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THE TWO-PARAMETER QUANTUM GROUP OF EXCEPTIONAL TYPE G_2 AND LUSZTIG'S SYMMETRIES

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We give the defining structure of the two-parameter quantum group of type G_2 defined over a field $\mathbb{Q}(r, s)$ (with $r \neq s$), and prove the Drinfel'd double structure as its upper and lower triangular parts, extending a result of Benkart and Witherspoon in type A and a recent result of Bergeron, Gao, and Hu in types B, C, D. We further discuss Lusztig's $\mathbb{Q}$-isomorphisms from $U_{r,s}(G_2)$ to its associated object $U_{s^{-1},r^{-1}}(G_2)$, which give rise to the usual Lusztig symmetries defined not only on $U_q(G_2)$ but also on the centralized quantum group $U_q^c(G_2)$ only when $r = s^{-1} = q$. (This also reflects the distinguishing difference between our newly defined two-parameter object and the standard Drinfel'd–Jimbo quantum groups.) Some interesting (r, s)-identities holding in $U_{r,s}(G_2)$ are derived from this discussion.

1. The two-parameter quantum group $U_{r,s}(G_2)$

Let $\mathbb{K} = \mathbb{Q}(r, s)$ be a field of rational functions with two indeterminates r, s.

Let Φ be a finite root system of G_2 with Π a base of simple roots, which is a subset of a Euclidean space $E = \mathbb{R}^3$ with an inner product $(\,,\,)$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ denote an orthonormal basis of E. Then $\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 + \epsilon_3 - 2\epsilon_1\}$ and $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1\}$. In this case, we set $r_1 = r^{(\alpha_1,\alpha_1)/2} = r$, $r_2 = r^{(\alpha_2,\alpha_2)/2} = r^3$ and $s_1 = s^{(\alpha_1,\alpha_1)/2} = s$, $s_2 = s^{(\alpha_2,\alpha_2)/2} = s^3$.

We begin by defining the two-parameter quantum group of type G_2, which is new.

Definition 1.1. Let $U = U_{r,s}(G_2)$ be the associative algebra over $\mathbb{Q}(r, s)$ generated by the symbols e_i, f_i, $\omega_i^{\pm 1}$ and $\omega_i'^{\pm 1}$ $(1 \le i \le 2)$, subject to the relations

(G1) $[\,\omega_i^{\pm 1}, \omega_j^{\pm 1}\,] = [\,\omega_i^{\pm 1}, \omega_j'^{\pm 1}\,] = [\,\omega_i'^{\pm 1}, \omega_j'^{\pm 1}\,] = 0, \quad \omega_i\omega_i^{-1} = 1 = \omega_j'\omega_j'^{-1};$

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(G2)
$$\begin{aligned} &\omega_1\, e_1\, \omega_1^{-1} = (rs^{-1})\, e_1, && \omega_1\, f_1\, \omega_1^{-1} = (r^{-1}s)\, f_1,\\ &\omega_1\, e_2\, \omega_1^{-1} = s^3\, e_2, && \omega_1\, f_2\, \omega_1^{-1} = s^{-3}\, f_2,\\ &\omega_2\, e_1\, \omega_2^{-1} = r^{-3}\, e_1, && \omega_2\, f_1\, \omega_2^{-1} = r^3\, f_1,\\ &\omega_2\, e_2\, \omega_2^{-1} = (r^3s^{-3})\, e_2, && \omega_2\, f_2\, \omega_2^{-1} = (r^{-3}s^3)\, f_2; \end{aligned}$$

(G3)
$$\begin{aligned} &\omega_1'\, e_1\, \omega_1'^{-1} = (r^{-1}s)\, e_1, && \omega_1'\, f_1\, \omega_1'^{-1} = (rs^{-1})\, f_1,\\ &\omega_1'\, e_2\, \omega_1'^{-1} = r^3\, e_2, && \omega_1'\, f_2\, \omega_1'^{-1} = r^{-3}\, f_2,\\ &\omega_2'\, e_1\, \omega_2'^{-1} = s^{-3}\, e_1, && \omega_2'\, f_1\, \omega_2'^{-1} = s^3\, f_1,\\ &\omega_2'\, e_2\, \omega_2'^{-1} = (r^{-3}s^3)\, e_2, && \omega_2'\, f_2\, \omega_2'^{-1} = (r^3s^{-3})\, f_2; \end{aligned}$$

(G4)
$$[\, e_i, f_j\,] = \delta_{ij}\,\frac{\omega_i - \omega_i'}{r_i - s_i} \text{ for } 1 \le i,\ j \le 2;$$

(G5) ((r, s)-Serre relations in positive part)

$$e_2^2e_1 - (r^{-3}+s^{-3})\, e_2e_1e_2 + (rs)^{-3}\, e_1e_2^2 = 0,$$
$$\begin{aligned} e_1^4e_2 - (r+s)(r^2+s^2)\, e_1^3e_2e_1 + rs(r^2+s^2)(r^2+rs+s^2)\, e_1^2e_2e_1^2\\ -(rs)^3(r+s)(r^2+s^2)\, e_1e_2e_1^3 + (rs)^6e_2e_1^4 = 0; \end{aligned}$$

(G6) ((r, s)-Serre relations in negative part)

$$f_1f_2^2 - (r^{-3}+s^{-3})\, f_2f_1f_2 + (rs)^{-3}\, f_2^2 f_1 = 0,$$
$$\begin{aligned} f_2f_1^4 - (r+s)(r^2+s^2)\, f_1f_2f_1^3 + rs(r^2+s^2)(r^2+rs+s^2)\, f_1^2f_2f_1^2\\ -(rs)^3(r+s)(r^2+s^2)\, f_1^3f_2f_1 + (rs)^6\, f_1^4f_2 = 0. \end{aligned}$$

Proposition 1.2. *The algebra $U_{r,s}(G_2)$ is a Hopf algebra with comultiplication, counit and antipode given by*

$$\begin{aligned} &\Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1}\otimes\omega_i^{\pm 1}, && \Delta(e_i) = e_i\otimes 1 + \omega_i\otimes e_i,\\ &\Delta(\omega_i'^{\pm 1}) = \omega_i'^{\pm 1}\otimes\omega_i'^{\pm 1}, && \Delta(f_i) = 1\otimes f_i + f_i\otimes\omega_i',\\ &\varepsilon(\omega_i^{\pm}) = \varepsilon(\omega_i'^{\pm 1}) = 1, && S(\omega_i^{\pm 1}) = \omega_i^{\mp 1},\ \ S(e_i) = -\omega_i^{-1}e_i,\\ &\varepsilon(e_i) = \varepsilon(f_i) = 0, && S(\omega_i'^{\pm 1}) = \omega_i'^{\mp 1},\ S(f_i) = -f_i\,\omega_i'^{-1}. \end{aligned}$$

Remark 1.3. (I) When $r = q = s^{-1}$, the quotient Hopf algebra of $U_{r,s}(G_2)$ modulo the Hopf ideal generated by elements $\omega_i' - \omega_i^{-1}$ ($1 \le i \le 2$) is just the standard quantum group $U_q(G_2)$ of Drinfel'd–Jimbo type; the quotient modulo the Hopf ideal generated by elements $\omega_i' - z_i\omega_i^{-1}$ ($1 \le i \le 2$), where z_i runs over the center, is the *centralized quantum group* $U_q^c(G_2)$.

(II) In any Hopf algebra $\mathscr{H}$, there exist left-adjoint and right-adjoint actions defined by the Hopf algebra structure:

$$\mathrm{ad}_l\, a\,(b) = \sum_{(a)} a_{(1)}\, b\, S(a_{(2)}), \qquad \mathrm{ad}_r\, a\,(b) = \sum_{(a)} S(a_{(1)})\, b\, a_{(2)},$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \in \mathcal{H} \otimes \mathcal{H}$, for any $a, b \in \mathcal{H}$.

