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**REIDEMEISTER TORSION, THE THURSTON NORM AND
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Cochran introduced Alexander polynomials over noncommutative Laurent polynomial rings. Their degrees were studied by Cochran, Harvey and Turaev, who gave lower bounds on the Thurston norm. We extend Cochran's definition to twisted Alexander polynomials, and show how Reidemeister torsion relates to these invariants, giving lower bounds on the Thurston norm in terms of the Reidemeister torsion. This yields a concise formulation of the bounds of Cochran, Harvey and Turaev. The Reidemeister torsion approach also provides a natural approach to proving and extending certain monotonicity results of Cochran and Harvey.

1. Introduction

The following algebraic setup allows us to define twisted noncommutative Alexander polynomials. Let \mathbb{K} be a (skew) field and $\gamma : \mathbb{K} \rightarrow \mathbb{K}$ a ring homomorphism. Denote by $\mathbb{K}_\gamma[t^{\pm 1}]$ the skew Laurent polynomial ring over \mathbb{K} , so the elements in $\mathbb{K}_\gamma[t^{\pm 1}]$ are formal sums $\sum_{i=m}^n a_i t^i$ ($m \leq n \in \mathbb{Z}$) with $a_i \in \mathbb{K}$. Addition is given by addition of the coefficients, and multiplication is defined using the rule $t^i a = \gamma^i(a) t^i$ for any $a \in \mathbb{K}$.

Let X be a connected CW complex with finitely many cells in dimension i . Given a representation $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ we can consider the $\mathbb{K}_\gamma[t^{\pm 1}]$ -modules $H_i^\alpha(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ and we define twisted noncommutative Alexander polynomials $\Delta_i^\alpha(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ (see [Section 3.3](#) for details). Twisted Alexander polynomials over commutative Laurent polynomial rings were first introduced in [[Lin 2001](#)], Alexander polynomials over skew Laurent polynomial rings in [[Cochran 2004](#)]. Our definition is a combination of the definitions in [[Kirk and Livingston 1999](#)] and [[Cochran 2004](#)]. In [Theorem 3.1](#) we describe the indeterminacy of these polynomials.

Denote by $\mathbb{K}_\gamma(t)$ the quotient field of $\mathbb{K}_\gamma[t^{\pm 1}]$. We denote the induced representation $\pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d) \rightarrow \text{GL}(\mathbb{K}_\gamma(t), d)$ by α as well. If the homology groups $H_i^\alpha(X; \mathbb{K}_\gamma(t)^d)$ vanish and if X is a finite connected CW complex, then

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we can define the Reidemeister torsion $\tau(X, \alpha) \in K_1(\mathbb{K}_\gamma(t))/\pm \alpha(\pi_1(X))$. An important tool is the Dieudonné determinant, which defines an isomorphism

$$\det : K_1(\mathbb{K}_\gamma(t)) \rightarrow \mathbb{K}_\gamma(t)_{ab}^\times,$$

where $\mathbb{K}_\gamma(t)_{ab}^\times$ denotes the abelianization of the multiplicative group $\mathbb{K}_\gamma(t)^\times = \mathbb{K}_\gamma(t) \setminus \{0\}$. We can therefore study $\det \tau(X, \alpha) \in \mathbb{K}_\gamma(t)_{ab}^\times / \pm \det \alpha(\pi_1(X))$. We refer to Sections 2.3 and 3.1 for details.

The following result generalizes commutative results of Turaev [1986; 2001] and Kirk and Livingston [1999].

Theorem 1.1. *Let X be a finite connected CW complex of dimension n . Let $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a representation such that $H_*^\alpha(X; \mathbb{K}_\gamma(t)^d) = 0$. Then $\Delta_i^\alpha(t) \neq 0$ for all i and*

$$\det \tau(X, \alpha) = \prod_{i=0}^{n-1} \Delta_i^\alpha(t)^{(-1)^{i+1}} \in \mathbb{K}_\gamma(t)_{ab}^\times / \{kt^e \mid k \in \mathbb{K} \setminus \{0\}, e \in \mathbb{Z}\}.$$

For $f(t) = \sum_{i=m}^n a_i t^i \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$ with $a_m \neq 0, a_n \neq 0$, we define its degree to be $\text{deg } f(t) = n - m$. We can extend this to a degree function $\text{deg} : \mathbb{K}_\gamma(t) \setminus \{0\} \rightarrow \mathbb{Z}$. We denote $\text{deg } \det \tau(X, \alpha)$ by $\text{deg } \tau(X, \alpha)$. Theorem 1.1 then implies that the degree of $\tau(X, \alpha)$ is the alternating sum of the degrees of the twisted Alexander polynomials (see Corollary 3.6).

We now turn to the study of 3-manifolds. Here and throughout the paper we will assume that all manifolds are compact, orientable and connected. Recall that given a 3-manifold M and $\phi \in H^1(M; \mathbb{Z})$ the Thurston norm [1986] of ϕ is defined as

$$\|\phi\|_T = \min\{-\chi(\hat{S}) \mid S \subset M \text{ properly embedded surface dual to } \phi\}$$

where \hat{S} denotes the result of discarding all connected components of S with positive Euler characteristic. As an example, consider $X(K) = S^3 \setminus \nu K$, where $K \subset S^3$ is a knot and νK denotes an open tubular neighborhood of K in S^3 . Let $\phi \in H^1(X(K); \mathbb{Z})$ be a generator, then $\|\phi\|_T = 2 \text{ genus}(K) - 1$.

Let X be a connected CW complex and let $\phi \in H^1(X; \mathbb{Z})$. We identify henceforth $H^1(X; \mathbb{Z})$ with $\text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z})$ and $\text{Hom}(\pi_1(X), \mathbb{Z})$. A representation $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ is called ϕ -compatible if for any $g \in \pi_1(X)$ we have $\alpha(g) = At^{\phi(g)}$ for some $A \in \text{GL}(\mathbb{K}, d)$. This generalizes a notion of Turaev [2002a].

The following theorem gives lower bounds on the Thurston norm using Reidemeister torsion. It contains the lower bounds of McMullen [2002], Cochran [2004], Harvey [2005], Turaev [2002a] and of the paper [Friedl and Kim 2006]. To our knowledge this theorem is the strongest of its kind. Not only does it contain these results, the formulation of the inequalities given in the papers just cited in terms

of the degrees of Reidemeister torsion also gives a very concise reformulation of their results.

Theorem 1.2. *Let M be a 3-manifold with empty or toroidal boundary. Let $\phi \in H^1(M; \mathbb{Z})$ and $\alpha : \pi_1(M) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ a ϕ -compatible representation. Then $\tau(M, \alpha)$ is defined if and only if $\Delta_1^\alpha(t) \neq 0$. If $\tau(M, \alpha)$ is defined, then*

$$\|\phi\|_T \geq \frac{1}{d} \deg \tau(M, \alpha).$$

If (M, ϕ) fibers over S^1 , then

$$\|\phi\|_T = \max\left\{0, \frac{1}{d} \deg \tau(M, \alpha)\right\}.$$

The most commonly used skew fields are the quotient fields $\mathbb{K}(G)$ of group rings $\mathbb{F}[G]$ (\mathbb{F} a commutative field) for certain torsion-free groups G , we refer to [Section 5.1](#) for details. The following theorem says roughly that larger groups give better bounds on the Thurston norm. The main idea of the proof is to use the fact that Reidemeister torsion behaves well under ring homomorphisms, in contrast to Alexander polynomials. See [Section 6](#) or [\[Harvey 2006\]](#) for the definition of an admissible triple.

Theorem 1.3. *Let M be a 3-manifold with empty or toroidal boundary or let M be a 2-complex with $\chi(M) = 0$. Let $\phi \in H^1(M; \mathbb{Z})$. Let $\alpha : \pi_1(M) \rightarrow \text{GL}(\mathbb{F}, d)$, \mathbb{F} a commutative field, be a representation and $(\varphi_G : \pi \rightarrow G, \varphi_H : \pi \rightarrow H, \phi)$ an admissible triple for $\pi_1(M)$, in particular we have epimorphisms $G \rightarrow H \rightarrow \mathbb{Z}$. Write $G' = \text{Ker}\{G \rightarrow \mathbb{Z}\}$ and $H' = \text{Ker}\{H \rightarrow \mathbb{Z}\}$.*

If $\tau(M, \varphi_H \otimes \alpha) \in K_1(\mathbb{K}(H')(t))$ is defined, then $\tau(M, \varphi_G \otimes \alpha) \in K_1(\mathbb{K}(G')(t))$ is defined. Furthermore in that case

$$\deg \tau(M, \varphi_G \otimes \alpha) \geq \deg \tau(M, \varphi_H \otimes \alpha).$$

A similar theorem holds for 2-complexes with Euler characteristic zero. As a special case consider the case that α is the trivial representation. Using [Theorem 1.1](#) we can recover the monotonicity results of [\[Cochran 2004; Harvey 2006\]](#). We hope that our alternative proof using Reidemeister torsion will contribute to the understanding of their results.

In the next section we recall the definition of Reidemeister torsion. In [Section 3](#) we introduce twisted noncommutative Alexander polynomials, compute their indeterminacies in [Theorem 3.1](#) and prove [Theorem 1.1](#). Beginning with [Section 4](#) we concentrate on 3-manifolds: [Section 4](#) gives the proof of [Theorem 1.2](#), [Section 5](#) contains examples of ϕ -compatible representations, in [Section 6](#) we prove [Theorem 1.3](#), and in [Section 7](#) we show that it implies Cochran's and Harvey's monotonicity results. We conclude with a few open questions in [Section 8](#).

2. Reidemeister torsion

2.1. Definition of $K_1(R)$. For the remainder of the paper we will only consider associative rings R with $1 \neq 0$ and with the property that if $r \neq s \in \mathbb{N}_0$, then R^r is not isomorphic to R^s as an R -module.

For such a ring R define $GL(R) = \varinjlim GL(R, d)$, where the maps $GL(R, d) \rightarrow GL(R, d+1)$ in the direct system are given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then $K_1(R)$ is defined as $GL(R)/[GL(R), GL(R)]$. Note that $K_1(R)$ is an abelian group. For details we refer to [Milnor 1966; Turaev 2001]. There exists a canonical map $GL(R, d) \rightarrow K_1(R)$ for every d , in particular there exists a homomorphism from the units of R into $K_1(R)$. By abuse of notation we denote the image of $A \in GL(R, d)$ in $K_1(R)$ by A as well. We denote by $-A$ the product of $A \in K_1(R)$ by the image of (-1) under the map $GL(R, 1) \rightarrow K_1(R)$.

