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Let $\gamma_0 : [a, b] \to \mathbb{R}^{1+k}$ be Lipschitz. Our main result provides a sufficient condition, expressed in terms of further accessory Lipschitz maps, for the C^3-rectifiability of $\gamma_0([a, b])$.

1. Introduction

A set in $\mathbb{R}^n$ is C^3-*rectifiable* if $\mathscr{H}^1$-almost all of it can be covered by countably many curves of class C^3 embedded in $\mathbb{R}^n$. The main goal of this paper is to prove the following result.

Theorem 1.1. *Let there be given Lipschitz maps*

$$\gamma_0, \gamma_1 : [a, b] \to \mathbb{R}^{1+k} \qquad \textit{and} \qquad \gamma_2 = (\gamma_{2\top}, \gamma_{2\perp}) : [a, b] \to \mathbb{R}^{1+k} \times \mathbb{R}^{1+k}$$

and a function $\omega : [a, b] \to \{\pm 1\}$ *such that*

$$\gamma_0'(t) = \omega(t)\, \|\gamma_0'(t)\|\, \gamma_1(t), \tag{1-1}$$

$$(\gamma_0'(t), \gamma_1'(t)) = \omega(t)\, \|(\gamma_0'(t), \gamma_1'(t))\|\, \gamma_2(t) \tag{1-2}$$

for almost every $t \in [a, b]$. *Then* $\gamma_0([a, b])$ *is a* C^3-*rectifiable set.*

Remark. In the special case when $\omega := 1$ while γ_0 is regular and at least of class C^2, the conditions (1-1) and (1-2) say that $\gamma_1(t)$ and $\gamma_2(t)$ are, respectively, the unit tangent vector of γ_0 at t and the unit tangent vector of (γ_0, γ_1) at t. This remark is at the root of the applications to geometric variational problems mentioned below.

Theorem 1.1 should be considered a step forward in a project, stated in [Delladio 2005], aimed at providing sufficient conditions for the C^H-rectifiability of a n-dimensional rectifiable set. Results concerning the case $H = 2$ were first obtained in [Anzellotti and Serapioni 1994; Delladio 2003; Fu 1998], but subtle mistakes seriously invalidating their proofs were discovered later [Delladio 2004; Fu 2004]. Then the paper [Delladio 2005], cleaning-up the simplest case $n = 1$ and $H = 2$, followed. Our future efforts will be aimed at extending the theory to any value of n

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and H. Joint work with Joseph Fu on C^2-rectifiability in all dimensions (invoking slicing in order to reduce to dimension one) is in progress.

Further work is in progress to apply these results to geometric variational problems via geometric measure theory and more precisely through the notion, first introduced in [Anzellotti et al. 1990], of a generalized Gauss graph. Former achievements in this direction include [Delladio 2001] (a somehow surprising application to differential geometry context), [Anzellotti and Delladio 1995] (an application to Willmore problem) and [Delladio 1997] (an application to a problem introduced in [Bellettini et al. 1993]). These last two papers followed the idea by De Giorgi of relaxing the functional with respect to L^1-convergence of the domains of integration. Now we expect that our results can be applied to handle functionals with integrands involving curvatures with their derivatives and, in particular, to get explicit representation formulas after relaxation.

The proof of Theorem 1.1 starts from the C^2-rectifiability of $\gamma_0([a,b])$, which is guaranteed by condition (1-1), as shown in [Delladio 2005]. The problem is reduced in Section 2 to proving that $\gamma_0([a,b])$ intersects the graph of any C^2 map

$$f:\mathbb{R}\to(\mathbb{R}u)^\perp \qquad (u\in\mathbb{R}^{1+k},\ \|u\|=1)$$

in a C^3-rectifiable set. From the first and second derivatives of f expressed in terms of the γ_i, we obtain in Section 3 a second order Taylor-type formula for f with the remainder in terms of the γ_i. Theorem 1.1 then follows by the Whitney Extension Theorem, also involving a Lusin-type argument (Section 4). Finally, the absolute curvature for a one-dimensional C^2-rectifiable set P is defined and proved to be approximately differentiable almost everywhere whenever P is C^3-rectifiable (Section 5).

2. Reduction to graphs

By virtue of the main result stated in [Delladio 2005], the equality (1-1) implies that $\gamma_0([a,b])$ is C^2-rectifiable. As a consequence, there must be countably many unit vectors

$$u_j\in\mathbb{R}^{1+k}$$

and maps of class C^2

$$f_j:\mathbb{R}\to(\mathbb{R}u_j)^\perp$$

such that

$$\mathcal{H}^1\bigl(\gamma_0([a,b])\setminus\textstyle\bigcup_j G_{f_j}\bigr)=0$$

where

$$G_{f_j}:=\bigl\{xu_j+f_j(x)\,|\,x\in\mathbb{R}\bigr\}.$$

Hence we need only show that the sets $\gamma_0([a,b]) \cap G_{f_j}$ are C^3-rectifiable. In other words, Theorem 1.1 becomes an immediate corollary of the following result.