From the viewpoint of adjoint actions, the (r, s)-Serre relations (G5) and (G6) take on the simpler forms

$$(\mathrm{ad}_l\, e_i)^{1-a_{ij}}\,(e_j) = 0 \qquad \textit{for any } i \neq j,$$
$$(\mathrm{ad}_r\, f_i)^{1-a_{ij}}\,(f_j) = 0 \qquad \textit{for any } i \neq j.$$

2. The Drinfel'd quantum double

Definition 2.1. A (Hopf) dual pairing of two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ (see [Bergeron et al. 2006] or [Klimyk and Schmüdgen 1997]) is a bilinear form $\langle\,,\rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{K}$ such that

(1) $$\langle f, 1_{\mathcal{A}}\rangle = \varepsilon_{\mathcal{U}}(f), \quad \langle 1_{\mathcal{U}}, a\rangle = \varepsilon_{\mathcal{A}}(a),$$

(2) $$\langle f, a_1 a_2\rangle = \langle \Delta_{\mathcal{U}}(f), a_1 \otimes a_2\rangle, \quad \langle f_1 f_2, a\rangle = \langle f_1 \otimes f_2, \Delta_{\mathcal{A}}(a)\rangle,$$

for all $f, f_1, f_2 \in \mathcal{U}$, and $a, a_1, a_2 \in \mathcal{A}$, where $\varepsilon_{\mathcal{U}}, \varepsilon_{\mathcal{A}}$ denote the counits of $\mathcal{U}$, $\mathcal{A}$ and $\Delta_{\mathcal{U}}, \Delta_{\mathcal{A}}$ the comultiplications.

A direct consequence of the defining properties above is that

$$\langle S_{\mathcal{U}}(f), a\rangle = \langle f, S_{\mathcal{A}}(a)\rangle, \quad f \in \mathcal{U}, a \in \mathcal{A},$$

where $S_{\mathcal{U}}, S_{\mathcal{A}}$ denote the respective antipodes of $\mathcal{U}$ and $\mathcal{A}$.

Definition 2.2. A bilinear form $\langle\,,\rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{K}$ is called a skew-dual pairing of two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ (see [Bergeron et al. 2006]) if $\langle\,,\rangle : \mathcal{U}^{\mathrm{cop}} \times \mathcal{A} \to \mathbb{K}$ is a Hopf dual pairing of $\mathcal{A}$ and $\mathcal{U}^{\mathrm{cop}}$, where $\mathcal{U}^{\mathrm{cop}}$ is the Hopf algebra having the opposite comultiplication to $\mathcal{U}$, and $S_{\mathcal{U}^{\mathrm{cop}}} = S_{\mathcal{U}}^{-1}$ is invertible.

Denote by $\mathcal{B} = B(G_2)$ the Hopf subalgebra of $U_{r,s}(G_2)$ generated by $e_j, \omega_j^{\pm 1}$, and by $\mathcal{B}' = B'(G_2)$ the one generated by $f_j, \omega_j'^{\pm 1}$, where $j = 1, 2$.

Proposition 2.3. *There exists a unique skew-dual pairing $\langle\,,\rangle : \mathcal{B}' \times \mathcal{B} \to \mathbb{Q}(r, s)$ of the Hopf subalgebras $\mathcal{B}$ and $\mathcal{B}'$ such that*

(3) $$\langle f_i, e_j\rangle = \delta_{ij} \frac{1}{s_i - r_i} \qquad (1 \le i, j \le 2),$$

(4) $$\langle \omega_1', \omega_1\rangle = r s^{-1}, \quad \langle \omega_1', \omega_2\rangle = r^{-3},$$
$$\langle \omega_2', \omega_1\rangle = s^3, \qquad \langle \omega_2', \omega_2\rangle = r^3 s^{-3},$$

(5) $$\langle \omega_i'^{\pm 1}, \omega_j^{-1}\rangle = \langle \omega_i'^{\pm 1}, \omega_j\rangle^{-1} = \langle \omega_i', \omega_j\rangle^{\mp 1} \quad (1 \le i, j \le 2),$$

and all other pairs of generators yield 0*. Furthermore*, $\langle S(a), S(b)\rangle = \langle a, b\rangle$ *for* $a \in \mathcal{B}'$, $b \in \mathcal{B}$.

Proof. Since any skew-dual pairing of bialgebras is determined by its values on generators, uniqueness is clear. We proceed to prove the existence of the pairing.

We begin by defining a bilinear form $\langle\,,\rangle : \mathcal{B}'^{\mathrm{cop}} \times \mathcal{B} \to \mathbb{Q}(r, s)$ first on the generators satisfying (3), (4), and (5). Then we extend it to a bilinear form on $\mathcal{B}'^{\mathrm{cop}} \times \mathcal{B}$ by requiring that (1) and (2) hold for $\triangle_{\mathcal{B}'^{\mathrm{cop}}} = \triangle^{\mathrm{op}}_{\mathcal{B}'}$. We will verify that the relations in $\mathcal{B}$ and $\mathcal{B}'$ are preserved, ensuring that the form is well-defined and so is a dual pairing of $\mathcal{B}$ and $\mathcal{B}'^{\mathrm{cop}}$ by definition.

It is direct to check that the bilinear form preserves all the relations among the $\omega_i^{\pm 1}$ in $\mathcal{B}$ and the $\omega_i'^{\pm 1}$ in $\mathcal{B}'$. Next, the structure constants (4) ensure the compatibility of the form defined above with those relations of (G2) and (G3) in $\mathcal{B}$ or $\mathcal{B}'$, respectively. We are left to verify that the form preserves the (r, s)-Serre relations in $\mathcal{B}$ and $\mathcal{B}'$. It suffices to show that the form on $\mathcal{B}'^{\mathrm{cop}} \times \mathcal{B}$ preserves the (r, s)-Serre relations in $\mathcal{B}$; the verification for $\mathcal{B}'^{\mathrm{cop}}$ is similar.

First, let us show that the form preserves the (r, s)-Serre relation of degree 2 in $\mathcal{B}$, that is,

$$\langle X,\, e_2^2 e_1 - (r^{-3} + s^{-3})\, e_2 e_1 e_2 + r^{-3} s^{-3}\, e_1 e_2^2 \,\rangle = 0,$$

where X is any word in the generators of $\mathcal{B}'$. It suffices to consider three monomials: $X = f_2^2 f_1,\ f_2 f_1 f_2,\ f_1 f_2^2$. However, in the degree 2's situation for type G_2, its proof is the same as that of type C_2 (see [Bergeron et al. 2006, (7C) and thereafter]).

Next, we verify that the (r, s)-Serre relation of degree 4 in $\mathcal{B}$ is preserved by the form; that is, we show that

$$\Big\langle X,\ e_1^4 e_2 - (r+s)(r^2+s^2)\, e_1^3 e_2 e_1 + rs(r^2+s^2)(r^2+rs+s^2)\, e_1^2 e_2 e_1^2 \\ - (rs)^3 (r+s)(r^2+s^2)\, e_1 e_2 e_1^3 + (rs)^6\, e_2 e_1^4 \Big\rangle$$

vanishes, where X is any word in the generators of $\mathcal{B}'$. By definition, this expression equals

$$\begin{aligned}(6)\quad \Big\langle \triangle^{(4)}(X),\ & e_1\otimes e_1\otimes e_1\otimes e_1\otimes e_2 - (r+s)(r^2+s^2)\, e_1\otimes e_1\otimes e_1\otimes e_2\otimes e_1 \\ & + rs(r^2+s^2)(r^2+rs+s^2)\, e_1\otimes e_1\otimes e_2\otimes e_1\otimes e_1 \\ & - (rs)^3(r+s)(r^2+s^2)\, e_1\otimes e_2\otimes e_1\otimes e_1\otimes e_1 + (rs)^6\, e_2\otimes e_1\otimes e_1\otimes e_1\otimes e_1 \Big\rangle,\end{aligned}$$

where $\triangle$ in the left-hand side of the pairing $\langle\,,\rangle$ indicates $\triangle^{\mathrm{op}}_{\mathcal{B}'}$. In order for any one of these terms to be nonzero, X must involve exactly four f_1 factors, one f_2 factor, and arbitrarily many $\omega_j'^{\pm 1}$ factors $(j = 1, 2)$.

