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HANS SCHOUTENS

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# ASYMPTOTIC HOMOLOGICAL CONJECTURES IN MIXED CHARACTERISTIC

HANS SCHOUTENS

In this paper, various Homological Conjectures are studied for local rings which are locally finitely generated over a discrete valuation ring  $V$  of mixed characteristic. Typically, we can only conclude that a particular conjecture holds for such a ring provided the residual characteristic of  $V$  is sufficiently large in terms of the complexity of the data, where the complexity is primarily given in terms of the degrees of the polynomials over  $V$  that define the data, but possibly also by some additional invariants such as (homological) multiplicity. Thus asymptotic versions of the Improved New Intersection Theorem, the Monomial Conjecture, the Direct Summand Conjecture, the Hochster–Roberts Theorem and the Vanishing of Maps of Tors Conjecture are given.

That the results only hold asymptotically is due to the fact that nonstandard arguments are used, relying on the Ax–Kochen–Ershov Principle, to infer their validity from their positive characteristic counterparts. A key role in this transfer is played by the Hochster–Huneke canonical construction of big Cohen–Macaulay algebras in positive characteristic via absolute integral closures.

## 1. Introduction

In the last three decades, all the so-called Homological Conjectures have been settled completely for noetherian local rings containing a field by work of Peskine and Szpiro [1973], Hochster and Roberts [1974], Hochster [1975b; 1983], Evans and Griffith [1981] and others, to cite just some of the key papers. More recently, Hochster and Huneke have given more simplified proofs of most of these results by means of their tight closure theory, including their canonical construction of big Cohen–Macaulay algebras in positive characteristic (see [Hochster and Huneke 1992; 1989; 2000; Huneke 1996]; for further discussion and proofs, see [Bruns and Herzog 1993, §9] or [Strooker 1990]).

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In sharp contrast is the development in mixed characteristic, where only sporadic results (often in low dimensions) are known, apart from the breakthrough by Roberts [1987], settling the New Intersection Theorem for all noetherian local rings, and the recent work of Heitmann [2002] in dimension three. Some attempts have been made by Hochster, either by finding a suitable substitute for tight closure in mixed characteristic [1994], or by constructing big Cohen–Macaulay modules in mixed characteristic [1975a]. These approaches have yet to bear fruit and the best result to date in this direction is the existence of big Cohen–Macaulay algebras in dimension three [Hochster 2002], which in turn relies on the positive solution of the Direct Summand Conjecture in dimension three by Heitmann [2002].

In this paper, we will follow the big Cohen–Macaulay algebra approach, but instead of trying to work with rings of Witt vectors, we will use the Ax–Kochen–Ershov Principle [Ax and Kochen 1965; Ershov 1965; 1966], linking complete discrete valuation rings in mixed characteristic with complete discrete valuation rings in positive characteristic via an equicharacteristic zero (nondiscrete) valuation ring (see Theorem 2.3 below). This intermediate valuation ring is obtained by a construction which originates from logic, but is quite algebraic in nature, to wit, the *ultraproduct* construction. Roughly speaking, this construction associates to an infinite collection of rings  $C_w$  their ultraproduct  $C_\infty$ , which should be thought of as a kind of “limit” or “average” (realized as a certain homomorphic image of the product). An ultraproduct inherits many of the algebraic properties of its components. The correct formulation of this transfer principle is Łos’ Theorem, which makes precise when a property carries over (namely, when it is first order definable in some suitable language). Properties that carry over are those of being *a domain, a field, a valuation ring, local, henselian*; among the properties that do not carry over is *noetherianness*, so that almost no ultraproduct is noetherian (except an ultraproduct of fields or of artinian rings of bounded length). This powerful tool is used in [Schmidt–Göttsch 1987; van den Dries and Schmidt 1984; Schoutens 2000a; 2007], to obtain uniform bounds in polynomial rings over fields; in [Schoutens 2000a; 2000b; 2003a; 2003c], to transfer properties from positive to zero characteristic; and in [Aschenbrenner and Schoutens 2007; Schoutens 2003d; 2004a; 2005a; 2005b], to give an alternative treatment of tight closure theory in equicharacteristic zero. The key fact in the first set of papers is a certain flatness result about ultraproducts (see Theorem 2.2 below for a precise formulation), and in the two last sets, the so-called *Lefschetz Principle for algebraically closed fields* (the ultraproduct of the algebraic closures of the  $p$ -element fields  $\mathbb{F}_p$  is isomorphic to  $\mathbb{C}$ ).

The Ax–Kochen–Ershov Principle is a kind of *Lefschetz Principle for henselian valued fields*, and its most concrete form states that the ultraproduct of all  $\mathbb{F}_p[[t]]$ , with  $t$  a single indeterminate, is isomorphic to the ultraproduct of all rings of  $p$ -adic

integers  $\mathbb{Z}_p$ . We will identify both ultraproducts and denote the resulting ring by  $\mathfrak{D}$ . It follows that  $\mathfrak{D}$  is an equicharacteristic zero henselian valuation ring with principal maximal ideal, whose separated quotient (i.e., reduction modulo the intersection of all powers of the maximal ideal) is an equicharacteristic zero excellent complete discrete valuation ring.

***Z-affine algebras.*** To explain the underlying idea in this paper, we introduce some notation. Let  $(Z, \mathfrak{p})$  be a (not necessarily noetherian) local ring. A *Z-affine algebra*  $C$  is any  $Z$ -algebra of the form  $C = Z[X]/I$  where  $X$  is a finite tuple of indeterminates and  $I$  a finitely generated ideal in  $Z[X]$ . A *local Z-affine algebra* is any localization  $R = C_{\mathfrak{m}}$  of a  $Z$ -affine algebra  $C$  with respect to a prime ideal  $\mathfrak{m}$  of  $C$  lying above  $\mathfrak{p}$ . In particular, the natural homomorphism  $Z \rightarrow R$  is local. We denote the category of all local  $Z$ -affine algebras by  $\underline{\text{Aff}}(Z)$ .

The objective is to transfer algebraic properties (such as the homological Conjectures) from the positive characteristic categories  $\underline{\text{Aff}}(\mathbb{F}_p[[t]])$  to the mixed characteristic categories  $\underline{\text{Aff}}(\mathbb{Z}_p)$ . This will be achieved through the intermediate equicharacteristic zero category  $\underline{\text{Aff}}(\mathfrak{D})$ . As this latter category consists mainly of nonnoetherian rings, we will have to find analogues in this setting of many familiar notions from commutative algebra, such as dimension, depth, Cohen–Macaulayness or regularity (see Sections 5 and 6).

The following example is paradigmatic: let  $X$  be a finite tuple of indeterminates and let  $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(A)$  be the ultraproduct of all  $\mathbb{F}_p[[t]][X]$ , and  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(A)$ , the ultraproduct of all  $\mathbb{Z}_p[X]$ . Note that both rings contain  $\mathfrak{D}$ , and in fact, contain  $\mathfrak{D}[X]$ . The key algebraic fact, which is equivalent to a result on effective bounds by Aschenbrenner [2001a], is that both inclusions  $\mathfrak{D}[X] \subseteq \mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(A)$  and  $\mathfrak{D}[X] \subseteq \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(A)$  are flat. Suppose we have in each  $\mathbb{F}_p[[t]][X]$  a polynomial  $f_p$ , and let  $f_{\infty}$  be their ultraproduct. A priori, we can only say that  $f_{\infty} \in \mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(A)$ . However, if all  $f_p$  have  $X$ -degree  $d$ , for some  $d$  independent from  $p$ , then  $f_{\infty}$  itself is a polynomial over  $\mathfrak{D}$  of degree  $d$  (since an ultraproduct commutes with finite sums by Łos’ Theorem). Hence, as  $f_{\infty}$  lies in  $\mathfrak{D}[X]$ , we can also view it as an element in  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(A)$ . Therefore, there are polynomials  $\tilde{f}_p \in \mathbb{Z}_p[X]$  whose ultraproduct is equal to  $f_{\infty}$ . The choice of the  $\tilde{f}_p$  is not unique, but any two choices will be equal for almost all  $p$ , by Łos’ Theorem. In conclusion, to a collection of polynomials defined over the various  $\mathbb{F}_p[[t]]$ , of uniformly bounded degree, we can associate, albeit not uniquely, a collection of polynomials defined over the various  $\mathbb{Z}_p$  (of uniformly bounded degree), and of course, this also works the other way. Instead of doing this for just one polynomial in each component, we can now do this for a finite tuple of polynomials of fixed length. If at the same time, we can maintain certain algebraic relations among them (characterizing one of the properties we seek to transfer), we will have achieved our goal.

Unfortunately, it is the nature of an ultraproduct that it only captures the “average” property of its components. In the present context, this means that the desired property does not necessarily hold in all  $\mathbb{Z}_p[X]$ , but only in almost all. In conclusion, we cannot hope for a full solution of the Homological Conjectures by this method, but only an *asymptotic* solution. In view of the above, the following definition is now natural.

**Complexity.** Let  $C$  be a  $Z$ -affine algebra, say, of the form  $C = Z[X]/I$ , with  $X$  a finite tuple of indeterminates and  $I$  a finitely generated ideal, and let  $R = C_{\mathfrak{m}}$  be a local  $Z$ -affine algebra (so that  $\mathfrak{p} \subseteq \mathfrak{m}$ ). We say that  $C$  has  $Z$ -complexity at most  $c$ , if  $|X| \leq c$  and  $I$  is generated by polynomials of degree at most  $c$ ; we say that  $R$  has  $Z$ -complexity at most  $c$ , if, moreover, also  $\mathfrak{m}$  is generated by polynomials of degree at most  $c$ . An element  $r \in C$  is said to have  $Z$ -complexity at most  $c$ , if  $C$  has  $Z$ -complexity at most  $c$  and  $r$  is the image of a polynomial in  $Z[X]$  of degree at most  $c$ . An element  $r \in R$  has  $Z$ -complexity at most  $c$ , if  $R$  has  $Z$ -complexity at most  $c$  and if  $r$  is (the image of) a quotient  $P/Q$  of polynomials of degree at most  $c$  with  $Q \notin \mathfrak{m}$ . We say that a tuple or a matrix has  $Z$ -complexity at most  $c$ , if each of its entries has  $Z$ -complexity at most  $c$  and the number of entries is also bounded by  $c$ . Note that in a  $Z$ -affine algebra, the sum of two elements of  $Z$ -complexity at most  $c$ , has again  $Z$ -complexity at most  $c$ , whereas in a *local*  $Z$ -affine algebra, the sum has  $Z$ -complexity at most  $2c$ .

An ideal  $J$  in  $C$  or  $R$  has  $Z$ -complexity at most  $c$ , if it is generated by a tuple of  $Z$ -complexity at most  $c$ . A  $Z$ -algebra homomorphism  $C \rightarrow C'$  or a local  $Z$ -algebra homomorphism  $R \rightarrow R'$  is said to have  $Z$ -complexity at most  $c$ , if  $C$  and  $C'$  (respectively,  $R$  and  $R'$ ) are (local)  $Z$ -affine algebras of  $Z$ -complexity at most  $c$  and the homomorphism is given by sending each indeterminate  $X_i$  to an element of  $Z$ -complexity at most  $c$ .

**Asymptotic properties.** Let  $\mathbf{P}$  be a property of noetherian local rings (possibly involving some additional data). We will use the phrase  $\mathbf{P}$  holds asymptotically in mixed characteristic, to express that for each  $c$ , we can find a bound  $c'$ , such that if  $V$  is a complete discrete valuation ring of mixed characteristic and  $C$  a local  $V$ -affine algebra of  $V$ -complexity at most  $c$  (and a similar bound on the additional data), then property  $\mathbf{P}$  holds for  $C$ , provided the characteristic of the residue field of  $V$  is at least  $c'$ . Sometimes, we have to control some additional invariants in terms of the bound  $c$ . In this paper, we will prove that in this sense, many Homological Conjectures hold asymptotically in mixed characteristic.

**A final note.** Its asymptotic nature is the main impediment of the present method to carry out Hochster’s program of obtaining tight closure and big Cohen–Macaulay algebras in mixed characteristic. For instance, despite the fact that we are able

to define an analogue of a balanced big Cohen–Macaulay algebra for  $\mathfrak{D}$ -affine domains, this object cannot be realized as an ultraproduct of  $\mathbb{Z}_p$ -algebras, so that there is no candidate so far for a big Cohen–Macaulay in mixed characteristic. Although I will not pursue this line of thought in this paper, one could also define some nonstandard closure operation on ideals in  $\mathfrak{D}$ -affine algebras, but again, such an operation will only partially descend to any component.

**Notation.** A tuple  $\mathbf{x}$  over a ring  $Z$  is always understood to be finite. Its length is denoted by  $|\mathbf{x}|$  and the ideal it generates is denoted  $\mathbf{x}Z$ . When we say that  $(Z, \mathfrak{p})$  is *local*, we mean that  $\mathfrak{p}$  is its (unique) maximal ideal, but we do not imply that  $Z$  has to be noetherian.

For a survey of the results and methods in this paper, see [Schoutens 2003b]. In the forthcoming [Schoutens 2004b] some of the present asymptotic versions will be generalized through a further investigation of the algebraic properties of ultraproducts using the notions introduced in Sections 5 and 6.

## 2. Ultraproducts

In this preliminary section, I state some generalities about ultraproducts and then briefly review the situation in equicharacteristic zero and the Ax–Kochen–Ershov Principle. The next section lays out the essential tools for conducting the transfer discussed in the introduction, to wit, approximations, protoproducts and nonstandard hulls, whose properties are then studied in Sections 5 and 6. The subsequent sections contain proofs of various asymptotic results, using these tools.

Whenever we have an infinite index set  $W$ , we will equip it with some (unnamed) countably incomplete nonprincipal ultrafilter; ultraproducts will always be taken with respect to this ultrafilter and we will write

$$\operatorname{ulim}_{w \rightarrow \infty} O_w \quad \text{or simply} \quad O_\infty$$

for the ultraproduct of objects  $O_w$  (this will apply to rings, ideals and elements alike). A first introduction to ultraproducts, including Łos’ Theorem, sufficient to understand the present paper, can be found in [Schoutens 2003d, §2]; for a more detailed treatment, see [Hodges 1993]. Łos’ Theorem states essentially that if a fixed algebraic relation holds among finitely many elements  $f_{1w}, \dots, f_{sw}$  in each ring  $C_w$ , then the same relation holds among their ultraproducts  $f_{1\infty}, \dots, f_{s\infty}$  in the ultraproduct  $C_\infty$ , and conversely, if such a relation holds in  $C_\infty$ , then it holds in almost all  $C_w$ . Here *almost all* means “for all  $w$  in a subset of the index set which belongs to the ultrafilter” (the idea is that sets belonging to the ultrafilter are *large*, whereas the remaining sets are *small*).

An immediate, but important application of Łos' Theorem is that the ultraproduct of algebraically closed fields of different prime characteristics is an (uncountable) algebraically closed field of characteristic zero, and any sufficiently large algebraically closed field of characteristic zero, including  $\mathbb{C}$ , can be realized thus.<sup>1</sup> This simple observation, in combination with work of van den Dries on nonstandard polynomials (see below), was exploited in [Schoutens 2003d] to define an alternative version of tight closure for  $\mathbb{C}$ -affine algebras, called *nonstandard tight closure*, which was then further generalized to arbitrary noetherian local rings containing the rationals in [Aschenbrenner and Schoutens 2007]. The ensuing notions of F-regularity and F-rationality have proved to be more versatile [Schoutens 2004a; 2005a; 2005b] than those defined in [Hochster and Huneke 2000].

Let me briefly recall the results in [van den Dries and Schmidt 1984; van den Dries 1979] on nonstandard polynomials mentioned above. Let  $K_w$  be fields (of arbitrary characteristic) with ultraproduct  $K_\infty$  (which is again a field by Łos' Theorem). Let  $X$  be a fixed finite tuple of indeterminates and set  $A := K_\infty[X]$  and  $A_w := K_w[X]$ . Let  $A_\infty$  be the ultraproduct of the  $A_w$ . As in the example discussed in the introduction, we have a canonical embedding of  $A$  inside  $A_\infty$ . In fact, the following easy observation, valid over arbitrary rings, describes completely the elements in  $A_\infty$  that lie in  $A$  (the proof is straightforward and left to the reader).

**Lemma 2.1.** *Let  $X$  be a finite tuple of indeterminates. Let  $C_w$  be rings and let  $C_\infty$  be their ultraproduct. If  $f_w$  is a polynomial in  $C_w[X]$  of degree at most  $c$ , for each  $w$  and for some  $c$  independent from  $w$ , then their ultraproduct in  $\text{ulim}_{w \rightarrow \infty} C_w[X]$  belongs already to the subring  $C_\infty[X]$ , and conversely, every element in  $C_\infty[X]$  is obtained in this way.*

This result also motivates the notion of *complexity* from the introduction. Returning to the results of Schmidt and van den Dries, the following two properties of the embedding  $A \subseteq A_\infty$  do not only imply the uniform bounds from [van den Dries and Schmidt 1984; Schoutens 2000a], but also play an important theoretical role in the development of nonstandard tight closure [Aschenbrenner and Schoutens 2007; Schoutens 2003d].

**Theorem 2.2** (Schmidt and van den Dries). *The embedding  $A \subseteq A_\infty$  is faithfully flat and every prime ideal in  $A$  extends to a prime ideal in  $A_\infty$ .*

To carry out the present program, we have to replace the base fields  $K_w$  by complete discrete valuation rings  $\mathfrak{D}_w$ . Unfortunately, we now have to face the following complications. Firstly, the ultraproduct  $\mathfrak{D}$  of the  $\mathfrak{D}_w$  is no longer noetherian, and

<sup>1</sup>To be more precise, any algebraically closed field of characteristic zero whose cardinality is of the form  $2^\lambda$  for some infinite cardinal  $\lambda$ , is an ultraproduct of algebraically closed fields of prime characteristic; under the generalized continuum hypothesis this means every uncountable algebraically closed field of characteristic zero.

so in particular the corresponding  $A := \mathfrak{D}[X]$  is nonnoetherian. Moreover, the embedding  $A \subseteq A_\infty$ , where  $A_\infty$  is now the ultraproduct of the  $A_w := \mathfrak{D}_w[X]$ , although flat (see [Theorem 4.2](#) below), is no longer faithfully flat (this is related to Kronecker’s problem; see [[Aschenbrenner 2001a](#)] or [[Schoutens  \$\geq\$  2007](#)] for details). Furthermore, not every prime ideal extends to a prime ideal. However, by working locally, we can circumvent all the latter complications (see [Theorem 4.2](#) and [Remark 4.5](#)).