Theorem 2.1. *Let $\gamma_0, \gamma_1, \gamma_2$ be as in Theorem 1.1. Consider a map*

$$f : \mathbb{R} \to (\mathbb{R}u)^\perp \qquad (u \in \mathbb{R}^{1+k}, \|u\| = 1)$$

of class C^2 and define

$$G_f := \{xu + f(x) \mid x \in \mathbb{R}\}.$$

Then the set $G_f \cap \gamma_0([a,b])$ is C^3-rectifiable.

In this section we take the first step toward the proof of Theorem 2.1, which will be concluded later in Section 4. Define

$$L := \gamma_0^{-1}(G_f) \cap \bigl\{ t \in [a,b] \bigm| \gamma_0'(t), \gamma_1'(t) \text{ exist, } \gamma_0'(t) \neq 0, \text{ (1-1) and (1-2) hold} \bigr\}.$$

By Lusin's Theorem, for any given real number $\varepsilon > 0$, there exists a closed subset L_ε of L such that

$$\gamma_0'|L_\varepsilon \text{ and } \omega|L_\varepsilon \text{ are continuous and } \mathscr{L}^1(L \backslash L_\varepsilon) \le \varepsilon. \tag{2-1}$$

If L_ε^* denotes the set of the density points of L_ε, then

$$L_\varepsilon^* \subset L_\varepsilon \tag{2-2}$$

since L_ε is closed. The equality

$$\mathscr{L}^1(L_\varepsilon \backslash L_\varepsilon^*) = 0 \tag{2-3}$$

also holds by a celebrated result of Lebesgue. In the special case that L has measure zero, we take $L_\varepsilon := \varnothing$, hence $L_\varepsilon^* := \varnothing$.

Now observe that

$$G_f \cap \gamma_0([a,b]) \backslash \gamma_0(L_\varepsilon^*) \subset \gamma_0\bigl(\gamma_0^{-1}(G_f) \cap [a,b] \backslash L_\varepsilon^*\bigr)$$

hence

$$\begin{aligned} \mathscr{H}^1\bigl(G_f \cap \gamma_0([a,b]) \backslash \gamma_0(L_\varepsilon^*)\bigr) &\le \mathscr{H}^1\bigl(\gamma_0\bigl(\gamma_0^{-1}(G_f) \cap [a,b] \backslash L_\varepsilon^*\bigr)\bigr) \\ &\le \int_{\gamma_0^{-1}(G_f) \cap [a,b] \backslash L_\varepsilon^*} \|\gamma_0'\| = \int_{L \backslash L_\varepsilon^*} \|\gamma_0'\| \le \varepsilon \, Lip(\gamma_0), \end{aligned}$$

which implies

$$\mathscr{H}^1\bigl(G_f \cap \gamma_0([a,b]) \setminus \textstyle\bigcup_{j=1}^\infty \gamma_0(L_{1/j}^*)\bigr) = 0.$$

Hence, to prove Theorem 2.1, it will be enough to verify that

$$\gamma_0(L_\varepsilon^*) \text{ is } C^3\text{-rectifiable} \tag{2-4}$$

for all $\varepsilon > 0$.

3. Second order Taylor formula and estimates

Proposition 3.1 below gives formulas for the first and second derivatives of f in terms of the γ_i. This yields a suitable second order Taylor formula in Theorem 3.1.

Throughout this section we shall assume $\mathscr{L}^1(L) > 0$. Notice that

$$\gamma_{2\top}(s) \neq 0 \quad \text{for all } s \in L \tag{3-1}$$

by (1-2), so the map

$$\mu : \{t \in [a,b] \mid \gamma_{2\top}(t) \neq 0\} \to \mathbb{R}^{1+k}, \qquad \mu(t) := \frac{\gamma_{2\perp}(t)}{\|\gamma_{2\top}(t)\|}$$

is well-defined in L.

Lemma 3.1. *Let $A, B, u \in \mathbb{R}^{1+k}$, with $\|u\| = 1$. Then*

$$(A \wedge B) \llcorner u = (A \cdot u)B - (B \cdot u)A.$$

Proof. Let $\{e_j\}$ be an orthonormal basis of $\mathbb{R}^{1+k}$ such that $e_1 = u$. One has

$$\big((A \wedge B) \llcorner u\big) \cdot e_i = \langle A \wedge B, u \wedge e_i \rangle = \sum_{j<l} (A_j B_l - A_l B_j)\langle e_j \wedge e_l, e_1 \wedge e_i \rangle$$
$$= A_1 B_i - A_i B_1 = \big((A \cdot u)B - (B \cdot u)A\big) \cdot e_i$$

for all $i = 1, 2, \ldots, 1+k$. □

Proposition 3.1. *Set*

$$x(t) := \gamma_0(t) \cdot u, \qquad t \in \mathbb{R}.$$

Then, for all $s \in L^*_\varepsilon$, one has

$$x'(s) = \gamma_0'(s) \cdot u \neq 0 \qquad (\textit{that is, } \gamma_1(s) \cdot u \neq 0) \tag{3-2}$$

and

$$f'(x(s)) = \frac{\gamma_1(s)}{\gamma_1(s) \cdot u} - u. \tag{3-3}$$

Moreover

$$f''(x(s)) = \frac{\big(\gamma_1(s) \wedge \mu(s)\big) \llcorner u}{(\gamma_1(s) \cdot u)^3}. \tag{3-4}$$