We will often make use of the observation (see [Rosenberg 1994, p. 61]) that for $A \in GL(R, d_1)$, $B \in GL(R, d_2)$ the product $AB \in K_1(R)$ is given by

$$AB = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in K_1(R).$$

2.2. Definition of Reidemeister torsion. Let C_* be a finite free chain complex of R -modules. By this we mean a chain complex of free finite right R -modules such that $C_i = 0$ for all but finitely many $i \in \mathbb{Z}$. Let $\mathcal{C}_i \subset C_i$ be a basis for all i with $C_i \neq 0$. Assume that $B_i = \text{Im}(C_{i+1}) \subset C_i$ is free, pick a basis \mathcal{B}_i of B_i and a lift $\tilde{\mathcal{B}}_i$ of \mathcal{B}_i to C_{i+1} . We write $\mathcal{B}_i \tilde{\mathcal{B}}_{i-1}$ for the collection of elements given by \mathcal{B}_i and $\tilde{\mathcal{B}}_{i-1}$. Since C_* is acyclic this is indeed a basis for C_i . Then we define the Reidemeister torsion of the based acyclic complex $(C_*, \{\mathcal{C}_i\})$ to be

$$\tau(C_*, \{\mathcal{C}_i\}) = \prod [\mathcal{B}_i \tilde{\mathcal{B}}_{i-1} / \mathcal{C}_i]^{(-1)^{i+1}} \in K_1(R),$$

where $[d/e]$ denotes the matrix of a basis change, i.e. $[d/e] = (a_{ij})$ where $d_i = \sum_j a_{ji} e_j$. (In contrast to Turaev’s book we view vectors as column vectors, so our matrix is the transpose of the matrix in [Turaev 2001, p. 1].)

It is easy to see that $\tau(C_*, \{\mathcal{C}_i\})$ is independent of the choice of $\{\mathcal{B}_i\}$ and of the choice of the lifts $\tilde{\mathcal{B}}_i$. If the R -modules B_i are not free, then one can show that they are stably free and a stable basis will then make the definition work again. See [Milnor 1966, p. 369] or [Turaev 2001, p. 13] for details.

2.3. Reidemeister torsion of a CW complex. Let X be a connected CW complex. Denote the universal cover of X by \tilde{X} . We view $C_*(\tilde{X})$, the chain complex of the universal cover, as a chain complex of right $\mathbb{Z}[\pi_1(X)]$ -modules, where the $\mathbb{Z}[\pi_1(X)]$ -module structure is given via deck transformations.

Let R be a ring. Let $\alpha : \pi_1(X) \rightarrow GL(R, d)$ be a representation. This equips R^d with a left $\mathbb{Z}[\pi_1(X)]$ -module structure. We can therefore consider the chain

complex $C_*^\alpha(X; R^d) = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^d$. This is a finite free chain complex of (right) R -modules. We denote its homology by $H_*^\alpha(X; R^d)$, we drop α from the notation if it is clear from the context.

Now assume that X is a finite connected CW complex. If

$$H_i^\alpha(X; R^d) = H_i(C_*^\alpha(X; R^d)) \neq 0$$

for some i , we write $\tau(X, \alpha) = 0$. Otherwise denote the i -cells of X by $\sigma_i^1, \dots, \sigma_i^{r_i}$ and denote by e_1, \dots, e_d the standard basis of R^d . Pick an orientation for each cell σ_i^j , and also pick a lift $\tilde{\sigma}_i^j$ for each cell σ_i^j to the universal cover \tilde{X} . We get a basis

$$\mathcal{C}_i = \{\tilde{\sigma}_i^1 \otimes e_1, \dots, \tilde{\sigma}_i^1 \otimes e_d, \dots, \tilde{\sigma}_i^{r_i} \otimes e_1, \dots, \tilde{\sigma}_i^{r_i} \otimes e_d\}$$

for $C_i^\alpha(X; R^d)$. Then we can define

$$\tau(C_*^\alpha(X; R^d), \{\mathcal{C}_i\}) \in K_1(R).$$

This element depends only on the ordering and orientation of the cells and on the choice of lifts of the cells to the universal cover. Therefore

$$\tau(X, \alpha) = \tau(C_*^\alpha(X, R^d), \{\mathcal{C}_i\}) \in K_1(R) / \pm \alpha(\pi_1(X))$$

is a well defined invariant of the CW complex X .

Now let M be a compact PL manifold. Pick any finite CW structure for M to define $\tau(M, \alpha) \in K_1(R) / \pm \alpha(\pi_1(M))$. By Chapman's theorem [1974] this is a well defined invariant of the manifold (independent of the choice of the CW structure).

2.4. Computation of Reidemeister torsion. We explain an algorithm for computing Reidemeister torsion formulated in [Turaev 2001, Section 2.1] in the commutative case.

Assume that we have a finite free chain complex of R -modules

$$0 \rightarrow C_m \xrightarrow{\partial_m} C_{m-1} \rightarrow \dots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0.$$

Let $A_i = (a_{jk}^i)$ be the matrix representing ∂_i corresponding to the given bases. Again, in contrast with Turaev, we view the elements in $R^{\text{rank}(C_i)}$ as column vectors.

Following [Turaev 2001, p. 8] we define a *matrix chain* for C to be a collection of sets $\xi = (\xi_0, \xi_1, \dots, \xi_m)$ where $\xi_i \subset \{1, 2, \dots, \text{rank}(C_i)\}$ so that $\xi_0 = \emptyset$. Given a matrix chain ξ we define $A_i(\xi)$, $i = 1, \dots, m$ to be the matrix formed by the entries a_{jk}^i with $j \notin \xi_{i-1}$ and $k \in \xi_i$. Put differently the matrix $(a_{jk}^i)_{jk}$ is given by considering only the ξ_i -columns of A_i and with the ξ_{i-1} -rows removed.

We say that a matrix chain ξ is a τ -chain if $A_1(\xi), \dots, A_m(\xi)$ are square matrices. The following is the generalization of Turaev's Theorem 2.2 to the noncommutative setting. His proof can easily be generalized to this more general setting.

Theorem 2.1. *Let ξ be a τ -chain such that $A_i(\xi)$ is invertible for all odd i . Then $A_i(\xi)$ is invertible for all even i if and only if $H_*(C) = 0$. If $H_*(C) = 0$, then*

$$\tau(C) = \epsilon \prod_{i=1}^m A_i(\xi)^{(-1)^i} \in K_1(R) \quad \text{for some } \epsilon \in \{\pm 1\}.$$

This proposition is the reason why Reidemeister torsion behaves in general well under ring homomorphisms.

3. Reidemeister torsion and Alexander polynomials

3.1. Laurent polynomial rings and the Dieudonné determinant. For the remainder of this paper let \mathbb{K} be a (skew) field and let $\mathbb{K}_\gamma[t^{\pm 1}]$ be a skew Laurent polynomial ring. By [Dodziuk et al. 2003, Corollary 6.3] the ring $\mathbb{K}_\gamma[t^{\pm 1}]$ has a classical quotient field $\mathbb{K}_\gamma(t)$ which is flat over $\mathbb{K}_\gamma[t^{\pm 1}]$ (compare [Ranicki 1998, p. 99]). In particular we can view $\mathbb{K}_\gamma[t^{\pm 1}]$ as a subring of $\mathbb{K}_\gamma(t)$ and any element in $\mathbb{K}_\gamma(t)$ is of the form $f(t)g(t)^{-1}$ for some $f(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ and $g(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$. We refer to Theorem 5.1 for a related result. Recall that we write $\mathbb{K}_\gamma(t)^\times = \mathbb{K}_\gamma(t) \setminus \{0\}$.

In the following we mean by an elementary column (row) operation the addition of a right multiple (left multiple) of one column (row) to a different column (row). Let A be an invertible $k \times k$ matrix over the skew field $\mathbb{K}_\gamma(t)$. After elementary row operations we can turn A into a diagonal matrix $D = (d_{ij})$. Then the Dieudonné determinant $\det A \in \mathbb{K}_\gamma(t)^\times_{ab} = \mathbb{K}_\gamma(t)^\times / [\mathbb{K}_\gamma(t)^\times, \mathbb{K}_\gamma(t)^\times]$ is defined to be $\prod_{i=1}^k d_{ii}$. This is a well defined map. The Dieudonné determinant is invariant under elementary row and column operations, and induces an isomorphism $\det : K_1(\mathbb{K}_\gamma(t)) \rightarrow \mathbb{K}_\gamma(t)^\times_{ab}$. Using the last observation in Section 2.1 it is easy to see that $A = \det A \in K_1(\mathbb{K}_\gamma(t))$. We will often make use of this equality. We refer to [Rosenberg 1994, Theorem 2.2.5 and Corollary 2.2.6] for more details.

In the introduction we defined $\deg : \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\} \rightarrow \mathbb{N}$. This can be extended to a homomorphism $\deg : \mathbb{K}_\gamma(t)^\times \rightarrow \mathbb{Z}$ via $\deg(f(t)g(t)^{-1}) = \deg f(t) - \deg g(t)$ for $f(t), g(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$. Clearly the degree map vanishes on $[\mathbb{K}_\gamma(t)^\times, \mathbb{K}_\gamma(t)^\times]$ and we get an induced homomorphism $K_1(\mathbb{K}_\gamma(t)) \rightarrow \mathbb{K}_\gamma(t)^\times_{ab} \rightarrow \mathbb{Z}$ which we also denote by \deg .

3.2. Orders of $\mathbb{K}_\gamma[t^{\pm 1}]$ -modules. Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$ -module. The ring $\mathbb{K}_\gamma[t^{\pm 1}]$ is a principal ideal domain (PID) since \mathbb{K} is a skew field. We can therefore find an isomorphism

$$H \cong \bigoplus_{i=1}^l \mathbb{K}_\gamma[t^{\pm 1}] / p_i(t) \mathbb{K}_\gamma[t^{\pm 1}]$$

for $p_i(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ for $i = 1, \dots, l$. Following [Cochran 2004] we define $\text{ord}(H) = \prod_{i=1}^l p_i(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$. This is called the *order of H* .