It suffices to consider five key cases:

(i) If $X = f_1^4 f_2$, we have

$$\begin{aligned}\triangle^{(4)}(X) = \big(&\omega_1' \otimes \omega_1' \otimes \omega_1' \otimes \omega_1' \otimes f_1 + \omega_1' \otimes \omega_1' \otimes \omega_1' \otimes f_1 \otimes 1 \\ &+ \omega_1' \otimes \omega_1' \otimes f_1 \otimes 1 \otimes 1 + \omega_1' \otimes f_1 \otimes 1 \otimes 1 \otimes 1 + f_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1\big)^4 \\ &\cdot \big(\omega_2' \otimes \omega_2' \otimes \omega_2' \otimes \omega_2' \otimes f_2 + \omega_2' \otimes \omega_2' \otimes \omega_2' \otimes f_2 \otimes 1 \\ &+ \omega_2' \otimes \omega_2' \otimes f_2 \otimes 1 \otimes 1 + \omega_2' \otimes f_2 \otimes 1 \otimes 1 \otimes 1 + f_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1\big).\end{aligned}$$

Expanding $\triangle^{(4)}(X)$, we get 120 relevant terms having a nonzero contribution to (6). They are listed in Table 1, together with their pairing values, where we have

Table 1. Terms of $\Delta^{(4)}(f_1^4 f_2)$ in (6) and their pairing values. We write β instead of ω' and $\cdot$ instead of $\otimes$ to save space. We have also set $a = \langle f_1, e_1\rangle^4 \langle f_2, e_2\rangle$, $x = \langle \omega_1', \omega_1\rangle$, $\bar{x} = \langle \omega_1', \omega_2\rangle$.

Summand in (6)	**1**	Summand in (6)	**2**
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	a	$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}a$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	xa	$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}xa$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	xa	$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^5a	$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^5a$
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	xa	$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^5a	$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^5a$
$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^5a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^5a$
$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^6a	$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^6a$

Table 1. (Continued)

Summand in (6)	**3**	Summand in (6)	**4**
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2xa$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^2a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^4a$
$f_1\beta_1^3\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^2a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^4a$
$\beta_1^3 f_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^5a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^5a$
$f_1\beta_1^3\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^3a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^4a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^5a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^5a$
$\beta_1^3 f_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^6a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^6a$
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3a$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2xa$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3xa$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^2a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^3a$
$f_1\beta_1^3\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2xa$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^2a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^3 f_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^4a$
$f_1\beta_1^3\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^2a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^4a$
$\beta_1^3 f_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^5a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^5a$

Summand in (6)	**5**	Summand in (6)	**5**
$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4xa$	$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4a$
$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^2a$	$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4xa$
$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^3a$	$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^2a$
$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^4a$	$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^3a$
$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^2a$	$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4xa$
$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^3a$	$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^2a$
$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^4a$	$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^3a$
$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^5a$	$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^4a$
$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^3a$	$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^2a$
$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^4a$	$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1^2 f_1\cdot' f_1\beta_1\cdot f_1$	$\bar{x}^4x^3a$
$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^5a$	$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^4a$
$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^6a$	$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^5a$

introduced

$$a = \langle f_1, e_1\rangle^4 \langle f_2, e_2\rangle, \quad x = \langle \omega_1', \omega_1\rangle, \quad \bar{x} = \langle \omega_1', \omega_2\rangle.$$

The expression in (6) equals

(sum of expressions in part 1 of Table 1)

$$\begin{aligned}&- (\text{sum of expressions in part 2}) \cdot (r+s)(r^2+s^2)\\ &+ (\text{sum of expressions in part 3}) \cdot rs(r^2+s^2)(r^2+rs+s^2)\\ &- (\text{sum of expressions in part 4}) \cdot (rs)^3(r+s)(r^2+s^2)\\ &+ (\text{sum of expressions in part 5}) \cdot (rs)^6.\end{aligned}$$

Thus, if we sum up all the pairing values listed in each part of Table 1 and multiply by the appropriate factor, we obtain the pairing value of (6):

$$\begin{aligned}&a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)\cdot\big(1-(r+s)(r^2+s^2)\bar{x}\\ &\quad + rs(r^2+s^2)\cdot(r^2+rs+s^2)\bar{x}^2 - (rs)^3(r+s)(r^2+s^2)\bar{x}^3 + (rs)^6\bar{x}^4\big)\\ &= a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)(1-r^3\bar{x})(1-r^2s\bar{x})(1-rs^2\bar{x})(1-s^3\bar{x})\\ &= 0 \qquad\qquad (\text{because } \bar{x} = \langle \omega_1', \omega_2\rangle = r^{-3}).\end{aligned}$$

(ii) $X = f_2 f_1^4$.

(iii) $X = f_1^2 f_2 f_1^2$.

(iv) $X = f_1^3 f_2 f_1$.

(v) $X = f_1 f_2 f_1^3$.

These four other cases are handled similarly, using the formulas given in the Appendix of [Hu and Shi 2006]. This takes care of $\Delta^{(4)}(X)$. The proof is completed by checking that the relations in B'^{cop} are preserved for G_2. □

Definition 2.4. For any two Hopf algebras $\mathscr{A}$ and $\mathscr{U}$ connected by a skew-dual pairing $\langle\,,\rangle$, one may form the Drinfel'd quantum double $\mathscr{D}(\mathscr{A}, \mathscr{U})$ as in [Klimyk and Schmüdgen 1997, 3.2], which is a Hopf algebra whose underlying coalgebra is $\mathscr{A}\otimes\mathscr{U}$ with the tensor product coalgebra structure, whose algebra structure is defined by

$$(7) \qquad (a\otimes f)(a'\otimes f') = \sum \langle \mathscr{S}_{\mathscr{U}}(f_{(1)}), a'_{(1)}\rangle\langle (f_{(3)}), a'_{(3)}\rangle a a'_{(2)} \otimes f_{(2)} f'$$

for $a, a' \in \mathscr{A}$ and $f, f' \in \mathscr{U}$, and whose antipode S is given by

$$(8) \qquad S(a\otimes f) = (1\otimes \mathscr{S}_{\mathscr{U}}(f))(\mathscr{S}_{\mathscr{A}}(a)\otimes 1).$$

Clearly, both mappings $\mathscr{A} \ni a \mapsto a\otimes 1 \in \mathscr{D}(\mathscr{A}, \mathscr{U})$ and $\mathscr{U} \ni f \mapsto 1\otimes f \in \mathscr{D}(\mathscr{A}, \mathscr{U})$ are injective Hopf algebra homomorphisms. Denote the image $a\otimes 1$ of a in $\mathscr{D}(\mathscr{A}, \mathscr{U})$ by $\hat{a}$, and the image $1\otimes f$ of f by $\hat{f}$. By (7), we have the following cross relations between elements $\hat{a}$ (for $a \in \mathcal{A}$) and $\hat{f}$ (for $f \in \mathcal{U}$) in the algebra $\mathcal{D}(\mathcal{A}, \mathcal{U})$:

$$\hat{f}\hat{a} = \sum \langle \mathcal{S}_{\mathcal{U}}(f_{(1)}), a_{(1)} \rangle \langle (f_{(3)}), a_{(3)} \rangle \hat{a}_{(2)} \hat{f}_{(2)}, \tag{9}$$

$$\sum \langle f_{(1)}, a_{(1)} \rangle \hat{f}_{(2)} \hat{a}_{(2)} = \sum \hat{a}_{(1)} \hat{f}_{(1)} \langle f_{(2)}, a_{(2)} \rangle. \tag{10}$$

In fact, as an algebra the double $\mathcal{D}(\mathcal{A}, \mathcal{U})$ is the universal algebra generated by the algebras $\mathcal{A}$ and $\mathcal{U}$ with cross relations (9) or, equivalently, (10).