To obtain the desired transfer, we will realize  $\mathfrak{D}$  in two different ways, as an ultraproduct of complete discrete valuation rings in positive characteristic and as an ultraproduct of complete discrete valuation rings in mixed characteristic, and then pass from one set to the other via  $\mathfrak{D}$ , as explained in the introduction (for more details, see [Section 6.9](#) below). This is the celebrated Ax–Kochen–Ershov Principle [[Ax and Kochen 1965](#); [Ershov 1965](#); [Ershov 1966](#)], and I will discuss this now. For each  $p$ , let  $\mathfrak{D}_p^{\text{mix}}$  be a complete discrete valuation ring of mixed characteristic with residue field  $\kappa_p$  of characteristic  $p$ . To each  $\mathfrak{D}_p^{\text{mix}}$ , we associate a corresponding equicharacteristic complete discrete valuation ring with the same residue field, by letting

$$(1) \quad \mathfrak{D}_p^{\text{eq}} := \kappa_p[[t]]$$

where  $t$  is a single indeterminate.

**Theorem 2.3** (Ax–Kochen–Ershov). *The ultraproduct of the  $\mathfrak{D}_p^{\text{eq}}$  is isomorphic (as a local ring) with the ultraproduct of the  $\mathfrak{D}_p^{\text{mix}}$ .*

**Remark 2.4.** As stated, we need to assume the continuum hypothesis. Otherwise, by the Keisler–Shelah Theorem [[Hodges 1993](#), Theorem 9.5.7], one might need to take further ultrapowers, that is to say, over a larger index set. In order to not complicate the exposition, I will nonetheless make the set-theoretic assumption, so that our index set can always be taken to be the set of prime numbers. The reader can convince himself that all proofs in this paper can be adjusted so that they hold without any set-theoretic assumption.

To conclude this section, I state a variant of Prime Avoidance which also works in mixed characteristic (note that for nonprime ideals one normally has to assume that the ring contains a field, see for instance [[Eisenbud 1995](#), Lemma 3.3]).

**Proposition 2.5.** *Let  $Z$  be a local ring with infinite residue field  $\kappa$ . Let  $C$  be an arbitrary  $Z$ -algebra and let  $W$  be a finitely generated  $Z$ -submodule of  $C$ . If  $\mathfrak{a}_1, \dots, \mathfrak{a}_t$  are ideals in  $C$  not containing  $W$ , then there exists  $f \in W$  not contained in any of the  $\mathfrak{a}_j$ .*

*Proof.* We induct on the number  $t$  of ideals to be avoided, where the case  $t = 1$  holds by assumption. Hence assume  $t > 1$ . By induction, we can find elements



$g_i \in W$ , for  $i = 1, 2$ , which lie outside any  $\mathfrak{a}_j$  for  $j \neq i$ . If either  $g_1 \notin \mathfrak{a}_1$  or  $g_2 \notin \mathfrak{a}_2$  we are done, so assume  $g_i \in \mathfrak{a}_i$ . Therefore, every element of the form  $g_1 + zg_2$  with  $z$  a unit in  $Z$  does not lie in  $\mathfrak{a}_1$  nor in  $\mathfrak{a}_2$ . Since  $\kappa$  is infinite, we can find  $t - 1$  units  $z_1, z_2, \dots, z_{t-1}$  in  $Z$  whose residues in  $\kappa$  are all distinct. I claim that at least one of the  $g_1 + z_i g_2$  lies outside all  $\mathfrak{a}_j$ , so that we found our desired element in  $W$ . Indeed, if not, then each  $g_1 + z_i g_2$  lies in one of the  $t - 2$  ideals  $\mathfrak{a}_3, \dots, \mathfrak{a}_t$ , by our previous remark. By the Pigeon Hole Principle, for some  $j$  and some  $l \neq k$ , we have that  $g_1 + z_k g_2$  and  $g_1 + z_l g_2$  lie both in  $\mathfrak{a}_j$ . Hence so does their difference  $(z_k - z_l)g_2$ . However,  $z_k - z_l$  is a unit in  $Z$ , by choice of the  $z_i$ , so that  $g_2 \in \mathfrak{a}_j$ , contradiction.  $\square$

**Corollary 2.6** (Controlled Ideal Avoidance). *Let  $Z$  be a local ring with infinite residue field and let  $C$  be a (local)  $Z$ -affine algebra. If  $I$  and  $\mathfrak{a}_1, \dots, \mathfrak{a}_t$  are ideals in  $C$  with  $I$  not contained in any  $\mathfrak{a}_i$ , then  $I$  contains an element outside every  $\mathfrak{a}_i$ . More precisely, if  $c$  is an upper bound for the  $Z$ -complexity of  $I$ , then there exists an element  $f \in I$  of  $Z$ -complexity at most  $c^2$ , not contained in any  $\mathfrak{a}_i$ .*

*Proof.* Let  $(x_1, \dots, x_n)$  be a tuple of  $Z$ -complexity at most  $c$  generating  $I$  and let  $W$  be the  $Z$ -submodule of  $C$  generated by  $(x_1, \dots, x_n)$ . In particular,  $W$  is not contained in any  $\mathfrak{a}_i$ , so that we may apply Proposition 2.5 to obtain an element  $f \in W$ , outside each  $\mathfrak{a}_i$ . Write  $f = z_1 x_1 + \dots + z_n x_n$  with  $z_i \in Z$ . After putting on a common denominator, we see that  $f$  has  $Z$ -complexity at most  $cn \leq c^2$  (in case  $C$  is not local, the  $Z$ -complexity of  $f$  is in fact at most  $c$ ).  $\square$

It is clear from the proof of Proposition 2.5 that in both results, we only need the residue field to have a larger cardinality than the number of ideals to be avoided.

### 3. Approximations, protoproducts and nonstandard hulls

In this section, some general results on ultraproducts of finitely generated algebras over discrete valuation rings will be derived. We start with introducing some general terminology, over arbitrary noetherian local rings, but once we start proving some nontrivial properties in the next sections, we will specialize to the case that the base rings are discrete valuation rings. For some results in the general case, we refer to [Schoutens 2007; 2004b;  $\geq$  2007].

For each  $w$ , we fix a noetherian local ring  $\mathfrak{D}_w$  and let  $\mathfrak{D}$  be its ultraproduct. If the  $\mathfrak{p}_w$  are the maximal ideals of the  $\mathfrak{D}_w$ , then their ultraproduct  $\mathfrak{p}$  is the maximal ideal of  $\mathfrak{D}$ . We will write  $\mathfrak{I}_{\mathfrak{D}}$  for the ideal of infinitesimals of  $\mathfrak{D}$ , that is to say, the intersection of all the powers  $\mathfrak{p}^k$  (note that in general  $\mathfrak{I}_{\mathfrak{D}} \neq (0)$  and therefore,  $\mathfrak{D}$  is in particular nonnoetherian).

By saturatedness of ultraproducts,  $\mathfrak{D}$  is quasicomplete in its  $\mathfrak{p}$ -adic topology in the sense that any Cauchy sequence has a (nonunique) limit. Hence the completion of  $\mathfrak{D}$  is  $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$  (see also Lemma 5.3 below). Moreover, we will assume that all  $\mathfrak{D}_w$

have embedding dimension at most  $\epsilon$ . Hence so do  $\mathfrak{D}$  and  $\mathfrak{D}/\mathfrak{J}_{\mathfrak{D}}$ . Since a complete local ring with finitely generated maximal ideal is noetherian [Matsumura 1986, Theorem 29.4], we showed that  $\mathfrak{D}/\mathfrak{J}_{\mathfrak{D}}$  is a noetherian complete local ring. For more details in the case of interest to us, where each  $\mathfrak{D}_w$  is a discrete valuation ring or a field, see [Becker et al. 1979].

We furthermore fix throughout a tuple of indeterminates  $X = (X_1, \dots, X_n)$ , and we set  $A := \mathfrak{D}[X]$  and  $A_w := \mathfrak{D}_w[X]$ .

**Definition 3.1.** The *nonstandard  $\mathfrak{D}$ -hull* of  $A$  is by definition the ultraproduct of the  $A_w$  and is denoted  $\mathfrak{L}_{\mathfrak{D}}(A)$ .

This terminology is a little misleading, because  $\mathfrak{L}_{\mathfrak{D}}(A)$  does not only depend on  $\mathfrak{D}$  but also on the choice of  $\mathfrak{D}_w$  whose ultraproduct is  $\mathfrak{D}$ . In fact, we will exploit this dependence when applying the Ax–Kochen–Ershov principle, in which case we have to declare more precisely which nonstandard  $\mathfrak{D}$ -hull is meant. Nonetheless, whenever  $\mathfrak{D}$  and  $\mathfrak{D}_w$  are clear from the context, we will denote the nonstandard  $\mathfrak{D}$ -hull of  $A$  simply by  $\mathfrak{L}(A)$ .

By Łos’ Theorem, we have an inclusion  $\mathfrak{D} \subseteq \mathfrak{L}(A)$ . Let us continue to write  $X_i$  for the ultraproduct in  $\mathfrak{L}(A)$  of the constant sequence  $X_i \in A_w$ . By Łos’ Theorem, the  $X_i$  are algebraically independent over  $\mathfrak{D}$ . In other words,  $A$  is a subring of  $\mathfrak{L}(A)$ . In the next section, we will prove the key algebraic property of the extension  $A \subseteq \mathfrak{L}(A)$  when the base rings  $\mathfrak{D}_w$  are discrete valuation rings, to wit, its flatness. We start with extending the notions of nonstandard hull and approximation from [Schoutens 2003d], to arbitrary local  $\mathfrak{D}$ -affine algebras (recall that a *local  $\mathfrak{D}$ -affine algebra* is a localization of a finitely presented  $\mathfrak{D}$ -algebra at a prime ideal containing  $\mathfrak{p}$ ).

**$\mathfrak{D}$ -approximations and nonstandard  $\mathfrak{D}$ -hulls.** An  *$\mathfrak{D}$ -approximation* of a polynomial  $f \in A$  is a sequence of polynomials  $f_w \in A_w$ , such that their ultraproduct is equal to  $f$ , viewed as an element in  $\mathfrak{L}(A)$ . Note that according to Lemma 2.1, we can always find such an  $\mathfrak{D}$ -approximation. Moreover, any two  $\mathfrak{D}$ -approximations are equal for almost all  $w$ , by Łos’ Theorem. Similarly, an  *$\mathfrak{D}$ -approximation* of a finitely generated ideal  $I := \mathfrak{f}A$  with  $\mathfrak{f}$  a finite tuple, is a sequence of ideals  $I_w := \mathfrak{f}_w A_w$ , where  $\mathfrak{f}_w$  is an  $\mathfrak{D}$ -approximation of  $\mathfrak{f}$  (meaning that each entry in  $\mathfrak{f}_w$  is an  $\mathfrak{D}$ -approximation of the corresponding entry in  $\mathfrak{f}$ ). Łos’ Theorem gives once more that any two  $\mathfrak{D}$ -approximations are almost all equal. Moreover, if  $I_w$  is some  $\mathfrak{D}$ -approximation of  $I$  then

$$(2) \quad \text{ulim}_{w \rightarrow \infty} I_w = I \mathfrak{L}(A).$$

Assume now that  $C$  is an  $\mathfrak{D}$ -affine algebra, say  $C = A/I$  with  $I$  a finitely generated ideal. We define an  *$\mathfrak{D}$ -approximation* of  $C$  to be the sequence of finitely

generated  $\mathfrak{D}_w$ -algebras  $C_w := A_w/I_w$ , where  $I_w$  is some  $\mathfrak{D}$ -approximation of  $I$ . We define the *nonstandard  $\mathfrak{D}$ -hull* of  $C$  to be the ultraproduct of the  $C_w$  and denote it  $\mathfrak{L}_{\mathfrak{D}}(C)$  or simply  $\mathfrak{L}(C)$ . It is not hard to show that  $\mathfrak{L}(C)$  is uniquely defined up to  $C$ -algebra isomorphism (for more details see [Schoutens 2003d] or [Schoutens 2007]). From (2), it follows that  $\mathfrak{L}(C) = \mathfrak{L}(A)/I\mathfrak{L}(A)$ . In particular, there is a canonical homomorphism  $C \rightarrow \mathfrak{L}(C)$  obtained from the base change  $A \rightarrow \mathfrak{L}(A)$ .

When  $I$  is not finitely generated,  $I\mathfrak{L}(A)$  might not be realizable as an ultraproduct of ideals, and consequently, has no  $\mathfrak{D}$ -approximation. Although one can find special cases of infinitely generated ideals admitting  $\mathfrak{D}$ -approximations, we will never have to do this in the present paper. Similarly, we only define  $\mathfrak{D}$ -approximations for  $\mathfrak{D}$ -affine algebras.

Although  $A \rightarrow \mathfrak{L}(A)$  is injective, this is not necessarily the case for  $C \rightarrow \mathfrak{L}(C)$ , if the  $\mathfrak{D}_w$  are not fields. For instance, if  $W$  is the set of prime numbers,  $\mathfrak{D}_p := \mathbb{Z}_p$  for each  $p \in W$  and  $I = (1 - \pi X, \gamma)A$  where  $\pi := \text{ulim}_{p \rightarrow \infty} p$  and  $\gamma := \text{ulim}_{p \rightarrow \infty} p^p$ , then  $I \neq (1)$  but  $I\mathfrak{L}(A) = (1)$ . However, when the  $\mathfrak{D}_w$  are discrete valuation rings, we will see shortly, that this phenomenon disappears if we localize at prime ideals containing  $\mathfrak{p}$ . Next we define a process which is converse to taking  $\mathfrak{D}$ -approximations.

**Protoproducts.** Fix some  $c$ . For each  $w$ , let  $I_w$  be an ideal in  $A_w$  of  $\mathfrak{D}_w$ -complexity at most  $c$ . In other words, we can write  $I_w = \mathbf{f}_w A_w$ , for some tuple  $\mathbf{f}_w$  of  $\mathfrak{D}_w$ -complexity at most  $c$ . Let  $\mathbf{f}$  be the ultraproduct of these tuples. By Lemma 2.1, the tuple  $\mathbf{f}$  is already defined over  $A$ . We call  $I := \mathbf{f}A$  the *protoproduct* of the  $I_w$ . It follows that the  $I_w$  are an  $\mathfrak{D}$ -approximation of  $I$  and that  $I\mathfrak{L}(A)$  is the ultraproduct of the  $I_w$ .

With  $C_w := A_w/I_w$  and  $C := A/I$ , we call  $C$  the *protoproduct* of the  $C_w$ . The  $C_w$  are an  $\mathfrak{D}$ -approximation of  $C$  and their ultraproduct  $\mathfrak{L}(C)$  is the nonstandard  $\mathfrak{D}$ -hull of  $C$ . We can now extend the previous definition to the image in  $C_w$  of an element  $c_w \in A_w$  (respectively, to the extension  $J_w C_w$  of a finitely generated ideal  $J_w \subseteq A_w$ ) of  $\mathfrak{D}_w$ -complexity at most  $c$  and define similarly their *protoproduct*  $c \in C$  and  $J C$  as the image in  $C$  of the respective protoproduct of the  $c_w$  and the  $J_w$ .

**Functoriality.** We have a commutative diagram

$$(3) \quad \begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ \downarrow & & \downarrow \\ \mathfrak{L}(C) & \xrightarrow{\mathfrak{L}(\varphi)} & \mathfrak{L}(D) \end{array}$$

where  $C \rightarrow D$  is an  $\mathfrak{D}$ -algebra homomorphism of finite type between  $\mathfrak{D}$ -affine algebras and  $\mathfrak{L}(C) \rightarrow \mathfrak{L}(D)$  is its base change over  $\mathfrak{L}(A)$ . Alternatively, we may view this diagram coming from a sequence of  $\mathfrak{D}_w$ -algebras homomorphisms  $C_w \rightarrow D_w$  of  $\mathfrak{D}_w$ -complexity at most  $c$ , for some  $c$  independent from  $w$ , in which case  $C \rightarrow D$  and  $\mathfrak{L}(C) \rightarrow \mathfrak{L}(D)$  are the respective protoproduct and ultraproduct of these homomorphisms.

**Lemma 3.2.** *Any prime ideal  $\mathfrak{m}$  of  $A$  containing  $\mathfrak{p}$  is finitely generated and its extension  $\mathfrak{m}\mathfrak{L}(A)$  is again prime.*

*Proof.* Since  $A/\mathfrak{p}A = \kappa[X]$  is noetherian, where  $\kappa$  is the residue field of  $\mathfrak{D}$ , the ideal  $\mathfrak{m}(A/\mathfrak{p}A)$  is finitely generated. Therefore so is  $\mathfrak{m}$ , since by assumption  $\mathfrak{p}$  is finitely generated. Moreover,  $\mathfrak{L}(A)/\mathfrak{p}\mathfrak{L}(A)$  is the ultraproduct of the  $\kappa_w[X]$ , so that by [Theorem 2.2](#), the extension  $\mathfrak{m}(\mathfrak{L}(A)/\mathfrak{p}\mathfrak{L}(A))$  is prime, whence so is  $\mathfrak{m}\mathfrak{L}(A)$ .  $\square$

In particular, if  $\mathfrak{m}_w$  is an  $\mathfrak{D}$ -approximation of  $\mathfrak{m}$ , then almost all  $\mathfrak{m}_w$  are prime ideals. Therefore, the following notions are well-defined (with the convention that we put  $B_n$  equal to zero whenever  $n$  is not a prime ideal of the ring  $B$ ). Let  $R$  be a local  $\mathfrak{D}$ -affine algebra, say, of the form  $C_{\mathfrak{m}}$ , with  $C$  an  $\mathfrak{D}$ -affine algebra and  $\mathfrak{m}$  a prime ideal containing  $\mathfrak{p}$ .

**Definition 3.3.** We call  $\mathfrak{L}(C)_{\mathfrak{m}\mathfrak{L}(C)}$  the *nonstandard  $\mathfrak{D}$ -hull* of  $R$  and denote it  $\mathfrak{L}_{\mathfrak{D}}(R)$  or simply  $\mathfrak{L}(R)$ . Moreover, if  $C_w$  and  $\mathfrak{m}_w$  are  $\mathfrak{D}$ -approximations of  $C$  and  $\mathfrak{m}$  respectively, then the collection  $R_w := (C_w)_{\mathfrak{m}_w}$  is an  *$\mathfrak{D}$ -approximation* of  $R$ .

One easily checks that the ultraproduct of the  $\mathfrak{D}$ -approximations  $R_w$  is precisely the nonstandard  $\mathfrak{D}$ -hull  $\mathfrak{L}(R)$ .

#### 4. Flatness of nonstandard $\mathfrak{D}$ -hulls

In this section, we specialize the notions from the previous result to the situation where each  $\mathfrak{D}_w$  is a discrete valuation ring. We fix throughout the following notation. For each  $w$ , let  $\mathfrak{D}_w$  be a discrete valuation ring with uniformizing parameter  $\pi_w$  and with residue field  $\kappa_w$ . Let  $\mathfrak{D}$ ,  $\pi$  and  $\kappa$  be their respective ultraproducts, so that  $\pi\mathfrak{D}$  is the maximal ideal of  $\mathfrak{D}$  and  $\kappa$  its residue field. We call any ring of this form an *ultra-DVR*. The intersection of all  $\pi^m\mathfrak{D}$  is called the *ideal of infinitesimals* of  $\mathfrak{D}$  and is denoted  $\mathfrak{I}_{\mathfrak{D}}$ . Using [[Schoutens 1999](#)], one sees that  $\mathfrak{D}/\pi^m\mathfrak{D}$  is an artinian local Gorenstein  $\kappa$ -algebra of length  $m$ .