Note that $\text{ord}(H) \in \mathbb{K}_\gamma[t^{\pm 1}]$ has a high degree of indeterminacy. For example writing the $p_i(t)$ in a different order will change $\text{ord}(H)$. Furthermore we can change $p_i(t)$ by multiplication by any element of the form kt^e where $k \in \mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ and $e \in \mathbb{Z}$. The following theorem can be viewed as saying that these are all possible indeterminacies.

Theorem 3.1. *Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$ -module. Then $\text{ord}(H) = 0$ if and only if H is not $\mathbb{K}_\gamma[t^{\pm 1}]$ -torsion. If $\text{ord}(H) \neq 0$, then $\text{ord}(H) \in \mathbb{K}_\gamma[t^{\pm 1}]$ is well defined considered as an element in $\mathbb{K}_\gamma(t)_{ab}^\times$ up to multiplication by an element of the form kt^e , $k \in \mathbb{K}^\times$ and $e \in \mathbb{Z}$.*

The first statement is clear. We postpone the proof of the second statement of the theorem to Section 3.4. We refer to [Cochran 2004, p. 367] for an alternative discussion of the indeterminacy of $\text{ord}(H)$; the idea of considering $\text{ord}(H)$ as an element in $\mathbb{K}_\gamma(t)_{ab}^\times$ is already present there.

It follows from Theorem 3.1 that $\text{deg ord}(H)$ is well defined. In fact we have the following interpretation of $\text{ord}(H)$.

Lemma 3.2 [Cochran 2004, p. 368]. *Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$ -torsion module. Then*

$$\text{deg ord}(H) = \dim_{\mathbb{K}} H.$$

Here we used that by [Stenström 1975, Proposition I.2.3] and [Cohn 1985, p. 48] every right \mathbb{K} -module V is free and has a well defined dimension $\dim_{\mathbb{K}}(V)$.

Proof. It is easy to see that for $f(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$ we have

$$\text{deg } f(t) = \dim_{\mathbb{K}}(\mathbb{K}_\gamma[t^{\pm 1}]/f(t)\mathbb{K}_\gamma[t^{\pm 1}]).$$

The lemma is now immediate. □

3.3. Alexander polynomials. Let X be a connected CW complex with finitely many cells in dimension i . Let $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a representation. The right $\mathbb{K}_\gamma[t^{\pm 1}]$ -module $H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ is called twisted (noncommutative) Alexander module. Similar modules were studied in [Cochran 2004; Harvey 2005; Turaev 2002a]. Note that $H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ is a finitely generated $\mathbb{K}_\gamma[t^{\pm 1}]$ -module since we assumed that X has only finitely many cells in dimension i and since $\mathbb{K}_\gamma[t^{\pm 1}]$ is a PID. We now define $\Delta_i^\alpha(t) = \text{ord}(H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)) \in \mathbb{K}_\gamma[t^{\pm 1}]$, this is called the (twisted) i -th Alexander polynomial of (X, α) .

The degrees of these polynomials (corresponding to one-dimensional representations) have been studied recently in various contexts [Cochran 2004; Harvey 2005; 2006; Turaev 2002a; Leidy and Maxim 2006; Friedl and Kim 2005; Friedl and Harvey 2006]. We hope that by determining the indeterminacy of the Alexander polynomials (Theorem 3.1) more information can be extracted from the Alexander polynomials than just the degrees.

3.4. Proof of Theorem 3.1. We first point out that $\mathbb{K}_\gamma[t^{\pm 1}]$ is a Euclidean ring with respect to the degree function. This means that given $f(t), g(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$ we can find $a(t), r(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ such that $f(t) = g(t)a(t) + r(t)$ and such that either $r(t) = 0$ or $\deg r(t) < \deg g(t)$.

Let A be an $r \times s$ matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ of rank r . Here and in the following the rank of a matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ will be understood as the rank of the matrix considered as a matrix over the skew field $\mathbb{K}_\gamma(t)$. Note that $\text{rank}(A) = r$ implies that in particular $s \geq r$. Since $\mathbb{K}_\gamma[t^{\pm 1}]$ is a Euclidean ring we can perform a sequence of elementary row and column operations to turn A into a matrix of the form $(D \ 0_{r \times (s-r)})$ where D is an $r \times r$ matrix and $0_{r \times (s-r)}$ stands for the $r \times (s-r)$ matrix consisting only of zeros. Since A is of rank r it follows that D has rank r as well, in particular D is a square matrix which is invertible over $\mathbb{K}_\gamma(t)$ and we can consider its Dieudonné determinant $\det D$. We define $\det A = \det D \in \mathbb{K}_\gamma(t)_{ab}^\times$.

Lemma 3.3. *Let A be a (square) matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ which is invertible over $\mathbb{K}_\gamma(t)$.*

- (1) *The Dieudonné determinant $\det A \in \mathbb{K}_\gamma(t)_{ab}^\times$ can be represented by an element in $\mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$.*
- (2) *If $A \in \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$, then $\det A \in \mathbb{K}_\gamma(t)_{ab}^\times$ can be represented by an element of the form kt^e , $k \in \mathbb{K}^\times, e \in \mathbb{Z}$.*
- (3) *The Dieudonné determinant induces a homomorphism*

$$\det : K_1(\mathbb{K}_\gamma(t)) \rightarrow \mathbb{K}_\gamma(t)_{ab}^\times$$

sending $K_1(\mathbb{K}_\gamma[t^{\pm 1}])$ to $\{kt^e \mid k \in \mathbb{K}^\times, e \in \mathbb{Z}\} / [\mathbb{K}_\gamma(t)^\times, \mathbb{K}_\gamma(t)^\times] \subset \mathbb{K}_\gamma(t)_{ab}^\times$.

Proof. The first statement follows from the discussion preceding the lemma. Now let $A \in \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], r)$. Then $\det A$ has degree zero, by Lemma 3.2 applied to $H = \mathbb{K}_\gamma[t^{\pm 1}]^r / A\mathbb{K}_\gamma[t^{\pm 1}]^r$. This proves the second statement. The last statement follows from the second statement and the fact that the Dieudonné determinant induces a homomorphism $\det : K_1(\mathbb{K}_\gamma(t)) \rightarrow \mathbb{K}_\gamma(t)_{ab}^\times$. □

Proposition 3.4. *Let A be an $r \times s$ matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ of rank r . Then $\det A \in \mathbb{K}_\gamma(t)_{ab}^\times$ is well defined up to multiplication by an element of the form kt^e , $k \in \mathbb{K}^\times, e \in \mathbb{Z}$. Furthermore $\det A$ is invariant under elementary row and column operations.*

Proof. First note that the effect of an elementary row operation on A over $\mathbb{K}_\gamma[t^{\pm 1}]$ can be described by left multiplication by a matrix $P \in \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], r)$. Similarly an elementary column operation on A over $\mathbb{K}_\gamma[t^{\pm 1}]$ can be described by right multiplication by an $s \times s$ matrix $Q \in \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], s)$.

Now assume we have $P_1, P_2 \in \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], r)$ and $Q_1, Q_2 \in \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], s)$ such that $P_i A Q_i = (D_i \ 0_{r \times (s-r)})$, $i = 1, 2$ where D_i is an $r \times r$ matrix. We are

done once we show that $\det D_1 = kt^e \det D_2 \in \mathbb{K}_\gamma(t)_{ab}^\times$ for some $k \in \mathbb{K}^\times, e \in \mathbb{Z}$. Let $E_i = P_i^{-1} D_i$. Then by [Lemma 3.3](#) we only have to show that $E_1 = E_2 \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}])$.

We have $(E_1 \ 0) Q_1^{-1} = (E_2 \ 0) Q_2^{-1}$. Let $Q := Q_2^{-1} Q_1 \in \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], s)$, we therefore get the equality $(E_1 \ 0) = (E_2 \ 0) Q$. Now write

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

where Q_{ij} is a $n_i \times n_j$ matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$ with $n_1 = r$ and $n_2 = s - r$. We get the equality

$$\begin{pmatrix} E_1 & 0 \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} E_2 & 0 \\ 0 & \text{id}_{s-r} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

It follows in particular that Q_{22} is invertible over $\mathbb{K}_\gamma(t)$. Furthermore we have

$$E_1 \cdot Q_{22} = E_2 \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}]).$$

Note that $\text{deg} : K_1(\mathbb{K}_\gamma(t)) \rightarrow \mathbb{Z}$ vanishes on $K_1(\mathbb{K}_\gamma[t^{\pm 1}])$ by [Lemma 3.3](#). We therefore get $\text{deg det } E_1 + \text{deg det } Q_{22} = \text{deg det } E_2$, in particular $\text{deg det } E_1 \leq \text{deg det } E_2$. But by symmetry we have $\text{deg det } E_2 \leq \text{deg det } E_1$. In particular $\text{deg det } Q_{22} = 0$. The proposition now follows immediately from [Lemma 3.3](#) since $\text{deg } f(t) = 0$ for $f(t) \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}$ if and only if $f(t) = kt^e$ for some $k \in \mathbb{K}^\times, e \in \mathbb{Z}$.

The last statement is immediate. □

Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$ -module. We say that an $r \times s$ matrix A is a presentation matrix for H if the following sequence is exact:

$$\mathbb{K}_\gamma[t^{\pm 1}]^s \xrightarrow{A} \mathbb{K}_\gamma[t^{\pm 1}]^r \rightarrow H \rightarrow 0.$$

We say that A has full rank if the rank of A equals r . Note that A has full rank if and only if $H \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t) = 0$.

The following lemma clearly implies [Theorem 3.1](#).

Lemma 3.5. *Let H be a finitely generated right $\mathbb{K}_\gamma[t^{\pm 1}]$ -module and let A_1, A_2 be presentation matrices for H . Then A_1 has full rank if and only if A_2 has full rank. Furthermore if A_i has full rank, then*

$$\det A_1 = \det A_2 \in \mathbb{K}_\gamma(t)_{ab}^\times / \{kt^e \mid k \in \mathbb{K} \setminus \{0\}, e \in \mathbb{Z}\}.$$

Proof. Two presentation matrices for H differ by a sequence of matrix moves of the following forms and their inverses:

- (1) Permutation of rows or columns.
- (2) Replacement of the matrix A by $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

- (3) Addition of an extra column of zeros to the matrix A .
- (4) Addition of a right scalar multiple of a column to another column.
- (5) Addition of a left scalar multiple of a row to another row.

This result coincides with [Lickorish 1997, Theorem 6.1] in the commutative case, but the proof there carries through in the case of the base ring $\mathbb{K}_\gamma[t^{\pm 1}]$ as well (compare [Harvey 2005, Lemma 9.2]).