Proof. Observe that

$$f(x(t)) = \gamma_0(t) - (\gamma_0(t) \cdot u)u = \gamma_0(t) - x(t)u$$

for all $t \in \gamma_0^{-1}(G_f)$. The sides of this equality are both differentiable in L^*_ε and since each point in $L^*_\varepsilon \subset \gamma_0^{-1}(G_f)$ is a limit point of $L_\varepsilon \subset \gamma_0^{-1}(G_f)$, the derivatives have to coincide in L_ε^*. Thus

$$x'(s)f'(x(s)) = \gamma_0'(s) - (\gamma_0'(s)\cdot u)u = \gamma_0'(s) - x'(s)u \tag{3-5}$$

for all $s \in L_\varepsilon^*$. We obtain (3-2) by recalling that $\gamma_0'(s) \neq 0$ at all $s \in L_\varepsilon^*$.

Formula (3-3) follows at once from (3-5) and (1-1).

By virtue of (3-2), the sides of (3-3) are both differentiable in L_ε^*. The derivatives must coincide in L_ε^*, since each point of L_ε^* is a limit point of L_ε^*. In view of Lemma 3.1, we then get

$$x'(s)f''(x(s)) = \frac{(\gamma_1(s)\cdot u)\gamma_1'(s) - (\gamma_1'(s)\cdot u)\gamma_1(s)}{(\gamma_1(s)\cdot u)^2} = \frac{(\gamma_1(s)\wedge\gamma_1'(s))\llcorner u}{(\gamma_1(s)\cdot u)^2}$$

for all $s \in L_\varepsilon^*$. Formula (3-4) finally follows from (3-2), (1-1) and (1-2). □

Now set

$$\Delta_s(t) := \gamma_0(t) - \gamma_0(s), \qquad s, t \in [a, b].$$

The map

$$\Sigma_s(t) := \Delta_s(t) - (\Delta_s(t)\cdot\gamma_1(s))\,\gamma_1(s) - \frac{(\Delta_s(t)\cdot u)^2}{2(\gamma_1(s)\cdot u)^2}\mu(s), \qquad t \in [a, b],$$

is well-defined for any given $s \in L_\varepsilon^*$, by Proposition 3.1.

If $s \in L_\varepsilon^*$, hence $s \in (a, b)$ and (3-1) holds, one has

$$\|\gamma_{2\top}(\sigma)\| \geq \tfrac{1}{2}\|\gamma_{2\top}(s)\| > 0 \quad \text{for all } \sigma \in I_s,$$

where I_s denotes a certain nontrivial open interval centered at s and included in $[a, b]$, existing by the continuity of $\gamma_{2\top}$. In particular, this inequality shows that $\mu|I_s$ is Lipschitz, so the map given, for $\sigma \in I_s$, by

$$\Psi_s(\sigma) := \mu(\sigma) - (\mu(\sigma)\cdot\gamma_1(s))\gamma_1(s) - \frac{\mu(s)}{(\gamma_1(s)\cdot u)^2}\big((\gamma_1(\sigma)\cdot u)^2 + (\Delta_s(\sigma)\cdot u)(\mu(\sigma)\cdot u)\big)$$

is well-defined and Lipschitz, provided $s \in L_\varepsilon^*$. Moreover

$$\Psi_s(s) = 0,$$

as follows at once from (1-2) and from the following simple result.

Proposition 3.2. *If $s \in L_\varepsilon^*$ then $\gamma_1(s)\cdot\gamma_1'(s) = 0$.*

Proof. Let $\{s_j\}$ be a sequence in L_ε converging to s, with $s_j \neq s$ for all j. Since

$$\|\gamma_1(s_j)\| = \|\gamma_1(s)\| = 1 \quad \text{for all } j,$$

by (1-1) and (2-2), we have

$$0 = \frac{\|\gamma_1(s_j)\|^2 - \|\gamma_1(s)\|^2}{s_j - s} = \frac{\gamma_1(s_j) - \gamma_1(s)}{s_j - s}\cdot\big(\gamma_1(s_j) + \gamma_1(s)\big).$$

The conclusion follows by letting $j \to \infty$. □

Theorem 3.1. *Let $s \in L_\varepsilon^*$.*

(1) *For all $t \in \gamma_0^{-1}(G_f)$,*

$$f(x(t)) - f(x(s)) - f'(x(s))\big(x(t) - x(s)\big) - \tfrac{1}{2} f''(x(s))\big(x(t) - x(s)\big)^2 = \frac{1}{\gamma_1(s)\cdot u}\,(\gamma_1(s) \wedge \Sigma_s(t)) \llcorner u. \tag{3-6}$$