Fix a finite tuple of indeterminates  $X$  and let  $A := \mathfrak{D}[X]$ . As before, we denote the nonstandard  $\mathfrak{D}$ -hull of  $A$  by  $\mathfrak{L}(A)$ ; recall that it is given as the ultraproduct of the  $\mathfrak{D}$ -approximations  $A_w := \mathfrak{D}_w[X]$ .

**Proposition 4.1.** *For  $I$  an ideal in  $A$ , the residue ring  $A/I$  is noetherian if and only if  $\mathfrak{I}_{\mathfrak{D}} \subseteq I$ . In particular, every maximal ideal of  $A$  contains  $\mathfrak{I}_{\mathfrak{D}}$  and is of the form  $\mathfrak{I}_{\mathfrak{D}}A + J$  with  $J$  a finitely generated ideal.*

*Proof.* Let  $C := A/I$  for some ideal  $I$  of  $A$ . If  $C$  is noetherian, then the intersection of all  $\pi^n C$  is zero by Krull’s Intersection Theorem. Hence  $\mathfrak{J}_\mathfrak{D} \subseteq I$ . Conversely, if  $\mathfrak{J}_\mathfrak{D} \subseteq I$ , then since  $A/\mathfrak{J}_\mathfrak{D}A = (\mathfrak{D}/\mathfrak{J}_\mathfrak{D})[X]$  is noetherian, so is  $C$ . The last assertion is now clear.  $\square$

In spite of [Lemma 3.2](#), there are even maximal ideals of  $A$  (necessarily not containing  $\pi$ ) which do not extend to a proper ideal in  $\mathfrak{L}(A)$ . For instance with  $X$  a single indeterminate and  $W = \mathbb{N}$ , the ideal  $\mathfrak{J}_\mathfrak{D}A + (1 - \pi X)A$  is maximal (with residue field the field of fractions of  $\mathfrak{D}/\mathfrak{J}_\mathfrak{D}$ ), but  $\mathfrak{J}_\mathfrak{D}\mathfrak{L}(A) + (1 - \pi X)\mathfrak{L}(A)$  is the unit ideal. Indeed, let  $f_\infty$  be the ultraproduct of the

$$f_w := (1 - (\pi_w X)^w)/(1 - \pi_w X).$$

Since  $(1 - \pi_w X)f_w \equiv 1$  modulo  $(\pi_w)^w A_w$ , we get by Łos’ Theorem that  $(1 - \pi X)f_\infty \equiv 1$  modulo  $\mathfrak{J}_\mathfrak{D}\mathfrak{L}(A)$ . Therefore, we cannot hope for  $A \rightarrow \mathfrak{L}(A)$  to be faithfully flat. Nonetheless, using for instance a result of Aschenbrenner on bounds of syzygies, we do have this property for local affine algebras. This result will prove to be crucial in what follows.

**Theorem 4.2.** *The canonical homomorphism  $A \rightarrow \mathfrak{L}(A)$  is flat. In particular, the canonical homomorphism of a local  $\mathfrak{D}$ -affine algebra to its nonstandard  $\mathfrak{D}$ -hull is faithfully flat, whence in particular injective.*

*Proof.* The last assertion is clear from the first, since the homomorphism  $R \rightarrow \mathfrak{L}(R)$  is obtained as a base change of  $A \rightarrow \mathfrak{L}(A)$  followed by a suitable localization, for any local  $\mathfrak{D}$ -affine algebra  $R$ . I will provide two different proofs for the first assertion

For the first proof, we use a result of Aschenbrenner [[Aschenbrenner 2001a](#)] in order to verify the equational criterion for flatness, that is to say, given a linear equation  $L = 0$ , with  $L$  a linear form over  $A$ , and given a solution  $\mathbf{f}_\infty$  over  $\mathfrak{L}(A)$ , we need to show that there exist solutions  $\mathbf{b}_i$  in  $A$  such that  $\mathbf{f}_\infty$  is an  $\mathfrak{L}(A)$ -linear combination of the  $\mathbf{b}_i$ . Choose  $L_w$  and  $\mathbf{f}_w$  with respective ultraproducts  $L$  and  $\mathbf{f}_\infty$ . In particular, almost all  $L_w$  have  $\mathfrak{D}_w$ -complexity at most  $c$ , for some  $c$  independent from  $w$ . By Łos’ Theorem,  $\mathbf{f}_w$  is a solution of the linear equation  $L_w = 0$ , for almost all  $w$ . Therefore, by [[Aschenbrenner 2001a](#), Corollary 4.27], there is a bound  $c'$ , only depending on  $c$ , such that  $\mathbf{f}_w$  is an  $A_w$ -linear combination of solutions  $\mathbf{b}_{1w}, \dots, \mathbf{b}_{sw}$  of  $\mathfrak{D}_w$ -complexity at most  $c$ . Note that  $s$  can be chosen independent from  $w$  as well by [[Schoutens 2007](#), Lemma 1]. In particular, the ultraproduct  $\mathbf{b}_i$  of the  $\mathbf{b}_{iw}$  lies in  $A$  by [Lemma 2.1](#). By Łos’ Theorem, each  $\mathbf{b}_i$  is a solution of  $L = 0$  in  $\mathfrak{L}(A)$ , whence in  $A$ , and  $\mathbf{f}_\infty$  is an  $\mathfrak{L}(A)$ -linear combination of the  $\mathbf{b}_i$ , proving flatness.

If we want to avoid the use of Aschenbrenner’s result, we can reason as follows. By [Theorem 2.2](#), both extensions  $A/\pi A \rightarrow \mathfrak{L}(A)/\pi \mathfrak{L}(A)$  and  $A \otimes Q \rightarrow \mathfrak{L}(A) \otimes Q$

are faithfully flat, where  $Q$  is the field of fractions of  $\mathfrak{D}$ . Let  $M$  be an  $A$ -module. Since  $\pi$  is  $A$ -regular, the standard spectral sequence

$$\mathrm{Tor}_p^{A/\pi A}(\mathfrak{L}(A)/\pi\mathfrak{L}(A), \mathrm{Tor}_q^A(M, A/\pi A)) \implies \mathrm{Tor}_{p+q}^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M)$$

degenerates into short exact sequences

$$\mathrm{Tor}_{i-1}^{A/\pi A}(\mathfrak{L}(A)/\pi\mathfrak{L}(A), (0 :_M \pi)) \rightarrow \mathrm{Tor}_i^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M) \rightarrow \mathrm{Tor}_i^{A/\pi A}(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M/\pi M),$$

for all  $i \geq 2$ . For  $i = 2$ , since  $A/\pi A \rightarrow \mathfrak{L}(A)/\pi\mathfrak{L}(A)$  is flat, the middle module  $\mathrm{Tor}_2^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M)$  vanishes. Applying this to the short exact sequence

$$0 \rightarrow \mathfrak{L}(A) \xrightarrow{\pi} \mathfrak{L}(A) \rightarrow \mathfrak{L}(A)/\pi\mathfrak{L}(A) \rightarrow 0$$

we get a short exact sequence

$$(4) \quad 0 = \mathrm{Tor}_2^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M) \rightarrow \mathrm{Tor}_1^A(\mathfrak{L}(A), M) \xrightarrow{\pi} \mathrm{Tor}_1^A(\mathfrak{L}(A), M).$$

On the other hand, flatness of  $A \otimes Q \rightarrow \mathfrak{L}(A) \otimes Q$  yields

$$(5) \quad \mathrm{Tor}_1^A(\mathfrak{L}(A), M) \otimes Q = \mathrm{Tor}_1^{A \otimes Q}(\mathfrak{L}(A) \otimes Q, M \otimes Q) = 0.$$

In order to prove that  $A \rightarrow \mathfrak{L}(A)$  is flat, it suffices by [Matsumura 1986, Theorem 7.8] to show that  $\mathrm{Tor}_1^A(\mathfrak{L}(A), A/I)$  vanishes, for every finitely generated ideal  $I$  of  $A$ . Towards a contradiction, suppose that  $\mathrm{Tor}_1^A(\mathfrak{L}(A), A/I)$  contains a nonzero element  $\tau$ . By (5), we have  $a\tau = 0$ , for some nonzero  $a \in \mathfrak{D}$ . As observed in [Sabbagh 1974, Proposition 3], every polynomial ring over a valuation ring is coherent, so that in particular  $I$  is finitely presented (namely, since  $I$  is torsion-free over  $\mathfrak{D}$ , it is  $\mathfrak{D}$ -flat, and therefore finitely presented by [Raynaud and Gruson 1971, Theorem 3.4.6]). Hence we have some exact sequence

$$A^{a_2} \xrightarrow{\varphi_2} A^{a_1} \xrightarrow{\varphi_1} A \rightarrow A/I \rightarrow 0.$$

Therefore  $\mathrm{Tor}_1^A(\mathfrak{L}(A), A/I)$  is calculated as the homology of the complex

$$\mathfrak{L}(A)^{a_2} \xrightarrow{\varphi_2} \mathfrak{L}(A)^{a_1} \xrightarrow{\varphi_1} \mathfrak{L}(A).$$

Suppose  $\tau$  is the image of a tuple  $\mathbf{x} \in \mathfrak{L}(A)^{a_1}$  with  $\varphi_1(\mathbf{x}) = 0$ . Hence  $\mathbf{x}$  does not belong to  $\varphi_2(\mathfrak{L}(A)^{a_2})$  but  $a\mathbf{x}$  does. Choose  $\mathbf{x}_w, a_w$  and  $\varphi_{i_w}$  with respective ultraproduct  $\mathbf{x}, a$  and  $\varphi_i$ . By Łoś' Theorem, almost all  $\mathbf{x}_w$  lie in the kernel of  $\varphi_{1_w}$  but not in the image of  $\varphi_{2_w}$ , yet  $a_w\mathbf{x}_w$  lies in the image of  $\varphi_{2_w}$ . Choose  $n_w \in \mathbb{N}$  maximal such that  $\mathbf{y}_w := (\pi_w)^{n_w}\mathbf{x}_w$  does not lie in the image of  $\varphi_{2_w}$ . Since almost all  $a_w$  are nonzero, this maximum exists for almost all  $w$ . Therefore, if  $\mathbf{y}$  is the ultraproduct of the  $\mathbf{y}_w$ , then  $\varphi_1(\mathbf{y}) = 0$  and  $\mathbf{y}$  does not lie in  $\varphi_2(\mathfrak{L}(A)^{a_2})$ , but  $\pi\mathbf{y}$

lies in  $\varphi_2(\mathcal{L}(A)^{a_2})$ . Therefore, the image of  $y$  in  $\text{Tor}_1^A(\mathcal{L}(A), A/I)$  is a nonzero element annihilated by  $\pi$ , contradicting (4).  $\square$

**Remark 4.3.** In [Schoutens  $\geq$  2007], I exhibit a general connection between the flatness of an ultraproduct over certain canonical subrings and the existence of bounds on syzygies. In particular, using these ideas, the second argument in the above proof of flatness reproves the result in [Aschenbrenner 2001a]. In fact, the role played here by coherence is not accidental either; see [Aschenbrenner 2001b] or [Schoutens  $\geq$  2007] for more details.

**Theorem 4.4.** *Let  $R$  be a local  $\mathfrak{D}$ -affine algebra with nonstandard  $\mathfrak{D}$ -hull  $\mathcal{L}(R)$  and  $\mathfrak{D}$ -approximation  $R_w$ .*

- *Almost all  $R_w$  are flat over  $\mathfrak{D}_w$  if and only if  $R$  is torsion-free over  $\mathfrak{D}$  if and only if  $\pi$  is  $R$ -regular.*
- *Almost all  $R_w$  are domains if and only if  $R$  is.*

*Proof.* Suppose first that almost all  $R_w$  are flat over  $\mathfrak{D}_w$ , which amounts in this case, to almost all  $R_w$  being torsion-free over  $\mathfrak{D}_w$ . By Łos’ Theorem,  $\mathcal{L}(R)$  is torsion-free over  $\mathfrak{D}$ , and since  $R \subseteq \mathcal{L}(R)$ , so is  $R$ . Conversely, assume  $\pi$  is  $R$ -regular. By faithful flatness,  $\pi$  is  $\mathcal{L}(R)$ -regular, whence almost all  $\pi_w$  are  $R_w$ -regular by Łos’ Theorem. Since the  $\mathfrak{D}_w$  are discrete valuation rings, this means that almost all  $\mathfrak{D}_w \rightarrow R_w$  are flat.

If almost all  $R_w$  are domains, then so is  $\mathcal{L}(R)$  by Łos’ Theorem, and hence so is  $R$ , since it embeds in  $\mathcal{L}(R)$ . Conversely, assume  $R$  is a domain. If  $\pi = 0$  in  $R$ , then  $\mathcal{L}(R)$  is a domain by Lemma 3.2, whence so are almost all  $R_w$  by Łos’ Theorem. So assume  $\pi$  is nonzero in  $R$ , whence  $R$ -regular. By what we just proved,  $R$  is then torsion-free over  $\mathfrak{D}$ . Let  $Q$  be the field of fractions of  $\mathfrak{D}$ . Write  $R$  in the form  $S/\mathfrak{p}$ , where  $S$  is some localization of  $A$  at a prime ideal containing  $\pi$  and  $\mathfrak{p}$  is a finitely generated prime ideal in  $S$ . Since  $S/\mathfrak{p}$  is torsion-free over  $\mathfrak{D}$ , the extension  $\mathfrak{p}(S \otimes_{\mathfrak{D}} Q)$  is again prime and its contraction in  $S$  is  $\mathfrak{p}$ . By Theorem 2.2, since we are now over a field,  $\mathfrak{p}(\mathcal{L}(S) \otimes_{\mathfrak{D}} Q)$  is a prime ideal, where  $\mathcal{L}(S)$  is the nonstandard  $\mathfrak{D}$ -hull of  $S$  (note that  $\mathcal{L}(S) \otimes_{\mathfrak{D}} Q$  is then the nonstandard hull of  $S \otimes_{\mathfrak{D}} Q$  in the sense of [Schoutens 2003d]). Moreover, since  $S/\mathfrak{p}$  is torsion-free over  $\mathfrak{D}$ , so is  $\mathcal{L}(S)/\mathfrak{p}\mathcal{L}(S)$  by the first assertion. This in turn means that

$$\mathfrak{p}\mathcal{L}(S) = \mathfrak{p}(\mathcal{L}(S) \otimes_{\mathfrak{D}} Q) \cap \mathcal{L}(S),$$

showing that  $\mathfrak{p}\mathcal{L}(S)$  is prime. It follows then from Łos’ Theorem that almost all  $\mathfrak{p}_w$  are prime, where  $\mathfrak{p}_w$  is an  $\mathfrak{D}$ -approximation of  $\mathfrak{p}$ , and hence almost all  $R_w$  are domains.  $\square$

**Remark 4.5.** The last assertion is equivalent with saying that any prime ideal in  $R$  extends to a prime ideal in  $\mathfrak{L}(R)$ . Indeed, let  $\mathfrak{q}$  be a prime ideal in  $R$  with  $\mathfrak{D}$ -approximation  $\mathfrak{q}_w$ . By the preceding result (applied to  $R/\mathfrak{q}$  and its  $\mathfrak{D}$ -approximation  $R_w/\mathfrak{q}_w$ ), we see that almost all  $\mathfrak{q}_w$  are prime, whence so is their ultraproduct  $\mathfrak{q}\mathfrak{L}(R)$ , by Łos’ Theorem.

### 5. Geometric dimension

In this and the next section, we will study the local algebra of the category  $\text{Aff}(\mathfrak{D})$ . Although part of the theory can be developed for arbitrary base rings  $\mathfrak{D}$ , or even for arbitrary local rings of finite embedding dimension (see [Schoutens 2004b]), we will only deal with the case that  $\mathfrak{D}$  is a local domain of embedding dimension one. Recall that the *embedding dimension* of a local ring  $(Z, \mathfrak{p})$  is by definition the minimal number of generators of  $\mathfrak{p}$ , and its *ideal of infinitesimals*  $\mathfrak{I}_Z$  is the intersection of all powers  $\mathfrak{p}^n$ . Of course, if  $Z$  is moreover noetherian, then its ideal of infinitesimals is zero. In general, we call  $\tilde{Z} := Z/\mathfrak{I}_Z$  the *separated quotient* of  $Z$ .

For the duration of the next two sections, let  $\mathfrak{D}$  denote a local domain of embedding dimension one, with generator of the maximal ideal  $\pi$ , with ideal of infinitesimals  $\mathfrak{I}_{\mathfrak{D}}$  and with residue field  $\kappa$ . We will work in the category  $\text{Aff}(\mathfrak{D})$  of local  $\mathfrak{D}$ -affine algebras, that is to say, the category of algebras of the form  $R := (A/I)_{\mathfrak{m}}$ , where as before  $A := \mathfrak{D}[X]$  for some finite tuple of indeterminates  $X$ , where  $I$  is a finitely generated ideal in  $A$  and where  $\mathfrak{m}$  is a prime ideal containing  $\pi$  and  $I$ . Nonetheless, some results can be stated even for local algebras which are locally finitely generated over  $\mathfrak{D}$ , that is, without the assumption that  $I$  is finitely generated. We call  $R$  a *torsion-free  $\mathfrak{D}$ -algebra* if it is torsion-free over  $\mathfrak{D}$  (that is to say, if  $ar = 0$  for some  $r \in R$  and some nonzero  $a \in \mathfrak{D}$ , then  $r = 0$ ). Recall from [Theorem 4.4](#) that a local  $\mathfrak{D}$ -affine algebra  $R$  is torsion-free if and only if  $\pi$  is  $R$ -regular.