Clearly none of the moves changes the status of being of full rank, and if a representation is of full rank, then it is easy to see that none of the moves changes the determinant. □

3.5. Proof of Theorem 1.1. Now let X be a finite connected CW complex of dimension n . Let $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a representation such that $H_*^\alpha(X; \mathbb{K}_\gamma(t)^d) = 0$. (Recall that we denote the induced representation $\pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma(t), d)$ by α as well). Furthermore recall that $\mathbb{K}_\gamma(t)$ is flat over $\mathbb{K}_\gamma[t^{\pm 1}]$, in particular $H_i(X; \mathbb{K}_\gamma(t)^d) = H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)$. It follows that $H_i(X; \mathbb{K}_\gamma(t)^d) = 0$ if and only if $H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ is $\mathbb{K}_\gamma[t^{\pm 1}]$ -torsion, which is equivalent to $\Delta_i^\alpha(t) \neq 0$. This proves the first statement of Theorem 1.1. To conclude the proof of Theorem 1.1 it remains to prove the following claim.

Claim. *If $H_*^\alpha(X; \mathbb{K}_\gamma(t)^d) = 0$, then*

$$\det \tau(X, \alpha) = \prod_{i=0}^{n-1} \Delta_i^\alpha(t)^{(-1)^{i+1}} \in \mathbb{K}_\gamma(t)_{ab}^\times / \{kt^e \mid k \in \mathbb{K} \setminus \{0\}, e \in \mathbb{Z}\}.$$

Proof. Let $C_* = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{K}_\gamma[t^{\pm 1}]^d$. Any $\mathbb{K}_\gamma[t^{\pm 1}]$ -basis for C_* also gives a basis for $C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)$, which we will always denote by the same symbol.

Denote by \mathcal{C}_* the $\mathbb{K}_\gamma[t^{\pm 1}]$ -basis of C_* as in Section 2.3. Let

$$r_i := \dim_{\mathbb{K}(t)}(C_i \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)),$$

$$s_i := \dim_{\mathbb{K}(t)}(\text{Ker}\{C_i \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t) \xrightarrow{\partial_i} C_{i-1} \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)\}).$$

Note that $s_i + s_{i-1} = r_i$ since $H_*(X; \mathbb{K}_\gamma(t)^d) = 0$. Note also that $\text{Ker}\{C_i \xrightarrow{\partial_i} C_{i-1}\}$ is a free direct summand of C_i of rank s_i since $\mathbb{K}_\gamma[t^{\pm 1}]$ is a PID. We can therefore pick $\mathbb{K}_\gamma[t^{\pm 1}]$ -bases $\mathcal{C}'_i = \{v_1, \dots, v_{r_i}\}$ for C_i such that $\{v_1, \dots, v_{s_i}\}$ is a basis for the kernel in question. Base changes from \mathcal{C}_i to \mathcal{C}'_i are given by matrices which are invertible over $\mathbb{K}_\gamma[t^{\pm 1}]$, so

$$\tau(C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t), \{\mathcal{C}_i\}) = \tau(C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t), \{\mathcal{C}'_i\}) \in K_1(\mathbb{K}_\gamma(t)) / K_1(\mathbb{K}_\gamma[t^{\pm 1}]).$$

Let A_i be the $r_{i-1} \times r_i$ matrix representing $\partial_i : C_i \rightarrow C_{i-1}$ with respect to the bases \mathcal{C}'_i and \mathcal{C}'_{i-1} . Let $\xi_i := \{s_i + 1, \dots, r_i\}$, $i = 1, \dots, n$ and $\xi_0 := \emptyset$. Let $A_i(\xi)$ as in

Theorem 2.1. Note that $A_i(\xi)$ is an $s_{i-1} \times s_{i-1}$ matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$. In particular $\xi := (\xi_0, \dots, \xi_n)$ is a τ -chain. It is easy to see that

$$A_i = \begin{pmatrix} 0_{s_{i-1} \times s_i} & A_i(\xi) \\ 0_{s_{i-2} \times s_i} & 0_{s_{i-2} \times s_{i-1}} \end{pmatrix}.$$

Since A_i has rank s_{i-1} it follows that $A_i(\xi)$ is invertible over $\mathbb{K}_\gamma(t)$. It follows from **Theorem 2.1** that

$$\tau(C_* \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t), \{\mathcal{C}'_i\}) = \prod_{i=1}^n A_i(\xi)^{(-1)^i} \in K_1(\mathbb{K}_\gamma(t)).$$

We also have short exact sequences

$$0 \rightarrow \mathbb{K}_\gamma[t^{\pm 1}]^{s_{i-1}} \xrightarrow{A_i(\xi)} \mathbb{K}_\gamma[t^{\pm 1}]^{s_i} \rightarrow H_{i-1}(C_*) = H_{i-1}(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \rightarrow 0.$$

In particular $(A_i(\xi))$ is a presentation matrix for $H_{i-1}(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$. It therefore follows from **Lemma 3.5** that $\det A_i(\xi) = \Delta_{i-1}^\alpha(t)$. □

The following corollary now follows from the fact that $\deg : \mathbb{K}_\gamma(t)^\times \rightarrow \mathbb{Z}$ is a homomorphism and from **Lemma 3.2**.

Corollary 3.6. *Let X be a finite connected CW complex of dimension n . Let $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a representation such that $H_*^\alpha(X; \mathbb{K}_\gamma(t)^d) = 0$. Then*

$$\deg \tau(X, \alpha) = \sum_{i=0}^{n-1} (-1)^{i+1} \deg \Delta_i^\alpha(t) = \sum_{i=0}^{n-1} (-1)^{i+1} \dim_{\mathbb{K}}(H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)).$$

Remark. In the case that $H_*(X; \mathbb{K}_\gamma(t)^d) \neq 0$ we can pick $\mathbb{K}_\gamma[t^{\pm 1}]$ -bases \mathcal{H}_i for the $\mathbb{K}_\gamma[t^{\pm 1}]$ -free parts of $H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$. These give bases for $H_i(X; \mathbb{K}_\gamma(t)^d) = H_i(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)$ and we can consider

$$\tau(X, \alpha, \{\mathcal{H}_i\}) = \tau(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{K}_\gamma(t)^d, \{\mathcal{H}_i\}) \in K_1(\mathbb{K}_\gamma(t))$$

(see [**Milnor 1966**] for details). As an element of $K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}])$, this $\tau(X, \alpha, \{\mathcal{H}_i\})$ is independent of the choice of $\{\mathcal{H}_i\}$. The proof of **Theorem 1.1** can be generalized to show that it is the alternating product of the orders of the $\mathbb{K}_\gamma[t^{\pm 1}]$ -torsion submodules of $H_*(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ (compare [**Kirk and Livingston 1999**] in the commutative case).

4. 3-manifolds and 2-complexes

We now restrict ourselves to ϕ -compatible representations since these have a closer connection to the topology of a space.

Lemma 4.1. *Let X be a connected CW complex with finitely many cells in dimensions 0 and 1. Let $\phi \in H^1(X; \mathbb{Z})$ nontrivial and let $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a ϕ -compatible representation. Then $\Delta_0^\alpha(t) \neq 0$. If X is in fact an k -manifold, then $\Delta_k^\alpha(t) = 1$.*

We need the following notation. If $A = (a_{ij})$ is an $r \times s$ matrix over $\mathbb{Z}[\pi_1(X)]$ and $\alpha : \pi_1(X) \rightarrow \text{GL}(R, d)$ a representation. Then we denote by $\alpha(A)$ the $rd \times sd$ matrix over R obtained by replacing each entry $a_{ij} \in \mathbb{Z}[\pi_1(X)]$ of A by the $d \times d$ matrix $\alpha(a_{ij})$.

Proof. First equip X with a CW structure with one 0-cell and n 1-cells g_1, \dots, g_n . We denote the corresponding elements in $\pi_1(X)$ by g_1, \dots, g_n as well. Since ϕ is nontrivial there exists at least one i such that $\phi(g_i) \neq 0$. Write $C_* = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{K}_\gamma[t^{\pm 1}]^d$. The boundary map $\partial_1 : C_1 \rightarrow C_0$ is represented by the matrix

$$(\alpha(1 - g_1), \dots, \alpha(1 - g_n)) = (\text{id} - \alpha(g_1), \dots, \text{id} - \alpha(g_n)).$$

Since α is ϕ -compatible it follows that $\alpha(1 - g_i) = \text{id} - At^{\phi(g_i)}$ for some matrix $A \in \text{GL}(\mathbb{K}, d)$. The first statement of the lemma now follows from [Lemma 4.2](#).

If X is a closed k -manifold then equip X with a CW structure with one k -cell. Since ϕ is primitive and ϕ -compatible an argument as above shows that $\partial_k : C_k \rightarrow C_{k-1}$ has full rank, i.e. $H_k(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) = 0$. Hence $\Delta_k^\alpha(t) = 1$. If X is a k -manifold with boundary, then it is homotopy equivalent to a $(k-1)$ -complex, and hence $H_k(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) = 0$. □

Lemma 4.2. *Let $\mathbb{K}_\gamma[t^{\pm 1}]$ be a skew Laurent polynomial ring and let A, B be invertible $d \times d$ matrices over \mathbb{K} and $r \neq 0$. Then $\text{deg det}(A + Bt^r) = dr$. In particular $A + Bt^r$ is invertible over $\mathbb{K}_\gamma(t)$.*

Harvey has proved a related result [[2005](#), Proposition 9.1].

Proof. We can clearly assume that $r > 0$. Let $\{e_1, \dots, e_d\}$ be a basis for \mathbb{K}^d . Consider the projection map $p : \mathbb{K}_\gamma[t^{\pm 1}]^d \rightarrow P = \mathbb{K}_\gamma[t^{\pm 1}]^d / (A + Bt^r)\mathbb{K}_\gamma[t^{\pm 1}]^d$. By [Lemma 3.2](#) we are done once we show that $p(e_it^j), i \in \{1, \dots, d\}, j \in \{0, \dots, r - 1\}$ form a basis for P as a right \mathbb{K} -vector space.