(2) *For all $t \in I_s$,*

$$\Sigma_s(t) = \int_s^t \omega(\rho)\,\|\gamma_0'(\rho)\| \left(\int_s^\rho \omega(\sigma)\,\|\gamma_0'(\sigma)\|\,\Psi_s(\sigma)\,d\sigma\right) d\rho.$$

Proof. (1) By recalling Proposition 3.1 and Lemma 3.1, we get

$$\begin{aligned}
f(x(t)) &- f(x(s)) - f'(x(s))(x(t) - x(s)) - \tfrac{1}{2} f''(x(s))(x(t) - x(s))^2 \\
&= \gamma_0(t) - x(t)u - (\gamma_0(s) - x(s)u) - \left(\frac{\gamma_1(s)}{\gamma_1(s)\cdot u} - u\right)(x(t) - x(s)) \\
&\qquad - \frac{(\gamma_1(s) \wedge \gamma_{2\perp}(s)) \llcorner u}{2\,\|\gamma_{2\top}(s)\|\,(\gamma_1(s)\cdot u)^3}(x(t) - x(s))^2 \\
&= \Delta_s(t) - \frac{\gamma_1(s)}{\gamma_1(s)\cdot u}(\Delta_s(t)\cdot u) - \frac{(\gamma_1(s) \wedge \gamma_{2\perp}(s)) \llcorner u}{2\,\|\gamma_{2\top}(s)\|\,(\gamma_1(s)\cdot u)^3}(\Delta_s(t)\cdot u)^2 \\
&= \frac{1}{\gamma_1(s)\cdot u}\left((\gamma_1(s) \wedge \Delta_s(t)) \llcorner u - \frac{(\gamma_1(s) \wedge \gamma_{2\perp}(s)) \llcorner u}{2\,\|\gamma_{2\top}(s)\|\,(\gamma_1(s)\cdot u)^2}(\Delta_s(t)\cdot u)^2\right).
\end{aligned}$$

This is just (3-6), in view of the definition of $\Sigma_s(t)$.

(2) Since Δ_s is Lipschitz and $\Delta_s(s) = 0$, one has

$$\begin{aligned}
\Sigma_s(t) &= \int_s^t \gamma_0'(\rho) - (\gamma_0'(\rho)\cdot\gamma_1(s))\gamma_1(s) - \frac{\mu(s)}{2(\gamma_1(s)\cdot u)^2}\frac{d}{d\rho}(\Delta_s(\rho)\cdot u)^2 d\rho \\
&= \int_s^t \gamma_0'(\rho) - (\gamma_0'(\rho)\cdot\gamma_1(s))\gamma_1(s) - \frac{\mu(s)}{(\gamma_1(s)\cdot u)^2}(\Delta_s(\rho)\cdot u)(\gamma_0'(\rho)\cdot u)\,d\rho
\end{aligned}$$

namely

$$\Sigma_s(t) = \int_s^t \omega(\rho)\|\gamma_0'(\rho)\|\,\Phi_s(\rho)\,d\rho \tag{3-7}$$

by (1-1), where Φ_s is the Lipschitz map defined by

$$\Phi_s(\rho) := \gamma_1(\rho) - (\gamma_1(\rho)\cdot\gamma_1(s))\gamma_1(s) - \frac{\mu(s)}{(\gamma_1(s)\cdot u)^2}(\Delta_s(\rho)\cdot u)(\gamma_1(\rho)\cdot u)$$

for $\rho \in [a, b]$. Observe that

$$\|\gamma_0'(\sigma)\|\gamma_{2\perp}(\sigma) = \|(\gamma_0'(\sigma), \gamma_1'(\sigma))\| \, \|\gamma_{2\top}(\sigma)\| \, \gamma_{2\perp}(\sigma) = \omega(\sigma) \, \|\gamma_{2\top}(\sigma)\| \, \gamma_1'(\sigma)$$

for a.e. $\sigma \in [a, b]$, by (1-2). Hence

$$\gamma_1'(\sigma) = \omega(\sigma)\|\gamma_0'(\sigma)\| \frac{\gamma_{2\perp}(\sigma)}{\|\gamma_{2\top}(\sigma)\|} = \omega(\sigma)\|\gamma_0'(\sigma)\|\mu(\sigma)$$

for a.e. $\sigma \in [a, b]$ such that $\gamma_{2\top}(\sigma) \neq 0$ — in particular, for a.e. $\sigma \in I_s$. By recalling the definition of Ψ_s, it follows at once that

$$\Phi_s'(\sigma) = \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma) \tag{3-8}$$

for a.e. $\sigma \in I_s$. We conclude using (3-7), (3-8) and noting that $\Phi_s(s) = 0$. □

As a consequence, we get the following integral representation of Σ_s' and the related first order Taylor formula for f'.