**Lemma 5.1.** *The separated quotient  $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$  of  $\mathfrak{D}$  is a discrete valuation ring with uniformizing parameter  $\pi$ .*

*Proof.* For each element  $a \in \mathfrak{D}$  outside  $\mathfrak{I}_{\mathfrak{D}}$ , there is a smallest  $e \in \mathbb{N}$  for which  $a \notin \pi^{e+1}\mathfrak{D}$ . Hence  $a = u\pi^e$  with  $u$  a unit in  $\mathfrak{D}$ . It is now straightforward to check that the assignment  $a \mapsto e$  induces a discrete valuation on  $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$ . □

Note that we do not even need  $\mathfrak{D}$  to be domain, having positive depth (that is to say, assuming that  $\pi\mathfrak{D}$  is not an associated prime of  $\mathfrak{D}$ ; see [Bruns and Herzog 1993, Proposition 9.1.4]) would suffice, for then  $\pi$  is necessarily  $\mathfrak{D}$ -regular. However, we do not need this amount of generality as in all our applications  $\mathfrak{D}$  will be an ultra-DVR, that is to say, an ultraproduct of discrete valuation rings  $\mathfrak{D}_w$ . If we are in this situation, then as before, we let  $A_w := \mathfrak{D}_w[X]$  and we let  $\mathfrak{L}(A)$  be their ultraproduct. Moreover, for  $R = (A/I)_{\mathfrak{m}}$  as above, we let  $\mathfrak{L}(R) := (\mathfrak{L}(A)/I\mathfrak{L}(A))_{\mathfrak{m}\mathfrak{L}(A)}$



be its nonstandard  $\mathfrak{D}$ -hull and we let  $R_w := (A_w/I_w)_{\mathfrak{m}_w}$  be an  $\mathfrak{D}$ -approximation of  $R$ , where  $I_w$  and  $\mathfrak{m}_w$  are  $\mathfrak{D}$ -approximations of  $I$  and  $\mathfrak{m}$  respectively. Note that  $\mathfrak{m}$  is finitely generated, as it contains by definition  $\pi$ .

**Lemma 5.2.** *Let  $(R, \mathfrak{m})$  be a local ring which is locally finitely generated over  $\mathfrak{D}$ . If  $I$  is a proper ideal in  $R$  containing some power  $\pi^m$ , then the intersection of all  $I^n$  for  $n \in \mathbb{N}$  is equal to  $\mathfrak{J}_{\mathfrak{D}}R$ . In particular,  $\mathfrak{J}_R = \mathfrak{J}_{\mathfrak{D}}R$  and the separated quotient of  $R$  is equal to  $\tilde{R} := R/\mathfrak{J}_{\mathfrak{D}}R$  whence is noetherian.*

*Proof.* Suppose  $\pi^m \in I \subseteq \mathfrak{m}$ . Let  $J$  be the intersection of all  $I^n$  for  $n \in \mathbb{N}$ . Since  $\pi^m \in I$ , we get  $\mathfrak{J}_{\mathfrak{D}}R \subseteq J$ . Since  $\tilde{R}$  is locally finitely generated over the discrete valuation ring  $\mathfrak{D}/\mathfrak{J}_{\mathfrak{D}}$  (see Lemma 5.1), it is noetherian. Applying Krull’s Intersection Theorem (see for instance [Matsumura 1986, Theorem 8.10]), we get  $J\tilde{R} = (0)$ , and hence that  $J = \mathfrak{J}_{\mathfrak{D}}R$ . The last assertion follows by letting  $I := \mathfrak{m}$ .  $\square$

**Lemma 5.3.** *Let  $\mathfrak{D}$  be an ultra-DVR. A local  $\mathfrak{D}$ -affine algebra  $(R, \mathfrak{m})$  has the same  $\mathfrak{m}$ -adic completion as its separated quotient, and this is also isomorphic to  $\mathfrak{L}(R)/\mathfrak{J}_{\mathfrak{L}(R)}$ . In particular, the completion is noetherian.*

*Proof.* Let  $\tilde{R} := R/\mathfrak{J}_R$  be the separated quotient. For every  $n$ , we have

$$R/\mathfrak{m}^n \cong \tilde{R}/\mathfrak{m}^n \tilde{R} \cong \mathfrak{L}(R)/\mathfrak{m}^n \mathfrak{L}(R),$$

where the second isomorphism follows from the fact that length is a first order invariant (see for instance [Schoutens 1999]). Hence  $R$ ,  $\tilde{R}$  and  $\mathfrak{L}(R)$  have the same completion  $\widehat{R}$ . Noetherianness now follows from Lemma 5.2. By saturatedness of ultraproducts (with respect to a countably incomplete nonprincipal ultrafilter),  $\mathfrak{L}(R)$  is quasicomplete in the sense that every Cauchy sequence has a (nonunique) limit. Therefore, its separated quotient  $\mathfrak{L}(R)/\mathfrak{J}_{\mathfrak{L}(R)}$  is complete, whence equal to  $\widehat{R}$ . For a more detailed proof, see [Schoutens 2004b, Lemma 5.2].  $\square$

Our first goal is to introduce a good notion of dimension. Below, the *dimension* of a ring will always mean its *Krull dimension*. Recall that it is always finite for noetherian local rings.

**Theorem 5.4.** *For a local ring  $(R, \mathfrak{m})$  which is locally finitely generated over  $\mathfrak{D}$ , the following numbers are all equal:*

- the least possible length  $d$  of a tuple in  $R$  generating some  $\mathfrak{m}$ -primary ideal;
- the dimension  $\widehat{d}$  of the completion  $\widehat{R}$ ;
- the dimension  $\tilde{d}$  of the separated quotient  $\tilde{R} := R/\mathfrak{J}_{\mathfrak{D}}R$ ;
- the degree  $\underline{d}$  of the Hilbert–Samuel polynomial  $\chi_R$ , which is defined as the unique polynomial with rational coefficients for which  $\chi_R(n)$  equals the length of  $R/\mathfrak{m}^{n+1}$  for all large  $n$ .

If  $\pi$  is  $R$ -regular, then  $R/\pi R$  has dimension  $d - 1$ .

If, moreover,  $\mathfrak{D}$  is an ultra-DVR and  $R$  a torsion-free local  $\mathfrak{D}$ -affine algebra with  $\mathfrak{D}$ -approximation  $R_w$ , then almost all  $R_w$  have dimension  $d$ .

*Proof.* By Lemmas 5.2 and 5.3, the separated quotient  $\tilde{R}$  is noetherian, with completion equal to  $\widehat{R}$ . Hence  $\widehat{d} = \tilde{d}$ . Moreover,  $\chi_R = \chi_{\widehat{R}}$ , so that by the Hilbert–Samuel theory,  $\underline{d} = \widehat{d}$ .

Let  $\mathbf{x}$  be a tuple of length  $\tilde{d}$  such that its image in  $\tilde{R}$  is a system of parameters of  $\tilde{R}$ . Hence, for some  $n$ , we have that  $\mathfrak{m}^n \subseteq \mathbf{x}R + \mathfrak{I}_{\mathfrak{D}}R$ . In particular, since  $\mathfrak{I}_{\mathfrak{D}}R \subseteq \pi^{n+1}R$ , we can find  $x \in \mathbf{x}R$  and  $r \in R$ , such that  $\pi^n = x + r\pi^{n+1}$ . Therefore,  $\pi^n \in \mathbf{x}R$ , since  $1 - r\pi$  is a unit. Since  $\mathfrak{I}_{\mathfrak{D}} \subseteq \pi^n \mathfrak{D}$ , we get  $\mathfrak{m}^n \subseteq \mathbf{x}R$ , showing that  $\mathbf{x}R$  is an  $\mathfrak{m}$ -primary ideal and hence that  $d \leq \tilde{d}$ . On the other hand, if  $\mathbf{y}$  is a tuple of length  $d$  such that  $\mathbf{y}R$  is  $\mathfrak{m}$ -primary, then  $\mathbf{y}\tilde{R}$  is an  $\mathfrak{m}\tilde{R}$ -primary ideal, and hence  $\tilde{d} \leq d$ . This concludes the proof of the first assertion.

Assume that  $\pi$  is moreover  $R$ -regular. I claim that  $\pi$  is  $\tilde{R}$ -regular. Indeed, suppose  $\pi\tilde{r} = 0$ , for some  $\tilde{r} \in \tilde{R}$ . Take a preimage  $r \in R$ , so that  $\pi r \in \mathfrak{I}_{\mathfrak{D}}R \subseteq \pi^n R$ , for every  $n$ . Since  $\pi$  is  $R$ -regular, we get  $r \in \pi^{n-1}R$ , for all  $n$ . Therefore,  $r \in \mathfrak{I}_{\mathfrak{D}}R$  by Lemma 5.2, whence  $\tilde{r} = 0$  in  $\tilde{R}$ , as we needed to show. Since  $\pi$  is  $\tilde{R}$ -regular and  $\tilde{R}/\pi\tilde{R} = R/\pi R$ , the dimension of  $R/\pi R$  is  $\tilde{d} - 1$ .

Suppose finally that  $\mathfrak{D}$  is moreover an ultra-DVR. We already observed that  $R_w/\pi_w R_w$  is an approximation of  $R/\pi R$  in the sense of [Schoutens 2003d]. In particular, by [Schoutens 2003d, Theorem 4.5], almost all  $R_w/\pi_w R_w$  have dimension  $\tilde{d} - 1$ . Since  $\pi$  is  $\mathfrak{L}(R)$ -regular by flatness, whence  $\pi_w$  is  $R_w$ -regular by Łos’ Theorem, we get that  $R_w$  has dimension  $\tilde{d}$ , for almost all  $w$ . □

**5.5. Geometric dimension.** The common value given by the theorem is called the *geometric dimension* of  $R$ . We call a tuple  $\mathbf{x}$  in  $R$  *generic*, if it generates an  $\mathfrak{m}$ -primary ideal and has length equal to the geometric dimension of  $R$ . Note that if  $(x_1, \dots, x_d)$  is a generic sequence, then  $R/(x_1, \dots, x_d)R$  has geometric dimension  $d - e$ .

**Corollary 5.6.** *In a local ring  $(R, \mathfrak{m})$  which is locally finitely generated over  $\mathfrak{D}$ , every  $\mathfrak{m}$ -primary ideal contains a generic sequence.*

*Proof.* Let  $\tilde{R} := R/\mathfrak{I}_{\mathfrak{D}}R$  and let  $d$  be the geometric dimension of  $R$ . Let  $\mathfrak{n}$  be an  $\mathfrak{m}$ -primary ideal of  $R$ . Since  $\mathfrak{n}\tilde{R}$  is  $\mathfrak{m}\tilde{R}$ -primary and  $\tilde{R}$  is noetherian, we can find a tuple  $\mathbf{y}$  with entries in  $\mathfrak{n}$  so that its image in  $\tilde{R}$  is a system of parameters. In particular,  $\mathbf{y}$  has length  $d$  by Theorem 5.4. Let  $S := R/\mathbf{y}R$  and  $\tilde{S} := S/\mathfrak{I}_{\mathfrak{D}}S$ . By Theorem 5.4, the geometric dimension of  $S$  is equal to the dimension of  $\tilde{S}$ , whence is zero since  $\tilde{S} = \tilde{R}/\mathbf{y}\tilde{R}$ . In particular,  $\mathbf{y}R$  is  $\mathfrak{m}$ -primary. Since  $\mathbf{y}$  has length equal to the geometric dimension of  $R$ , it is therefore a generic sequence. □

In fact the above proof shows that there is a one-one correspondence between generic sequences in  $R$  and systems of parameters in  $R/\mathfrak{J}_\mathfrak{D}R$ . In general, the last assertion in [Theorem 5.4](#) is false when  $R$  is not torsion-free. For instance, let  $R := \mathfrak{D}/a\mathfrak{D}$  with  $a$  a nonzero infinitesimal, so that each  $R_w = \mathfrak{D}_w/a_w\mathfrak{D}_w$  has dimension zero, but  $R/\mathfrak{J}_R$  is the (one-dimensional) discrete valuation ring  $\mathfrak{D}/\mathfrak{J}_\mathfrak{D}$ .

In the following definition, let  $\mathfrak{D}$  be an ultra-DVR and let  $R$  be a local  $\mathfrak{D}$ -affine algebra of geometric dimension  $d$ , with  $\mathfrak{D}$ -approximation  $R_w$ . Note that the  $R_w$  have almost all dimension at most  $d$ . Indeed, if  $\mathbf{y}$  has length  $d$  and generates an  $\mathfrak{m}$ -primary ideal, then almost all  $\mathbf{y}_w$  are  $\mathfrak{m}_w$ -primary by Łos' Theorem, for  $\mathbf{y}_w$  an  $\mathfrak{D}$ -approximation of  $\mathbf{y}$ .

**Definition 5.7.** We say that  $R$  is *isodimensional* if almost all  $R_w$  have dimension equal to the geometric dimension of  $R$ .

[Theorem 5.4](#) shows that every torsion-free local  $\mathfrak{D}$ -affine algebra is isodimensional. In particular, over an ultra-DVR, the protoproduct  $R$  of domains  $R_w$  of uniformly bounded  $\mathfrak{D}_w$ -complexity is isodimensional, since  $\mathfrak{L}(R)$  is then a domain by Łos' Theorem, whence so is  $R$  as it embeds in  $\mathfrak{L}(R)$ . The next result shows that generic sequences in an isodimensional ring are the analog of systems of parameters.

**Corollary 5.8.** *Let  $\mathfrak{D}$  be an ultra-DVR and  $R$  an isodimensional local  $\mathfrak{D}$ -affine algebra with  $\mathfrak{D}$ -approximation  $R_w$ . Let  $\mathbf{x}$  be a tuple in  $R$  with  $\mathfrak{D}$ -approximation  $\mathbf{x}_w$ .*

*If  $\mathbf{x}$  is generic, then  $\mathbf{x}_w$  is a system of parameters of  $R_w$ , for almost all  $w$ . Conversely, if  $(\pi_w)^c \in \mathbf{x}_w R_w$ , for some  $c$  and almost all  $w$ , then  $\mathbf{x}$  is generic.*

*Proof.* Let  $\mathfrak{m}$  be the maximal ideal of  $R$ , with  $\mathfrak{D}$ -approximation  $\mathfrak{m}_w$ . Let  $d$  be the geometric dimension of  $R$ , so that almost all  $R_w$  have dimension  $d$ . Suppose first that  $\mathbf{x}$  is generic, so that  $|\mathbf{x}| = d$  and  $\mathbf{x}R$  is  $\mathfrak{m}$ -primary. Since  $\mathbf{x}\mathfrak{L}(R)$  is then  $\mathfrak{m}\mathfrak{L}(R)$ -primary,  $\mathbf{x}_w R_w$  is  $\mathfrak{m}_w$ -primary by Łos' Theorem, showing that  $\mathbf{x}_w$  is a system of parameters for almost all  $w$ .

Conversely, suppose  $\mathbf{x}_w$  is a system of parameters of  $R_w$ , generating an ideal containing  $(\pi_w)^c$ . By Łos' Theorem and faithful flatness,  $\pi^c \in \mathbf{x}R$ . Applying [[Schoutens 2007](#), Corollary 4] to the artinian base ring  $\mathfrak{D}_w/(\pi_w)^c$ , we can find a bound  $c'$ , only depending on  $c$ , such that  $(\mathfrak{m}_w)^{c'} \subseteq \mathbf{x}_w R_w$ , for almost all  $w$ . Hence  $\mathfrak{m}^{c'}\mathfrak{L}(R) \subseteq \mathbf{x}\mathfrak{L}(R)$ , so that by faithful flatness,  $\mathbf{x}R$  is  $\mathfrak{m}$ -primary. This shows that  $\mathbf{x}$  is generic. □

The additional requirement in the converse is necessary: indeed, for arbitrary  $n_w > 0$ , the element  $(\pi_w)^{n_w}$  is a parameter in  $\mathfrak{D}_w$  and has  $\mathfrak{D}_w$ -complexity zero, but if  $n_w$  is unbounded, its ultraproduct is an infinitesimal whence not generic. To characterize isodimensional rings, we use the following notion introduced in [[Schoutens 2006](#)].

**Definition 5.9** (Parameter degree). The *parameter degree* of a noetherian local ring  $C$  is by definition the smallest possible length of a residue ring  $C/\mathbf{x}C$ , where  $\mathbf{x}$  runs over all systems of parameters of  $C$ .

In general, the parameter degree is larger than the multiplicity, with equality precisely when  $C$  is Cohen–Macaulay, provided the residue field is infinite (see [Matsumura 1986, Theorem 17.11]). The homological degree of  $C$  is an upper bound for its parameter degree (see [Schoutens 2006, Corollary 4.6]). A priori, being isodimensional is a property of the  $\mathfrak{D}$ -approximations of  $R$ , of for that matter, of its nonstandard  $\mathfrak{D}$ -hull. However, the last equivalent condition in the next result shows that it is in fact an intrinsic property.

**Proposition 5.10.** *Let  $\mathfrak{D}$  be an ultra-DVR and let  $R$  be a local  $\mathfrak{D}$ -affine algebra with  $\mathfrak{D}$ -approximation  $R_w$ . The following are equivalent:*

- (i)  $R$  is isodimensional;
- (ii) there exists a  $c \in \mathbb{N}$ , such that for almost all  $w$ , we can find a system of parameters  $\mathbf{x}_w$  of  $R_w$  of  $\mathfrak{D}_w$ -complexity at most  $c$ , generating an ideal containing  $(\pi_w)^c$ ;
- (iii) there exists an  $e \in \mathbb{N}$ , such that almost all  $R_w$  have parameter degree at most  $e$ ;
- (iv) for every generic sequence in  $R$  of the form  $(\pi, \mathbf{y})$ , the contracted ideal  $\mathbf{y}R \cap \mathfrak{D}$  is zero.

*Proof.* Let  $\mathfrak{m}$  be the maximal ideal of  $R$ , with  $\mathfrak{D}$ -approximation  $\mathfrak{m}_w$ . Let  $d$  be the geometric dimension of  $R$  and let  $d'$  be the dimension of almost all  $R_w$ . Suppose first that  $d = d'$ . Let  $\mathbf{x}$  be any generic sequence in  $R$  with  $\mathfrak{D}$ -approximation  $\mathbf{x}_w$ . By Łos' Theorem, almost all  $\mathbf{x}_w$  generate an  $\mathfrak{m}_w$ -primary ideal. Since their length is equal to the dimension of  $R_w$ , they are almost all systems of parameters of  $R_w$ . Choose  $c$  large enough so that  $\pi^c \in \mathbf{x}R$ . Enlarging  $c$  if necessary, we may moreover assume by Lemma 2.1 that almost all  $\mathbf{x}_w$  have  $\mathfrak{D}_w$ -complexity at most  $c$ . By Łos' Theorem,  $(\pi_w)^c \in \mathbf{x}_w R_w$ , so (ii) holds.

Assume next that  $c$  and the  $\mathbf{x}_w$  are as in (ii). Let  $\overline{R}_w := R_w/(\pi_w)^c R_w$ . We can apply [Schoutens 2007, Corollary 2] over  $\mathfrak{D}_w/(\pi_w)^c \mathfrak{D}_w$  to the  $\mathfrak{m}_w \overline{R}_w$ -primary ideal  $\mathbf{x}_w \overline{R}_w$ , to conclude that there is some  $c'$ , depending only on  $c$ , such that  $\overline{R}_w/\mathbf{x}_w \overline{R}_w$  has length at most  $c'$ . Since the latter residue ring is just  $R_w/\mathbf{x}_w R_w$  by assumption, the parameter degree of  $R_w$  is at most  $c'$ , and hence (iii) holds.