It follows easily from the fact that A, B are invertible that this is indeed a generating set. Let $v \in \mathbb{K}_\gamma[t^{\pm 1}]^d \setminus \{0\}$. We can write $v = \sum_{i=-n}^m v_i t^i, v_i \in \mathbb{K}^d$ with $v_n \neq 0, v_m \neq 0$. Since A, B are invertible it follows that $(A + Bt^r)v$ has terms with t -exponent n and terms with t -exponent $m + r$. This observation can be used to show that the vectors in question are linearly independent in P . □

We can now give the proof of [Theorem 1.2](#).

Proof of Theorem 1.2. Now let M be a 3-manifold whose boundary is empty or consists of tori. A standard duality argument shows that $2\chi(M) = \chi(\partial M) = 0$.

Let $\phi \in H^1(M; \mathbb{Z})$ be nontrivial, and $\alpha : \pi_1(M) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ a ϕ -compatible representation.

We first show that $H_*(M; \mathbb{K}_\gamma(t)^d) = 0$ if and only if $\Delta_1^\alpha(t) \neq 0$. In Section we showed that $H_i(M; \mathbb{K}_\gamma(t)^d) = 0$ if and only if $\Delta_i^\alpha(t) \neq 0$. It now follows from Lemma 4.1 that $H_i(M; \mathbb{K}_\gamma(t)^d) = 0$ for $i = 0, 3$. If $\Delta_1^\alpha(t) \neq 0$, then $H_1(M; \mathbb{K}_\gamma(t)^d) = 0$. Since $\chi(H_i(M; \mathbb{K}_\gamma(t)^d)) = d\chi(M) = 0$ it follows that $H_2(M; \mathbb{K}_\gamma(t)^d) = 0$.

Claim. $\|\phi\|_T$ is bounded below by

$$\frac{1}{d}(-\dim_{\mathbb{K}} H_0^\alpha(M; \mathbb{K}_\gamma[t^{\pm 1}]^d) + \dim_{\mathbb{K}} H_1^\alpha(M; \mathbb{K}_\gamma[t^{\pm 1}]^d) - \dim_{\mathbb{K}} H_2^\alpha(M; \mathbb{K}_\gamma[t^{\pm 1}]^d)).$$

This inequality becomes an equality if (M, ϕ) fibers over S^1 and if $M \neq S^1 \times D^2$, $M \neq S^1 \times S^2$.

Proof. If ϕ vanishes on $X \subset M$, the restriction of α to $\pi_1(X)$ lies in $\text{GL}(\mathbb{K}, d) \subset \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ since α is ϕ -compatible. Therefore

$$H_i^\alpha(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \cong H_i^\alpha(X; \mathbb{K}^d) \otimes_{\mathbb{K}} \mathbb{K}_\gamma[t^{\pm 1}].$$

The proofs of [Friedl and Kim 2006, Theorem 3.1 and Theorem 6.1] can now easily be translated to this noncommutative setting. This proves the claim. \square

Combining the results of the claim with Lemma 3.2 and Corollary 3.6 we immediately get a proof for Theorem 1.2. \square

In order to relate Theorem 1.2 to the results of [Cochran 2004; Harvey 2005; Turaev 2002a] we need the following computations for one-dimensional ϕ -compatible representations. Recall that $\phi \in H^1(X; \mathbb{Z})$ is called primitive if the corresponding map $\phi : H_1(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ is surjective.

Lemma 4.3. *Let X be a connected CW complex with finitely many cells in dimensions zero and one. Let $\phi \in H^1(X; \mathbb{Z})$ primitive. Let $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ be a ϕ -compatible one-dimensional representation. If $\alpha(\pi_1(X)) \subset \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ is cyclic, then $\Delta_0^\alpha(t) = at - 1$ for some $a \in \mathbb{K} \setminus \{0\}$. Otherwise $\Delta_0^\alpha(t) = 1$.*

Proof. Equip X with a CW structure with one 0-cell and then consider the chain complex for X as in Lemma 4.1. The lemma now follows easily from the observation that in $\mathbb{K}_\gamma[t^{\pm 1}]$ we have $\text{gcd}(1-at, 1-bt) = 1$ if $a \neq b \in \mathbb{K}$. \square

Lemma 4.4. *Let X be a 3-manifold with empty or toroidal boundary or let X be a 2-complex with $\chi(X) = 0$. Let $\phi \in H^1(M; \mathbb{Z})$ nontrivial. Let $\alpha : \pi_1(M) \rightarrow \text{GL}(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ be a ϕ -compatible one-dimensional representation. Assume that $\Delta_1^\alpha(t) \neq 0$. If X is a closed 3-manifold, then $\text{deg } \Delta_2^\alpha(t) = \text{deg } \Delta_0^\alpha(t)$, otherwise $\Delta_2^\alpha(t) = 1$.*

Proof. First assume that X is a 3-manifold. Then the lemma follows from combining [Turaev 2002a, Sections 4.3 and 4.4] with [Friedl and Kim 2006, Lemmas 4.7 and 4.9]. (The latter results also hold in the noncommutative setting.) If X is a 2-complex then the argument in the proof of Theorem 1.2 shows that $H_2(X; \mathbb{K}_\gamma(t)^d) = 0$. But since X is a 2-complex we have $H_2(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) \subset H_2(X; \mathbb{K}_\gamma(t)^d)$, hence $H_2(X; \mathbb{K}_\gamma[t^{\pm 1}]^d) = 0$ and $\Delta_2^\alpha(t) = 1$. \square

Remark. We cannot apply the duality results of [Friedl and Kim 2006, Lemma 4.12 and Proposition 4.13] since the natural involution on $\mathbb{Z}[G]$ does not necessarily extend to an involution on $\mathbb{K}_\gamma[t^{\pm 1}]$, i.e., the representation $\mathbb{Z}[G] \rightarrow \mathbb{K}_\gamma[t^{\pm 1}]$ is not necessarily unitary.

It now follows immediately from Lemma 4.3 and 4.4 and the discussion in Section 5 that Theorem 1.2 contains the results of [McMullen 2002; Cochran 2004; Harvey 2005; Turaev 2002a; Friedl and Kim 2006].

Remark. Given a 2-complex X , Turaev [2002b] defined a norm

$$\| \cdot \|_X : H^1(X; \mathbb{R}) \rightarrow \mathbb{R},$$

modeled on the definition of the Thurston norm of a 3-manifold. He then gave lower bounds for the Turaev norm (see also [Turaev 2002a]) which have the same form as certain lower bounds for the Thurston norm. Going through the proofs in [Friedl and Kim 2006] it is not hard to see that the obvious version of Theorem 1.2 for 2-complexes also holds.

If M is a 3-manifold with boundary, then it is homotopy equivalent to a 2-complex X . It is not known whether the Thurston norm of M agrees with the Turaev norm on X . But the fact that Theorem 1.2 holds in both cases, and the observation that $\deg \tau(X, \alpha)$ is a homotopy invariant by Theorem 1.1 suggests that they do in fact agree.

5. Examples for skew fields and ϕ -compatible representations

5.1. Skew fields of group rings. A group G is called locally indicable if for every finitely generated subgroup $U \subset G$ there exists a nontrivial homomorphism $U \rightarrow \mathbb{Z}$.

Theorem 5.1. *Let G be a locally indicable and amenable group and let \mathbb{F} be a commutative field. Then the following hold.*

- (1) $\mathbb{F}[G]$ is an Ore domain, in particular it embeds in its classical right ring of quotients $\mathbb{K}(G)$.
- (2) $\mathbb{K}(G)$ is flat over $\mathbb{F}[G]$.

It follows from [Higman 1940] that $\mathbb{F}[G]$ has no zero divisors. The first part now follows from [Tamari 1957] or [Dodziuk et al. 2003, Corollary 6.3]. Part (b)

is a property of Ore localizations (see [Ranicki 1998, p. 99], for example). We call $\mathbb{K}(G)$ the *Ore localization* of $\mathbb{F}[G]$.

A group G is called poly-torsion-free-abelian (PTFA) if there exists a filtration

$$1 = G_0 \subset G_1 \subset \dots \subset G_{n-1} \subset G_n = G$$

such that G_i/G_{i-1} is torsion free abelian. PTFA groups are amenable and locally indicable [Strebel 1974]. The group rings of PTFA groups played an important role in [Cochran et al. 2003; Cochran 2004; Harvey 2005].

5.2. Examples for ϕ -compatible representations. Let X be a connected CW complex and $\phi \in H^1(X; \mathbb{Z})$. We give examples of ϕ -compatible representations.

Let \mathbb{F} be a commutative field. The element $\phi \in H^1(X; \mathbb{Z}) \cong \text{Hom}(H_1(X; \mathbb{Z}), \langle t \rangle)$ induces a ϕ -compatible representation $\phi : \mathbb{Z}[\pi_1(X)] \rightarrow \mathbb{F}[t^{\pm 1}]$. Furthermore if $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{F}, d)$ is a representation, then $\pi_1(X)$ acts via $\alpha \otimes \phi$ on the $\mathbb{F}[t^{\pm 1}]$ -module $\mathbb{F}^d \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \cong \mathbb{F}[t^{\pm 1}]^d$. We therefore get a representation $\alpha \otimes \phi : \pi_1(X) \rightarrow \text{GL}(\mathbb{F}[t^{\pm 1}], d)$, which is clearly ϕ -compatible. In this particular case Theorem 1.2 was proved in [Friedl and Kim 2006].

To describe the ϕ -compatible representations of Cochran [2004] and Harvey [2005; 2006] we need the following definition.

Definition. Let π be a group, $\phi : \pi \rightarrow \mathbb{Z}$ an epimorphism and $\varphi : \pi \rightarrow G$ an epimorphism to a locally indicable and amenable group G such that there exists a map $\phi_G : G \rightarrow \mathbb{Z}$ (which is necessarily unique) such that

$$\begin{array}{ccc} \pi & \xrightarrow{\varphi} & G \\ & \searrow \phi & \downarrow \phi_G \\ & & \mathbb{Z} \end{array}$$

commutes. Following [Harvey 2006, Definition 1.4] we call (φ, ϕ) an *admissible pair*. If ϕ_G is an isomorphism, then (φ, ϕ) is called *initial*.