Theorem 2.5. *The two-parameter quantum group $U_{r,s}(G_2)$ is isomorphic to the Drinfel'd quantum double $\mathcal{D}(\mathcal{B}, \mathcal{B}')$.*

The proof is the same as that of [Bergeron et al. 2006, Theorem 2.5].

Remark 2.6. The proofs of Proposition 2.3 and Theorem 2.5 show the compatibility of the defining relations of $U_{r,s}(G_2)$. The proof of Theorem 2.5 indicates that the cross relations between $\mathcal{B}$ and $\mathcal{B}'$ are precisely half the ones appearing in (G1)–(G4), and the proof of Proposition 2.3 then shows the compatibility of the remaining relations appearing in $\mathcal{B}$ and $\mathcal{B}'$, including the other half of (G1)–(G4) and the (r, s)-Serre relations (G5)–(G6).

3. Lusztig's symmetries from $U_{r,s}(G_2)$ to $U_{s^{-1},r^{-1}}(G_2)$

As we did in [Bergeron et al. 2006] for the classical types A, B, C, D, we call $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$ the quantum group associated to $(U_{r,s}(G_2), \langle\,,\,\rangle)$, where the pairing $\langle \omega_i' \,|\, \omega_j \rangle$ is defined by replacing (r, s) with (s^{-1}, r^{-1}) in the defining formula for $\langle \omega_i', \omega_j \rangle$. Obviously,

$$\langle \omega_i' \,|\, \omega_j \rangle = \langle \omega_j', \omega_i \rangle.$$

We now study Lusztig's symmetry property between $(U_{r,s}(G_2), \langle\,,\,\rangle)$ and its associated object $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$, which indeed indicates the difference in structures between the two-parameter quantum group introduced above and the usual one-parameter quantum group of Drinfel'd–Jimbo type.

To define the Lusztig symmetries, we introduce the notation of divided-power elements (in $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$). For any nonnegative integer $k \in \mathbb{N}$, set

$$\langle k \rangle_i = \frac{s_i^{-k} - r_i^{-k}}{s_i^{-1} - r_i^{-1}}, \qquad \langle k \rangle_i! = \langle 1 \rangle_i \langle 2 \rangle_i \cdots \langle k \rangle_i,$$

and for any element $e_i, f_i \in (U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$, define the divided-power elements

$$e_i^{(k)} = e_i^k / \langle k \rangle_i!\,, \qquad f_i^{(k)} = f_i^k / \langle k \rangle_i!\,.$$

Definition 3.1. To every i $(i=1,2)$, there corresponds a $\mathbb{Q}$-linear mapping $\mathcal{T}_i$: $(U_{r,s}(G_2), \langle\,,\,\rangle) \to (U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$ such that $\mathcal{T}_i(r)=s^{-1}$, $\mathcal{T}_i(s)=r^{-1}$, which acts on the generators ω_j, ω_j', e_j, f_j $(1\le j\le 2)$ as

$$\mathcal{T}_i(\omega_j)=\omega_j\,\omega_i^{-a_{ij}}, \qquad \mathcal{T}_i(\omega_j')=\omega_j'\,\omega_i'^{\,-a_{ij}},$$
$$\mathcal{T}_i(e_i)=-\,\omega_i'^{\,-1}f_i, \qquad \mathcal{T}_i(f_i)=-(r_is_i)\,e_i\,\omega_i^{-1},$$

and for $i\neq j$,

$$\mathcal{T}_i(e_j)=\sum_{\nu=0}^{-a_{ij}}(-1)^{\nu}(rs)^{\frac{\nu}{2}(-a_{ij}-\nu)}\langle\omega_j',\omega_i\rangle^{-\nu}\langle\omega_i',\omega_i\rangle^{\frac{\nu}{2}(1+a_{ij})}e_i^{(\nu)}e_j\,e_i^{(-a_{ij}-\nu)},$$

$$\mathcal{T}_i(f_j)=(r_js_j)^{\delta_{ij}^{+}}\sum_{\nu=0}^{-a_{ij}}(-1)^{\nu}(rs)^{\frac{\nu}{2}(-a_{ij}-\nu)}\langle\omega_i',\omega_j\rangle^{\nu}\langle\omega_i',\omega_i\rangle^{-\frac{\nu}{2}(1+a_{ij})}f_i^{(-a_{ij}-\nu)}f_jf_i^{(\nu)},$$

where (a_{ij}) is the Cartan matrix of the simple Lie algebra $\mathfrak{g}$ of type G_2, and for any $i\neq j$,

$$\delta_{ij}^{+}=\begin{cases}2, & \text{if } i<j \text{ and } a_{ij}\neq 0,\\ 1, & \text{otherwise .}\end{cases}$$

Lemma 3.2. $\mathcal{T}_i$ $(i=1,2)$ *preserves the defining relations* (G1)–(G3) *of* $U_{r,s}(G_2)$ *into its associated object* $U_{s^{-1},r^{-1}}(G_2)$.

Proof. For G_2, we have

$$\langle\omega_1',\omega_1\rangle=rs^{-1}=\langle\omega_1'|\,\omega_1\rangle, \qquad \langle\omega_1',\omega_2\rangle=r^{-3}=\langle\omega_2'|\,\omega_1\rangle,$$
$$\langle\omega_2',\omega_1\rangle=s^{3}=\langle\omega_1'|\,\omega_2\rangle, \qquad \langle\omega_2',\omega_2\rangle=r^{3}s^{-3}=\langle\omega_2'|\,\omega_2\rangle.$$

We show that $\mathcal{T}_1$, $\mathcal{T}_2$ preserve the defining relations (G1)–(G3). (G1) are automatically satisfied. To check (G2) and (G3): first of all, by direct calculation, we have $\mathcal{T}_k(\langle\omega_i',\omega_j\rangle)=\langle\mathcal{T}_k(\omega_i'),\mathcal{T}_k(\omega_j)\rangle=\langle\omega_j',\omega_i\rangle=\langle\omega_i'|\,\omega_j\rangle$, for $i,j,k\in\{1,2\}$. This fact ensures that $\mathcal{T}_k(k=1,2)$ preserve (G2) and (G3), that is,

$$\mathcal{T}_k(\omega_j)\mathcal{T}_k(e_i)\mathcal{T}_k(\omega_j)^{-1}=\langle\omega_i'|\,\omega_j\rangle\mathcal{T}_k(e_i),$$
$$\mathcal{T}_k(\omega_j)\mathcal{T}_k(f_i)\mathcal{T}_k(\omega_j)^{-1}=\langle\omega_i'|\,\omega_j\rangle^{-1}\mathcal{T}_k(f_i),$$
$$\mathcal{T}_k(\omega_j')\mathcal{T}_k(e_i)\mathcal{T}_k(\omega_j')^{-1}=\langle\omega_j'|\,\omega_i\rangle^{-1}\mathcal{T}_k(e_i),$$
$$\mathcal{T}_k(\omega_j')\mathcal{T}_k(f_i)\mathcal{T}_k(\omega_j')^{-1}=\langle\omega_j'|\,\omega_i\rangle\mathcal{T}_k(f_i),$$

where checking the other three identities is equivalent to checking the first one. □

Lemma 3.3. $\mathcal{T}_i$ $(i=1,2)$ *preserves the defining relations* (G4) *into its associated object* $U_{s^{-1},r^{-1}}(G_2)$.