To show that (iii) implies (i), assume that almost all  $R_w$  have parameter degree at most  $e$ . Let  $\mathbf{y}_w$  be a system of parameters of  $R_w$  such that  $R_w/\mathbf{y}_w R_w$  has length at most  $e$ , for almost all  $w$ . It follows that  $(\mathfrak{m}_w)^e$  is contained in  $\mathbf{y}_w R_w$ . Let  $\mathbf{y}_\infty$  be the ultraproduct of the  $\mathbf{y}_w$ . By Łos' Theorem,  $\mathfrak{m}^e \mathfrak{L}(R) \subseteq \mathbf{y}_\infty \mathfrak{L}(R)$  whence  $\mathfrak{m}^e \widehat{R} \subseteq \mathbf{y}_\infty \widehat{R}$ , by Lemma 5.3, showing that  $\mathbf{y}_\infty \widehat{R}$  is  $\mathfrak{m} \widehat{R}$ -primary. Since  $\mathbf{y}_\infty$  has

length at most  $d'$  (some entries might be zero in  $\widehat{R}$ ), the dimension of  $\widehat{R}$  is at most  $d'$ . Since we already remarked that  $d' \leq d$ , we get from [Theorem 4.4](#) that  $d' = d$ .

So remains to show that (iv) is equivalent to the other conditions. Assume first that it holds but that  $R$  is not isodimensional. Since we have inequalities  $d - 1 \leq d' \leq d$ , this means that  $d' = d - 1$ . Moreover,  $R/\pi R$  must have geometric dimension also equal to  $d - 1$ , for if not, its geometric dimension would be  $d$ , whence almost all  $R_w/\pi_w R_w$  would have dimension  $d$  by [[Schoutens 2003d](#), Theorem 4.5], which is impossible. Since there is a uniform bound  $c$  on the  $\mathfrak{D}_w$ -complexity of each  $R_w$ , we can choose, using [Corollary 2.6](#), a system of parameters  $\mathbf{y}_w$  of  $R_w$  of  $\mathfrak{D}_w$ -complexity at most  $c^2$ . In particular, some power of  $\pi_w$  lies in  $\mathbf{y}_w R_w$ . Let  $a \in \mathfrak{D}$  be the ultraproduct of these powers. If  $\mathbf{y}$  is the ultraproduct of the  $\mathbf{y}_w$ , then  $\mathbf{y}$  is already defined over  $R$  by [Lemma 2.1](#). By Łos' Theorem,  $a \in \mathbf{y}\mathfrak{L}(R)$ , whence by faithful flatness,  $a$  is a nonzero element in  $\mathbf{y}R \cap \mathfrak{D}$ . Therefore, to reach the desired contradiction with (iv), we only need to show that  $(\pi, \mathbf{y})$  is generic. As we already established,  $R_w/\pi_w R_w$  has dimension  $d - 1$ , so that  $\mathbf{y}_w$  is also a system of parameters in that ring. Therefore,  $\mathbf{y}$  is a system of parameters in  $R/\pi R$  by [[Schoutens 2003d](#), Theorem 4.5]. This in turn implies that  $(\pi, \mathbf{y})$  generates an  $\mathfrak{m}$ -primary ideal in  $R$ . Since this tuple has length  $d$ , it is therefore generic, as we wanted to show.

Finally, assume  $R$  is isodimensional, and suppose  $(\pi, \mathbf{y})$  is generic. Let  $a \in \mathbf{y}R \cap \mathfrak{D}$  and choose  $\mathfrak{D}$ -approximations  $a_w$  and  $\mathbf{y}_w$  of  $a$  and  $\mathbf{y}$  respectively. By Łos' Theorem,  $a_w \in \mathbf{y}_w R_w$ . However, if  $a$  is nonzero, then  $a_w$  is, up to a unit, a power of  $\pi_w$ , which contradicts the assertion in [Corollary 5.8](#) that  $(\pi_w, \mathbf{y}_w)$  is a system of parameters. So  $a = 0$ , as we needed to show. □

**Corollary 5.11.** *For each  $c$ , there exists a bound  $PD(c)$  with the following property. Let  $V$  be a discrete valuation ring and let  $C$  be a local  $V$ -affine algebra of  $V$ -complexity at most  $c$ . If  $C$  is torsion-free over  $V$ , then the parameter degree of  $C$  is at most  $PD(c)$ .*

*Proof.* If the statement is false for some  $c$ , then we can find for each  $w$  a discrete valuation ring  $\mathfrak{D}_w$  and a torsion-free local  $\mathfrak{D}_w$ -affine algebra  $R_w$  of  $\mathfrak{D}_w$ -complexity at most  $c$ , whose parameter degree is at least  $w$ . Let  $R$  be the protoproduct of the  $R_w$  and let  $\mathfrak{L}(R)$  be their ultraproduct. Since  $\pi_w$  is  $R_w$ -regular,  $\pi$  is  $\mathfrak{L}(R)$ -regular, whence  $R$ -regular. Hence  $R$  is isodimensional by [Theorem 5.4](#). Therefore, there is a bound on the parameter degree of almost all  $R_w$  by [Proposition 5.10](#), contradicting our assumption. □

Our next goal is to introduce a notion similar to height. Let  $I$  be an arbitrary ideal of  $R$ .

**Definition 5.12** (Geometric height). We call the *geometric height* of  $I$  the maximum of all  $h$  such that there exists a generic sequence whose first  $h$  entries belong to  $I$ .

For noetherian rings, we cannot expect a good relationship between the height of an ideal and the dimension of its residue ring, unless the ring is a catenary domain; the following is the analogue over ultra-DVR's.

**Theorem 5.13.** *Let  $\mathfrak{D}$  be an ultra-DVR and let  $R$  be a local  $\mathfrak{D}$ -affine domain with  $\mathfrak{D}$ -approximation  $R_w$ . Let  $I$  be a finitely generated ideal in  $R$  with  $\mathfrak{D}$ -approximation  $I_w$ .*

*If  $R/I$  is isodimensional, then the geometric height of  $I$  is equal to the geometric dimension of  $R$  minus the geometric dimension of  $R/I$ , and this is also equal to the height of almost all  $I_w$ .*

*Proof.* Let  $d$  be the geometric dimension of  $R$  and  $e$  the geometric dimension of  $R/I$ . Since a domain is isodimensional, almost all  $R_w$  have dimension  $d$  by [Theorem 5.4](#), and by assumption, almost all  $R_w/I_w$  have dimension  $e$ . Let  $h$  be the geometric height of  $I$ . Let  $\mathbf{z}$  be a generic sequence in  $R$  with its first  $h$  entries in  $I$ , and let  $\mathbf{z}_w$  be an  $\mathfrak{D}$ -approximation of  $\mathbf{z}$ . By [Corollary 5.8](#), almost all  $\mathbf{z}_w$  are a system of parameters in  $R_w$ . Since by Łos' Theorem the first  $h$  entries of  $\mathbf{z}_w$  lie in  $I_w$ , we get that  $R_w/I_w$  has dimension at most  $d - h$ . In other words,  $h \leq d - e$ . Since almost all  $R_w$  are catenary domains, almost all  $I_w$  have height  $d - e$ .

So remains to show that  $d - e \leq h$ . By [Lemma 5.2](#), the separated quotient of  $R/I$  is equal to  $\tilde{R}/I\tilde{R}$ . Therefore, by the remark following [Corollary 5.6](#), we can find a generic sequence  $(x_1, \dots, x_d)$  in  $R$  such that (the image of)  $(x_1, \dots, x_e)$  is a generic sequence in  $R/I$ . By definition of generic sequence,  $S := R/(x_1, \dots, x_e)R$  has geometric dimension  $d - e$ . If  $x_{iw}$  is an  $\mathfrak{D}$ -approximation of  $x_i$ , then almost each  $\mathbf{x}_w := (x_{1w}, \dots, x_{ew})$  is a system of parameters in  $R_w/I_w$  by [Corollary 5.8](#). Since  $\mathbf{x}_w$  is therefore part of a system of parameters in  $R_w$ , almost each  $S_w := R_w/\mathbf{x}_w R_w$  has dimension  $d - e$  by [[Matsumura 1986](#), Theorem 14.1]. By choice of the  $x_i$ , the ideal  $I + (x_1, \dots, x_e)R$  is  $\mathfrak{m}$ -primary and hence  $IS$  is  $\mathfrak{m}S$ -primary. Therefore, by [Corollary 5.6](#), we can find a tuple  $\mathbf{y}$  of length  $d - e$  in  $I$ , so that its image in  $S$  is a generic sequence. It follows that  $(x_1, \dots, x_e)R + \mathbf{y}R$  is  $\mathfrak{m}$ -primary. Since  $(\mathbf{y}, x_1, \dots, x_e)$  has length  $d$ , it is a generic sequence, showing that  $d - e \leq h$ .  $\square$

## 6. Pseudo singularities

In this section, we maintain the notation introduced in the previous section. Our goal is to extend several singularity notions of noetherian local rings to the category of local  $\mathfrak{D}$ -affine algebras.

**Grade and depth.** Let  $B$  be an arbitrary ring and  $I := (x_1, \dots, x_n)B$  a finitely generated ideal. The *grade* of  $I$ , denoted  $\text{grade}(I)$ , is by definition equal to  $n - h$ , where  $h$  is the largest value  $i$  for which the  $i$ -th Koszul homology  $H_i(x_1, \dots, x_n)$  is nonzero. For a local ring  $R$  of finite embedding dimension, we define its *depth* as the grade of its maximal ideal.

If  $B$  is moreover noetherian, then we can define the grade of  $I$  alternatively as the minimal  $i$  for which  $\text{Ext}_B^i(B/I, B)$  is nonzero (for all this see for instance [Bruns and Herzog 1993, §9.1]). An arbitrary local ring has positive depth if and only if its maximal ideal is not an associated prime. Grade, and hence depth, *deforms well*, in the sense that the

$$\text{grade}(I(B/\mathbf{x}B)) = \text{grade}(I) - |\mathbf{x}|$$

for every  $B$ -regular sequence  $\mathbf{x}$  in  $I$ . For a locally finitely generated  $\mathfrak{D}$ -algebra  $(R, \mathfrak{m})$ , its depth never exceeds its geometric dimension. Indeed, by definition, the grade of a finitely generated ideal never exceeds its minimal number of generators, and by [Bruns and Herzog 1993, Proposition 9.1.3], the depth of  $R$  is equal to the grade of any of its  $\mathfrak{m}$ -primary ideals. It follows that the depth of  $R$  is at most its geometric dimension.

In general, the grade of a finitely generated ideal might be positive without it containing a  $B$ -regular element. However, the next lemma shows that this is not the case for ultraproducts of noetherian local rings.

**Lemma 6.1.** *Let  $C_\infty$  be the ultraproduct of noetherian local rings  $C_w$  and let  $I_\infty$  be a finitely generated ideal of  $C_\infty$  obtained as the ultraproduct of ideals  $I_w \subseteq C_w$ .*

*If  $I_\infty$  has grade  $n$ , then there exists a  $C_\infty$ -regular sequence of length  $n$  with all of its entries in  $I_\infty$ . Moreover, any permutation of a  $C_\infty$ -regular sequence is again  $C_\infty$ -regular.*

*Proof.* By [Bruns and Herzog 1993, Proposition 9.1.3], there exists a finite tuple of indeterminates  $Y$  and a  $C_\infty[Y]$ -regular sequence  $\mathbf{f}_\infty$  of length  $n$ , with all of its entries in  $I_\infty C_\infty[Y]$ . Choose tuples  $\mathbf{f}_w$  in  $C_w[Y]$  so that their ultraproduct is  $\mathbf{f}_\infty$ . By Łos' Theorem,  $\mathbf{f}_w$  is  $C_w[Y]$ -regular and has all of its entries in  $I_w C_w[Y]$ , for almost all  $w$ . This shows that  $I_w C_w[Y]$  has grade at least  $n$ . Since  $C_w \rightarrow C_w[Y]$  is faithfully flat,  $I_w$  has grade at least  $n$  by [Bruns and Herzog 1993, Proposition 9.1.2]. Hence, since  $C_w$  is noetherian, we can find a  $C_w$ -regular sequence  $\mathbf{x}_w$  of length  $n$  with all of its entries in  $I_w$ . By Łos' Theorem, the ultraproduct  $\mathbf{x}_\infty$  of the  $\mathbf{x}_w$  is  $C_\infty$ -regular and has all of its entries in  $I_\infty$ .

The last assertion follows from Łos' Theorem and the fact that in a noetherian local ring, any permutation of a regular sequence is again regular ([Matsumura 1986, Theorem 16.3]).  $\square$



Recall that a noetherian local ring for which its dimension and its depth (respectively, its dimension and its embedding dimension) coincide is Cohen–Macaulay (respectively, regular). We will shortly see that upon replacing dimension by geometric dimension, we get equally well behaved notions. Let us therefore make the following definitions, for  $R$  a local  $\mathfrak{D}$ -affine algebra.

**Definition 6.2.** We say that  $R$  is *pseudo-Cohen–Macaulay*, if its geometric dimension is equal to its depth, and *pseudoregular*, if its geometric dimension is equal to its embedding dimension.

**Theorem 6.3.** *Let  $\mathfrak{D}$  be an ultra-DVR and let  $R$  be an isodimensional local  $\mathfrak{D}$ -affine algebra with  $\mathfrak{D}$ -approximation  $R_w$ . In order for  $R$  to be pseudo-Cohen–Macaulay it is necessary and sufficient that almost all  $R_w$  are Cohen–Macaulay.*

*Proof.* Let  $d$  be the geometric dimension of  $R$  and  $\delta$  its depth. Suppose first that  $d = \delta$ . Since  $R \rightarrow \mathcal{L}(R)$  is faithfully flat,  $\mathcal{L}(R)$  has depth  $\delta$  as well by [Bruns and Herzog 1993, Proposition 9.1.2]. By Lemma 6.1, there exists an  $\mathcal{L}(R)$ -regular sequence  $\mathbf{x}_\infty$  of length  $d$ . If  $\mathbf{x}_w$  is an  $\mathfrak{D}$ -approximation of  $\mathbf{x}_\infty$ , then almost each  $\mathbf{x}_w$  is  $R_w$ -regular by Los’ Theorem. Since almost all  $R_w$  have dimension  $d$  by isodimensionality, almost all are Cohen–Macaulay.

Conversely, assume almost all  $R_w$  are Cohen–Macaulay. It follows by reversing the above argument that  $\mathcal{L}(R)$  has depth  $d$  and hence, so has  $R$ , by faithful flatness.  $\square$

Since every system of parameters is a regular sequence in a local Cohen–Macaulay ring, we expect a similar behavior for generic sequences, and this indeed holds.

**Theorem 6.4.** *Let  $\mathfrak{D}$  be an ultra-DVR and let  $R$  be an isodimensional local  $\mathfrak{D}$ -affine algebra. If  $R$  is pseudo-Cohen–Macaulay, then any generic sequence is  $R$ -regular.*

*Proof.* Let  $\mathbf{x}$  be a generic sequence with  $\mathfrak{D}$ -approximation  $\mathbf{x}_w$ . Almost each  $\mathbf{x}_w$  is a system of parameters in  $R_w$ , by Corollary 5.8. Since almost all  $R_w$  are Cohen–Macaulay by Theorem 6.3, almost each  $\mathbf{x}_w$  is  $R_w$ -regular. Hence  $\mathbf{x}$  is  $\mathcal{L}(R)$ -regular, by Los’ Theorem, whence  $R$ -regular, by faithful flatness.  $\square$

**Theorem 6.5.** *Let  $\mathfrak{D}$  be an ultra-DVR. An isodimensional local  $\mathfrak{D}$ -affine algebra  $R$  with  $\mathfrak{D}$ -approximation  $R_w$  is pseudoregular if and only if almost all  $R_w$  are regular local rings.*

*Proof.* Let  $\mathfrak{m}$  be the maximal ideal of  $R$ , with  $\mathfrak{D}$ -approximation  $\mathfrak{m}_w$ . Let  $\mathcal{L}(R)$  be the nonstandard  $\mathfrak{D}$ -hull of  $R$ . Let  $\epsilon$  be the embedding dimension of  $R$  and  $d$  its geometric dimension. Suppose that  $R$  is pseudoregular, that is to say, that  $\epsilon = d$ . Hence  $\mathfrak{m} = \mathbf{x}R$  for some  $d$ -tuple  $\mathbf{x}$ , necessarily generic. Since  $\mathfrak{m}\mathcal{L}(R) = \mathbf{x}\mathcal{L}(R)$ ,



Łos’ Theorem yields that  $\mathfrak{m}_w = \mathbf{x}_w R$ , where  $\mathbf{x}_w$  is an  $\mathfrak{D}$ -approximation of  $\mathbf{x}$ . Since almost all  $R_w$  have dimension  $d$ , almost all are regular local rings.

Conversely, suppose almost all  $R_w$  are regular. Since the  $\mathfrak{D}_w$ -complexity of almost all  $R_w$  is at most  $c$ , for some  $c$ , we can find a regular system of parameters  $\mathbf{x}_w$  of  $\mathfrak{D}_w$ -complexity at most  $c$  (as part of a minimal system of generators of  $\mathfrak{m}_w$ ). By Lemma 2.1, their ultraproduct  $\mathbf{x}$  belongs to  $R$ , and is a generic sequence by Corollary 5.8. By Łos’ Theorem and faithful flatness,  $\mathbf{x}R = \mathfrak{m}$  whence  $\epsilon \leq d$ . Since geometric dimension never exceeds embedding dimension,  $\epsilon = d$  and  $R$  is pseudoregular.  $\square$

The following is now immediate from the previous result and Theorem 4.4.

**Corollary 6.6.** *Let  $\mathfrak{D}$  be an ultra-DVR. If  $R$  is a pseudoregular local  $\mathfrak{D}$ -affine algebra, then  $R$  is a domain if and only if it is isodimensional. Moreover, if this is the case, then every localization of  $R$  with respect to a prime ideal containing  $\pi$  is again pseudoregular.*

In fact, the protoproduct  $R$  of regular local  $\mathfrak{D}_w$ -affine algebras  $R_w$  of uniformly bounded  $\mathfrak{D}_w$ -complexity is pseudoregular and isodimensional. Indeed, we already observed that then  $R$  is isodimensional, and therefore by Theorem 6.5, pseudoregular. For a homological characterization of pseudoregularity, see Corollary 11.5 below.

**Example 6.7.** If  $R$  denotes the localization of  $\mathfrak{D}[X, Y]/(X^2 + Y^3 + \pi)$  at the maximal ideal generated by  $X, Y$  and  $\pi$ , then  $R$  is pseudoregular (namely  $X$  and  $Y$  generate the maximal ideal, so  $\epsilon = 2$ , and since  $R/\pi R$  has dimension one,  $d = 2$  as well). Note though that  $R/\pi R$  is not regular.

