pi,t)}$ implies $\omega_J \subset \omega_{(\pi,t)}$ by Lemma 11.2. Hence by Theorem 7.1, there exists a surjective $*$-homomorphism $\rho' \colon \mathcal{O}_{X_{\omega_J}} \to \mathcal{O}(J, X)$ such that $\pi = \rho' \circ \pi_{\omega_J}$ and $t = \rho' \circ t_{\omega_J}$. Clearly ρ and ρ' are the inverses of each others. Hence $\mathcal{O}(J, X)$ is isomorphic to $\mathcal{O}_{X_{\omega_J}}$. $\qquad\square$

By Proposition 11.3, the T-pair $\omega_{(\pi,t)}$ arising from the representation (π, t) on $\mathcal{O}(J, X)$ coincides with $\omega_J = (J_{-\infty}, J)$. From this fact, we have the following corollaries.

Corollary 11.4. *Let (π, t) be the representation of X on $\mathcal{O}(J, X)$. Then the kernel of the map $\pi \colon A \to \mathcal{O}(J, X)$ is $J_{-\infty}$, and we have*

$$\{a \in A \mid \varphi_X(a) \in \mathcal{K}(X), \ and \ \pi(a) = \psi_t(\varphi_X(a))\} = J.$$

Proof. This easily follows from $\omega_{(\pi,t)} = \omega_J$. $\qquad\square$

Corollary 11.5. *The relative Cuntz–Pimsner algebra $\mathbb{O}(J, X)$ is zero if and only if $J_{-\infty} = A$.*

Proof. Clear by Proposition 11.3. □

Corollary 11.6 [Muhly and Solel 1998, Proposition 2.21]. *The map $\pi \colon A \to \mathbb{O}(J, X)$ is injective if and only if $J \cap \ker \varphi_X = 0$.*

Proof. By Corollary 11.4, $\pi \colon A \to \mathbb{O}(J, X)$ is injective if and only if $J_{-\infty} = 0$, which is equivalent to the condition $J \cap \ker \varphi_X = 0$ as we saw above. □

We now state a gauge-invariant uniqueness theorem for relative Cuntz–Pimsner algebras.

Corollary 11.7. *For a representation (π', t') of X satisfying $\pi'(a) = \psi_{t'}(\varphi_X(a))$ for $a \in J$, the natural surjection $\mathbb{O}(J, X) \to C^*(\pi', t')$ is an isomorphism if and only if (π', t') admits a gauge action, $\ker \pi' = J_{-\infty}$, and*

$$\{a \in A \mid \pi'(a) \in \psi_{t'}(\mathcal{K}(X))\} = J.$$

Proof. By Proposition 11.3, $\mathbb{O}(J, X)$ is canonically isomorphic to $\mathbb{O}_{X_{\omega_J}}$. By Theorem 7.1, the surjection from $\mathbb{O}_{X_{\omega_J}}$ to $C^*(\pi', t')$ is injective if and only if (π', t') admits a gauge action and $\omega_{(\pi', t')} = \omega_J$. The last two conditions in the statement just rephrase the condition $\omega_{(\pi', t')} = \omega_J$. □

Note that we automatically have $\ker \pi' \supset J_{-\infty}$ and $\{a \in A \mid \pi'(a) \in \psi_{t'}(\mathcal{K}(X))\} \supset J$. Note also that in general we cannot replace the condition

$$\{a \in A \mid \pi'(a) \in \psi_{t'}(\mathcal{K}(X))\} = J.$$

with the condition

$$\{a \in A \mid \varphi_X(a) \in \mathcal{K}(X), \text{ and } \pi'(a) = \psi_{t'}(\varphi_X(a))\} = J,$$

which seems natural at first glance. This is because there may exist $a \in A$ with $\varphi_X(a) \notin \mathcal{K}(X)$ satisfying $[\varphi_X(a)]_{J_{-\infty}} \in \mathcal{K}(X_{J_{-\infty}})$ and $\pi'(a) = \psi_{t'}([\varphi_X(a)]_{J_{-\infty}}) \in \psi_{t'}(\mathcal{K}(X))$ (see Lemma 5.10 (iv) and (v)). In the case that $J \cap \ker \varphi_X = 0$, the statement of Corollary 11.7 has the following simple forms.

Corollary 11.8. *Let us assume $J \cap \ker \varphi_X = 0$. For a representation (π', t') of X satisfying $\pi'(a) = \psi_{t'}(\varphi_X(a))$ for $a \in J$, the natural surjection $\mathbb{O}(J, X) \to C^*(\pi', t')$ is an isomorphism if and only if (π', t') is injective, admits a gauge action, and satisfies*

$$\{a \in A \mid \varphi_X(a) \in \mathcal{K}(X), \text{ and } \pi'(a) = \psi_{t'}(\varphi_X(a))\} = J.$$

We remark that an ideal J of A satisfies $\varphi_X(J) \subset \mathcal{K}(X)$ and $J \cap \ker \varphi_X = 0$ if and only if $J \subset J_X$. As we saw in Corollary 11.6, the maps from A and X to the relative Cuntz–Pimsner algebra $\mathbb{O}(J, X)$ is injective only when J satisfies

$J \subset J_X$. Thus it is not a good idea to examine the structure of $\mathbb{O}(J, X)$ in terms of A, X and J unless J satisfies $J \subset J_X$. Anyway, the following result on the ideal structure of relative Cuntz–Pimsner algebras $\mathbb{O}(J, X)$ can be easily obtained similarly as Theorem 8.6 or Proposition 8.8.

Proposition 11.9. *Let X be a C^*-correspondence over a C^*-algebra A, and J be an ideal of A with $\varphi_X(J) \subset \mathcal{K}(X)$. Then there exists a one-to-one correspondence between the set of all gauge-invariant ideals of $\mathbb{O}(J, X)$ and the set of all T-pairs $\omega = (I, I')$ of X satisfying $J \subset I'$, which preserves inclusions and intersections.*

We note that a T-pair $\omega = (I, I')$ satisfies $J \subset I'$ if and only if $\omega_J \subset \omega$ by Lemma 11.2.

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