Corollary 3.1. *Let $s \in L_\varepsilon^*$ and $t \in L_\varepsilon^* \cap I_s$. Then*

(1) *The map Σ_s is differentiable at t and*

$$\Sigma_s'(t) = \omega(t)\|\gamma_0'(t)\| \int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\, d\sigma.$$

(2) *One has*

$$f'(x(t)) - f'(x(s)) - f''(x(s))(x(t) - x(s)) \\ = \frac{1}{(\gamma_1(t)\cdot u)\,(\gamma_1(s)\cdot u)} \left(\gamma_1(s) \wedge \int_s^t \omega(\sigma)\,\|\gamma_0'(\sigma)\|\,\Psi_s(\sigma) d\sigma \right) \llcorner\, u.$$

Proof. (1) Observe that $t + h \in I_s \subset (a, b)$ provided $|h|$ is small enough. By Theorem 3.1(2), then,

$$\begin{aligned} \frac{\Sigma_s(t+h) - \Sigma_s(t)}{h} &= \frac{1}{h} \int_t^{t+h} \omega(\rho)\,\|\gamma_0'(\rho)\| \left(\int_s^\rho \omega(\sigma)\,\|\gamma_0'(\sigma)\|\,\Psi_s(\sigma)\, d\sigma \right) d\rho \\ &= I_1(h) + I_2(h) \end{aligned}$$

for all small enough values of $|h|$, where we have set — with a harmless abuse of notation and recalling that $\omega|L_\varepsilon^*$ is continuous, by (2-1) and (2-2) —

$$\begin{aligned} I_1(h) &:= \frac{\omega(t)}{h} \int_{[t,t+h]\cap L_\varepsilon^*} \|\gamma_0'(\rho)\| \left(\int_s^\rho \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma) d\sigma \right) d\rho, \\ I_2(h) &:= \frac{1}{h} \int_{[t,t+h]\setminus L_\varepsilon^*} \omega(\rho)\|\gamma_0'(\rho)\| \left(\int_s^\rho \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma) d\sigma \right) d\rho. \end{aligned}$$

We have

$$I_1(h) = \frac{\omega(t)}{h} \int_{[t,t+h]\cap L^*_\varepsilon} \big(\|\gamma_0'(\rho)\| - \|\gamma_0'(t)\|\big)\bigg(\int_s^\rho \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\,d\sigma\bigg)\,d\rho$$
$$+ \frac{\omega(t)\|\gamma_0'(t)\|}{h}$$
$$\times \int_{[t,t+h]\cap L^*_\varepsilon} \bigg(\int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)d\sigma + \int_t^\rho \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\,d\sigma\bigg)\,d\rho.$$

Recalling that

(i) $\gamma_0'|L^*_\varepsilon$ is continuous, by (2-1) and (2-2),

(ii) γ_0 is Lipschitz and Ψ_s is bounded (in fact it is Lipschitz!), and

(iii) t is a density point of L_ε (hence of L^*_ε, by (2-3)),

we see that

$$\lim_{h\to 0} I_1(h) = \omega(t)\|\gamma_0'(t)\| \int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\,d\sigma.$$

The conclusion follows now by observing that, as an easy consequence of (ii) and (iii), one also has

$$\lim_{h\to 0} I_2(h) = 0.$$

(2) The two members of (3-6) are differentiable at t, by (1). Since t is a limit point of $L_\varepsilon \subset \gamma_0^{-1}(G_f)$ the derivatives have to coincide, by Theorem 3.1(1), namely

$$\Big(f'(x(t)) - f'(x(s)) - f''(x(s))\big(x(t)-x(s)\big)\Big)x'(t) = \frac{1}{\gamma_1(s)\cdot u}\big(\gamma_1(s)\wedge\Sigma_s'(t)\big)\llcorner u.$$

We conclude by recalling Proposition 3.1, part (1) of the corollary and (1-1). □

4. Conclusion of the proof of Theorem 1.1

To complete the proof of Theorem 2.1, hence of Theorem 1.1, we have to verify (2-4). For $i = 1, 2, \dots$, define $\Gamma^{(i)}$ as the set of the points $s \in L^*_\varepsilon$ satisfying, for all $t \in L^*_\varepsilon$ such that $|t-s| \le (b-a)/i$, the estimates

$$\big\| f(x(t)) - f(x(s)) - f'(x(s))(x(t)-x(s)) - \tfrac{1}{2} f''(x(s))(x(t)-x(s))^2 \big\| \le i\,|x(t)-x(s)|^3,$$
$$\| f'(x(t)) - f'(x(s)) - f''(x(s))(x(t)-x(s))\| \le i\,|x(t)-x(s)|^2,$$
$$\| f''(x(t)) - f''(x(s))\| \le i\,|x(t)-x(s)|.$$

Obviously,

$$\Gamma^{(i)} \subset \Gamma^{(i+1)} \subset L^*_\varepsilon$$

for all i, and it is easy to verify that

$$\bigcup_i \Gamma^{(i)} = L_\varepsilon^*; \tag{4-1}$$

indeed, for $s \in L_\varepsilon^*$, Theorem 3.1 and the equality $\Psi_s(s) = 0$ for the Lipschitz function Ψ_s imply that

$$\begin{aligned}\Big\| f(x(t)) - f(x(s)) - f'(x(s))(x(t)-x(s)) - \tfrac{1}{2} f''(x(s))(x(t)-x(s))^2 \Big\| \\ \le \frac{\|\Sigma_s(t)\|}{|\gamma_1(s)\cdot u|} \le \frac{\operatorname{Lip}(\gamma_0)^2 \operatorname{Lip}(\Psi_s)}{|\gamma_1(s)\cdot u|} \left| \int_s^t \left(\int_s^\rho |\sigma - s|\, d\sigma \right) d\rho \right| \\ = A(s)\, |t-s|^3\end{aligned}$$

for all $t \in L_\varepsilon^* \cap I_s$, where

$$A(s) := \frac{\operatorname{Lip}(\gamma_0)^2 \operatorname{Lip}(\Psi_s)}{6\, |\gamma_1(s)\cdot u|}.$$