Now let $(\varphi : \pi_1(X) \rightarrow G, \phi)$ be an admissible pair for $\pi_1(X)$. In the following we denote $\text{Ker}\{\phi : G \rightarrow \mathbb{Z}\}$ by $G'(\phi)$. When the homomorphism ϕ is understood we will write G' for $G'(\phi)$. Clearly G' is still a locally indicable and amenable group. Let \mathbb{F} be any commutative field and $\mathbb{K}(G')$ the Ore localization of $\mathbb{F}[G]$. Pick an element $\mu \in G$ such that $\phi(\mu) = 1$. Let $\gamma : \mathbb{K}(G') \rightarrow \mathbb{K}(G')$ be the homomorphism given by $\gamma(a) = \mu a \mu^{-1}$. Then we get a representation

$$\begin{aligned} G &\rightarrow \text{GL}(\mathbb{K}(G')_{\gamma}[t^{\pm 1}], 1) \\ g &\mapsto (g\mu^{-\phi(g)}t^{\phi(g)}). \end{aligned}$$

It is clear that $\alpha : \pi_1(X) \rightarrow G \rightarrow \text{GL}(\mathbb{K}(G')_{\gamma}[t^{\pm 1}], 1)$ is ϕ -compatible. Note that the ring $\mathbb{K}(G')_{\gamma}[t^{\pm 1}]$, and hence the representation above, depend on the

choice of μ . We will nonetheless suppress γ from the notation since different choices of splittings give isomorphic rings. We often make use of the fact that $f(t)g(t)^{-1} \rightarrow f(\mu)g(\mu)^{-1}$ defines an isomorphism $\mathbb{K}(G')(t) \rightarrow \mathbb{K}(G)$ (see [Harvey 2005, Proposition 4.5]). Similarly $\mathbb{Z}[G'][t^{\pm 1}] \xrightarrow{\cong} \mathbb{Z}[G]$.

An important example of admissible pairs is provided by Harvey’s rational derived series of a group G ; see [Harvey 2005, Section 3]. Let $G_r^{(0)} = G$ and define inductively

$$G_r^{(n)} = \{g \in G_r^{(n-1)} \mid g^k \in [G_r^{(n-1)}, G_r^{(n-1)}] \text{ for some } k \in \mathbb{Z} \setminus \{0\}\}.$$

Note that $G_r^{(n-1)}/G_r^{(n)} \cong (G_r^{(n-1)})/[G_r^{(n-1)}, G_r^{(n-1)}]/\mathbb{Z}$ -torsion. By [Harvey 2005, Corollary 3.6] the quotients $G/G_r^{(n)}$ are PTFA groups for any G and any n . If $\phi : G \rightarrow \mathbb{Z}$ is an epimorphism, then $(G \rightarrow G/G_r^{(n)}, \phi)$ is an admissible pair for (G, ϕ) for any $n > 0$.

For example, if K is a knot and $G = \pi_1(S^3 \setminus K)$, it follows from [Strebel 1974] that $G_r^{(n)} = G^{(n)}$, i.e. the rational derived series equals the ordinary derived series (compare [Cochran 2004; Harvey 2005]).

Remark. Recall that for a knot K the knot exterior $S^3 \setminus \nu K$ is denoted by $X(K)$. Let $\pi = \pi_1(X(K))$ and let $\phi \in H^1(X(K); \mathbb{Z})$ primitive. Then

$$\bar{\delta}_n(K) = \dim_{\mathbb{K}(\pi'/(\pi')_r^{(n)})} (H_1(X(K), \mathbb{K}(\pi'/(\pi')_r^{(n)})[t^{\pm 1}]))$$

is a knot invariant for $n > 0$. Cochran [2004, p. 395, Question 5] asked whether $K \mapsto \bar{\delta}_n(K)$ is of finite type.

Eisermann [2000, Lemma 7] has shown that the genus is not a finite type knot invariant. Cochran [2004] showed that $\bar{\delta}_n(K) \leq 2 \text{ genus}(K)$ (see also Theorem 1.2 together with Corollary 3.6 and Lemmas 4.3 and 4.4). Eisermann’s argument can now be used to show that $K \mapsto \bar{\delta}_n(K)$ is not of finite type either.

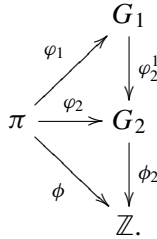
Let X be again be a connected CW complex and $\phi \in H^1(X; \mathbb{Z})$. The two types of ϕ -compatible representations given above can be combined as follows. Let $\alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{F}, d)$ be a representation and let $\varphi : \pi_1(X) \rightarrow G$ be a homomorphism such that (φ, ϕ) is an admissible pair. Denote the Ore localization of $\mathbb{F}[G']$ by $\mathbb{K}(G')$. Then $\pi_1(X)$ acts via $\varphi \otimes \alpha$ on $\mathbb{K}(G')[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}^d \cong \mathbb{K}(G')[t^{\pm 1}]^d$. We therefore get a ϕ -compatible representation $\varphi \otimes \alpha : \pi_1(X) \rightarrow \text{GL}(\mathbb{K}(G')[t^{\pm 1}], d)$.

6. Comparing different ϕ -compatible maps

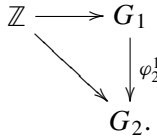
Definition [Harvey 2006]. Let π be a group and $\phi : \pi \rightarrow \mathbb{Z}$ an epimorphism. Furthermore let $\varphi_1 : \pi \rightarrow G_1$ and $\varphi_2 : \pi \rightarrow G_2$ be epimorphisms to locally indicable and amenable groups G_1 and G_2 . We call $(\varphi_1, \varphi_2, \phi)$ an *admissible triple* for π

if there exist epimorphisms $\varphi_2^1 : G_1 \rightarrow G_2$ (which is not an isomorphism) and $\phi_2 : G_2 \rightarrow \mathbb{Z}$ such that $\varphi_2 = \varphi_2^1 \circ \varphi_1$, and $\phi = \phi_2 \circ \varphi_2$.

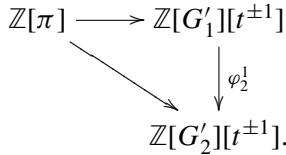
The situation can be summarized in the diagram



In particular, (φ_i, ϕ) , $i = 1, 2$, are admissible pairs for π . Given an admissible triple we can pick splittings $\mathbb{Z} \rightarrow G_i$ of φ_i , $i = 1, 2$ which make the following diagram commute:



We therefore get an induced commutative diagram of ring homomorphisms



(We are suppressing the notation for the twisting in the skew Laurent polynomial rings.) Denote the ϕ -compatible maps $\mathbb{Z}[\pi] \rightarrow \mathbb{K}(G_i)[t^{\pm 1}]$, $i = 1, 2$ by φ_i as well. For convenience we recall [Theorem 1.3](#).

Theorem 1.3. *Let M be a 3-manifold whose boundary is a (possibly empty) collection of tori or let M be a 2-complex with $\chi(M) = 0$. Let $\alpha : \pi_1(M) \rightarrow \text{GL}(\mathbb{F}, d)$ be a representation and $(\varphi_1, \varphi_2, \phi)$ an admissible triple for $\pi_1(M)$. If $\tau(M, \varphi_2 \otimes \alpha) \neq 0$, then $\tau(M, \varphi_1 \otimes \alpha) \neq 0$. Furthermore in that case*

$$\deg \tau(M, \varphi_1 \otimes \alpha) \geq \deg \tau(M, \varphi_2 \otimes \alpha).$$

6.1. Proof of Theorem 1.3 for closed 3-manifolds. Let M be a closed 3-manifold. Choose a triangulation of M . Let T be a maximal tree in the 1-skeleton of the triangulation and let T' be a maximal tree in the dual 1-skeleton. Following [\[McMullen 2002, Section 5\]](#) we collapse T to form a single 0-cell and join the 3-simplices along T' to form a single 3-cell. Since $\chi(M) = 0$ the number n of 1-cells equals

the number of 2-cells. Consider the chain complex of the universal cover \tilde{M} :

$$0 \rightarrow C_3(\tilde{M})^1 \xrightarrow{\partial_3} C_2(\tilde{M})^n \xrightarrow{\partial_2} C_1(\tilde{M})^n \xrightarrow{\partial_1} C_0(\tilde{M})^1 \rightarrow 0,$$

where the superscript indicates the rank over $\mathbb{Z}[\pi_1(M)]$. Picking appropriate lifts of the (oriented) cells of M to cells of \tilde{M} we get bases $\tilde{\sigma}_i = \{\tilde{\sigma}_i^1, \dots, \tilde{\sigma}_i^{r_i}\}$ for the $\mathbb{Z}[\pi_1(M)]$ -modules $C_i(\tilde{M})$, such that if A_i denotes the matrix corresponding to ∂_i , then A_1 and A_3 are of the form

$$\begin{aligned} A_3 &= (1 - g_1, \dots, 1 - g_n)^t, & g_i &\in \pi_1(M) \\ A_1 &= (1 - h_1, \dots, 1 - h_n), & h_i &\in \pi_1(M). \end{aligned}$$

Clearly $\{h_1, \dots, h_n\}$ is a generating set for $\pi_1(M)$. Since M is a closed 3-manifold $\{g_1, \dots, g_n\}$ is a generating set for $\pi_1(M)$ as well. In particular we can find $k, l \in \{1, \dots, n\}$ such that $\phi(g_k) \neq 0, \phi(h_l) \neq 0$.

In the following we write $\alpha_i = \varphi_i \otimes \alpha : \pi_1(M) \rightarrow \text{GL}(\mathbb{K}(G'_i)[t^{\pm 1}] \otimes \mathbb{F}^d) \rightarrow \text{GL}(\mathbb{K}(G'_i)[t^{\pm 1}], d), i = 1, 2$ and we write $\varphi = \varphi_2^1$.

Lemma 6.1. *We have*

$$\begin{aligned} \text{deg}(\alpha_1(1 - h_l)) &= \text{deg}(\alpha_2(1 - h_l)) = d|\phi(h_l)|, \\ \text{deg}(\alpha_1(1 - g_k)) &= \text{deg}(\alpha_2(1 - g_k)) = d|\phi(g_k)|. \end{aligned}$$

In particular, the matrices $\alpha_i(1 - h_l), \alpha_i(1 - g_k)$ are invertible over $\mathbb{K}(G'_i)(t)$ for $i = 1, 2$.

Proof. Note that $\alpha_i(1 - h_l) = \text{id} - \alpha_i(h_l), \alpha_i(1 - g_k) = \text{id} - \alpha_i(g_k)$ and that $\phi(h_l) \neq 0, \phi(g_k) \neq 0$. The lemma now follows from [Lemma 4.2](#) since α_1 and α_2 are ϕ -compatible. □

Denote by B the result of deleting the k -column and the l -row of A_2 .