Proof. Put $\Delta = r^2+rs+s^2$. To check (G4): for $i=1,2$, we have

$$[\mathcal{T}_i(e_i),\,\mathcal{T}_i(f_i)] = (r_is_i)\omega_i'^{-1}(f_ie_i-e_if_i)\omega_i^{-1} = \mathcal{T}_i([e_i,f_i]),$$

$$\begin{aligned}[\mathcal{T}_2(e_1),\,\mathcal{T}_2(f_1)] &= [e_1e_2-r^3e_2e_1,\; rs(f_2f_1-s^3f_1f_2)]\\ &= rs\big(f_2[e_1,f_1]e_2+e_1[e_2,f_2]f_1-r^3([e_2,f_2]f_1e_1+e_2f_2[e_1,f_1])\\ &\qquad -s^3([e_1,f_1]f_2e_2+e_1f_1[e_2,f_2])+(rs)^3(e_2[e_1,f_1]f_2+f_1[e_2,\,f_2]e_1)\big)\\ &= \frac{\omega_2\omega_1-\omega_2'\omega_1'}{s^{-1}-r^{-1}} = \frac{\mathcal{T}_2(\omega_1)-\mathcal{T}_2(\omega_1')}{s^{-1}-r^{-1}} = \mathcal{T}_2([e_1,f_1]),\end{aligned}$$

and as for

$$\begin{aligned}[\mathcal{T}_1(e_2),\,\mathcal{T}_1(f_2)] = \frac{r^3s^3}{(r+s)^2\Delta^2}\big[&(rs^2)^3e_2e_1^3-rs^3\Delta e_1e_2e_1^2+s\Delta e_1^2e_2e_1-e_1^3e_2,\\ &(r^2s)^3f_1^3f_2-sr^3\Delta f_1^2f_2f_1+r\Delta f_1f_2f_1^2-f_2f_1^3\big],\end{aligned}$$

we have to show that the bracket on the right-hand side is equal to

$$\Delta(r+s)^2\,\frac{\omega_2\omega_1^3-\omega_2'\omega_1'^3}{r-s}.$$

To do so, we introduce the notations of "quantum root vectors" in terms of adjoint actions, as follows:

$$\begin{aligned}E_{12} &= (\mathrm{ad}_l e_1)(e_2) = e_1e_2-s^3e_2e_1,\\ F_{12} &= (\mathrm{ad}_r f_1)(f_2) = f_2f_1-r^3f_1f_2,\\ E_{112} &= (\mathrm{ad}_l e_1)^2(e_2) = e_1E_{12}-rs^2E_{12}e_1,\\ F_{112} &= (\mathrm{ad}_r f_1)^2(f_2) = F_{12}f_1-r^2sf_1F_{12},\\ E_{1112} &= (\mathrm{ad}_l e_1)^3(e_2) = e_1^3e_2-s\Delta e_1^2e_2e_1+rs^3\Delta e_1e_2e_1^2-(rs^2)^3e_2e_1^3,\\ F_{1112} &= (\mathrm{ad}_r f_1)^3(f_2) = f_2f_1^3-r\Delta f_1f_2f_1^2+sr^3\Delta f_1^2f_2f_1-(r^2s)^3f_1^3f_2.\end{aligned}$$

That is, we need to verify that

$$[\,E_{1112},\,F_{1112}\,] = \Delta(r+s)^2\,\frac{\omega_2\omega_1^3-\omega_2'\omega_1'^3}{r-s}.$$

By direct calculation using the Leibniz rule, we have

$$\begin{aligned}&[e_1,\,F_{12}] = -\Delta\omega_1f_2, &&[e_2,\,F_{12}] = f_1\omega_2',\\ &[E_{12},f_1] = -\Delta e_2\omega_1', &&[E_{12},f_2] = \omega_2e_1,\end{aligned}$$

$$[E_{12},\,F_{12}] = \frac{\omega_1\omega_2-\omega_1'\omega_2'}{r-s},$$

$$[e_1, F_{112}] = -(r+s)^2\omega_1 F_{12}, \quad [e_2, F_{112}] = s(s^2-r^2) f_1^2\omega_2',$$
$$[E_{112}, f_1] = -(r+s)^2 E_{12}\omega_1', \quad [E_{112}, f_2] = r(r^2-s^2)\omega_2 e_1^2,$$
$$[E_{112}, F_{12}] = (r+s)^2\omega_1\omega_2 e_1, \ [E_{12}, F_{112}] = (r+s)^2 f_1\omega_1'\omega_2',$$
$$[E_{112}, F_{112}] = (r+s)^2 \, \frac{\omega_1^2\omega_2 - \omega_1'^2\omega_2'}{r-s},$$

as well as

$$[e_1, F_{1112}] = [e_1, F_{112} f_1 - rs^2 f_1 F_{112}] = -\Delta\omega_1 F_{112},$$

$$\begin{aligned}[E_{112}, F_{1112}] &= [E_{112}, F_{112} f_1 - rs^2 f_1 F_{112}] \\ &= [E_{112}, F_{112}] f_1 - rs^2 f_1 [E_{112}, F_{112}] + F_{112}[E_{112}, f_1] \\ &\qquad - rs^2 [E_{112}, f_1] F_{112} \\ &= \Delta(r+s)^2 f_1 \omega_1'^2\omega_2',\end{aligned}$$

$$\begin{aligned}[E_{1112}, F_{1112}] &= [e_1 E_{112} - r^2 s E_{112} e_1, F_{1112}] \\ &= [e_1, F_{1112}] E_{112} - r^2 s E_{112} [e_1, F_{1112}] + e_1 [E_{112}, F_{1112}] \\ &\qquad - r^2 s [E_{112}, F_{1112}] e_1 \\ &= \Delta\omega_1 [E_{112}, F_{112}] + \Delta(r+s)^2 [e_1, f_1] \omega_1'^2\omega_2' \\ &= \Delta(r+s)^2 \, \frac{\omega_2\omega_1^3 - \omega_2'\omega_1'^3}{r-s}.\end{aligned}$$

Thus, we arrive at $[\mathcal{T}_1(e_2), \mathcal{T}_1(f_2)] = \mathcal{T}_1([e_2, \ f_2]) \in U_{s^{-1}, r^{-1}}(G_2)$. □

Lemma 3.4. *$\mathcal{T}_2$ preserves the (r, s)-Serre relations* (G5)$_1$, (G6)$_1$ *into its associated object $U_{s^{-1}, r^{-1}}(G_2)$:*

$$\mathcal{T}_2(e_2)^2\mathcal{T}_2(e_1) - (r^3{+}s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + (rs)^3\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)^2 = 0, \tag{11}$$

$$\mathcal{T}_2(f_1)\mathcal{T}_2(f_2)^2 - (r^3{+}s^3)\mathcal{T}_2(f_2)\mathcal{T}_2(f_1)\mathcal{T}_2(f_2) + (rs)^3\mathcal{T}_2(f_1)\mathcal{T}_2(f_2)^2 = 0. \tag{12}$$

Proof. For the degree 2 (r, s)-Serre relation (G5)$_1$

$$e_2^2 e_1 - (r^{-3} + s^{-3}) e_2 e_1 e_2 + r^{-3} s^{-3} e_1 e_2^2 = 0,$$

observe that

$$\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) = r^{-3}\mathcal{T}_2(e_2)\mathcal{T}_2(e_1) - r^{-3} e_1, \quad \mathcal{T}_2(e_2) e_1 = s^3 e_1 \mathcal{T}_2(e_2). \tag{13}$$

Making $\mathcal{T}_2$ act algebraically on the left-hand side of (G5)$_1$, we have

$$\begin{aligned}&\mathcal{T}_2(e_2)^2\mathcal{T}_2(e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + (rs)^3\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)^2 \\ &\quad = \mathcal{T}_2(e_2) r^3 (\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + r^{-3} e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) \\ &\qquad\qquad + (rs)^3 (r^{-3}\mathcal{T}_2(e_2)\mathcal{T}_2(e_1) - r^{-3} e_1)\mathcal{T}_2(e_2) \\ &\quad = 0,\end{aligned}$$

proving (11). The proof of (12) is similar. □

To prove that $\mathcal{T}_1$ preserves the Serre relations, we need three auxiliary lemmas.