Since

$$\frac{x(t)-x(s)}{t-s} \to x'(s) \qquad (\text{as } t \to s)$$

and $x'(s) \neq 0$ by Proposition 3.1, it follows that

$$\left| \frac{x(t)-x(s)}{t-s} \right| \ge \frac{|x'(s)|}{2} > 0 \tag{4-2}$$

provided $|t-s|$ is small enough. Then

$$\begin{aligned}\Big\| f(x(t)) - f(x(s)) - f'(x(s))(x(t)-x(s)) - \tfrac{1}{2} f''(x(s))(x(t)-x(s))^2 \Big\| \\ \le \frac{8A(s)}{|x'(s)|^3} |x(t)-x(s)|^3\end{aligned} \tag{4-3}$$

whenever t lies in L_ε^* and $|t-s|$ is small enough.

Analogously, from Corollary 3.1(2) we get

$$\begin{aligned}\| f'(x(t)) - f'(x(s)) - f''(x(s))(x(t)-x(s)) \| \\ \le \frac{1}{|\gamma_1(t)\cdot u|\, |\gamma_1(s)\cdot u|} \left\| \int_s^t \omega(\sigma) \|\gamma_0'(\sigma)\| \Psi_s(\sigma)\, d\sigma \right\| \\ \le \frac{\operatorname{Lip}(\gamma_0) \operatorname{Lip}(\Psi_s)}{|\gamma_1(t)\cdot u|\, |\gamma_1(s)\cdot u|} \left| \int_s^t |\sigma - s|\, d\sigma \right| = \frac{B(s)}{|\gamma_1(t)\cdot u|} |t-s|^2\end{aligned}$$

for all $t \in L_\varepsilon^* \cap I_s$, where

$$B(s) := \frac{\operatorname{Lip}(\gamma_0) \operatorname{Lip}(\Psi_s)}{2\, |\gamma_1(s)\cdot u|}.$$

Since $\gamma_1(t) \to \gamma_1(s)$ as $t \to s$ and since $\gamma_1(s) \cdot u \neq 0$ by Proposition 3.1, one also has

$$|\gamma_1(t) \cdot u| \geq \frac{|\gamma_1(s) \cdot u|}{2} > 0 \tag{4-4}$$

provided $|t - s|$ is small enough. Recalling (4-2), we obtain

$$\begin{aligned}\|f'(x(t)) - f'(x(s)) - f''(x(s))(x(t) - x(s))\| \\ \leq \frac{8\,B(s)}{|\gamma_1(s) \cdot u|\,|x'(s)|^2}|x(t) - x(s)|^2\end{aligned} \tag{4-5}$$

on condition that $t \in L_\varepsilon^*$ and $|t - s|$ is small enough.

Since $\mu|I_s$ is Lipschitz and by (4-4), it follows that the map

$$t \mapsto \frac{\big(\gamma_1(t) \wedge \mu(t)\big) \llcorner u}{(\gamma_1(t) \cdot u)^3}$$

is Lipschitz in a neighborhood of s. Then, by also recalling Proposition 3.1, a number $C(s)$ has to exist such that

$$\|f''(x(t)) - f''(x(s))\| \leq C(s)\,|t - s|$$

provided $t \in L_\varepsilon^*$ and $|t - s|$ is small enough. By (4-2) one has

$$\|f''(x(t)) - f''(x(s))\| \leq \frac{2\,C(s)}{|x'(s)|}\,|x(t) - x(s)| \tag{4-6}$$

whenever $t \in L_\varepsilon^*$ and $|t - s|$ is small enough.

Now (4-3), (4-5) and (4-6) imply that $s \in \Gamma^{(i)}$, for i big enough. Hence (4-1) follows.