Lemma 6.2. *$\tau(M, \alpha_i) \neq 0$ if and only if $\alpha_i(B)$ is invertible. Furthermore if $\tau(M, \alpha_i) \neq 0$, then*

$$\tau(M, \alpha_i) = \alpha_i(1 - g_k)^{-1} \alpha_i(B) \alpha_i(1 - h_l)^{-1} \in K_1(\mathbb{K}(G'_i)(t)) / \pm \alpha_i(\pi_1(M)).$$

Proof. Denote the standard basis of $\mathbb{K}(G'_i)(t)^d$ by e_1, \dots, e_d . We equip $C_j = C_j^{\alpha_i}(M; \mathbb{K}(G'_i)(t)^d)$ with the ordered bases

$$\mathcal{C}_j = \{\tilde{\sigma}_j^1 \otimes e_1, \dots, \tilde{\sigma}_j^1 \otimes e_d, \dots, \tilde{\sigma}_j^{r_j} \otimes e_1, \dots, \tilde{\sigma}_j^{r_j} \otimes e_d\}.$$

Now let

$$\begin{aligned} \xi_0 &= \emptyset, \\ \xi_1 &= \{ld+1, \dots, l(d+1)\}, \\ \xi_2 &= \{1, \dots, nd\} \setminus \{kd+1, \dots, (k+1)d\}, \\ \xi_3 &= \{1, \dots, d\}. \end{aligned}$$

Then $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$ is a τ -chain for C_* . We have $A_1(\xi) = \alpha_i(1 - h_l)$, $A_2(\xi) = \alpha_i(B)$ and $A_3(\xi) = \alpha_i(1 - g_k)$. Clearly $A_1(\xi)$ and $A_3(\xi)$ are invertible by [Lemma 6.1](#). The proposition now follows immediately from [Theorem 2.1](#). \square

Now assume that $\tau(M, \alpha_2) \neq 0$. Then $\alpha_2(B)$ is invertible over $\mathbb{K}(G'_2)(t)$ by [Lemma 6.2](#). Note that $\alpha_i(B)$ is defined over $\mathbb{Z}[G'_i][t^{\pm 1}] \subset \mathbb{K}(G'_i)(t)$. In particular $\alpha_2(B) = \varphi(\alpha_1(B))$. It follows from the following lemma that $\alpha_1(B)$ is invertible as well.

Lemma 6.3. *Let P be an $r \times s$ matrix over $\mathbb{Z}[G'_1][t^{\pm 1}]$. If*

$$\begin{aligned} \mathbb{Z}[G_2]^s &\rightarrow \mathbb{Z}[G_2]^r \\ v &\mapsto \varphi(P)v \end{aligned}$$

is invertible over $\mathbb{K}(G'_2)(t)$, then P is invertible over $\mathbb{K}(G'_1)(t)$. The same holds with “invertible” replaced by “injective”.

Proof. Assume that multiplication by $\varphi(P)$ is injective over $\mathbb{K}(G'_2)(t)$. Since $\mathbb{Z}[G_2] \rightarrow \mathbb{K}(G'_2)(t) = \mathbb{K}(G_2)$ is injective it follows that $\varphi(P) : \mathbb{Z}[G_2]^s \rightarrow \mathbb{Z}[G_2]^r$ is injective. By [Proposition 6.4](#) the map $P : \mathbb{Z}[G_1]^s \rightarrow \mathbb{Z}[G_1]^r$ is injective as well. Since $\mathbb{K}(G'_1)(t) = \mathbb{K}(G_1)$ is flat over $\mathbb{Z}[G_1]$ it follows that $P : \mathbb{K}(G'_1)(t)^s \rightarrow \mathbb{K}(G'_1)(t)^r$ is injective.

If $\varphi(P)$ is invertible over the skew field $\mathbb{K}(G'_2)(t)$, then $r = s$. But an injective homomorphism between vector spaces of the same dimension over a skew field is in fact an isomorphism. This shows that P is invertible over $\mathbb{K}(G'_1)(t)$. \square

Proposition 6.4. *If G_1 is locally indicable, and if $\mathbb{Z}[G_1]^s \rightarrow \mathbb{Z}[G_1]^r$ is a map such that $\mathbb{Z}[G_1]^s \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2] \rightarrow \mathbb{Z}[G_1]^r \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2]$ is injective, then $\mathbb{Z}[G_1]^s \rightarrow \mathbb{Z}[G_1]^r$ is injective as well.*

Proof. Let $K = \text{Ker}\{\varphi : G_1 \rightarrow G_2\}$. Clearly K is again locally indicable. Note that $\mathbb{Z}[G_1]^s \rightarrow \mathbb{Z}[G_1]^r$ can also be viewed as a map between free $\mathbb{Z}[K]$ -modules. Pick any right inverse $\lambda : G_2 \rightarrow G_1$ of φ . It is easy to see that $g \otimes h \mapsto g\lambda(h) \otimes 1$, $g \in G_1, h \in G_2$ induces an isomorphism

$$\mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2] \rightarrow \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[K]} \mathbb{Z}.$$

By assumption $\mathbb{Z}[G_1]^s \otimes_{\mathbb{Z}[K]} \mathbb{Z} \rightarrow \mathbb{Z}[G_2]^r \otimes_{\mathbb{Z}[K]} \mathbb{Z}$ is injective. Since K is locally indicable it follows immediately from [\[Gersten 1983\]](#) or [\[Howie and Schneebeli 1983\]](#) (see also [\[Strebel 1974\]](#) for the case of PTFA groups) that $\mathbb{Z}[G_1]^s \rightarrow \mathbb{Z}[G_1]^r$ is injective. \square

By [Lemma 6.2](#) we have now showed that if $\tau(M, \alpha_2) \neq 0$, then $\tau(M, \alpha_1) \neq 0$. Furthermore

$$\deg \tau(M, \alpha_i) = \deg \alpha_i(B) - \deg \alpha_i(1 - g_k) - \deg \alpha_i(1 - h_l), \quad i = 1, 2.$$

Theorem 1.3 now follows immediately from [Lemma 6.1](#) and from the following proposition.

Proposition 6.5. *Let P be an $r \times r$ matrix over $\mathbb{Z}[G'_1][t^{\pm 1}]$. If $\varphi(P)$ is invertible,*

$$\deg P \geq \deg \varphi(P).$$

Remark. (1) If $\varphi : R \rightarrow S$ is a homomorphism of commutative rings, and if P is a matrix over $R[t^{\pm 1}]$, then clearly

$$\deg P = \deg \det P \geq \deg \varphi(\det P) = \deg \det \varphi(P) = \deg \varphi(P).$$

Similarly, several other results in this paper, e.g. [Theorem 3.1](#) and [Lemma 6.1](#) are clear in the commutative world, but require more effort in our noncommutative setting.

(2) If $(\mathbb{Z}[G'_1], \{f \in \mathbb{Z}[G'_1] \mid \varphi(f) \neq 0 \in \mathbb{Z}[G'_2]\})$ has the Ore property, one can give an elementary proof of the proposition by first diagonalizing over $\mathbb{K}(G'_2)$ and then over $\mathbb{K}(G'_1)$. Since this is not known to be the case, we have to give a more indirect proof.

The following proof is based on arguments in [[Cochran 2004](#)] and [[Harvey 2006](#)].

Proof of Proposition 6.5. Let $s = \deg \varphi(P)$. Pick a map $f : \mathbb{Z}[G'_1]^s \rightarrow \mathbb{Z}[G'_1][t^{\pm 1}]^r$ such that the induced map

$$\mathbb{K}(G'_2)^s \rightarrow \mathbb{K}(G'_2)[t^{\pm 1}]^r \rightarrow \mathbb{K}(G'_2)[t^{\pm 1}]^r / \varphi(P)\mathbb{K}(G'_2)[t^{\pm 1}]^r$$

is an isomorphism. Denote by $0 \rightarrow C_1 \xrightarrow{P} C_0 \rightarrow 0$ the complex

$$0 \rightarrow \mathbb{Z}[G'_1][t^{\pm 1}]^r \xrightarrow{P} \mathbb{Z}[G'_1][t^{\pm 1}]^r \rightarrow 0,$$

and denote by $0 \rightarrow D_0 \rightarrow 0$ the complex with $D_0 = \mathbb{Z}[G'_1]^s$. We have a chain map $D_* \rightarrow C_*$ given by $f : D_0 \rightarrow C_0$. Denote by $\text{Cyl}(D_* \xrightarrow{f} C_*)$ the mapping cylinder of the complexes. We then get a short exact sequence of complexes

$$0 \rightarrow D_* \rightarrow \text{Cyl}(D_* \xrightarrow{f} C_*) \rightarrow \text{Cyl}(D_* \xrightarrow{f} C_*)/D_* \rightarrow 0.$$

More explicitly, we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_1 \oplus D_0 & \xrightarrow{(\text{id} \text{ id})} & C_1 \oplus D_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & D_0 & \xrightarrow{(0)} & C_0 \oplus D_0 & \longrightarrow & C_0 \longrightarrow 0 \\ & & & & \downarrow \begin{pmatrix} P & -f \\ 0 & \text{id} \end{pmatrix} & & \downarrow (P \ -f) \end{array}$$

Recall that $\text{Cyl}(D_* \xrightarrow{f} C_*)$ and C_* are chain homotopic. Using the definition of f we therefore see that

$$f : H_0(D_*; \mathbb{K}(G'_2)) \rightarrow H_0(\text{Cyl}(D_* \xrightarrow{f} C_*), \mathbb{K}(G'_2))$$

is an isomorphism. Since P is invertible over $\mathbb{K}(G'_2)(t)$ it follows that

$$H_1(\text{Cyl}(D_* \xrightarrow{f} C_*); \mathbb{K}(G'_2)) = 0.$$

It follows from the long exact homology sequence corresponding to the short exact sequence of chain complexes above that

$$H_1(\text{Cyl}(D_* \xrightarrow{f} C_*)/D_*; \mathbb{K}(G'_2)) = 0;$$

thus the matrix $(P - f)$ is injective over $\mathbb{K}(G'_2)$. It follows from [Lemma 6.3](#) that

$$H_1(\text{Cyl}(D_* \xrightarrow{f} C_*)/D_*; \mathbb{K}(G'_1)) = 0$$

as well. Again looking at the long exact homology sequence we get that

$$f : H_0(D_*; \mathbb{K}(G'_1)) \rightarrow H_0(\text{Cyl}(D_* \xrightarrow{f} C_*); \mathbb{K}(G'_1)) = H_0(C_*; \mathbb{K}(G'_1))$$

is an injection. Hence

$$\begin{aligned} \deg \varphi(P) &= s = \dim_{\mathbb{K}(G'_2)}(H_0(D_*; \mathbb{K}(G'_2))) \\ &= \dim_{\mathbb{K}(G'_1)}(H_0(D_*; \mathbb{K}(G'_1))) \\ &\leq \dim_{\mathbb{K}(G'_1)}(H_0(C_*; \mathbb{K}(G'_1))) = \deg P. \end{aligned} \quad \square$$