**Corollary 6.8.** *Let  $\mathfrak{D}$  be an ultra-DVR and let  $R$  be an isodimensional local  $\mathfrak{D}$ -affine algebra. If  $R$  is pseudoregular, then it is pseudo-Cohen–Macaulay.*

*Proof.* Let  $R_w$  be an  $\mathfrak{D}$ -approximation of  $R$ . By Theorem 6.5, almost all  $R_w$  are regular whence Cohen–Macaulay. This in turn implies that  $R$  is pseudo-Cohen–Macaulay by Theorem 6.3.  $\square$

Without the isodimensionality assumption, the result is false. For instance, let  $a$  be a nonzero element in the ideal of infinitesimals of  $\mathfrak{D}$  and put  $R := \mathfrak{D}/a\mathfrak{D}$ . It follows that  $R$  has geometric dimension one, whence is pseudoregular, but its depth is zero.

**6.9. Transfer.** Let me now elaborate on why the results in this section are instances of transfer between positive and mixed characteristic. Suppose  $\tilde{\mathfrak{D}}$  is a second ultra-DVR, realized as the ultraproduct of discrete valuation rings  $\tilde{\mathfrak{D}}_w$  and suppose  $\mathfrak{D} \cong \tilde{\mathfrak{D}}$ . Note that this does *not* imply that  $\mathfrak{D}_w$  and  $\tilde{\mathfrak{D}}_w$  are almost all pair-wise isomorphic. In fact, in the next sections, one set of discrete valuation

rings will be of mixed characteristic and the other set of prime characteristic. Let  $R$  be a local  $\mathfrak{D}$ -affine algebra. Since  $R$  is then also local  $\tilde{\mathfrak{D}}$ -affine, it admits a nonstandard  $\tilde{\mathfrak{D}}$ -hull and  $\tilde{\mathfrak{D}}$ -approximations with respect to this second set of discrete valuation rings; let us denote them by  $\mathfrak{L}_{\tilde{\mathfrak{D}}}(R)$  and  $\tilde{R}_w$  respectively. Suppose  $\mathfrak{D}_w$  and  $\tilde{\mathfrak{D}}_w$  have pair-wise isomorphic residue fields (as will be the case below). Since the  $R_w/\pi_w R_w$  are an approximation of the  $\kappa$ -algebra  $R/\pi R$  (in the sense of [Schoutens 2003d]) and, mutatis mutandis, so are the  $\tilde{R}_w/\tilde{\pi}_w \tilde{R}_w$ , where  $\tilde{\pi}_w$  is a uniformizing parameter of  $\tilde{\mathfrak{D}}_w$ , we get from [Schoutens 2003d, 3.2.3] that almost all  $R_w/\pi_w R_w$  are isomorphic to  $\tilde{R}_w/\tilde{\pi}_w \tilde{R}_w$ . Therefore, if we assume that there is no torsion, then  $R_w$  and  $\tilde{R}_w$  have the same dimension, and one set consists of almost all Cohen–Macaulay local rings if and only if the other set does (note that this argument does not yet use the above *pseudo* notions). However, this argument breaks down in the presence of torsion, or, when we want to transfer the regularity property. This can be overcome by using the notions defined in this section, provided we have a uniform upper bound on the parameter degree.

Suppose, for some  $d, e \in \mathbb{N}$ , that almost all  $R_w$  have dimension  $d$  and parameter degree at most  $e$ . Note that in view of Corollary 5.11 this last condition is automatically satisfied if almost all  $R_w$  are torsion-free over  $\mathfrak{D}_w$ ; and that it is implied by the assumption that almost all  $R_w$  have uniformly bounded homological multiplicity (see [Schoutens 2006, Corollary 4.6]). Applying Proposition 5.10 twice gives first that  $R$  is isodimensional, with geometric dimension  $d$ , and then that almost all  $\tilde{R}_w$  have dimension  $d$  and uniformly bounded parameter degree. Now, Theorems 6.3 and 6.5 tell us that almost all  $R_w$  are respectively Cohen–Macaulay or regular, if and only if almost all  $\tilde{R}_w$  are.

## 7. Big Cohen–Macaulay algebras

In [Aschenbrenner and Schoutens 2007; Schoutens 2004a], ultraproducts of absolute integral closures in characteristic  $p$  were used to define big Cohen–Macaulay algebras in equicharacteristic zero. This same process can be used in the current mixed characteristic setting. Recall that for an arbitrary domain  $B$ , we define its *absolute integral closure* as the integral closure of  $B$  in some algebraic closure of its field of fractions and denote it  $B^+$ . This is uniquely defined up to  $B$ -algebra isomorphism.

For each prime number  $p$ , let  $\mathfrak{D}_p^{\text{mix}}$  be a mixed characteristic complete discrete valuation ring with uniformizing parameter  $\pi_p$  and residue field  $\kappa_p$  of characteristic  $p$ , and let  $\mathfrak{D}$ ,  $\pi$  and  $\kappa$  be their respective ultraproducts. Put  $\mathfrak{D}_p^{\text{eq}} := \kappa_p[[t]]$ , for  $t$  a single indeterminate. By Theorem 2.3, the Ax–Kochen–Ershov Theorem,  $\mathfrak{D}$  is isomorphic to the ultraproduct of the  $\mathfrak{D}_p^{\text{eq}}$ . As before,  $\mathfrak{I}_{\mathfrak{D}}$  denotes the ideal of infinitesimals of  $\mathfrak{D}$ . Put  $A := \mathfrak{D}[X]$ , for a fixed tuple of indeterminates  $X$ ,

and let  $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(A)$  and  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(A)$  be its respective equicharacteristic and mixed characteristic nonstandard  $\mathfrak{D}$ -hull, that is to say, the ultraproduct of respectively the  $A_p^{\text{eq}} := \mathfrak{D}_p^{\text{eq}}[X]$  and the  $A_p^{\text{mix}} := \mathfrak{D}_p^{\text{mix}}[X]$ .

Throughout,  $R$  will be a local  $\mathfrak{D}$ -affine domain with  $R_p^{\text{eq}}$  and  $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(R)$  respectively an equicharacteristic  $\mathfrak{D}$ -approximation and the equicharacteristic nonstandard  $\mathfrak{D}$ -hull of  $R$  (so that  $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(R)$  is the ultraproduct of the  $R_p^{\text{eq}}$ ). By [Theorem 4.4](#), almost all  $R_p^{\text{eq}}$  are local domains.

**Definition 7.1.** Define  $\mathfrak{B}(R)$  as the ultraproduct of the  $(R_p^{\text{eq}})^+$ .

Since  $(R_p^{\text{eq}})^+$  is well-defined up to  $R_p^{\text{eq}}$ -algebra isomorphism, we have that  $\mathfrak{B}(R)$  is well-defined up to  $R$ -algebra isomorphism. Moreover, this construction is weakly functorial in the following sense. Let  $R \rightarrow S$  be an  $\mathfrak{D}$ -algebra homomorphism between local  $\mathfrak{D}$ -affine domains. This induces  $\mathfrak{D}_p^{\text{eq}}$ -algebra homomorphisms  $R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$  of the corresponding equicharacteristic  $\mathfrak{D}$ -approximations. These in turn yield homomorphisms  $(R_p^{\text{eq}})^+ \rightarrow (S_p^{\text{eq}})^+$  between the absolute integral closures. Taking ultraproducts, we get an  $\mathfrak{D}$ -algebra homomorphism  $\mathfrak{B}(R) \rightarrow \mathfrak{B}(S)$  and a commutative diagram

$$(6) \quad \begin{array}{ccc} R & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \\ \mathfrak{B}(R) & \xrightarrow{\quad} & \mathfrak{B}(S). \end{array}$$

**Theorem 7.2.** *If  $R$  is a local  $\mathfrak{D}$ -affine domain, then any generic sequence in  $R$  is  $\mathfrak{B}(R)$ -regular.*

*Proof.* Let  $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(R)$  and  $R_p^{\text{eq}}$  be respectively, the equicharacteristic nonstandard  $\mathfrak{D}$ -hull and an equicharacteristic  $\mathfrak{D}$ -approximation of  $R$ . Let  $\mathbf{x}$  be a generic sequence, and let  $\mathbf{x}_p$  be an  $\mathfrak{D}$ -approximation of  $\mathbf{x}$ . By [Corollary 5.8](#), almost each  $\mathbf{x}_p$  is a system of parameters in  $R_p^{\text{eq}}$ , whence is  $(R_p^{\text{eq}})^+$ -regular by [[Hochster and Huneke 1992](#)]. By Łos’ Theorem,  $\mathbf{x}$  is  $\mathfrak{B}(R)$ -regular. □

### 8. Improved New Intersection Theorem

The remaining sections will establish various asymptotic versions in mixed characteristic of the Homological Conjectures listed in the abstract. We start with discussing Intersection Theorems. By [[Roberts 1987](#)], we now know that the New Intersection Theorem holds for all noetherian local rings. However, this is not yet known for the Improved New Intersection Theorem. We need some terminology and notation (all taken from [[Bruns and Herzog 1993](#)]).

Let  $C$  be an arbitrary noetherian local ring and  $\varphi: C^a \rightarrow C^b$  a linear map between finite free  $C$ -modules. We will always think of  $\varphi$  as an  $(a \times b)$ -matrix over  $C$ . For  $r > 0$ , recall that the  $r$ -th Fitting ideal of  $\varphi$ , denoted  $I_r(\varphi)$ , is the ideal in  $C$  generated by all  $(r \times r)$  minors of  $\varphi$ ; if  $r$  exceeds the size of the matrix, we put  $I_r(\varphi) := (0)$ .

By a *finite free complex* over  $C$  we mean a complex

$$(F_\bullet) \quad 0 \rightarrow C^{a_s} \xrightarrow{\varphi_s} C^{a_{s-1}} \xrightarrow{\varphi_{s-1}} \dots \xrightarrow{\varphi_2} C^{a_1} \xrightarrow{\varphi_1} C^{a_0} \rightarrow 0.$$

We call  $s$  the *length* of the complex, and for each  $i$ , we define

$$r_i := \sum_{j=i}^s (-1)^{j-i} a_j.$$

We will refer to  $r_i$  as the *expected rank* of  $\varphi_i$ . We will call the residue ring  $C/I_{r_i}(\varphi_i)$  the  $i$ -th Fitting ring of  $F_\bullet$  and we will denote it  $\mathfrak{R}_i(F_\bullet)$ .

The  $i$ -th *homology* of  $F_\bullet$  is by definition the quotient module

$$H_i(F_\bullet) := \text{Ker}(\varphi_i) / \text{Im}(\varphi_{i+1}).$$

We call  $F_\bullet$  *acyclic*, if all  $H_i(F_\bullet) = 0$  for  $i > 0$ . In that case,  $F_\bullet$  yields a *finite free resolution* of  $H_0(F_\bullet)$ .

In case  $C$  is a  $Z$ -affine algebra with  $Z$  a local ring, we say that  $F_\bullet$  has  $Z$ -complexity at most  $c$ , if its length  $s$  is at most  $c$ , if all  $a_i \leq c$ , and if every entry of each  $\varphi_i$  has  $Z$ -complexity at most  $c$ . Below we will say that an element  $\tau$  in a homology module  $H_i(F_\bullet)$  has  $Z$ -complexity at most  $c$ , if it is the image of a tuple in  $\text{Ker}(\varphi_i)$  of  $Z$ -complexity at most  $c$  (for more details, see [Section 11](#) below).

**Theorem 8.1** (Asymptotic Improved New Intersection Theorem). *For each  $c$ , there exists a bound  $\text{INIT}(c)$  with the following property. Let  $V$  be a mixed characteristic discrete valuation ring and let  $(C, \mathfrak{m})$  be a local  $V$ -affine domain. Let  $F_\bullet$  be a finite free complex over  $C$ . Assume  $H_0(F_\bullet)$  has a minimal generator  $\tau$ , such that  $C\tau$  has finite length and assume that  $c$  simultaneously bounds the  $V$ -complexity of  $C$ ,  $\tau$  and  $F_\bullet$ , the parameter degree of each Fitting ring  $\mathfrak{R}_i(F_\bullet)$ , and the length of  $C\tau$ .*

*If  $\mathfrak{R}_i(F_\bullet)$  has dimension at most  $\dim C - i$ , for  $i = 1, \dots, s$ , then the dimension of  $C$  is at most the length of the complex  $F_\bullet$ , provided the characteristic of the residue field of  $\mathfrak{D}$  is bigger than  $\text{INIT}(c)$ .*

*Proof.* If  $\pi C = 0$ , then  $C$  contains the residue field of  $V$  and in that case the Theorem is known (see for instance [[Bruns and Herzog 1993](#), Theorem 9.4.1] or [[Evans and Griffith 1981](#); [Hochster 1983](#)]). So we may moreover assume that  $C$  is flat over  $V$ . By faithful flat descent, we may replace  $V$  and  $C$  by  $\widehat{V}$  and a suitable localization of  $\widehat{V} \otimes_V C$  respectively, where  $\widehat{V}$  is the completion of  $V$ . In other words, we only need to prove the result for a torsion-free local domain

over a complete discrete valuation ring of mixed characteristic. Suppose this last assertion is false for some  $c$ , so that there exists an infinite set  $U$  of prime numbers and for each  $p \in U$  a counterexample consisting of the following data:

- a mixed characteristic complete discrete valuation ring  $\mathfrak{D}_p^{\text{mix}}$  with uniformizing parameter  $\pi_p$ , whose residue field has characteristic  $p$ ;
- a local  $\mathfrak{D}_p^{\text{mix}}$ -affine domain  $R_p^{\text{mix}}$  of  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c$ ;
- a finite free complex

$$(F_{p\bullet}^{\text{mix}}) \quad 0 \rightarrow (R_p^{\text{mix}})^{a_s} \xrightarrow{\varphi_{s,p}} (R_p^{\text{mix}})^{a_{s-1}} \xrightarrow{\varphi_{s-1,p}} \dots \xrightarrow{\varphi_{2,p}} (R_p^{\text{mix}})^{a_1} \xrightarrow{\varphi_{1,p}} (R_p^{\text{mix}})^{a_0} \rightarrow 0$$

of length  $s$  and of  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c$ , such that the  $i$ -th Fitting ring  $\mathfrak{R}_i(F_{p\bullet}^{\text{mix}})$  has dimension at most  $d - i$  and parameter degree at most  $c$ ;

- a minimal generator  $\tau_p$  of  $H_0(F_{p\bullet}^{\text{mix}})$  of  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c$ , generating a module of length at most  $c$ ,

but such that  $s$  is strictly less than the dimension of  $R_p^{\text{mix}}$ . Choose some nonprincipal ultrafilter on the set of prime numbers which contains  $U$ . In particular, we have a counterexample with the above properties for almost all  $p$ . Without loss of generality, we may assume that the dimension of each  $R_p^{\text{mix}}$  and that the ranks of each  $F_{p\bullet}^{\text{mix}}$  are independent from  $p$ , since there are only finitely many possibilities, and hence precisely one such possibility almost always holds. In particular, the expected ranks do not depend on  $p$ .

Let  $\mathfrak{D}$  and  $\pi$  be the respective ultraproduct of the  $\mathfrak{D}_p^{\text{mix}}$  and the  $\pi_p$ . Let  $R$  and  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$  be the respective protoproduct and ultraproduct of the  $R_p^{\text{mix}}$ . It follows from [Theorem 4.4](#), that  $R$  is a local  $\mathfrak{D}$ -affine domain, and from [Theorem 4.2](#), that  $R \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$  is faithfully flat. Let  $d$  be the geometric dimension of  $R$ , so that almost all  $R_p^{\text{mix}}$  have dimension  $d$  by [Theorem 5.4](#). Let  $\varphi_i$  be the ultraproduct of the  $\varphi_{i,p}$ . It follows from [Lemma 2.1](#) that each  $\varphi_i$  is already defined over  $R$ . Hence by Łos' Theorem

$$(F_{\bullet}) \quad 0 \rightarrow R^{a_s} \xrightarrow{\varphi_s} R^{a_{s-1}} \xrightarrow{\varphi_{s-1}} \dots \xrightarrow{\varphi_2} R^{a_1} \xrightarrow{\varphi_1} R^{a_0} \rightarrow 0$$

is a finite free complex. Let  $M$  denote its zeroth homology and fix some  $i$ . By Łos' Theorem,  $I_{r_i}(\varphi_{i,p})$  is an  $\mathfrak{D}$ -approximation of  $I_{r_i}(\varphi_i)$ . By the uniform boundedness of the parameter degrees,  $\mathfrak{R}_i(F_{\bullet})$  is isodimensional by [Proposition 5.10](#). If  $d_i$  is the geometric dimension of  $\mathfrak{R}_i(F_{\bullet})$ , then  $d - d_i$  is equal to the height of almost all  $I_{r_i}(\varphi_{i,p})$  and to the geometric height of  $I_{r_i}(\varphi_i)$ , by [Theorem 5.13](#). In particular, by assumption,  $i \leq d - d_i$ , and therefore, by definition of geometric height, we can find a generic sequence  $\mathbf{x}_i$  in  $R$  whose first  $i$  entries belong to  $I_{r_i}(\varphi_i)$ .

Let  $B := \mathcal{B}(R)$ . Since  $\mathbf{x}_i$  is  $B$ -regular by [Theorem 7.2](#), the grade of  $I_{r_i}(\varphi_i)B$  is at least  $i$ . Since this holds for all  $i$ , the Buchsbaum–Eisenbud–Northcott Acyclicity Theorem ([\[Bruns and Herzog 1993, Theorem 9.1.6\]](#)) proves that  $F_\bullet \otimes_R B$  is acyclic. Since  $B$  has depth at least  $d$ , it follows from [Theorem 9.1.2](#) of the same reference that the zeroth homology of  $F_\bullet \otimes_R B$ , that is to say,  $M \otimes_R B$ , has depth at least  $d - s$ .

Let  $\tau$  be the ultraproduct of the  $\tau_p$ . Note that each  $\tau_p$  is by assumption the image of a tuple in  $(R_p^{\text{mix}})^{a_0}$  of  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c$ , so that  $\tau$  is already defined over  $R$  by [Lemma 2.1](#). By Łos’ Theorem,  $\tau$  is a minimal generator of

$$H_0(F_\bullet \otimes \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)) = M \otimes \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R),$$

and by [\[Schoutens 1999, Proposition 1.1\]](#) or [\[Jensen and Lenzing 1989, Proposition 9.1\]](#), the length of  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)\tau$  is at most  $c$ . By faithful flatness,  $\tau \in M - \mathfrak{m}M$ , where  $\mathfrak{m}$  is the maximal ideal of  $R$ , and  $R\tau$  has length at most  $c$ . In particular, the image of  $\tau \otimes 1$  in  $M/\mathfrak{m}M \otimes B/\mathfrak{m}B$  is nonzero, and therefore  $\tau \otimes 1$  itself is a nonzero element of  $M \otimes B$ . Since  $\mathfrak{m}^c$  annihilates  $\tau \otimes 1$ , we conclude that  $M \otimes B$  has depth zero. Together with the conclusion from the previous paragraph, we get  $d \leq s$ , contradiction. □

This type of argument by *reductio ad absurdum*, to obtain uniform bounds via ultraproducts, is very common and will be used constantly in the sequel. We will shorten the argument by saying from the start that by way of contradiction, we may assume that for some  $c$ , there exist for almost each  $p$  a counterexample with such and such properties.