As a consequence of (4-1), we are reduced to verifying that

$$\gamma_0(\Gamma^{(i)}) \text{ is } C^3\text{-rectifiable for all } i. \tag{4-7}$$

To prove this, we first set

$$\begin{aligned} a_j^{(i)} &:= a + \frac{(b-a)j}{i} && (j = 0, \dots, i), \\ \Gamma_j^{(i)} &:= \Gamma^{(i)} \cap \big(a_j^{(i)}, a_{j+1}^{(i)}\big) && (j = 0, \dots, i-1), \\ F_j^{(i)} &:= \overline{x\big(\Gamma_j^{(i)}\big)} && (j = 0, \dots, i-1). \end{aligned}$$

For any pair $\xi, \eta \in F_j^{(i)}$, there are two sequences $\{s_h\}, \{t_h\} \subset \Gamma_j^{(i)}$ such that

$$\lim_{h\to\infty} x(s_h) = \xi, \qquad \lim_{h\to\infty} x(t_h) = \eta.$$

Since the three estimates at the beginning of this section (page 264) hold with $s = s_h$, $t = t_h$, we obtain, by letting $h \to \infty$,

$$\begin{aligned}\left\| f(\eta) - f(\xi) - f'(\xi)(\eta - \xi) - \tfrac{1}{2} f''(\xi)(\eta - \xi)^2 \right\| &\le i\, |\eta - \xi|^3,\\ \left\| f'(\eta) - f'(\xi) - f''(\xi)(\eta - \xi) \right\| &\le i\, |\eta - \xi|^2,\\ \left\| f''(\eta) - f''(\xi) \right\| &\le i\, |\eta - \xi|.\end{aligned}$$

Therefore $f|F_j^{(i)}$ can be extended to a map of class $C^{2,1}$

$$f_j^{(i)} : \mathbb{R} \to (\mathbb{R}u)^{\perp}$$

by invoking the Whitney extension Theorem [Stein 1970, Chapter VI, §2.3].

Finally, a Lusin-type result [Federer 1969, §3.1.15] implies that $\gamma_0(\Gamma_j^{(i)})$ has to be C^3-rectifiable (compare [Anzellotti and Serapioni 1994, Proposition 3.2]). Hence (4-7) follows.

5. Approximately differentiable absolute curvature of a one-dimensional C^3-rectifiable set

We now extend the notion of absolute curvature to arbitrary one-dimensional C^2-rectifiable subsets P of $\mathbb{R}^{1+k}$. Consider a "C^2-covering of P", that is, a countable family

$$\mathscr{A} = \{C_i\},$$

where the C_i are compact curves of class C^2, embedded in the base space and such that

$$\mathscr{H}^1\big(P \setminus \textstyle\bigcup_i C_i\big) = 0.$$

Part (1) of the next proposition and the remark following it provide the argument proving the well-posedness of Definition 5.1 below.

Proposition 5.1. *Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}^{1+k}$ be maps of class C^2 and let x_0 be a density point of*

$$F := \big\{x \in \mathbb{R} \,\big|\, \varphi(x) = \psi(x)\big\}.$$

(1) *$\varphi'(x_0) = \psi'(x_0)$ and $\varphi''(x_0) = \psi''(x_0)$.*

(2) *$\varphi'''(x_0) = \psi'''(x_0)$ if φ and ψ are of class C^3.*

Proof. The set F^* of density points of F satisfies $F^* \subset F$ and $\mathscr{L}^1(F \setminus F^*) = 0$; hence every point in F^* is a limit point of F^*. The proposition follows. □

Remark. The following facts follow easily from Proposition 5.1(1).

(a) If x is a density point of both $P \cap C_i$ and $P \cap C_j$, then the absolute curvatures of C_i and C_j coincide at x. Hence, denoting by $(P \cap C_i)^*$ the set of density points of $P \cap C_i$, the function

$$\alpha_P^{\mathcal{A}} : \bigcup_i (P \cap C_i)^* \to \mathbb{R}, \quad x \mapsto \text{ the absolute curvature of } C_{i(x)} \text{ at } x$$

where $i(x)$ is any index such that $x \in (P \cap C_{i(x)})^*$, is well-defined. Moreover,

$$\mathcal{H}^1\big(P \setminus \bigcup_i (P \cap C_i)^*\big) = \mathcal{H}^1\big(P \setminus \bigcup_i (P \cap C_i)\big) = \mathcal{H}^1\big(P \setminus \bigcup_i C_i\big) = 0,$$

by a well-known result of Lebesgue.

(b) If $\mathcal{B}$ is another C^2-covering of P, then $\alpha_P^{\mathcal{A}}$ and $\alpha_P^{\mathcal{B}}$ are representatives of the same measurable function, with domain P.

Definition 5.1. The measurable real-valued function with domain P and having $\alpha_P^{\mathcal{A}}$ as a representative (see preceding remark) is said to be the *absolute curvature* of P and is denoted by α_P.

Proposition 5.2. *If P is C^3-rectifiable, then α_P is approximately differentiable; that is:*

(1) *For any given C^3-covering $\mathcal{A} = \{C_i\}$ of P, the function $\alpha_P^{\mathcal{A}}$ is approximately differentiable at every point in $(P \cap C_i)^*$, for all i.*

(2) *If $\mathcal{A}$ and $\mathcal{B}$ are C^3-coverings of P, then one has*

$$apD\alpha_P^{\mathcal{A}} = apD\alpha_P^{\mathcal{B}}, \quad \text{a.e. in } P.$$

Proof. (1) Consider any point $a \in (P \cap C_{i_0})^*$. Without loss of generality, we can assume that C_{i_0} is the graph of a function of class C^3, namely

$$C_{i_0} = \{tu + h(t) \mid t \in I\}$$

where u is a unit vector in $\mathbb{R}^{1+k}$, I is a closed interval centered at 0 and

$$h \in C^3(I\,,\ (\mathbb{R}u)^{\perp}), \qquad h(0) = a.$$

Set $U := I^\circ \times \mathbb{R}^k$ and let $g : U \to \mathbb{R}$ be defined as the function mapping $(t, v) \in U$ to the absolute curvature of C_{i_0} at $tu + h(t)$, that is,

$$g(t, v) = \frac{\|(u + h'(t)) \wedge h''(t)\|}{(1 + \|h'(t)\|^2)^{3/2}}, \quad (t, v) \in U \tag{5-1}$$

by (8.4.13.1) in [Berger and Gostiaux 1988].