Lemma 3.5. *In the notation in Lemma 3.3, we have*

$$[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_2] = 0.$$

Proof. Since $e_1 E_{1112} - r^3 E_{1112} e_1 = \mathrm{ad}_l(e_1)^4(e_2) = 0$ (Serre relation), and

$$\begin{aligned}[E_{1112},\ f_2] = [e_1 E_{112} - r^2 s E_{112} e_1,\ f_2] &= e_1[E_{112},\ f_2] - r^2 s [E_{112},\ f_2] e_1 \\ &= r^3(r-s)(r^2-s^2)\,\omega_2 e_1^3,\end{aligned}$$

we obtain

$$\begin{aligned}&[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_2] \\ &\quad = E_{1112}[E_{112},\ f_2] + [E_{1112},\ f_2]E_{112} - r^3\big(E_{112}[E_{1112},\ f_2] + [E_{112},\ f_2]E_{1112}\big) \\ &\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1^3 E_{112} - r\Delta e_1 E_{1112} e_1 - (r^2 s)^3 E_{112} e_1^3\big) \\ &\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1^3 E_{112} - r\Delta e_1^2 E_{112} e_1 + r^3 s \Delta e_1 E_{112} e_1^2 - (r^2 s)^3 E_{112} e_1^3\big) \\ &\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1 \cdot (\mathcal{SR}) - r s^2 (\mathcal{SR}) \cdot e_1\big) \\ &\quad = 0,\end{aligned}$$

where $(\mathcal{SR})$ denotes the left-hand-side presentation of the (r, s)-Serre relation $(\mathrm{G5})_2$

$$e_1^2 E_{112} - r^2(r+s) e_1 E_{112} e_1 + r^5 s E_{112} e_1^2 = 0,$$

and we used the replacement $E_{1112} = e_1 E_{112} - r^2 s E_{112} e_1$ in the third equality. □

Lemma 3.6. *In the notation of Lemma 3.3, we have*

$$[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_1] = 0.$$

Proof. It is easy to check that $[E_{1112}, f_1] = -\Delta E_{112}\omega_1'$. Thus

$$\begin{aligned}&[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_1] \\ &\quad = E_{1112}[E_{112},\ f_1] + [E_{1112},\ f_1]E_{112} - r^3\big(E_{112}[E_{1112},\ f_1] + [E_{112},\ f_1]E_{1112}\big) \\ &\quad = (r+s)\,\big((r+s)((rs)^3 E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2\big)\omega_1'.\end{aligned}$$

It suffices to show that

$$E_{1112}E_{12} = (rs)^3 E_{12}E_{1112} + r(r-s)(r+s)^{-1}\Delta E_{112}^2. \tag{14}$$

At first, we note that the (r, s)-Serre relation $(\mathrm{G5})_1$ is equivalent to

$$E_{12}e_2 = r^3 e_2 E_{12}.$$

Since $e_1e_2 = E_{12} + s^3e_2e_1$, we get

$$\begin{aligned}E_{112}e_2 &= (e_1E_{12} - rs^2E_{12}e_1)e_2 = r^3e_1e_2E_{12} - rs^2E_{12}e_1e_2\\ &= r^3(E_{12} + s^3e_2e_1)E_{12} - rs^2E_{12}(E_{12} + s^3e_2e_1)\\ &= r(r^2-s^2)E_{12}^2 + (rs)^3e_2(e_1E_{12} - rs^2E_{12}e_1)\\ &= r(r^2-s^2)E_{12}^2 + (rs)^3e_2E_{112}.\end{aligned}$$

Next, we claim

$$E_{1112}e_2 = (rs^2)^3e_2E_{1112} - r(rs-r^2+s^2)E_{112}E_{12} + (rs)^2(r^2+rs-s^2)E_{12}E_{112}.$$

Indeed, since $E_{1112} = e_1E_{112} - r^2sE_{112}e_1$, $E_{112} = e_1E_{12} - rs^2E_{12}e_1$, and $e_1e_2 = E_{12} + s^3e_2e_1$, we have

$$\begin{aligned}E_{1112}e_2 &= e_1(E_{112}e_2) - r^2sE_{112}(e_1e_2)\\ &= r(r^2-s^2)e_1E_{12}^2 + (rs)^3(e_1e_2)E_{112} - r^2sE_{112}(e_1e_2)\\ &= r(r^2-s^2)e_1E_{12}^2 + (rs)^3E_{12}E_{112} + (rs^2)^3e_2e_1E_{112} - r^2sE_{112}E_{12}\\ &\qquad - (rs^2)^2(E_{112}e_2)e_1\\ &= r(r^2-s^2)E_{112}E_{12} + (rs)^2(r^2-s^2)E_{12}e_1E_{12} + (rs)^3E_{12}E_{112}\\ &\qquad + (rs^2)^3e_2e_1E_{112} - r^2sE_{112}E_{12} - r^3s^4(r^2-s^2)E_{12}^2e_1 - r^5s^7e_2E_{112}e_1\\ &= (rs^2)^3e_2E_{1112} - r(rs-r^2+s^2)E_{112}E_{12} + (rs)^2(r^2+rs-s^2)E_{12}E_{112}.\end{aligned}$$

To prove (14), we first note that

$$\begin{aligned}&[(r+s)((rs)^3E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2,\ f_1]\\ &\quad= (r+s)(rs)^3\big(E_{12}[E_{1112},\ f_1] + [E_{12},\ f_1]E_{1112}\big)\\ &\qquad\qquad - (r+s)\big(E_{1112}[E_{12},\ f_1] + [E_{1112},\ f_1]E_{12}\big)\\ &\qquad\qquad + r(r-s)\Delta(E_{112}[E_{112}, f_1] + [E_{112},\ f_1]E_{112})\\ &\quad= -(r+s)(rs)^3\Delta\big(E_{12}E_{112} + s^3e_2E_{1112}\big)\omega_1'\\ &\qquad\qquad + (r+s)\Delta\big(E_{1112}e_2 + r^2sE_{112}E_{12}\big)\omega_1'\\ &\qquad\qquad - r(r-s)(r+s)^2\Delta\big(E_{112}E_{12} + rs^2E_{12}E_{112}\big)\omega_1',\end{aligned}$$

which vanishes by the preceding identity. A similar but longer computation (see [Hu and Shi 2006] for details) shows that the bracket

$$[(r+s)((rs)^3E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2,\ f_2].$$

also vanishes. Then, through an argument similar to the one used in the deduction of [Benkart et al. 2006, Lemma 3.4], we get (14). □

By [Benkart et al. 2006, Lemma 3.4], Lemmas 3.5 and 3.6 imply:

Lemma 3.7. $$E_{1112}E_{112}-r^3E_{112}E_{1112}=0.$$

Lemma 3.8. *$\mathcal{T}_1$ preserves the (r, s)-Serre relations* (G5)$_1$, (G6)$_1$ *into its associated object $U_{s^{-1},r^{-1}}(G_2)$:*

$$\mathcal{T}_1(e_2)^2\mathcal{T}_1(e_1)-(r^3+s^3)\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)+(rs)^3\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)^2=0, \tag{15}$$

$$\mathcal{T}_1(f_1)\mathcal{T}_1(f_2)^2-(r^3+s^3)\mathcal{T}_1(f_2)\mathcal{T}_1(f_1)\mathcal{T}_1(f_2)+(rs)^3\mathcal{T}_1(f_2)^2\mathcal{T}_1(f_1)=0. \tag{16}$$

Proof. By direct calculation, we have

$$\begin{aligned}\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)&=\Big(-\frac{1}{s^3(r+s)\Delta}E_{1112}\Big)(-\omega_1'^{-1}f_1)\\&=s^3\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)-\frac{1}{rs^2(r+s)}E_{112}.\end{aligned} \tag{17}$$

Hence, to prove (15) is equivalent to prove

$$\mathcal{T}_1(e_2)E_{112}-r^3E_{112}\mathcal{T}_1(e_2)=0.$$

However, the latter is given by Lemma 3.7.

The proof of (16) is analogous. □

To prove that $\mathcal{T}_2$ preserves the Serre relations, we also need auxiliary lemmas. Write

$$E_{21}:=(ad_le_2)(e_1)=e_2e_1-r^{-3}e_1e_2,$$

and note that (G5)$_1$ is equivalent to $(ad_le_2)(E_{21})=e_2E_{21}-s^{-3}E_{21}e_2=0$, i.e., $E_{21}e_2=s^3e_2E_{21}$.