### 9. Monomial and Direct Summand Conjectures

We keep notation as in the previous section, so that in particular  $\mathfrak{D}$  will denote the ultraproduct of mixed characteristic complete discrete valuation rings  $\mathfrak{D}_p^{\text{mix}}$ . In order to formulate a nonstandard version of the Monomial Conjecture, we need some terminology. Let  $\mathbb{N}_\infty$  be the ultrapower of  $\mathbb{N}$ . Let  $C_w$  be rings,  $X := (X_1, \dots, X_d)$  indeterminates and  $A_\infty$  the ultraproduct of the  $C_w[X]$ . Although each  $C_w[X]$  is  $\mathbb{N}$ -graded, it is not true that  $A_\infty$  is  $\mathbb{N}_\infty$ -graded, since we might have infinite sums of monomials in  $A_\infty$ . Nonetheless, for each  $\nu_\infty \in (\mathbb{N}_\infty)^d$ , the element  $X^{\nu_\infty}$  is well-defined, namely, if  $\nu_\infty$  is the ultraproduct of elements  $\nu_w \in \mathbb{N}$ , then

$$X^{\nu_\infty} := \text{ulim}_{w \rightarrow \infty} X^{\nu_w}.$$

In particular, if  $B_\infty$  is an arbitrary ultraproduct of rings  $B_w$  and if  $\mathbf{x}$  is a  $d$ -tuple in  $B_\infty$ , then  $\mathbf{x}^{\nu_\infty}$  is a well-defined element of  $B_\infty$ .

By a *cone*  $H$  in a semigroup  $\Gamma$  (e.g.,  $\Gamma = \mathbb{N}^d$  or  $\Gamma = \mathbb{N}_\infty^d$ ), we mean a subset  $H$  of  $\Gamma$  such that  $\nu + \Gamma \subseteq H$ , for every  $\nu \in H$ , where  $\nu + \Gamma$  stands for the collection of

all  $\nu + \gamma$  with  $\gamma \in \Gamma$ . A cone  $H$  is *finitely generated*, if there exist  $\nu_1, \dots, \nu_s \in H$ , called *generators* of the cone, such that

$$H = \bigcup_i \nu_i + \Gamma.$$

If  $H$  is a cone in  $\mathbb{N}^d$ , we let  $J_H$  be the monomial ideal in  $\mathbb{Z}[Y]$  generated by all  $Y^\nu$  with  $\nu \in H$ , where  $Y$  is a  $d$ -tuple of indeterminates. If  $H$  is generated by  $\nu_1, \dots, \nu_s$ , then  $J_H$  is generated by  $X^{\nu_1}, \dots, X^{\nu_s}$ . Conversely, if  $J$  is a monomial ideal in  $\mathbb{Z}[X]$ , then the collection of all  $\nu$  for which  $X^\nu \in J$  is a cone in  $\mathbb{N}^d$ . Since  $\mathbb{Z}[Y]$  is noetherian, every cone in  $\mathbb{N}^d$  is finitely generated. This is no longer true for a cone in  $\mathbb{N}^d_\infty$ .

Let  $B$  be an arbitrary ring. We will use the following well-known fact about regular sequences. If  $\mathbf{x}$  is a  $B$ -regular sequence (in fact, it suffices that  $\mathbf{x}$  is quasiregular),  $H$  a cone in  $\mathbb{N}^d$  and  $\nu \notin H$ , then  $\mathbf{x}^\nu$  does not lie in the ideal  $J_H(\mathbf{x})$  generated by all  $\mathbf{x}^\theta$  with  $\theta \in H$ .

**Corollary 9.1.** *Let  $R$  be a local  $\mathfrak{D}$ -affine domain with equicharacteristic nonstandard  $\mathfrak{D}$ -hull  $\mathfrak{L}^{\text{eq}}_{\mathfrak{D}}(R)$ . Let  $\mathbf{x}$  be a generic sequence in  $R$ , let  $H$  be a cone in  $\mathbb{N}^d_\infty$  and let  $\nu \in \mathbb{N}^d_\infty$ . If  $\nu \notin H$ , then*

$$(7) \quad \mathbf{x}^\nu \notin (\mathbf{x}^\mu \mid \mu \in H)\mathfrak{L}^{\text{eq}}_{\mathfrak{D}}(R).$$

*Proof.* Suppose (7) is false for some choice of cone  $H$  of  $\mathbb{N}^d_\infty$  and some  $\nu_0 \notin H$ . In other words, we can find  $f_{i\infty}$  in  $\mathfrak{L}^{\text{eq}}_{\mathfrak{D}}(R)$  and tuples  $\nu_i$  in  $H$ , such that

$$(8) \quad \mathbf{x}^{\nu_0} = f_{1\infty}\mathbf{x}^{\nu_1} + \dots + f_{s\infty}\mathbf{x}^{\nu_s}.$$

Since  $R \rightarrow \mathfrak{B}(R)$  factors through  $\mathfrak{L}^{\text{eq}}_{\mathfrak{D}}(R)$ , we can view (8) as a relation in  $\mathfrak{B}(R)$ , and we want to show that that is impossible. Let  $R_p^{\text{eq}}$  be an equicharacteristic  $\mathfrak{D}$ -approximation of  $R$ , so that  $\mathfrak{B}(R)$  is the ultraproduct of the  $(R_p^{\text{eq}})^+$ . Choose tuples  $\nu_{ip} \in \mathbb{N}$ , elements  $f_{ip} \in (R_p^{\text{eq}})^+$  and tuples  $\mathbf{x}_p$  in  $R_p^{\text{eq}}$  whose respective ultraproducts are  $\nu_i$ ,  $f_{i\infty}$  and  $\mathbf{x}$ . By Łos' Theorem, we get

$$(9) \quad \mathbf{x}^{\nu_{0p}} = f_{1p}\mathbf{x}_p^{\nu_{1p}} + \dots + f_{sp}\mathbf{x}_p^{\nu_{sp}}$$

in  $(R_p^{\text{eq}})^+$ , for almost all  $p$ . Łos' Theorem also yields that  $\nu_{0p}$  does not lie in the cone of  $\mathbb{N}^d$  generated by  $\nu_{1p}, \dots, \nu_{sp}$ , for almost all  $p$ . However,  $\mathbf{x}$  is  $\mathfrak{B}(R)$ -regular by Theorem 7.2, whence, almost all  $\mathbf{x}_p$  are  $(R_p^{\text{eq}})^+$ -regular by Łos' Theorem. By our above discussion on regular sequences, (9) cannot hold for those  $p$ .  $\square$

**Theorem 9.2** (Asymptotic Monomial Conjecture I). *For each  $c$ , there exists a bound  $MC(c)$  with the following property. Let  $Y$  be a tuple of indeterminates and  $J$  a monomial ideal in  $\mathbb{Z}[Y]$ . Let  $V$  be a mixed characteristic discrete valuation ring and let  $C$  be a local  $V$ -affine domain. Let  $\mathbf{y}$  be a system of parameters in  $C$*

and let  $J(\mathbf{y})C$  denote the ideal in  $C$  obtained from  $J$  by the substitution  $Y \mapsto \mathbf{y}$ . Assume  $JV[Y]$ ,  $C$  and  $\mathbf{y}$  have  $V$ -complexity at most  $c$  and  $\pi^c \in \mathbf{y}C$ .

If  $Y^v$  is a monomial of degree at most  $c$  not belonging to  $J$ , then  $\mathbf{y}^v \notin J(\mathbf{y})C$ , provided the characteristic of the residue field of  $V$  is bigger than  $MC(c)$ .

*Proof.* Note that since  $C$  has  $V$ -complexity at most  $c$ , its dimension  $d$  is at most  $c$ . By faithful flat descent, we may reduce to the case that  $V$  is complete. Suppose the result is false for some  $c$ , so that we can find for almost each prime number  $p$ ,

- a mixed characteristic complete discrete valuation ring  $\mathfrak{O}_p^{\text{mix}}$  with uniformizing parameter  $\pi_p$ , whose residue field has characteristic  $p$ ,
- a local  $\mathfrak{O}_p^{\text{mix}}$ -affine domain  $R_p^{\text{mix}}$  of  $\mathfrak{O}_p^{\text{mix}}$ -complexity at most  $c$ ,
- multi-indices  $\nu_{0p}, \dots, \nu_{tp}$  such that  $|v_{ip}| \leq c$  and  $\nu_{0p}$  is not in the cone generated by the remaining tuples,
- a system of parameters  $\mathbf{y}_p$  of  $\mathfrak{O}_p^{\text{mix}}$ -complexity at most  $c$  generating an ideal containing  $(\pi_p)^c$ ,

such that

$$(10) \quad \mathbf{y}_p^{\nu_{0p}} \in (\mathbf{y}_p^{\nu_{1p}}, \dots, \mathbf{y}_p^{\nu_{tp}})R_p^{\text{mix}}.$$

Note that the possible number  $t$  of tuples  $\nu_{ip}$  is bounded in terms of  $c$  and hence can be taken to be independent of  $p$ . Let  $\mathfrak{D}$  be the ultraproduct of the  $\mathfrak{O}_p^{\text{mix}}$  and let  $R$  and  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$  be the respective protoproduct and ultraproduct of the  $R_p^{\text{mix}}$ . Since  $R$  is then a domain, it is isodimensional. Let  $\mathbf{y}$  and  $\nu_i$  be the respective ultraproducts of  $\mathbf{y}_p$  and  $\nu_{ip}$ . In particular,  $|v_i| \leq c$ , so that  $\nu_i \in \mathbb{N}^d$ . Let  $H$  be the cone in  $\mathbb{N}^d$  generated by  $\nu_1, \dots, \nu_t$ . By Łos' Theorem,  $\nu_0 \notin H$ . The sequence  $\mathbf{y}$  is defined over  $R$ , by Lemma 2.1, and is generic in  $R$ , by Corollary 5.8. By an application of Łos' Theorem to (10) together with Theorem 4.2, we get

$$\mathbf{y}^{\nu_0} \in (\mathbf{y}^{\nu_1}, \dots, \mathbf{y}^{\nu_t})R.$$

However, this contradicts Corollary 9.1 for the cone  $H$ . □

**Remark 9.3.** In [Schoutens 2003b, Theorem 1.1], this result was stated erroneously without imposing a bound on the degrees of the monomials. I can only prove this more general result in the special case given by Corollary 9.5 below.

Using some results from [Schoutens  $\geq$  2007], we can remove the restriction on  $C$  to be a domain. Namely, by the usual argument, we reduce to the domain case by killing a minimal prime  $\mathfrak{p}$  of  $C$  of maximal dimension (that is to say, so that  $\dim C = \dim C/\mathfrak{p}$ ). However, in order to apply the theorem to the domain  $C/\mathfrak{p}$ , we must be guaranteed that its  $V$ -complexity is at most  $c'$ , for some  $c'$  only depending on  $c$ . Such a bound does indeed exist.



**Theorem 9.4** (Asymptotic Direct Summand Conjecture). *For each  $c$ , we can find a bound  $DS(c)$  with the following property. Let  $V$  be a mixed characteristic discrete valuation ring and let  $C \rightarrow D$  be a finite, injective local  $V$ -algebra homomorphism of  $V$ -complexity at most  $c$ .*

*If  $C$  is regular, then  $C$  is a direct summand of  $D$  (as a  $C$ -module), provided the characteristic of the residue field of  $V$  is bigger than the bound  $DS(c)$ .*

*Proof.* If  $\pi C = 0$ , we are in the equicharacteristic case and the result is well-known. So we may assume that  $V \subseteq C$ . We leave it to the reader to make the reduction to the case that  $V$  is complete and  $D$  is torsion-free over  $V$ . Towards a contradiction, suppose for some  $c$  and almost each  $p$ , we have a mixed characteristic complete discrete valuation ring  $\mathfrak{O}_p^{\text{mix}}$  with residue field of characteristic  $p$ , and a finite, injective local  $\mathfrak{O}_p^{\text{mix}}$ -algebra homomorphism  $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$  of  $\mathfrak{O}_p^{\text{mix}}$ -complexity at most  $c$ , such that  $R_p^{\text{mix}}$  is regular but not a direct summand of  $S_p^{\text{mix}}$ .

By the transfer described in [Section 6.9](#), these data in mixed characteristic yield corresponding data in equal characteristic. In particular, we have for almost each  $p$ , an equicharacteristic  $p$  complete discrete valuation ring  $\mathfrak{O}_p^{\text{eq}}$ , and a finite, injective local  $\mathfrak{O}_p^{\text{eq}}$ -algebra homomorphism  $R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$  of  $\mathfrak{O}_p^{\text{eq}}$ -complexity at most  $c$ , such that  $R_p^{\text{eq}}$  is regular. Although, we did not discuss transfer of homomorphisms and their properties, it is not hard to see, using faithfully flat descent, that almost no  $R_p^{\text{eq}}$  is a direct summand of  $S_p^{\text{eq}}$ . However, this is in violation of the Direct Summand theorem in equicharacteristic. □

**Corollary 9.5** (Asymptotic Monomial Conjecture II). *For each  $c$ , we can find a bound  $MC'(c)$  with the following property. Let  $V$  be a mixed characteristic discrete valuation ring, let  $D$  be a local  $V$ -affine algebra and let  $(x_1, \dots, x_d)$  be a system of parameters in  $D$ .*

*If there exists a finite, injective local  $V$ -algebra homomorphism  $C \subseteq D$  of  $V$ -complexity at most  $c$ , such that the  $x_i$  belong to  $C$  and generate its maximal ideal, then  $(x_1 \cdots x_d)^t$  does not belong to  $(x_1^{t+1}, \dots, x_d^{t+1})D$ , for all  $t \geq 0$ , provided the residue field of  $V$  is bigger than  $MC'(c)$ .*

*Proof.* We may take  $MC'(c)$  equal to the bound  $DS(c)$  from [Theorem 9.4](#). Indeed, since  $D$  has dimension  $d$ , so does  $C$ , showing that  $C$  is regular. Hence  $C$  is a direct summand of  $D$  by [Theorem 9.4](#), so that we are done by [[Bruns and Herzog 1993](#), Lemma 9.2.2]. □

Note that the bounds provided by [Theorem 9.2](#) for the problem at hand depend a priori also on the exponent  $t$ , so that the corollary gives a stronger result. Interestingly, by Cohen’s Structure Theorem, any system of parameters in a complete local  $V$ -affine domain arises as the image of a regular system of parameters under a finite extension. However, since we are forced to work with noncomplete  $V$ -affine algebras, it is not clear yet to which extent the above theorem applies.

## 10. Pure subrings of regular rings

We keep notation as in the previous section, so that in particular  $\mathfrak{D}$  will denote the ultraproduct of mixed characteristic complete discrete valuation rings  $\mathfrak{D}_p^{\text{mix}}$ . Our goal is to show an asymptotic version of the Hochster–Roberts Theorem in [Hochster and Roberts 1974]. Recall that a ring homomorphism  $C \rightarrow D$  is called *cyclically pure* if every ideal  $I$  in  $C$  is extended from  $D$ , that is to say, if  $I = ID \cap C$ .

**Theorem 10.1.** *If  $R$  is a pseudoregular isodimensional local  $\mathfrak{D}$ -affine algebra, then  $R \rightarrow \mathfrak{B}(R)$  is faithfully flat.*

*Proof.* Let  $L$  be a linear form in a finite number of indeterminates  $Y$  with coefficients in  $R$  and let  $\mathbf{b}$  be a solution in  $B := \mathfrak{B}(R)$  of  $L = 0$ . Let  $R_p^{\text{eq}}, L_p^{\text{eq}}$  and  $\mathbf{b}_p^{\text{eq}}$  be equicharacteristic  $\mathfrak{D}$ -approximations of  $R, L$  and  $\mathbf{b}$  respectively. By Łos’ Theorem,  $\mathbf{b}_p^{\text{eq}}$  is a solution in  $(R_p^{\text{eq}})^+$  of the linear equation  $L_p^{\text{eq}} = 0$ . By [Aschenbrenner 2001a, Corollary 4.27], we can find tuples  $\mathbf{a}_{1p}^{\text{eq}}, \dots, \mathbf{a}_{sp}^{\text{eq}}$  over  $R_p^{\text{eq}}$  generating the module of solutions of  $L_p^{\text{eq}} = 0$ , all of  $\mathfrak{D}_p^{\text{eq}}$ -complexity at most  $c$ , for some  $c$  independent from  $p$  and  $s$ . Let  $\mathbf{a}_1, \dots, \mathbf{a}_s$  be the respective ultraproducts, which are then defined over  $R$  by Lemma 2.1. By Łos’ Theorem,  $L(\mathbf{a}_i) = 0$ , for each  $i$ . On the other hand, almost all  $R_p^{\text{eq}}$  are regular, by Theorem 6.5. Therefore,  $R_p^{\text{eq}} \rightarrow (R_p^{\text{eq}})^+$  is flat by [Huneke 1996, Theorem 9.1]. Hence we can write  $\mathbf{b}_p^{\text{eq}}$  as a linear combination over  $(R_p^{\text{eq}})^+$  of the  $\mathbf{a}_{ip}^{\text{eq}}$ . By Łos’ Theorem,  $\mathbf{b}$  is a  $B$ -linear combination of the solutions  $\mathbf{a}_i$ , showing that  $R \rightarrow B$  is flat whence faithfully flat.  $\square$

**Proposition 10.2.** *Let  $R \rightarrow S$  be an injective homomorphism of local isodimensional  $\mathfrak{D}$ -affine algebras. If  $R/\pi R \rightarrow S/\pi S$  is cyclically pure and  $S$  is a pseudoregular local ring, then  $R$  is pseudo-Cohen–Macaulay.*

*Proof.* Since  $S$  is a domain by Corollary 6.6, so is  $R$ . If  $\pi R = 0$ , we are in an equicharacteristic noetherian situation and the statement becomes the Hochster–Roberts Theorem [Hochster and Roberts 1974]. Therefore, we may assume  $\pi$  is  $R$ -regular, so that we can choose a generic sequence  $\mathbf{x} := (x_1, \dots, x_d)$  in  $R$  with  $x_1 = \pi$ . For each  $n \leq d$ , let  $I_n := (x_1, \dots, x_n)R$ . Suppose  $rx_{n+1} \in I_n$ , for some  $r \in R$ . By Theorem 7.2, the sequence  $\mathbf{x}$  is a  $\mathfrak{B}(R)$ -regular. Therefore,  $r \in I_n \mathfrak{B}(R)$ . Since the homomorphism  $R \rightarrow S$  induces a homomorphism  $\mathfrak{B}(R) \rightarrow \mathfrak{B}(S)$ , we get  $r \in I_n \mathfrak{B}(S)$ . By Theorem 10.1, we have

$$I_n \mathfrak{B}(S) \cap S = I_n S,$$

so  $r \in I_n S$ . Using finally that  $R/\pi R \rightarrow S/\pi S$  is cyclically pure and  $\pi \in I_n$ , we get  $r \in I_n$ . This shows that  $\mathbf{x}$  is  $R$ -regular, so that  $R$  has depth at least  $d$  and hence is pseudo-Cohen–Macaulay.  $\square$

**Theorem 10.3** (Asymptotic Hochster–Roberts Theorem). *For each  $c$ , we can find a bound  $HR(c)$  with the following property. Let  $V$  be a mixed characteristic discrete valuation ring and let  $C \rightarrow D$  be a local  $V$ -algebra homomorphism of  $V$ -complexity at most  $c$ . If  $C \rightarrow D$  is cyclically pure and  $D$  is regular, then  $C$  is Cohen–Macaulay, provided the characteristic of the residue field of  $V$  is at least  $HR(c)$ .*

*Proof.* As before, we may reduce to the case that  $V$  is complete and that  $V \subseteq C$ . Suppose this assertion is then false for some  $c$ , so that we can find for almost each prime number  $p$ , a mixed characteristic complete discrete valuation ring  $\mathfrak{D}_p^{\text{mix}}$  with residue field of characteristic  $p$  and a cyclically pure  $\mathfrak{D}_p^{\text{mix}}$ -algebra homomorphism  $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$  of  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c$ , such that  $S_p^{\text{mix}}$  is regular but  $R_p^{\text{mix}}$  is not Cohen–Macaulay. Let  $R \rightarrow S$  and  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(S)$  be respectively the protoproduct and the ultraproduct of the  $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$ . **Theorem 6.3** implies that  $R$  is not pseudo-Cohen–Macaulay, and **Theorem 6.5**, that  $S$  is pseudoregular. I claim that  $R/\pi R \rightarrow S/\pi S$  is cyclically pure. Assuming this claim, we get from **Proposition 10.2** that  $R$  is pseudo-Cohen–Macaulay, contradiction.