Obviously, the function g is differentiable at a. Moreover, since

$$(P \cap C_{i_0})^* \subset E := \Big\{x \in \bigcup_i (P \cap C_i)^* \;\Big|\; \alpha_P^{\mathcal{A}}(x) = g(x)\Big\}$$

by the definition of $\alpha_P^{\mathscr{A}}$, the set E has density 1 at a. According to [Federer 1969, §3.2.16], the function $\alpha_P^{\mathscr{A}}$ is approximately differentiable at a and one has

$$\operatorname{ap}D\alpha_P^{\mathscr{A}}(a) = Dg(a)|\mathbb{R}\tau, \text{ with } \tau := (1, h'(0)). \tag{5-2}$$

(2) This follows easily from (5-1) and (5-2), by recalling Proposition 5.1. □

References

[Anzellotti and Delladio 1995] G. Anzellotti and S. Delladio, "Minimization of functionals of curvatures and the Willmore problem", pp. 33–43 in *Advances in geometric analysis and continuum mechanics* (Stanford, 1993), edited by P. Concus and K. Lancaster, International Press, Cambridge, MA, 1995. MR 96i:49052 Zbl 0840.49008

[Anzellotti and Serapioni 1994] G. Anzellotti and R. Serapioni, "$\mathscr{C}^k$-rectifiable sets", *J. Reine Angew. Math.* **453** (1994), 1–20. MR 95g:49078 Zbl 0799.49028

[Anzellotti et al. 1990] G. Anzellotti, R. Serapioni, and I. Tamanini, "Curvatures, functionals, currents", *Indiana Univ. Math. J.* **39**:3 (1990), 617–669. MR 91k:49059 Zbl 0718.49030

[Bellettini et al. 1993] G. Bellettini, G. Dal Maso, and M. Paolini, "Semicontinuity and relaxation properties of a curvature depending functional in 2D", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **20**:2 (1993), 247–297. MR 94g:49101 Zbl 0797.49013

[Berger and Gostiaux 1988] M. Berger and B. Gostiaux, *Differential geometry: manifolds, curves, and surfaces*, Graduate Texts in Mathematics **115**, Springer, New York, 1988. MR 88h:53001 Zbl 0629.53001

[Delladio 1997] S. Delladio, "Special generalized Gauss graphs and their application to minimization of functionals involving curvatures", *J. Reine Angew. Math.* **486** (1997), 17–43. MR 98f:49043 Zbl 0871.49034

[Delladio 2001] S. Delladio, "On hypersurfaces in $\mathbf{R}^{n+1}$ with integral bounds on curvature", *J. Geom. Anal.* **11**:1 (2001), 17–42. MR 2002a:49059 Zbl 1034.49043

[Delladio 2003] S. Delladio, "The projection of a rectifiable Legendrian set is C^2-rectifiable: a simplified proof", *Proc. Roy. Soc. Edinburgh Sect. A* **133**:1 (2003), 85–96. MR 2003m:53089 Zbl 1035.53010

[Delladio 2004] S. Delladio, "Taylor polynomials and non-homogeneous blow-ups", *Manuscripta Math.* **113**:3 (2004), 383–395. MR 2006a:41007 Zbl 02078018

[Delladio 2005] S. Delladio, "A result about C^2-rectifiability of one-dimensional rectifiable sets: Application to a class of one-dimensional integral currents", preprint, Università di Trento, 2005, Available at http://eprints.biblio.unitn.it/archive/00000783. To appear in *Boll. Un. Matem. Italiana*.

[Federer 1969] H. Federer, *Geometric measure theory*, Grundlehren der math. Wiss. **153**, Springer, New York, 1969. MR 41 #1976 Zbl 0176.00801

[Fu 1998] J. H. G. Fu, "Some remarks on Legendrian rectifiable currents", *Manuscripta Math.* **97**:2 (1998), 175–187. MR 2000g:49051 Zbl 0916.53038

[Fu 2004] J. H. G. Fu, "Erratum to: "Some remarks on Legendrian rectifiable currents" [Manuscripta Math. **97** (1998), no. 2, 175–187]", *Manuscripta Math.* **113**:3 (2004), 397–401. MR 2005k:49120 Zbl 1066.53014

[Stein 1970] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Series **30**, Princeton University Press, Princeton, NJ, 1970. MR 44 #7280 Zbl 0207.13501

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