Lemma 3.9. $\big[\,e_1E_{21}^3-s\Delta E_{21}e_1E_{21}^2+rs^3\Delta E_{21}^2e_1E_{21}-(rs^2)^3E_{21}^3e_1,\ f_1\,\big]=0.$

Proof. Since $[E_{21},\ f_1]=r^{-3}\Delta e_2\omega_1$, $\omega_1E_{21}=rs^2E_{21}\omega_1$, $[E_{21}^2,\ f_1]=r^{-3}s^{-1}(r+s)\cdot\Delta E_{21}e_2\omega_1$, $\omega_1'E_{21}=r^2sE_{21}\omega_1'$, and $[E_{21}^3,\ f_1]=r^{-3}s^{-2}\Delta^2E_{21}^2e_2\omega_1$, we get

$$\begin{aligned}\Sigma_1&=\frac{\omega_1-\omega_1'}{r-s}E_{21}^3-(rs^2)^3E_{21}^3\frac{\omega_1-\omega_1'}{r-s}-s\Delta E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}^2+rs^3\Delta E_{21}^2\frac{\omega_1-\omega_1'}{r-s}E_{21}\\&=-(rs)^3\Delta E_{21}^3\omega_1'+rs^2\Delta E_{21}^2\omega_1'E_{21}\\&=0\end{aligned}$$

and

$$
\begin{aligned}
&\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\ f_1 \,\big] \\
&\quad = e_1[E_{21}^3,\ f_1] + [e_1,\ f_1]E_{21}^3 - (rs^2)^3\big(E_{21}^3[e_1,\ f_1] + [E_{21}^3,\ f_1]e_1\big) \\
&\qquad - s\Delta\big(E_{21}e_1[E_{21}^2,\ f_1] + E_{21}[e_1,\ f_1]E_{21}^2 + [E_{21},\ f_1]e_1E_{21}^2\big) \\
&\qquad + rs^3\Delta\big(E_{21}^2e_1[E_{21},\ f_1] + E_{21}^2[e_1,\ f_1]E_{21} + [E_{21}^2,\ f_1]e_1E_{21}\big) \\
&\quad = r^{-3}s^{-2}\Delta^2e_1E_{21}^2e_2\omega_1 + \frac{\omega_1-\omega_1'}{r-s}E_{21}^3 \\
&\qquad - s\Delta\Big(r^{-3}s^{-1}(r+s)\Delta E_{21}e_1E_{21}e_2\omega_1 + E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}^2 + r^{-3}\Delta e_2\omega_1e_1E_{21}^2\Big) \\
&\qquad + rs^3\Delta\Big(r^{-3}\Delta E_{21}^2e_1e_2\omega_1 + E_{21}^2\frac{\omega_1-\omega_1'}{r-s}E_{21} + r^{-3}s^{-1}(r+s)\Delta E_{21}e_2\omega_1e_1E_{21}\Big) \\
&\qquad - (rs^2)^3\Big(E_{21}^3\frac{\omega_1-\omega_1'}{r-s} + r^{-3}s^{-2}\Delta^2E_{21}^2e_2\omega_1e_1\Big) \\
&\quad = \Sigma_1 + (r^{-3}s^{-2}\Delta^2)\,\Sigma_2\,\omega_1 = (r^{-3}s^{-2}\Delta^2)\,\Sigma_2\,\omega_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_2 = e_1E_{21}^2e_2 - s^2(r+s)E_{21}e_1E_{21}e_2 - (rs^2)^3e_2e_1E_{21}^2& \\
+ rs^5E_{21}^2e_1e_2 + r^3s^5(r+s)E_{21}e_2e_1E_{21} - r^4s^5E_{21}^2e_2e_1&.
\end{aligned}
$$

We next show $\Sigma_2 = 0$. As $E_{21}e_2 = s^3e_2E_{21}$ and $e_2e_1 - r^{-3}e_1e_2 = E_{21}$, we get

$$
\begin{aligned}
\Sigma_2 &= \big(e_1E_{21}^2e_2 - (rs^2)^3e_2e_1E_{21}^2\big) + \big(rs^5E_{21}^2e_1e_2 - r^4s^5E_{21}^2e_2e_1\big) \\
&\qquad - s^2(r+s)E_{21}e_1E_{21}e_2 + r^3s^5(r+s)E_{21}e_2e_1E_{21} \\
&= -(rs^2)^3E_{21}^3 - r^4s^5E_{21}^3 + r^3s^5(r+s)E_{21}^3 \\
&= 0.
\end{aligned}
$$

This completes the proof. □

Lemma 3.10. $\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\ f_2 \,\big] = 0.$

Proof. Noting that

$$
[E_{21},\ f_2] = -r^{-3}\omega_2'e_1, \qquad E_{21}\omega_2' = r^3\omega_2'E_{21},
$$
$$
[E_{21}^2,\ f_2] = -r^{-3}\omega_2'(e_1E_{21} + r^3E_{21}e_1),
$$
$$
[E_{21}^3,\ f_2] = -r^{-3}\omega_2'(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1),
$$

we obtain

$$\begin{aligned}
\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\ f_2\,\big]
&= e_1[E_{21}^3,\ f_2] - s\Delta\big(E_{21}e_1[E_{21}^2,\ f_2] + [E_{21},\ f_2]e_1E_{21}^2\big)\\
&\qquad + rs^3\Delta\big(E_{21}^2e_1[E_{21},\ f_2] + [E_{21}^2,\ f_2]e_1E_{21}\big) - (rs^2)^3[E_{21}^3,\ f_2]e_1\\
&= -r^{-3}\omega_2'\Big(s^3e_1\big(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1\big)\\
&\qquad - s\Delta\big((rs)^3E_{21}e_1\big(e_1E_{21} + r^3E_{21}e_1\big) + e_1^2E_{21}^2\big)\\
&\qquad + rs^3\Delta\big((r^2s)^3E_{21}^2e_1^2 + (e_1E_{21} + r^3E_{21}e_1)e_1E_{21}\big)\\
&\qquad - (rs^2)^3(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1)e_1\Big)\\
&= -r^{-2}s\,\omega_2'\,S,
\end{aligned}$$

where

$$\begin{aligned}
S = (rs)^2(r^3-s^3)\big(e_1E_{21}^2e_1 + E_{21}e_1^2E_{21}\big) + s^2(2r^2+rs+s^2)(e_1E_{21})^2\\
- r^5s^3(2s^2+rs+r^2)(E_{21}e_1)^2 - (r+s)\big(e_1^2E_{21}^2 - (rs)^6E_{21}^2e_1^2\big).
\end{aligned}$$

It remains to prove that $S = 0$, which by [Benkart et al. 2006, Lemma 3.4] is equivalent to showing that $[S,\ f_1] = 0 = [S,\ f_2]$. To this end, we first observe:

Lemma 3.11. $e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3 = 0.$

Proof. It is easy to see that

$$e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3 = r^{-3}(\mathrm{ad}_l e_1)^4(e_2),$$

which is in fact the (r, s)-Serre relation $(G5)_2$ up to a factor r^{-3}. □

Now set $S_i := [S,\ f_i]$. Using the equations at the bottom of page 341, we obtain after some manipulations (see [Hu and Shi 2006] for details)

$$\begin{aligned}
S_2 &= (rs)^2(r^3-s^3)\big(e_1[E_{21}^2, f_2]e_1 + [E_{21}, f_2]e_1^2E_{21} + E_{21}e_1^2[E_{21}, f_2]\big)\\
&\qquad + s^2(2r^2+rs+s^2)\big(e_1[E_{21}, f_2]e_1E_{21} + e_1E_{21}e_1[E_{21}, f_2]\big)\\
&\qquad - r^5s^3(2s^2+rs+r^2)\big([E_{21}, f_2]e_1E_{21}e_1 + E_{21}e_1[E_{21}, f_2]e_1\big)\\
&\qquad - (r+s)\big(e_1^2[E_{21}^2, f_2] - (rs)^6[E_{21}^2, f_2]e_1^2\big)\\
&= -r^{-3}(rs)^2(r^3+s^3)\,\omega_2'\big(e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3\big),
\end{aligned}$$

which vanishes by Lemma 3.11.