6.2. Proof of [Theorem 1.3](#) for 3-manifolds with boundary and for 2-complexes.

First let X be a finite connected 2-complex with $\chi(X) = 0$. We can give X a CW structure with one 0-cell. If n denotes the number n of 1-cells, then $n - 1$ equals the number of 2-cells. Now consider the chain complex of the universal cover \tilde{X} :

$$0 \rightarrow C_2(\tilde{X})^{n-1} \xrightarrow{\partial_2} C_1(\tilde{X})^n \xrightarrow{\partial_1} C_0(\tilde{X})^1 \rightarrow 0.$$

As in [Section 6.1](#) we pick lifts of the cells of X to cells of \tilde{X} to get bases such that if A_i denotes the matrix corresponding to ∂_i , then

$$A_1 = (1 - h_1, \dots, 1 - h_n).$$

Note that $\{h_1, \dots, h_n\}$ is a generating set for $\pi_1(X)$. Let $l \in \{1, \dots, n\}$ such that $\phi(l) \neq 0$. The proof of [Lemma 6.2](#) can easily be modified to prove the following.

Lemma 6.6. *Denote by B the result of deleting the l -row of A_2 . Then $\tau(X, \alpha) \neq 0$ if and only if $\alpha(B)$ is invertible. Furthermore if $\tau(X, \alpha) \neq 0$, then*

$$\tau(X, \alpha) = \alpha(B)\alpha(1 - h_l)^{-1} \in K_1(\mathbb{K}(G'_i)(t))/\pm \alpha_i(\pi_1(X)).$$

The proof of [Theorem 1.3](#) for closed manifolds can now easily be modified to cover the case of 2-complexes X with $\chi(X) = 0$.

Now let M be again a 3-manifold whose boundary consists of a nonempty set of tori. A duality argument shows that $\chi(M) = \frac{1}{2}\chi(\partial(M)) = 0$. Clearly M is homotopy equivalent to a 2-complex. Reidemeister torsion is not a homotopy invariant but the following lemma still allows us to reduce the case of a 3-manifold with boundary to the case of a 2-complex.

Lemma 6.7 [[Turaev 2001](#), p. 56 and [Theorem 9.1](#)]. *Let M be a 3-manifold with boundary. Then there exists a 2-complex X and a simple homotopy equivalence $M \rightarrow X$. In particular, if $\alpha : \pi_1(X) \cong \pi_1(M) \rightarrow \text{GL}(R, d)$ is a representation such that $H_*(X, R^d) = 0$, then*

$$\tau(M, \alpha) = \tau(X, \alpha) \in K_1(R) / \pm \alpha(\pi_1(M)).$$

[Theorem 1.3](#) for 3-manifolds with boundary now follows from [Theorem 1.3](#) for 2-complexes X with $\chi(X) = 0$.

7. Harvey’s monotonicity theorem for groups

Let π be a finitely presented group and let $(\varphi : \pi \rightarrow G, \phi : \pi \rightarrow \mathbb{Z})$ be an admissible pair for π . Consider $G' = G'(\phi_G)$ and pick a splitting $\mathbb{Z} \rightarrow G$ of ϕ_G . As in [Section 5.2](#) we can consider the skew Laurent polynomial ring $\mathbb{K}(G')[t^{\pm 1}]$ together with the ϕ -compatible representation $\pi \rightarrow \text{GL}(\mathbb{K}(G')[t^{\pm 1}], 1)$.

Following [[Harvey 2006](#), Definition 1.6] we define $\bar{\delta}_G(\phi)$ to be zero if the group $H_1(\pi, \mathbb{K}(G')[t^{\pm 1}])$ is not $\mathbb{K}(G')[t^{\pm 1}]$ -torsion and

$$\bar{\delta}_G(\phi) = \dim_{\mathbb{K}(G')}(H_1(\pi, \mathbb{K}(G')[t^{\pm 1}]))$$

otherwise. We give an alternative proof for the following result of Harvey [[2006](#), [Theorem 2.9](#)].

Theorem 7.1. *Let $\pi = \pi_1(M)$, where M is a closed 3-manifold. If $(\varphi_1 : \pi \rightarrow G_1, \varphi_2 : \pi \rightarrow G_2, \phi)$ is an admissible triple for π , then*

$$\begin{aligned} \bar{\delta}_{G_1}(\phi) &\geq \bar{\delta}_{G_2}(\phi) && \text{if } (\varphi_2, \phi) \text{ is not initial,} \\ \bar{\delta}_{G_1}(\phi) &\geq \bar{\delta}_{G_2}(\phi) - 2 && \text{otherwise.} \end{aligned}$$

Proof. We clearly only have to consider the case that $\bar{\delta}_{G_2}(\phi) > 0$. We can build $K(\pi, 1)$ by adding i -handles to M with $i \geq 3$. It therefore follows that for the admissible pairs $(\varphi_i : \pi \rightarrow G_i, \phi)$ we have

$$\bar{\delta}_{G_i}(\phi) = \dim_{\mathbb{K}(G'_i)}(H_1(K_1(\pi, 1); \mathbb{K}(G'_i)[t^{\pm 1}])) = \dim_{\mathbb{K}(G'_i)}(H_1(M; \mathbb{K}(G'_i)[t^{\pm 1}])).$$

We combine this equality with [Theorem 1.3](#), [Corollary 3.6](#) and [Lemmas 3.2, 4.3, 4.4](#). The theorem follows now immediately from the observation that $\text{Im}\{\pi_1(M) \rightarrow G_i \rightarrow \text{GL}(\mathbb{K}(G'_i)[t^{\pm 1}], 1)\}$ is cyclic if and only if $\phi : G_i \rightarrow \mathbb{Z}$ is an isomorphism. \square

This monotonicity result gives in particular an obstruction for a group π to be the fundamental group of a closed 3-manifold. For example, [Harvey \[2006, Example 3.2\]](#) shows that as an immediate consequence we get the well-known fact that \mathbb{Z}^m , $m \geq 4$, is not a 3-manifold group.

Remark. In [\[Friedl and Kim 2005\]](#) we considered the case $\pi = \pi_1(M)$, where M is a closed 3-manifold. Given an admissible pair $(\varphi : \pi \rightarrow G, \phi)$ we show (under a mild assumption) that $\bar{\delta}_G(\phi)$ is even, generalizing [\[Turaev 1986, p. 141\]](#). In [\[Friedl and Harvey 2006\]](#) it is shown that given $\pi \rightarrow G$, G locally indicable and amenable, the map

$$\begin{aligned} \text{Hom}(G, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ \phi &\mapsto \bar{\delta}_G(\phi) \end{aligned}$$

defines a seminorm on $\text{Hom}(G, \mathbb{Z})$.

Let π be a finitely presented group of deficiency at least one, for example $\pi = \pi_1(M)$ where M is a 3-manifold with boundary. Using a presentation of deficiency one we can build a 2-complex X with $\chi(X) = 0$ and $\pi_1(X) = \pi$. The same proof as the proof of [Theorem 7.1](#) now gives the following theorem of Harvey. (In the case that $\pi = \pi_1(S^3 \setminus K)$ for K a knot, this was first proved in [\[Cochran 2004\]](#).)

Theorem 7.2 [\[Harvey 2006, Theorem 2.2\]](#). *If π is a finitely presented group of deficiency one and if $(\varphi_1, \varphi_2, \phi)$ is an admissible triple for π , then*

$$\begin{aligned} \bar{\delta}_{G_1}(\phi) &\geq \bar{\delta}_{G_2}(\phi) && \text{if } (\varphi_2, \phi) \text{ is not initial,} \\ \bar{\delta}_{G_1}(\phi) &\geq \bar{\delta}_{G_2}(\phi) - 1 && \text{otherwise.} \end{aligned}$$

8. Open questions and problems

Let M be a 3-manifold and $\phi \in H^1(M; \mathbb{Z})$. We propose the following three problems for further study.

(1) If $(\varphi : \pi_1(M) \rightarrow G, \phi)$ is an admissible pair for $\pi_1(M)$ and if $\alpha : \pi_1(M) \rightarrow \text{GL}(\mathbb{F}, d)$ factors through φ , does it follow that

$$\frac{1}{d} \deg \tau(M, \alpha) \leq \deg \tau(M, \mathbb{Z}[\pi_1(M)] \rightarrow \mathbb{K}(G')(t))?$$

Put differently, are the Thurston norm bounds of Cochran and Harvey optimal, i.e., at least as good as the Thurston norm bounds of [\[Friedl and Kim 2006\]](#) for any representation factoring through G ?

(2) In many cases $\deg \tau(M, \mathbb{Z}[\pi_1(M)] \rightarrow \mathbb{K}(G')(t)) < \|\phi\|_T$, for any admissible pair $(\varphi : \pi_1(M) \rightarrow G, \phi)$. For example this is the case if K is a knot with $\Delta_K(t) = 1$ and $M = X(K)$. It is an interesting question whether invariants can be defined for any map $\pi_1(M) \rightarrow G$, G a (locally indicable) torsion-free group. For example it might be possible to work with $\mathcal{U}(G)$ the algebra of affiliated operators (see [Reich 1998], for instance) instead of $\mathbb{K}(G)$. If such an extension is possible, then it is a natural question whether the Thurston norm is determined by such more general bounds. This might be too optimistic in the general case, but it could be true in the case of a knot complement.

(3) If (M, ϕ) fibers over S^1 , the corresponding Alexander polynomial defined over $\mathbb{Z}[t^{\pm 1}]$ is monic, that is, the top coefficient is ± 1 . Because of the high degree of indeterminacy of Alexander polynomials over skew Laurent polynomial rings a corresponding statement is meaningless. Since Reidemeister torsion has a much smaller indeterminacy it is potentially possible to use it to extend the fiberedness obstruction as in [Goda et al. 2005].

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