To prove the claim, let  $I$  be an arbitrary ideal in  $R$  containing  $\pi$ . Let  $r \in IS \cap R$ , so that we need to show that  $r \in I$ . Note that  $I$  is finitely generated, as  $R/\pi R$  is noetherian. Let  $I_p^{\text{mix}}$  and  $r_p^{\text{mix}}$  be mixed characteristic  $\mathfrak{D}$ -approximations in  $R_p^{\text{mix}}$  of  $I$  and  $r$  respectively. By Łos’ Theorem, almost all  $r_p^{\text{mix}}$  lie in  $I_p^{\text{mix}} S_p^{\text{mix}} \cap R_p^{\text{mix}}$ , whence in  $I_p^{\text{mix}}$  by cyclical purity. By Łos’ Theorem,  $r \in I \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$ , so that  $r \in I$  by faithful flatness, as we needed to prove. □

### 11. Asymptotic vanishing for maps of Tor

**Proposition 11.1.** *If  $R \rightarrow S$  is an integral extension of local  $\mathfrak{D}$ -affine domains, then  $\mathfrak{B}(R) = \mathfrak{B}(S)$ .*

*Proof.* Since any integral extension is a direct limit of finite extensions, we may assume that  $R \rightarrow S$  is finite. Choose an equicharacteristic  $\mathfrak{D}$ -approximation

$$R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$$

of  $R \rightarrow S$ . By **Theorem 4.4** and Łos’ Theorem, almost all  $R_p^{\text{eq}}$  and  $S_p^{\text{eq}}$  are domains and the extension  $R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$  is finite. Therefore,  $(R_p^{\text{eq}})^+ = (S_p^{\text{eq}})^+$ , so that in the ultraproduct, we get  $\mathfrak{B}(R) = \mathfrak{B}(S)$ . □

**Theorem 11.2.** *Let  $R \rightarrow S \rightarrow T$  be local  $\mathfrak{D}$ -algebra homomorphisms between local  $\mathfrak{D}$ -affine domains. Assume that  $R$  and  $T$  are pseudoregular and that  $R \rightarrow S$  is integral and injective. For every  $R$ -module  $M$ , the induced map*

$$\text{Tor}_i^R(S, M) \rightarrow \text{Tor}_i^R(T, M)$$

*is zero for all  $i \geq 1$ .*

*Proof.* Since  $R \rightarrow S$  is integral, we have that  $\mathcal{B}(R) = \mathcal{B}(S)$  by [Proposition 11.1](#). Therefore,  $\mathrm{Tor}_i^R(\mathcal{B}(S), M) = 0$ , for all  $i \geq 1$ , by [Theorem 10.1](#). By weak functoriality, we have, for each  $i \geq 1$ , a commutative diagram

$$(11) \quad \begin{array}{ccc} \mathrm{Tor}_i^R(S, M) & \longrightarrow & \mathrm{Tor}_i^R(T, M) \\ \downarrow & & \downarrow \\ 0 = \mathrm{Tor}_i^R(\mathcal{B}(S), M) & \longrightarrow & \mathrm{Tor}_i^R(\mathcal{B}(T), M). \end{array}$$

In particular, the composite map in this diagram is zero, so that the statement follows once we have shown that the last vertical map is injective. However, this is clear, since  $T \rightarrow \mathcal{B}(T)$  is faithfully flat by [Theorem 10.1](#).  $\square$

To make use of this theorem, we need to incorporate modules in our present setup. I will not provide full details, since many results are completely analogous to the case where we work over a field, and this has been treated in detail in [[Schoutens 2000a](#)]. Of course, we do not have the full equivalent of [Theorem 2.2](#) to our disposal, but for most purposes, the flatness result in [Theorem 4.2](#) suffices.

Let  $C$  be an arbitrary noetherian local ring and  $M$  a finitely generated module over  $C$ . We say that a finite free complex  $F_\bullet$  is a *finite free resolution of  $M$  up to level  $n$* , if  $H_0(F_\bullet) = M$  and all  $H_j(F_\bullet) = 0$ , for  $j = 1, \dots, n$ . Hence, if  $n$  is strictly larger than the length of  $F_\bullet$ , then this just means that  $F_\bullet$  is a finite free resolution of  $M$  (compare with the terminology introduced in the beginning of [Section 8](#)).

Suppose moreover that  $Z$  is a noetherian local ring and  $C$  is a local  $Z$ -affine algebra. We say that  $M$  has  *$Z$ -complexity at most  $c$* , if  $C$  has  $Z$ -complexity at most  $c$  and if  $M$  can be realized as the cokernel of a matrix of  $Z$ -complexity at most  $c$  (meaning that its size is at most  $c$  and all its entries have  $Z$ -complexity at most  $c$ ).

**Proposition 11.3.** *For each pair  $(c, n)$ , there exist bounds  $\mathrm{RES}(c, n)$  and  $\mathrm{HOM}(c)$  with the following property. Let  $V$  be a mixed characteristic discrete valuation ring and let  $C$  be a local  $V$ -affine algebra of  $V$ -complexity at most  $c$ .*

- Any finitely generated  $C$ -module of  $V$ -complexity at most  $c$ , admits a (minimal) finite free resolution up to level  $n$  of  $V$ -complexity at most  $\mathrm{RES}(c, n)$ .
- Any finite free complex over  $C$  of  $V$ -complexity at most  $c$ , has homology modules of  $V$ -complexity at most  $\mathrm{HOM}(c)$ .

*Proof.* The first assertion follows by induction from the already quoted [[Aschenbrenner 2001a](#), Corollary 4.27] on bounds of syzygies (compare with the proof of [[Schoutens 2000a](#), Theorem 4.3]). It is also clear that we may take this resolution to be minimal (=every tuple in one of the kernels has its entries in the maximal ideal), if we choose to do so. The second assertion is derived from the flatness of

the nonstandard  $\mathfrak{D}$ -hull in exactly the same manner as the corresponding result for fields was obtained in [Schoutens 2000a, Lemma 4.2 and Theorem 4.3].  $\square$

Recall that the *weak global dimension* of a ring  $C$  is by definition the supremum (possibly infinite) of the weak homological dimensions (=flat dimensions) of all  $C$ -modules, that is to say, the supremum of all  $n$  for which  $\text{Tor}_n^C(\cdot, \cdot)$  is not identically zero.

**Corollary 11.4.** *A pseudoregular local  $\mathfrak{D}$ -affine domain has finite weak global dimension.*

*Proof.* Let  $R$  be a pseudoregular local  $\mathfrak{D}$ -affine domain. Given an arbitrary  $R$ -module  $M$ , we have to show that  $M$  has bounded flat dimension, that is to say, admits a flat resolution of bounded length. Assume first that  $M$  is finitely presented. Hence we can realize  $M$  as the cokernel of some matrix  $\Gamma$ . Let  $\mathfrak{L}(R)$  be the nonstandard  $\mathfrak{D}$ -hull of  $R$  and let  $R_w$  and  $\Gamma_w$  be  $\mathfrak{D}$ -approximations of  $R$  and  $\Gamma$  respectively. Let  $M_w$  be the cokernel of  $\Gamma_w$ . Let  $d$  be the geometric dimension of  $R$ . By Proposition 11.3, we can find a finite free resolution  $F_{w\bullet}$  up to level  $d$  of each  $M_w$ , of  $\mathfrak{D}_w$ -complexity at most  $c$ , for some  $c$  depending only on  $\Gamma$ , whence independent from  $w$ . Since almost each  $R_w$  is regular by Theorem 6.5 and has dimension  $d$  by Theorem 5.4, almost each  $M_w$  has projective dimension at most  $d$ , so that we can even assume that  $F_{w\bullet}$  is a finite free resolution of  $M_w$ . Let  $F_\bullet$  be the protoproduct of the  $F_{w\bullet}$  (that is to say, the finite free complex over  $R$  given by the protoproduct of the matrices in  $F_{w\bullet}$ ). By Łos’ Theorem,  $F_\bullet \otimes_R \mathfrak{L}(R)$  is a free resolution of  $M \otimes_R \mathfrak{L}(R)$ , and therefore by faithful flat descent,  $F_\bullet$  is a free resolution of  $M$ , proving that  $M$  has projective dimension at most  $d$ .

Assume now that  $M$  is arbitrary. By what we just proved, we have for every finitely generated ideal  $I$  of  $R$  that  $\text{Tor}_{d+1}^R(M, R/I)$  vanishes. Hence, if  $H$  is a  $d$ -th syzygy of  $M$ , then  $\text{Tor}_1^R(H, R/I) = 0$ . Since this holds for every finitely generated ideal of  $R$ , we get from [Matsumura 1986, Theorem 7.7] that  $H$  is flat over  $R$ . Hence  $M$  has finite flat dimension at most  $d$ .  $\square$

By [Jensen 1970], any flat  $R$ -module has projective dimension less than the finitistic global dimension of  $R$  (the supremum of all projective dimensions of modules of finite projective dimension). Therefore, if, moreover, the finitistic global dimension of  $R$  is finite, then so is its global dimension. For a noetherian local ring, its global dimension is finite if and only if its residue field has finite projective dimension (if and only if it is regular). Here is the pseudo analogue of this.

**Corollary 11.5.** *A local  $\mathfrak{D}$ -affine domain is pseudoregular if and only if it is a coherent regular ring in the sense of [Bertin 1971], if and only if its residue field has finite projective dimension.*

*Proof.* In [Bertin 1971] or [Glaz 1992, §5], a local ring  $R$  is called a *coherent regular ring*, if every finitely generated ideal of  $R$  has finite projective dimension. If  $R$  is a pseudoregular local  $\mathfrak{D}$ -affine domain, then this property was established in the course of the proof of Corollary 11.4. Conversely, suppose  $R$  is a local  $\mathfrak{D}$ -affine domain in which every finitely generated ideal has finite projective dimension. In particular, its residue field  $k$  admits a finite projective resolution, say of length  $n$ . Let  $R_w$  and  $k_w$  be  $\mathfrak{D}$ -approximations of  $R$  and  $k$  respectively. Since the  $k_w$  have uniformly bounded  $\mathfrak{D}_w$ -complexity, Proposition 11.3 allows us to take a minimal finite free resolution  $F_{w\bullet}$  of  $k_w$  up to level  $n$ , with the property that each  $F_{w\bullet}$  has  $\mathfrak{D}_w$ -complexity at most  $c$ , for some  $c$  independent from  $w$ . Let  $F_\bullet$  be the protoproduct of these resolutions. By Łos' Theorem and faithfully flat descent,  $F_\bullet$  is a minimal finite free resolution of  $k$  up to level  $n$ . Since  $F_\bullet$  is minimal and since  $k$  has by assumption projective dimension  $n$ , it follows that the final morphism (that is to say, the left most arrow) in  $F_\bullet$  is injective. By Łos' Theorem, so are almost all final morphisms in  $F_{w\bullet}$ , showing that almost all  $k_w$  have finite projective dimension. By Serre's characterization of regular local rings, we conclude that almost all  $R_w$  are regular. Theorem 6.5 then yields that  $R$  is pseudoregular, as we wanted to show.  $\square$

Closer inspection of the above argument shows that the residue field of a pseudoregular local  $\mathfrak{D}$ -affine domain  $R$  has projective dimension equal to the geometric dimension of  $R$ . In particular, the weak global dimension of  $R$  is equal to its geometric dimension.

**Theorem 11.6** (Asymptotic Vanishing for Maps of Tors). *For each  $c$ , we can find a bound  $VT(c)$  with the following property. Let  $V$  be a mixed characteristic discrete valuation ring, let  $C \rightarrow D \rightarrow E$  be local  $V$ -algebra homomorphisms of local  $V$ -affine domains and let  $M$  be a finitely generated  $R$ -module, all of  $V$ -complexity at most  $c$ .*

*If  $C$  and  $E$  are regular and  $C \rightarrow D$  is finite and injective, then the natural map  $\mathrm{Tor}_n^C(D, M) \rightarrow \mathrm{Tor}_n^C(E, M)$  is zero, for all  $n \geq 1$ , provided the characteristic of the residue field of  $V$  is at least  $VT(c)$ .*

*Proof.* Note that  $C$  has dimension at most  $c$  and therefore  $\mathrm{Tor}_n^C(\cdot, \cdot)$  vanishes identically for all  $n > c$  and the assertion trivially holds for these values of  $n$ . If  $\pi C = 0$ , we are in the equicharacteristic case, for which the result is known ([Huneke 1996, Theorem 9.7]). Hence we may assume that all rings are torsion-free over  $V$ . Moreover, without loss of generality, we may assume that  $V$  is complete. Suppose even in this restricted setting, there is no such bound for  $c$  and some  $1 \leq n \leq c$ . Hence, for almost each prime number  $p$ , we can find a counterexample consisting of the following data:

- a mixed characteristic complete discrete valuation ring  $\mathfrak{D}_p^{\text{mix}}$  of residual characteristic  $p$ ;
- local  $R_p^{\text{mix}}$ -algebra homomorphisms  $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}} \rightarrow T_p^{\text{mix}}$  of  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c$  between torsion-free local domains, with  $R_p^{\text{mix}}$  and  $T_p^{\text{mix}}$  regular and  $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$  finite and injective;
- a finitely generated  $R_p^{\text{mix}}$ -module  $M_p^{\text{mix}}$  of  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c$ ;

such that

$$\text{Tor}_n^{R_p^{\text{mix}}}(S_p^{\text{mix}}, M_p^{\text{mix}}) \rightarrow \text{Tor}_n^{R_p^{\text{mix}}}(T_p^{\text{mix}}, M_p^{\text{mix}})$$

is nonzero.

Let  $\mathfrak{D}$  be the ultraproduct of the  $\mathfrak{D}_p^{\text{mix}}$  and let  $M$  be the protoproduct of the  $M_p^{\text{mix}}$  (that is to say,  $M$  is the cokernel of the protoproduct of matrices whose cokernel is  $M_p^{\text{mix}}$ ). Let  $R \rightarrow S \rightarrow T$  and  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(S) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(T)$  be the respective protoproduct and mixed characteristic ultraproduct of the homomorphisms  $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}} \rightarrow T_p^{\text{mix}}$ . It follows from [Corollary 6.6](#) and [Theorems 4.4](#) and [6.5](#), that  $R$ ,  $S$  and  $T$  are local  $\mathfrak{D}$ -affine domains with  $R$  and  $T$  pseudoregular. By Łos’ Theorem, using that the  $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$  have bounded  $\mathfrak{D}_p^{\text{mix}}$ -complexity,  $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(S)$  is finite, whence so is  $R \rightarrow S$  by faithful flat descent. By [Theorem 11.2](#), the natural homomorphism  $\text{Tor}_n^R(S, M) \rightarrow \text{Tor}_n^R(T, M)$  is therefore zero.

By [Proposition 11.3](#), we can find a finite free resolution  $F_{p\bullet}^{\text{mix}}$  of  $M_p^{\text{mix}}$  up to level  $n$ , of  $\mathfrak{D}_p^{\text{mix}}$  complexity at most  $c'$ , for some  $c'$  only depending on  $c$  (note that  $n \leq c$ ). By definition of  $\text{Tor}$ , we have isomorphisms

$$\begin{aligned} \text{Tor}_n^{R_p^{\text{mix}}}(S_p^{\text{mix}}, M_p^{\text{mix}}) &\cong H_n(F_{p\bullet}^{\text{mix}} \otimes_{R_p^{\text{mix}}} S_p^{\text{mix}}) \\ \text{Tor}_n^{R_p^{\text{mix}}}(T_p^{\text{mix}}, M_p^{\text{mix}}) &\cong H_n(F_{p\bullet}^{\text{mix}} \otimes_{R_p^{\text{mix}}} T_p^{\text{mix}}) \end{aligned}$$

In particular, by [Proposition 11.3](#), both modules have  $\mathfrak{D}_p^{\text{mix}}$ -complexity at most  $c''$ , for some  $c''$  only depending on  $c'$ , whence only on  $c$ . Let  $H_S$  and  $H_T$  be their respective protoproduct, so that by Łos’ Theorem and our assumptions,  $H_S \rightarrow H_T$  is nonzero. Let  $F_\bullet$  be the protoproduct of the  $F_{p\bullet}^{\text{mix}}$ . By Łos’ Theorem and faithful flatness,  $H_S$  and  $H_T$  are isomorphic to  $H_n(F_\bullet \otimes_R S)$  and  $H_n(F_\bullet \otimes_R T)$  respectively. Since  $F_\bullet$  is a finite free resolution of  $M$  up to level  $n$  by another application of Łos’ Theorem and faithful flatness, these two modules are also isomorphic to  $\text{Tor}_n^R(S, M)$  and  $\text{Tor}_n^R(T, M)$  respectively. Hence the natural map between these two modules is nonzero, contradiction. □

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HANS SCHOUTENS  
DEPARTMENT OF MATHEMATICS  
CITY UNIVERSITY OF NEW YORK  
365 FIFTH AVENUE  
NEW YORK, NY 10016  
UNITED STATES

[hschoutens@citytech.cuny.edu](mailto:hschoutens@citytech.cuny.edu)

<http://websupport1.citytech.cuny.edu/Faculty/hschoutens>