Next we prove that $S_1 = 0$. Using the formulas at the very beginning of the proof of Lemma 3.9 and noting that

$$[e_1^2,\ f_1] = \frac{r+s}{rs}\cdot\frac{s\omega_1 - r\omega_1'}{r-s}e_1,$$

we can express S_1 as the sum $A+B+C+D$, where

$$\begin{aligned}
A &:= (rs)^2\Delta(\omega_1-\omega_1')E_{21}^2e_1\\
&\qquad -r^5s^3(2s^2+rs+r^2)E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1+(rs)^5\frac{(r+s)^2}{r-s}E_{21}^2(s\omega_1-r\omega_1')e_1,\\
B &:= (rs)(r+s)\Delta E_{21}(s\omega_1-r\omega_1')e_1E_{21}\\
&\qquad +s^2(2r^2+rs+s^2)\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1E_{21}-r^5s^3(2s^2+rs+r^2)E_{21}e_1E_{21}\frac{\omega_1-\omega_1'}{r-s},\\
C &:= (rs)^2\Delta e_1E_{21}^2(\omega_1-\omega_1')\\
&\qquad +s^2(2r^2+rs+s^2)e_1E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}-\frac{(r+s)^2}{rs}\frac{s\omega_1-r\omega_1'}{r-s}e_1E_{21}^2,\\
D &:= \frac{\Delta}{r^3}\Big((rs)^2(r^3-s^3)\big(s^{-1}(r+s)e_1E_{21}e_2\omega_1e_1+e_2\omega_1e_1^2E_{21}+E_{21}e_1^2e_2\omega_1\big)\\
&\qquad +s^2(2r^2+rs+s^2)\big(e_1e_2\omega_1e_1E_{21}+e_1E_{21}e_1e_2\omega_1\big)\\
&\qquad -r^5s^3(2s^2+rs+r^2)\big(e_2\omega_1e_1E_{21}e_1+E_{21}e_1e_2\omega_1e_1\big)\\
&\qquad -(r+s)^2s^{-1}\big(e_1^2E_{21}e_2\omega_1-(rs)^6E_{21}e_2\omega_1e_1^2\big)\Big).
\end{aligned}$$

Noting that $\omega_1E_{21}=rs^2E_{21}\omega_1$ and $\omega_1'E_{21}=r^2sE_{21}\omega_1'$, we obtain the simplified expressions

$$\begin{aligned}
A &= -r^5s^4(r^3-s^3)E_{21}^2e_1\omega_1,\\
B &= 0,\\
C &= rs^2(r^3-s^3)e_1E_{21}^2\omega_1.
\end{aligned}$$

For the last summand, a calculation using the equalities $E_{21}e_2=s^3e_2E_{21}$, $r^{-3}e_1e_2=e_2e_1-E_{21}$ and $e_2e_1=E_{21}+r^{-3}e_1e_2$ leads to

$$D=(r^3-s^3)\left(r^5s^4E_{21}^2e_1-rs^2e_1E_{21}^2\right)\omega_1$$

(see [Hu and Shi 2006] for details), showing that $S_1=A+B+C+D=0$. This completes the proof of Lemma 3.10. □

The next identity is a consequence of Lemmas 3.9, 3.10 and [Benkart et al. 2006, Lemma 3.4].

Lemma 3.12. $\quad e_1E_{21}^3-s\Delta E_{21}e_1E_{21}^2+rs^3\Delta E_{21}^2e_1E_{21}-(rs^2)^3E_{21}^3e_1=0.$

Lemma 3.13. *$\mathcal{T}_2$ preserves the (r,s)-Serre relations* (G5)$_2$, (G6)$_2$ *into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

Proof. For the fourth-degree (r, s)-Serre relation (G5)$_2$, we have to prove that

$$(rs)^6\mathcal{T}_2(e_1)^4\mathcal{T}_2(e_2) - (rs)^3(r+s)(r^2+s^2)\mathcal{T}_2(e_1)^3\mathcal{T}_2(e_2)\mathcal{T}_2(e_1) + (rs)(r^2+s^2)(r^2+rs+s^2)\mathcal{T}_2(e_1)^2\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^2 - (r+s)(r^2+s^2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^3 + \mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^4$$

vanishes. By virtue of the commutation relation in (13), this is equivalent to

$$e_1\mathcal{T}_2(e_1)^3 - s\Delta\mathcal{T}_2(e_1)e_1\mathcal{T}_2(e_1)^2 + rs^3\Delta\mathcal{T}_2(e_1)^2e_1\mathcal{T}_2(e_1) - (rs^2)^3\mathcal{T}_2(e_1)^3e_1 = 0.$$

However, since $\mathcal{T}_2(e_1) = e_1e_2 - r^3e_2e_1 = (-r^3)E_{21}$, the above identity is exactly the one given by Lemma 3.12.

Similarly, we can verify that $\mathcal{T}_2$ preserves the (r, s)-Serre relation (G6)$_2$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$. □

Lemma 3.14. *$\mathcal{T}_1$ preserves the (r, s)-Serre relations (G5)$_2$, (G6)$_2$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

Proof. For the fourth-degree (r, s)-Serre relation (G5)$_2$, we have to prove that

$$(rs)^6\mathcal{T}_1(e_1)^4\mathcal{T}_1(e_2) - (rs)^3(r+s)(r^2+s^2)\mathcal{T}_1(e_1)^3\mathcal{T}_1(e_2)\mathcal{T}_1(e_1) + (rs)(r^2+s^2)(r^2+rs+s^2)\mathcal{T}_1(e_1)^2\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^2 - (r+s)(r^2+s^2)\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^3 + \mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^4 = 0.$$

In view of the commutation relation in (17), this is equivalent to

$$E_{112}\mathcal{T}_1(e_1)^3 - r\Delta\mathcal{T}_1(e_1)E_{112}\mathcal{T}_1(e_1)^2 + r^3s\Delta\mathcal{T}_1(e_1)^2E_{112}\mathcal{T}_1(e_1) - (r^2s)^3\mathcal{T}_1(e_1)^3E_{112} = 0.$$

We can further reduce this condition to

$$E_{12}\mathcal{T}_1(e_1)^2 - r^2(r+s)\mathcal{T}_1(e_1)E_{12}\mathcal{T}_1(e_1) + r^5s\mathcal{T}_1(e_1)^2E_{12} = 0, \tag{18}$$

as a consequence of the commutative relation

$$E_{112}\mathcal{T}_1(e_1) = rs^2\mathcal{T}_1(e_1)E_{112} + r^{-1}s(r+s)^2E_{12},$$

itself arising from the equalities

$$[E_{112}f_1] = -(r+s)^2E_{12}\omega_1', \quad \omega_1'E_{112} = rs^2E_{112}\omega_1'.$$

Again, since $[E_{12}, f_1] = -\Delta e_2\omega_1'$, we have

$$E_{12}\mathcal{T}_1(e_1) = r^2s\mathcal{T}_1(e_1)E_{12} + r^{-1}s\Delta e_2,$$

by which (18) is finally reduced to $e_2\mathcal{T}_1(e_1) = r^3\mathcal{T}_1(e_1)e_2$, since $\mathcal{T}_1(e_1) = -\omega_1'^{-1}f_1$.

The proof of the second part is similar. □

Theorem 3.15. *$\mathcal{T}_1$ and $\mathcal{T}_2$ are the Lusztig symmetries from $U_{r,s}(G_2)$ to its associated quantum group $U_{s^{-1},r^{-1}}(G_2)$ as $\mathbb{Q}$-isomorphisms, inducing the usual Lusztig symmetries as $\mathbb{Q}(q)$-automorphisms not only on the quantum group $U_q(G_2)$ of Drinfel'd–Jimbo type but also on the centralized quantum group $U_q^c(G_2)$, only when $r = q = s^{-1}$.* □

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